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Properties of Connected Ortho-convex Sets in the Plane

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Abstract—Topological properties of connected ortho-convex sets in the plane, i.e., connected sets convex along the horizontal and vertical lines are studied. Several geometric statements concerning the ortho-separation of ortho-convex sets are proved.

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Let $\mathbb{R}^2$ be the Euclidean plane. A set $A \subset \mathbb{R}^2$ is said to be *ortho-convex* if the intersection of A with any horizontal or vertical line is empty or connected. Properties of ortho-convex sets in the plane were studied by many authors (see, e.g., the bibliography in [1] and [2]). In recent years, the attention of researchers has been attracted by topological and separation-type properties [1]–[3], which have turned out to be closely related to the theory of extremal problems. This paper studies the separation of two disjoint connected ortho-convex sets. We show that, under fairly natural constraints on these sets, they are separated by an "ortho-convex hyperplane." We also consider supporting properties of closed ortho-convex sets. The proof of the main results is preceded by a series of statements concerned with topological properties of connected ortho-convex sets.

Throughout, we use the following notation: $\mathbf{G}$ denotes the class of all ortho-convex sets in $\mathbb{R}^2$; $\mathbf{G^c}$ is the class of all connected ortho-convex sets in $\mathbb{R}^2$; $l_u(m_u)$ is the horizontal (vertical) straight line passing through a point u; $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$; cl, int, and fr are the operators of taking topological closure, interior, and boundary, respectively; $A_x(A_y)$ is the projection of a set A on the abscissa (ordinate) axis; for $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, the Δ_u^i are the closed quarter-planes determined by a point u and the lines l_u and m_u; $u\circ(\bullet)\!-\!\circ(\bullet)v$ is an undirected interval (open, half-open, or closed) joining points $u, v \in \mathbb{R}^2$ for which $u_y = v_y$, where a solid (empty) circle after the notation of the point of $\mathbb{R}^2$ indicates that this point (does not) belong to the interval (e.g., $u\circ\!-\!\circ v$ denotes an open interval); rays $l_u \cap \Delta_u^i$ and $l_u \cap \Delta_u^i \setminus \{u\}$ are denoted, like intervals, by $s\circ\!-\!\bullet u$ and $s\circ\!-\!\circ u$, respectively, where $s = (\pm\infty, u_y)$ and the sign in front of ∞ is determined by the quarter-plane Δ_u^i; if $u_y \neq v_y$, then we set $u\bullet\!-\!\bullet v = \varnothing$;

$$A\bullet\!-\!\bullet B := \{u\bullet\!-\!\bullet v \mid u \in A,\, v \in B,\, u_y = v_y\};$$

and the expression $A\bullet\!\not\!-\!\bullet B$ means that $A\bullet\!-\!\bullet B = \varnothing$. The symbols $\circ|\circ$ and $\bullet\nmid\bullet$ mean the same as $\circ\!-\!\circ$ and $\bullet\!\not\!-\!\bullet$, respectively, but with a horizontal interval replaced by a vertical one.

Using this notation, we can define an ortho-convex set A as

$$A \in \mathbf{G} \qquad \Longleftrightarrow \qquad \forall\, u, v\colon \quad (u_y = v_y \;\;\Rightarrow\;\; u\bullet\!-\!\bullet v \subset A) \wedge (u_x = v_x \;\;\Rightarrow\;\; u\bullet|\bullet v \subset A).$$

Following the terminology of [4], we say that a set $H \in \mathbf{G}$ is an *ortho-half-space* if its complement $\mathbb{R}^2 \setminus H$ belongs to $\mathbf{G}$ as well. To the boundary $\Gamma = \mathrm{fr}\, H$ of a half-space we refer to as an *ortho-hyperplane*.

The properties of ortho-convex sets given in the following two propositions are similar to those of sets in $\mathbb{R}^2$ convex in the ordinary sense.

Proposition 1. *If* $A \in \mathbf{G^c}$*, then* $\mathrm{cl}\, A \in \mathbf{G^c}$. *If* $A \in \mathbf{G}$*, then* $\mathrm{int}\, A \in \mathbf{G}$.

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Proof. First, we show, that the closure of $A \in \mathbf{G^c}$ is ortho-convex. Consider $u, v \in \operatorname{cl} A$ for which $u \bullet\!\!-\!\!\bullet v \neq \varnothing$. Obviously, it suffices to consider the case where $u \neq v$. Suppose that there exists an open disk $B_r = \{w \in \mathbb{R}^2 \mid \|w\| < r\}$ such that the sets

$$D^u = (u + B_r) \cap A, \qquad D^v = (v + B_r) \cap A$$

cannot be joined by a horizontal interval, i.e., $D^u \bullet\!\!\not\!-\!\!\bullet D^v$. We claim that this contradicts the connectedness of A.

Without loss of generality, we can assume that $(u + B_r) \cap (v + B_r) = \varnothing$ (this is so if $r \leq |u_x - v_x|/2$). Obviously, we have $D^u_y \cap D^v_y = \varnothing$. Since $D^u \bullet\!\!\not\!-\!\!\bullet D^v$, it follows that at least one of the points u and v, say v, does not belong to A. Consider the line l_u; clearly, $v \in l_u$. It divides the plane $\mathbb{R}^2$ into the two disjoint open half-planes π^+ and π^- above and below l_u, respectively. Consider the possible different locations of D^u and D^v with respect to the line l_u.

(1) Suppose that the sets D^u and D^v contain points on the same side of l_u, say in π^+, i.e.,

$$D^{u+} := D^u \cap \pi^+ \neq \varnothing \neq D^{v+} := D^v \cap \pi^+.$$

If $u \notin \operatorname{cl} D^{u+}$ or $v \notin \operatorname{cl} D^{v+}$, we can decrease the radius r so that one of the corresponding sets D^{u+} and D^{v+} becomes empty and, as a result, either we arrive at one of the situations analyzed in (2) and (3) or $u \in \operatorname{cl} D^{u-}$ and $v \in \operatorname{cl} D^{v-}$, i.e., the situation essentially coinciding with that considered below in the proof of (1) occurs. Thus, throughout the proof of (1), we assume that the points u and v are limit for D^{u+} and D^{v+}, respectively.

The sets D^u_y and D^v_y can be represented as disjoint unions of their connected components: $D^u_y = \bigcup_\alpha I^u_\alpha$, $D^v_y = \bigcup_\beta I^v_\beta$, and

$$I^u_\alpha \cap I^v_\beta = \varnothing$$

for all α and β. Clearly, the sets I^u_α and I^v_β are intervals of the vertical line and, therefore, can be ordered coordinatewise.

Let $J(\widetilde{y}) = D^u \cap l_{\widetilde{y}}$, where $\widetilde{y} \in D^u_y$; since D^u, it follows that the set $J(\widetilde{y})$, which is the intersection of the two ortho-convex sets $u + B_r$ and A, is an interval. We set $J(I) = \bigcup_{\widetilde{y} \in I} J(\widetilde{y})$ for $I \in \{I^u_\alpha, I^v_\beta\}$. Note that $J_x(I^u_{\alpha_1})$ and $J_x(I^u_{\alpha_2})$ with $\alpha_1 \neq \alpha_2$ are disjoint; indeed, otherwise, the components I^u_α would intersect (since D^u is ortho-convex, the existence of an $x \in J_x(I^u_{\alpha_1}) \cap J_x(I^u_{\alpha_2})$ would imply that of a vertical interval with abscissa x joining $J(I^u_{\alpha_1})$ with $J(I^u_{\alpha_2})$ and, hence, $I^u_{\alpha_1}$ with $I^u_{\alpha_2}$). Consider the set

$$\Phi = \{\alpha \mid J(I^u_\alpha) \cap m_u \setminus \{u\} \neq \varnothing\}.$$

It cannot contain more than one element, because if $(J(I^u_{\alpha'}) \bullet\!|\!\bullet J(I^u_{\alpha''})) \cap m_u \neq \varnothing$ for different $\alpha', \alpha'' \in \Phi$, then again $I^u_{\alpha_1}$ and $I^u_{\alpha_2}$ must intersect. Taking into account this property of Φ, we choose a radius of B_r so that $\Phi = \varnothing$. This ensures that any set $J(I^u_\alpha)$ above the line l_u is contained entirely in one of the open quarter-planes I or II with respect to u, i.e., on the right or on the left of the line m_u. Moreover, since $u \in \operatorname{cl} D^{u+}$, it follows that $u \in \operatorname{cl}(\bigcup_\alpha J(I^u_\alpha))$ for all sets $J(I^u_\alpha)$ contained in some of these open quarter-planes, say in quarter-plane I. Thanks to this observation, in case (1), it suffices to study only those sets I^u_α for which the corresponding sets $J(I^u_\alpha)$ are contained in quarter-plane I.

Now, consider any points $\overline{y}_1, \overline{y}_2 \in D^{v+}_y$ satisfying the conditions

$$\overline{y}_1 < \overline{y}_2, \qquad \exists\, \overline{I}^{\,u}_{\alpha_2}\colon\ \ \overline{y}_2 < \overline{I}^{\,u}_{\alpha_2}, \qquad \exists\, \overline{I}^{\,u}_{\alpha_1} \subset (\overline{y}_1, \overline{y}_2).$$

Such points exist, because, otherwise, the conditions $u \in \operatorname{cl}(\bigcup_\alpha J(I^u_\alpha))$ and $v \in \operatorname{cl} D^{v+}$ are violated. Let $\pi_{\overline{y}_1}$ denote the open upper half-plane determined by the line $l_{\overline{y}_1}$, and let $\pi_{\overline{y}_2}$ denote the open lower half-plane determined by $l_{\overline{y}_2}$. Obviously, we have $J(\overline{I}^{\,u}_{\alpha_1}) \subset \pi_{\overline{y}_1} \cap \pi_{\overline{y}_2}$.

Take any point $\overline{x}_2$ of $J_x(\overline{I}^{\,u}_{\alpha_2})$. By virtue of the observation made above, the set $J(\overline{I}^{\,u}_{\alpha_2})$ is contained entirely in open quarter-plane I with respect to u; hence $\overline{x}_2 < u_x$. The point $(\overline{x}_2, \overline{y}_2) \in \mathbb{R}^2$ belongs to $u + B_r$, i.e.,

$$\|(\overline{x}_2, \overline{y}_2) - u\| < r.$$

Indeed, if $\|(\overline{x}_2, \overline{y}_2) - u\| \geq r$, then there exists a $t \in J(\overline{I}^u_{\alpha_2})$ for which $t_x = \overline{x}_2, t_y > \overline{y}_2$ and, hence, $\|t - y\| \geq r$; however, $J(\overline{I}^u_{\alpha_2}) \subset u + B_r$. Moreover, this implies the inclusion

$$[u_x, \overline{x}_2] \times [u_y, \overline{y}_2] \subset u + B_r.$$

Consider the open left half-plane $\pi_{\overline{x}_2}$ determined by the line $m_{\overline{x}_2}$. Since $u \in \operatorname{cl}(\bigcup_\alpha J(I^u_\alpha))$ and $v \in \operatorname{cl} D^{v+}$, it follows that

$$\exists\, \overline{I}^u_{\alpha_1} \subset [\overline{y}_1, \overline{y}_2]: \quad J(\overline{I}^u_{\alpha_1}) \subset \pi_{\overline{y}_1} \cap \pi_{\overline{y}_2} \cap \pi_{\overline{x}_2}$$

if we choose sufficiently small $\overline{y}_1$. Next, take any point $\overline{x}_1 > u_x$ of a set $J_x(\overline{I}^u_{\alpha_3}) < \overline{x}_2$ for which $\overline{I}^u_{\alpha_3} \subset (u_y, \overline{y}_1)$. The existence of such a point $\overline{x}_1$ is again ensured by the conditions $u \in \operatorname{cl}(\bigcup_\alpha J(I^u_\alpha))$ and $v \in \operatorname{cl} D^{v+}$. The same conditions ensure also the existence of a set $J(\overline{I}^u_{\alpha_1})$ such that $\overline{I}^u_{\alpha_1} \subset (\overline{y}_1, \overline{y}_2)$ and $J_x(\overline{I}^u_{\alpha_1}) \subset (\overline{x}_1, \overline{x}_2)$, provided that $\overline{y}_1$ is sufficiently small. Let $\pi_{\overline{x}_1}$ be the open right half-plane determined by $m_{\overline{x}_1}$. We have

$$J(\overline{I}^u_{\alpha_1}) \subset \pi_{\overline{x}_1} \cap \pi_{\overline{y}_1} \cap \pi_{\overline{x}_2} \cap \pi_{\overline{y}_2} \subset u + B_r,$$

and $J(\overline{I}^u_{\alpha_1})$ is entirely contained in quarter-plane I.

We set

$$C = \pi_{\overline{x}_1} \cap \pi_{\overline{y}_1} \cap \pi_{\overline{x}_2} \cap \pi_{\overline{y}_2}, \qquad M = A \cap C, \qquad N = A \setminus C.$$

The sets M and N are open and closed in A. Indeed, let $s \in \operatorname{cl} M$; then either $s \in \operatorname{int} C = C$ and, therefore, $s \notin N$, or $s \in \operatorname{fr} C$. Obviously, in the latter case, it suffices to consider two cases:

(a) $s \in l_{\overline{y}_2}$, in which case $s_x \in [u_x, \overline{x}_2]$ and $s_y = \overline{y}_2$;

(b) $s \in m_{\overline{x}_2}$, in which case $s_x = \overline{x}_2$ and $s_y \in [u_y, \overline{y}_2]$.

Consider each of these cases in more detail.

Suppose that case (a) occurs and $s \in \operatorname{cl} M \cap N$. Then $s \in ([u_x, \overline{x}_2] \times [u_y, \overline{y}_2]) \cap A$, whence, by virtue of the remarks made above, we obtain $s \in (u + B_r) \cap A = D^u$. Thus, $\overline{y}_2 \in D^u_y \cap D^v_y$, which contradicts the requirement $D^u \bullet\!\!-\!\!/\!\!-\!\!\bullet D^v$, and, therefore, $s \notin N$. In case (b), we have $s \notin N$ from the following considerations. Suppose that $s \in N$, which again means that $s \in D^u$ and, by construction, s lies on the same vertical line as some of the points $t \in J(\overline{I}^u_{\alpha_2}) \subset D^u$, for which $t_x = \overline{x}_2$ and $t_y > \overline{y}_2$; then $(s\bullet|\bullet t) \cap l_{\overline{y}_2} \neq \varnothing$, and the ortho-convexity D^u implies $((s\bullet|\bullet t) \cap l_{\overline{y}_2})_y = \overline{y}_2 \in D^u_y$, so that the condition $D^u \bullet\!\!-\!\!/\!\!-\!\!\bullet D^v$ is again violated. Thus, $s \in \operatorname{cl} M$ implies $s \notin N$; precisely the same argument shows that $s \in \operatorname{cl} N$ implies $s \notin M$.

As a result, we obtain a contradiction: the set A is disconnected, although it is assumed to be connected in the proposition being proved. Therefore, for any disk B_r, we have $D^u \bullet\!-\!\bullet D^v$. Consider two sequences $\{u_n\}$ and $\{v_n\}$ satisfying the conditions

$$u_n \to u, \qquad v_n \to v, \qquad u_n \in D^u, \qquad v_n \in D^v, \qquad u_n \bullet\!-\!\bullet v_n \neq \varnothing.$$

In each of the intervals $u_n \bullet\!-\!\bullet v_n$, we take a point $w_n = \lambda u_n + (1 - \lambda) v_n \in A$, $\lambda \in (0, 1)$. Obviously,

$$w_n \to w = \lambda u + (1 - \lambda) v \in \operatorname{cl} A.$$

Thus, $u, v \in \operatorname{cl} A$, $u \bullet\!-\!\bullet v \neq \varnothing$ implies the inclusion $u \bullet\!-\!\bullet v \subset \operatorname{cl} A$.

(2) Suppose that the sets $D^u \setminus l_u$ and $D^v \setminus l_u$ are on opposite sides of the line l_u. To be specific, we assume that $D^{u+} \neq \varnothing \neq D^{v-}$. Then, obviously, $D^{u-} \cup D^{v+} = \varnothing$. Here, as in case (1), we either assume that the points u and v are limit for D^{u+} and D^{v-}, respectively, or arrive at a situation analyzed in (3).

Since the set A is connected, it follows that some of its intersection points with $l_u \setminus \{u\}$ must lie outside the disks $u + B_r$ and $v + B_r$. (Indeed, otherwise, A can be partitioned into two disjoint components A_1 and A_2:

$$A = A_1 \cup A_2, \qquad A_1 \subset \pi^+ \cup U(u), \qquad A_2 \subset \pi^- \setminus \operatorname{cl} U(u),$$

where $U(u)$ is a neighborhood (not necessarily circular) of u satisfying the conditions

$$D^u \subset U(u) \subset u + B_r \qquad \text{and} \qquad A \cap \pi^- \cap \operatorname{cl} U(u) = \varnothing).$$

Consider the possible locations of these points on the line l_u with respect to u and v; without the loss of generality, we assume that $u_x < v_x$.

(2) A. Suppose that there exists a point $e \in A \cap l_u$ outside the interval $u{\bullet\!-\!\bullet}v$ and the disk $u + B_r$ on the same side as u. This point e is nonisolated, because A is disconnected. Suppose that $e \in \operatorname{cl}(A \cap \pi^-)$. Then we can apply the argument of (1) to $D^e = A \cap (e + B_r)$ and $D^v = A \cap (v + B_r)$, which gives $e{\bullet\!-\!\bullet}v \subset \operatorname{cl} A$; moreover, since $u \in e{\bullet\!-\!\bullet}v$, it follows that $u{\bullet\!-\!\bullet}v \subset \operatorname{cl} A$. If a point with the properties specified above does not exist, i.e., $(s{\circ\!-\!\circ}u) \cap A \cap \operatorname{cl} A \cap \pi^- = \varnothing$ for $s = (-\infty, u_y)$, then either there exists a point in $A \cap l_u \setminus (s{\circ\!-\!\bullet}u)$ and case (2) B or (2) C occurs, or the set A is again disconnected:

$$A = A_1 \cup A_2, \qquad A_1 \subset \pi^+ \cup \Bigg(\bigcup_{\substack{e \in A \cap l_u,\\ e_x \le u_x}} U(e)\Bigg), \qquad A_2 \subset \pi^- \setminus \operatorname{cl}\Bigg(\bigcup_{\substack{e \in A \cap l_u,\\ e_x \le u_x}} U(e)\Bigg),$$

where the $U(e)$ are neighborhoods of the points e with $e_x \le u_x$ satisfying the condition

$$A \cap \pi^- \cap \operatorname{cl} U(e) = \varnothing$$

and $D^u \subset U(u) \subset u + B_r$.

(2) B. Suppose that there exists a point $e \in A \cap l_u$ on the right of v outside the disk $v + B_r$. As in case (2) A, we assume that $e \in \operatorname{cl}(A \cap \pi^+)$ and again show that $u{\bullet\!-\!\bullet}v \subset \operatorname{cl} A$. If the required points exist is none of the cases (2) A and (2) B, then either there is a point in $A \cap l_u \setminus ((s^-{\circ\!-\!\bullet}u) \cup (v{\bullet\!-\!\circ}s^+))$ for $s^\pm = (\pm\infty, u_y)$ and case (2) C occurs or the set A is disconnected:

$$A = A_1 \cup A_2,$$

$$A_1 \subset \pi^+ \cup \Bigg(\bigcup_{\substack{e \in A \cap l_u,\\ e_x \le u_x}} U(e)\Bigg) \setminus \operatorname{cl}\Bigg(\bigcup_{\substack{e \in A \cap l_u,\\ e_x \ge v_x}} U(e)\Bigg),$$

$$A_2 \subset \pi^- \cup \Bigg(\bigcup_{\substack{e \in A \cap l_u,\\ e_x \ge v_x}} U(e)\Bigg) \setminus \operatorname{cl}\Bigg(\bigcup_{\substack{e \in A \cap l_u,\\ e_x \le u_x}} U(e)\Bigg),$$

where the $U(e)$ are neighborhoods of points e chosen as in case (2) A if $e_x \le u_x$ and satisfying the condition

$$A \cap \pi^+ \cap \operatorname{cl} U(e) = \varnothing$$

if $e_x \ge v_x$ and $D^v \subset U(u) \subset v + B_r$.

(2) C. Suppose that there exists a point $e \in E := (u{\circ\!-\!\circ}v) \cap A$ outside the disks $u + B_r$ and $v + B_r$. If

$$\exists\, e_1' \in E \cap \operatorname{cl}(A \cap \pi^+) \qquad \text{and} \qquad \exists\, e_2' \in E \cap \operatorname{cl}(A \cap \pi^-),$$

then, applying the argument of (1) to the pairs u, e_1' and e_2', v and taking into account the inclusion $e_1'{\bullet\!-\!\bullet}e_2' \subset A$, we obtain $u{\bullet\!-\!\bullet}e_1' \subset \operatorname{cl} A$ and $e_2'{\bullet\!-\!\bullet}v \subset \operatorname{cl} A$, whence $u{\bullet\!-\!\bullet}v \subset \operatorname{cl} A$.

Now, consider the case where points with the properties specified in (2) A–C do not exist. As applied to (2) C, we can write

$$E \cap \operatorname{cl}(A \cap \pi^+) = \varnothing \qquad \text{or} \qquad E \cap \operatorname{cl}(A \cap \pi^-) = \varnothing.$$

Suppose, e.g., that $E \cap \operatorname{cl}(A \cap \pi^-) = \varnothing$. Then, as in cases (2) A and (2) B, we obtain a contradiction: the set A is disconnected, because

$$A = A_1 \cup A_2,$$

$$A_1 \subset \pi^+ \cup \Bigg(\bigcup_{\substack{e\in A\cap l_u,\\ e_x<\overline{e}_x}} U(e)\Bigg) \setminus \mathrm{cl}\Bigg(\bigcup_{\substack{e\in A\cap l_u,\\ e_x>\overline{e}_x}} U(e)\Bigg),$$

$$A_2 \subset \pi^- \cup \Bigg(\bigcup_{\substack{e\in A\cap l_u,\\ e_x>\overline{e}_x}} U(e)\Bigg) \setminus \mathrm{cl}\Bigg(\bigcup_{\substack{e\in A\cap l_u,\\ e_x<\overline{e}_x}} U(e)\Bigg),$$

where $\overline{e} = (\sup E_x, E_y)$ and the neighborhoods $U(e)$ are chosen so that

$$U(e)\cap\{\overline{e}\}=\varnothing, \quad A\cap\pi^-\cap \mathrm{cl}\,U(e)=\varnothing \quad \text{if } e_x<\overline{e}_x, \qquad A\cap\pi^+\cap \mathrm{cl}\,U(e)=\varnothing \quad \text{if } e_x>\overline{e}_x.$$

(3) It remains to consider the case where one of the sets D^u and D^v lies entirely on the line l_u; to be specific, suppose that $D^u \subset l_u$. Then, by virtue of the assumption $D^u \bullet\!\!\not\!-\!\!\bullet D^v$, we can assert that $D^v \cap l_u = \varnothing$, i.e., $D^v = D^{v+}\cup D^{v-} \neq \varnothing$. If the set $\{e \mid e \in A\cap l_u,\ e_x \le v_x\}$ contains a point e' belonging to $\mathrm{cl}(A\setminus l_u)$, then, applying the argument used in (1) and (2) to the points e' and v and taking into account the inclusion $u\bullet\!-\!\bullet e' \subset \mathrm{cl}\,A$, we obtain $u\bullet\!-\!\bullet v \subset \mathrm{cl}\,A$. The situation in which there is no point e' with the required properties cannot occur, because, in this situation, the set A is again disconnected, which is a contradiction:

$$A = A_1\cup A_2, \qquad A_1 \subset \bigcup_{\substack{e\in A\cap l_u,\\ e_x<\overline{e}_x}} U(e), \qquad A_2 \subset \mathbb{R}^2 \setminus \mathrm{cl}\Bigg(\bigcup_{\substack{e\in A\cap l_u,\\ e_x<\overline{e}_x}} U(e)\Bigg),$$

where $\overline{e} = (\sup\{e_x \mid e\in l_u\cap A,\ e_x\le v_x\},\ u_y)$, and the neighborhoods $U(e)$ are chosen so that $U(e)\cap\{v\}=\varnothing$ and $(A\setminus l_u)\cap \mathrm{cl}\,U(e)=\varnothing$.

Summarizing the study performed in (1)–(3), we conclude that the set $\mathrm{cl}\,A$ is horizontally convex. Similar considerations show that $\mathrm{cl}\,A$ is also vertically convex; therefore, this set is ortho-convex. Since the connectedness of any set is preserved by the closure operation, it follows that $\mathrm{cl}\,A$ is connected.

It remains to prove that the interior of each set $A\in\mathbf{G}$ is ortho-convex. Take any two points $u, v\in \mathrm{int}\,A$ for which $u\bullet\!-\!\bullet v\neq\varnothing$. There exists a disk B such that $u+B, v+B \in \mathrm{int}\,A$. Let w be any point in the interval $u\bullet\!-\!\bullet v$; then the disk $w+B$ is entirely contained in $\mathrm{int}\,A$. Indeed, if $w=\lambda u+(1-\lambda)v$ for $\lambda\in(0,1)$ and $w'\in w+B$, then the relations

$$u' = u+(w'-w)\in u+B, \qquad v' = v+(w'-w)\in v+B$$

imply the chain of equalities

$$w' = u'+(w-u) = u'+(1-\lambda)(v-u) = u'+(1-\lambda)(v'-u') = \lambda u'+(1-\lambda)v',$$

but since A is ortho-convex and $u'\bullet\!-\!\bullet v'\neq\varnothing$, it follows that $w'\in A$. Therefore, the set $\mathrm{int}\,A$ is horizontally convex. A similar argument proves that it is also vertically convex, which implies its ortho-convexity. This completes the proof of the proposition. □

Proposition 2. *Let $H\neq\varnothing$ be the ortho-half-space determined by an ortho-hyperplane $\Gamma\neq\varnothing$. Then $\mathrm{int}\,H\neq\varnothing$, $\mathrm{int}\,\Gamma=\varnothing$, and $\Gamma\in\mathbf{G}$. If, in addition, $H\in\mathbf{G^c}$ and $\mathbb{R}^2\setminus H\in\mathbf{G^c}$, then $\Gamma\in\mathbf{G^c}$.*

Proof. First, we show that the given half-space has nonempty interior; we argue by contradiction. Suppose that the half-space H has no interior points. Take any point $w\in H$; for any open disk B centered at zero, we have

$$(w+B)\cap H\neq\varnothing\neq(w+B)\cap(\mathbb{R}^2\setminus H).$$

Since H is ortho-convex, it follows that the line l_w intersects $(w+B)\cap H$ in a horizontal interval with endpoints u and v, which do not necessarily belong to H. Consider the three possible locations of the point w with respect to this interval.

Case 1: $w=u=v$, i.e., $(w+B)\cap H\cap l_w=\{w\}$. In this case, $(w+B)\cap l_w\setminus\{w\}\subset\mathbb{R}^2\setminus H$. Consider two points $s,t\in(w+B)\cap l_w$ on opposite sides of w. They belong to the set $\mathbb{R}^2\setminus H$, and the ortho-convexity of this set implies $s\bullet\!-\!\bullet t\subset\mathbb{R}^2\setminus H$.

Since $w \in s\bullet\!\!-\!\!\bullet t$, it follows that $w \in \mathbb{R}^2 \setminus H$, which is a contradiction.

Case 2: $w \in u\circ\!\!-\!\!\circ v$. Let us choose the radius of the disk B so small that $w + B$ does not contain u and v. Consider the line m_w; it again intersects the set $(w + B) \cap H$ in an interval with endpoints p and q, which is vertical this time. We again consider several subcases.

Case 2, A: $w = p = q$. As in Case 1, we obtain a contradiction.

Case 2, B: $w \in p\circ|\circ q$. We again decrease the radius of B, this time so that $w + B$ does not contain p and q. Next, we take any point $z \in (w + B) \cap m_w \setminus \{w\}$ and consider the line l_z. It intersects $(w + B) \cap H$ in an interval with endpoints u' and v'. If $z = u' = v'$, then

$$(w + B) \cap l_z \setminus \{z\} \subset \mathbb{R}^2 \setminus H;$$

taking two points $s, t \in (w + B) \cap l_z \setminus \{z\}$ on opposite sides of z, we obtain a contradiction: $z \in H$ and $z \in \mathbb{R}^2 \setminus H$. If $z \in u'\bullet\!\!-\!\!\bullet v'$ and $u' \neq v'$, then the points of $u'\circ\!\!-\!\!\circ v'$ can be joined by vertical intervals with the corresponding points of l_w, because H is ortho-convex; these intervals form a set (rectangle) with nonempty interior contained in H. However, this contradicts the assumption $\operatorname{int} H = \varnothing$.

Case 2, C: $w \in \{p, q\}$, $p \neq q$. Without loss of generality, we assume that $w = q$. Applying an argument similar to that used in Case 2, B to any point $z \in p\circ|\circ w$, we again obtain a contradiction.

Case 3: $w \in \{u, v\}$, $u \neq v$. Without loss of generality, we assume that $w = u$. Instead of w, take any point $w' \in w\circ\!\!-\!\!\circ v$. Obviously, the argument used in Case 2 applies to w' and again leads to a contradiction.

Thus, any ortho-half-space must have interior points.

The fact that the interior of the ortho-hyperplane Γ is empty readily follows from the definition of Γ: $\Gamma = \operatorname{fr} H$. The ortho-convexity of Γ follows from the ortho-convexity of the sets $\operatorname{cl} H$ and $\operatorname{cl}(\mathbb{R}^2 \setminus H)$, which, in turn, follows from Proposition 1, the relation $\Gamma = \operatorname{cl} H \cap \operatorname{cl}(\mathbb{R}^2 \setminus H)$, and the fact that the intersection of ortho-convex sets is ortho-convex.

It remains to prove the last assertion of the proposition, namely, that Γ is connected provided that so are the ortho-half-spaces determining it. Note that the plane $\mathbb{R}^2$ is contractible with respect to the circle; therefore, given any two closed connected sets E and F such that $\mathbb{R}^2 = E \cup F$, the intersection $E \cap F$ is connected as well [5]. Therefore, the relations

$$\mathbb{R}^2 = \operatorname{cl} H \cup \operatorname{cl}(\mathbb{R}^2 \setminus H) \qquad \text{and} \qquad \Gamma = \operatorname{cl} H \cap \operatorname{cl}(\mathbb{R}^2 \setminus H),$$

and the connectedness of $\operatorname{cl} H$ and $\operatorname{cl}(\mathbb{R}^2 \setminus H)$, which follows from Proposition 1, imply the connectedness of the ortho-hyperplane Γ. This completes the proof of the proposition. □

Remark. Our proof of the first assertion of Proposition 2 does not use the connectedness of the set H or $\mathbb{R}^2 \setminus H$. At the same time, simple examples show that the connectedness of an ortho-half-space does not imply that of its complement. We refer to ortho-half-spaces and ortho-hyperplanes satisfying the assumptions of the last assertion of Proposition 2 as *proper* ones. The importance of this definition is demonstrated by Propositions 6 and 7. It is easy to see that improper ortho-hyperplanes can be disconnected.

In what follows, we say that a map $t \overset{\gamma}{\longmapsto} (x(t), y(t))$ of a connected set $C \subset \mathbb{R}$ to $\mathbb{R}^2$ determines a continuous monotone curve γ, if γ is one-to-one and the functions $x(t)$ and $y(t)$ are continuous and monotone. By an abuse of language, we refer to the image of $\gamma(C)$ as a *continuous monotone curve*, too. If $C = [\alpha, \beta]$, then we say that the curve γ is *compact*.

In what follows, we use the following construction. Suppose that a point u and its closed quarter-plane Δ_u^j, $j \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, are fixed and either a point $s \in l_u \cap \Delta_u^j \setminus \{u\}$, a ray $s\circ\!\!-\!\!\bullet u \subset \Delta_u^j$ and a point $t \in m_u \cap \Delta_u^j \setminus \{u\}$, or a ray $t\circ|\bullet u \subset \Delta_u^j$ is chosen. (Recall that, in the case of rays, $s = (\pm\infty, u_y)$ and $t = (u_x, \pm\infty)$, where the sign in front of ∞ is determined by the quarter-plane Δ_u^j). We refer to the figure

$$T = \gamma^i \cup (s\circ(\bullet)\!\!-\!\!\bullet u) \cup (t\circ(\bullet)|\bullet u)$$

formed by the intervals $s\circ(\bullet)\!-\!\bullet u$ and $t\circ(\bullet)|\bullet u$ and a continuous monotone curve γ^i for which $C \cap \operatorname{fr} C \neq \varnothing$ with endpoint $u \in \gamma^i(C \cap \operatorname{fr} C)$ contained in the closed quarter-plane

$$\Delta_u^i,\ i = j + 2(-1)^{[j/3]} \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\},$$

opposite to Δ_u^j as a *tripod with node* u. (Here the sign after s and t is always $\circ$ in the case of rays, and $[\cdot]$ denotes the integer part of a number.)

Propositions 3 and 4 presented below describe the structure of connected ortho-convex sets with empty interior.

Proposition 3. *If $A \in \mathbf{G}^{\mathrm{c}}$ is a compact set with empty interior, then it has one of the following types*:

i) *a compact continuous monotone curve*;

ii) *a union of at most two tripods of the form described above.*

Proof. Since the set A is compact, it can be included in a rectangle; let Δ be the minimal rectangle containing A. (Here and in what follows, by a rectangle we mean the closed convex hull of points $a, b, c, d \in \mathbb{R}^2$ satisfying the conditions $a\bullet\!-\!\bullet b \neq \varnothing$, $c\bullet\!-\!\bullet d \neq \varnothing$, $a\bullet|\bullet d \neq \varnothing$, and $b\bullet|\bullet c \neq \varnothing$.) If $\operatorname{int} \Delta = \varnothing$, then A is an interval and, therefore, is of type (i). Suppose that $\operatorname{int} \Delta \neq \varnothing$ and take any point $u \in A$. The set A is closed and connected; hence any neighborhood of u contains infinitely many points of A. The lines l_u and m_u divide the rectangle Δ into four parts according to quarters; we denote these parts minus l_u and m_u by $\widetilde{\Delta}_u^i$, $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ ($\widetilde{\Delta}_u^i := \Delta \cap \Delta_u^i \setminus (l_u \cup m_u)$). Consider the possible intersections of these lines with A and the location of the point u with respect to these intersections.

Case 1: $A \cap l_u = A \cap m_u = \{u\}$, $u \in A \cap \operatorname{int} \Delta$. In this case, some of the rectangles $\widetilde{\Delta}_u^i$, $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, contains infinitely many points of A arbitrarily close to u. To be specific, suppose that this is the rectangle $\widetilde{\Delta}_u^{\mathrm{II}}$, i.e., $u \in \operatorname{cl} D^{\mathrm{II}}$, where $D^i := A \cap \widetilde{\Delta}_u^i$, $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$. Let us show that the rectangles $\widetilde{\Delta}_u^{\mathrm{I}}$ and $\widetilde{\Delta}_u^{\mathrm{III}}$ contain no points of A. Consider two cases.

Case 1, A: the rectangle $\widetilde{\Delta}_u^{\mathrm{I}}$ contains infinite many points of A arbitrarily close to u ($u \in \operatorname{cl} D^{\mathrm{I}}$). Suppose that $D^{\mathrm{I}}\bullet\!-\!\bullet D^{\mathrm{II}} \neq \varnothing$, i.e., there exists a $t \in D^{\mathrm{I}}$ and an $s \in D^{\mathrm{II}}$ such that $s\bullet\!-\!\bullet t \subset A$. Then

$$(s\bullet\!-\!\bullet t) \cap m_u = \{w\} \subset A;$$

by construction, we have $w \neq u$. At the same time, $A \cap m_u = \{u\}$. This contradiction shows that $D^{\mathrm{I}}\bullet\!\not\!-\!\bullet D^{\mathrm{II}}$. Applying the argument used in the proof of Proposition 1, case (1), to the sets D^{I} and D^{II}, we see that A is disconnected. Therefore, this case cannot occur.

Case 1, B: the point u has a neighborhood disjoint from the rectangle $\widetilde{\Delta}_u^{\mathrm{I}}$ and the set A ($u \notin \operatorname{cl} D^{\mathrm{I}}$). Without loss of generality, we can assume that the rectangle $\Delta \cap \Delta_u^{\mathrm{I}}$ contains a closed rectangle with sides parallel to l_u and m_u and vertex u which intersects A only in u. (We have taken into account the relations $A \cap l_u = A \cap m_u = \{u\}$.) Let us denote this rectangle by $\widetilde{\Delta}$. Consider the open half-planes π_{l_u} and π_{m_u} above l_u and on the right of m_u, respectively. Suppose that the rectangle $\widetilde{\Delta}_u^{\mathrm{I}}$ intersects A. We set

$$C = \pi_{l_u} \cap \pi_{m_u} \setminus \widetilde{\Delta}, \qquad M = A \cap C, \qquad N = A \setminus C.$$

Obviously, M is open and closed in A, and so is N, because $N \subset A \setminus \operatorname{cl} C$. Therefore, A is disconnected, which eliminates this case, too.

A similar argument proves that $A \cap \widetilde{\Delta}_u^{\mathrm{III}} = \varnothing$. Since $A \cap l_u = A \cap m_u = \{u\}$ and $u \in \operatorname{int} \Delta$, it follows that $A \cap \widetilde{\Delta}_u^{\mathrm{IV}} \neq \varnothing$, because otherwise $(\widetilde{\Delta}_u^{\mathrm{III}} \cup \widetilde{\Delta}_u^{\mathrm{IV}}) \cap A = \varnothing$ and the rectangle Δ is not minimal.

Thus, we have obtained the inclusion $A \subset \widetilde{\Delta}_u^{\mathrm{II}} \cup \widetilde{\Delta}_u^{\mathrm{IV}} \cup \{u\}$; moreover,

$$A \cap \widetilde{\Delta}_u^{\mathrm{II}} \neq \varnothing \neq A \cap \widetilde{\Delta}_u^{\mathrm{IV}}.$$

Case 2: $A \cap l_u = A \cap m_u = \{u\}$, $u \in A \cap \operatorname{fr}\Delta$. It is easy to show that, in this case, the point u must be a vertex of Δ. This is proved by analogy with the preceding case, where two of the four rectangles with respect to u were eliminated and the remaining two adjacent rectangles were analyzed, after which it was shown that only one rectangle can intersect A.

Case 3: $(A \cap l_u) \cup (A \cap m_u) \neq \{u\}$. Without loss of generality, taking into account the ortho-convexity of A, we can assume that $A \cap l_u = s\bullet\!-\!\bullet t$, $s \neq t$. Consider the possible locations of the point u in the interval $s\bullet\!-\!\bullet t$.

Case 3, A1: $u \in s\circ\!-\!\circ t$, $A \cap m_u \neq \{u\}$. Since A is ortho-convex, it follows that $A \cap m_u = v\bullet\!|\!\bullet w$, where $v \notin l_u$ or $w \notin l_u$. Without loss of generality, we assume that $s_x < t_x$ and $v_y < u_y$. We claim that $v \in \operatorname{fr}\Delta$. Suppose that, on the contrary, $v \in \operatorname{int}\Delta$. Consider the rectangles $\Delta_u^{\mathrm{III}} \cap \Delta^i_{l_v \cap m_s}$, $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$. We have

$$\Delta_u^{\mathrm{III}} \cap \Delta^{\mathrm{I}}_{l_v \cap m_s} \cap A = (s\bullet\!-\!\bullet u) \cup (u\bullet\!|\!\bullet v), \qquad \Delta_u^{\mathrm{III}} \cap \Delta^{\mathrm{II}}_{l_v \cap m_s} \cap A = \{s\},$$
$$\Delta_u^{\mathrm{III}} \cap \Delta^{\mathrm{III}}_{l_v \cap m_s} \cap A = \varnothing, \qquad \Delta_u^{\mathrm{III}} \cap \Delta^{\mathrm{IV}}_{l_v \cap m_s} \cap A = \{v\}.$$

Indeed, if the first relation is false, then the condition $\operatorname{int} A = \varnothing$ is violated; if the second or fourth one is false, then either the condition $\operatorname{int} A = \varnothing$ or one of the conditions $A \cap l_u = s\bullet\!-\!\bullet t$ and $A \cap m_u = v\bullet\!|\!\bullet w$, respectively, is violated; and if the third relation is false, then either the condition $\operatorname{int} A = \varnothing$ is violated or the set A is disconnected: for a set C separating A we can take, e.g., $C = \Delta \cap \Delta^{\mathrm{III}}_{l_v \cap m_s}$. Similar considerations apply to the rectangles $\Delta_u^{\mathrm{IV}} \cap \Delta^i_{l_v \cap m_t}$, $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$. It follows that the set

$$\Delta \cap \biggl(\bigcup_{\substack{i \in \{\mathrm{III}, \mathrm{IV}\},\\ a \in \{s,t\}}} \Delta^i_{l_v \cap m_a}\biggr) \setminus l_v,$$

which coincides with the rectangle lying in Δ strictly below l_v, is disjoint from A. However, this contradicts the minimality of Δ; therefore, $v \in \operatorname{fr}\Delta$.

Note that, as a byproduct, we have proved the relation $A \cap (\widetilde{\Delta}_u^{\mathrm{III}} \cup \widetilde{\Delta}_u^{\mathrm{IV}}) = \varnothing$. It remains to determine the location of A with respect to $\widetilde{\Delta}_u^{\mathrm{I}}$ and $\widetilde{\Delta}_u^{\mathrm{II}}$. Suppose that the rectangle $\widetilde{\Delta}_u^{\mathrm{II}}$ contains infinitely many points of A. Then, clearly, $\widetilde{\Delta}_u^{\mathrm{I}} \cap A = \varnothing$. Moreover, recalling that $u \in s\circ\!-\!\circ t$, we see that this relation can be strengthened:

$$A \cap \Delta_u^{\mathrm{I}} \setminus (s\bullet\!-\!\bullet t) = \varnothing.$$

Since $A \cap (\widetilde{\Delta}_u^{\mathrm{I}} \cup \widetilde{\Delta}_u^{\mathrm{IV}}) = \varnothing$ and $t \neq u \neq v$, it follows from the same considerations as above that $t \in \operatorname{fr}\Delta$, which implies the inclusion $A \subset \widetilde{\Delta}_u^{\mathrm{II}} \cup (s\bullet\!-\!\bullet t) \cup (u\bullet\!|\!\bullet v)$, in which $A \subset \widetilde{\Delta}_u^{\mathrm{II}} \neq \varnothing$. The set $A \cap \widetilde{\Delta}_u^{\mathrm{II}}$ cannot be finite, because if it is, then A is disconnected. The situation where $A \cap \widetilde{\Delta}_u^{\mathrm{I}} \neq \varnothing$ is considered in a similar way.

Now, suppose that $(\widetilde{\Delta}_u^{\mathrm{I}} \cup \widetilde{\Delta}_u^{\mathrm{II}}) \cap A = \varnothing$. Then $w \in \operatorname{fr}\Delta$, because otherwise the rectangle Δ is not minimal. By assumption, we have $u \in s\circ\!-\!\circ t$; hence $v, w \in \operatorname{fr}\Delta$, the minimality of Δ, and the relations $\widetilde{\Delta}_u^i \cap A = \varnothing$, $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, imply $A = (s\bullet\!-\!\bullet t) \cup (v\bullet\!|\!\bullet w)$ (i.e., $s, t, v, w \in \operatorname{fr}\Delta$).

Case 3, A2: $u \in s\circ\!-\!\circ t$, $A \cap m_u = \{u\}$. Obviously, in this case, the location of A with respect to the point u is characterized by its location with respect to the points of the set $s\bullet\!-\!\circ u\circ\!-\!\bullet t$. First, suppose that two rectangles adjacent along l_u, say $\widetilde{\Delta}_u^{\mathrm{II}}$ and $\widetilde{\Delta}_u^{\mathrm{III}}$, satisfy the condition $A \cap \widetilde{\Delta}_u^{\mathrm{II}} \neq \varnothing \neq A \cap \widetilde{\Delta}_u^{\mathrm{III}}$. Then, by analogy with Case 1, we have $A \cap (\widetilde{\Delta}_u^{\mathrm{I}} \cup \widetilde{\Delta}_u^{\mathrm{IV}}) = \varnothing$, and there exists a $v \in s\bullet\!-\!\circ u$: $A \cap m_v \neq \{v\}$; indeed, otherwise, we have $A \cap (\widetilde{\Delta}_u^{\mathrm{II}} \cup \widetilde{\Delta}_u^{\mathrm{III}}) = \varnothing$ if $s \in \operatorname{fr}\Delta$ and

$$(A \cap \widetilde{\Delta}_s^{\mathrm{II}}) \bullet\!|\!\bullet (A \cap \widetilde{\Delta}_s^{\mathrm{III}}) \neq \varnothing \qquad \text{and} \qquad ((A \cap \widetilde{\Delta}_s^{\mathrm{II}}) \bullet\!|\!\bullet (A \cap \widetilde{\Delta}_s^{\mathrm{III}})) \cap (s\bullet\!-\!\bullet t) = \varnothing$$

if $s \notin \operatorname{fr}\Delta$. Therefore, the location of A with respect to v is as described in Cases 3, A, 3, B1, and 3, B2, in which v is a node of some tripod. Moreover, it follows from the analysis of these cases that the number of such nodes in $s\bullet\!-\!\bullet t$ cannot exceed two, and two different nodes necessarily give two right-angled tripods on opposite sides of the line l_u.

Now, suppose, e.g., that $A \cap \widetilde{\Delta}_u^{\mathrm{III}} = \varnothing$. Then either $A \cap \widetilde{\Delta}_u^{\mathrm{I}} = \varnothing$ or $A \cap \widetilde{\Delta}_u^{\mathrm{IV}} = \varnothing$. If $A \cap \widetilde{\Delta}_u^{\mathrm{IV}} = \varnothing$, then $A \cap m_u = \{u\}$ implies $s{\circ}{-}{\circ}t \in \operatorname{fr} \Delta$; hence either $A \cap \widetilde{\Delta}_u^{\mathrm{II}} \neq \varnothing$ or $A \cap \widetilde{\Delta}_u^{\mathrm{I}} \neq \varnothing$; for example, if $A \cap \widetilde{\Delta}_u^{\mathrm{II}} \neq \varnothing$, then we obtain the inclusion

$$A \subset \widetilde{\Delta}_u^{\mathrm{II}} \cup (s{\bullet}{-}{\bullet}t),$$

where t is a vertex of Δ. If $A \cap \widetilde{\Delta}_u^{\mathrm{I}} = \varnothing$, then

$$A \subset \widetilde{\Delta}_u^{\mathrm{II}} \cup \widetilde{\Delta}_u^{\mathrm{IV}} \cup (s{\bullet}{-}{\bullet}t).$$

Case 3, B: $u \in \{s, t\}$. It is sufficient to consider the situation in which $u = s$ and $s_x < t_x$. Consider the three possible intersections of m_s with A.

Case 3, B1: $A \cap m_s = v{\bullet}|{\bullet}w$, $v \neq s \neq w$. Clearly, this case reduces to Case 3, A1.

Case 3, B2: $A \cap m_s = s{\bullet}|{\bullet}v$, $v \neq s$. Without loss of generality, we assume that $v_y < s_y$. Since the intervals $s{\bullet}{-}{\bullet}t$ and $s{\bullet}|{\bullet}v$ are nondegenerate, A is connected, and $\operatorname{int} A = \varnothing$, it follows that $A \cap \widetilde{\Delta}_u^{\mathrm{IV}} = \varnothing$. Suppose that the rectangle $\widetilde{\Delta}_s^{\mathrm{II}}$ contains infinitely many points of A. The assumptions made above imply

$$A \cap \Delta \cap \Delta_s^{\mathrm{II}} \setminus \{s\} = A \cap \widetilde{\Delta}_s^{\mathrm{II}}.$$

Using the considerations performed in Case 1, we obtain

$$A \cap \Delta_s^{\mathrm{III}} \setminus m_s = \varnothing = A \cap \Delta_s^{\mathrm{I}} \setminus l_s;$$

this means that the set A is contained in the rectangles $\widetilde{\Delta}_s^{\mathrm{II}}$ and Δ_s^{IV}. Moreover, taking into account the relation $A \cap \widetilde{\Delta}_s^{\mathrm{IV}} = \varnothing$, we obtain $A \subset \widetilde{\Delta}_s^{\mathrm{II}} \cup (\Delta_s^{\mathrm{IV}} \setminus \widetilde{\Delta}_s^{\mathrm{IV}})$ and $A \cap \widetilde{\Delta}_s^{\mathrm{II}} \neq \varnothing$. For the same reasons as in Case 3, A1, this is possible only if $t, v \in \operatorname{fr} \Delta$. As a result, we obtain

$$A \subset \widetilde{\Delta}_s^{\mathrm{II}} \cup (s{\bullet}{-}{\bullet}t) \cup (s{\bullet}|{\bullet}v), \qquad A \cap \widetilde{\Delta}_s^{\mathrm{II}} \neq \varnothing.$$

As in Case 3, A1, the set $A \cap \widetilde{\Delta}_s^{\mathrm{II}}$ cannot be finite. Now, suppose that $A \cap \widetilde{\Delta}_s^{\mathrm{II}} = \varnothing$; then, obviously,

$$A \subset \widetilde{\Delta}_s^{\mathrm{III}} \cup \widetilde{\Delta}_s^{\mathrm{I}} \cup (s{\bullet}{-}{\bullet}t) \cup (s{\bullet}|{\bullet}v).$$

Case 3, B3: $A \cap m_s = \{s\}$. Suppose that the rectangle $\widetilde{\Delta}_s^{\mathrm{II}}$ contains points of A. As in Case 3, B2, the number of such points is infinite. Using the argument from Case 1 yet again, we obtain

$$A \cap \Delta_s^{\mathrm{III}} \setminus \{s\} = \varnothing = A \cap \Delta_s^{\mathrm{I}} \setminus l_s.$$

This implies the inclusion

$$A \subset \widetilde{\Delta}_s^{\mathrm{II}} \cup \widetilde{\Delta}_s^{\mathrm{IV}} \cup (s{\bullet}{-}{\bullet}t), \qquad \text{where} \quad A \cap \widetilde{\Delta}_s^{\mathrm{II}} \neq \varnothing.$$

In particular, if $A \cap \widetilde{\Delta}_s^{\mathrm{IV}} = \varnothing$, then $s{\bullet}{-}{\bullet}t \subset \operatorname{fr} \Delta$ and the point t is a vertex of the rectangle Δ: otherwise, Δ would not be minimal. If $A \cap \widetilde{\Delta}_s^{\mathrm{III}} \neq \varnothing$, then, similarly,

$$A \subset \widetilde{\Delta}_s^{\mathrm{III}} \cup \widetilde{\Delta}_s^{\mathrm{I}} \cup (s{\bullet}{-}{\bullet}t), \qquad \text{where} \quad A \cap \widetilde{\Delta}_s^{\mathrm{III}} \neq \varnothing.$$

If $(\widetilde{\Delta}_s^{\mathrm{II}} \cup \widetilde{\Delta}_s^{\mathrm{III}}) \cap A = \varnothing$, then, necessarily, $s \in \operatorname{fr} \Delta$, and we must examine the points of $s{\circ}{-}{\bullet}t$. However, as is easy to see, to these points the considerations of all preceding Cases 3, * apply. It follows from these considerations that if $A \cap \widetilde{\Delta}_s^{\mathrm{I}} \neq \varnothing \neq A \cap \widetilde{\Delta}_s^{\mathrm{IV}}$, then the interval $s{\circ}{-}{\bullet}t$ contains the nodal points of one or two tripods; otherwise, when $A \cap \widetilde{\Delta}_s^{\mathrm{I}} = \varnothing$ or $A \cap \widetilde{\Delta}_s^{\mathrm{IV}} = \varnothing$, we have $s{\bullet}{-}{\bullet}t \subset \operatorname{fr} \Delta$, and the point s is a vertex of the rectangle Δ. In particular, if $A \cap \widetilde{\Delta}_s^{\mathrm{I}} = \varnothing$, then $A \subset \widetilde{\Delta}_s^{\mathrm{IV}} \cup (s{\bullet}{-}{\bullet}t)$.

It is easy to see that in Cases 3 A1, 3, B1, and 3, B2, the point u may be the node of a tripod. Moreover, if the set A contains a tripod, then it follows from the considerations in these cases that it cannot contain more than two different nodes of tripods.

Suppose that A contains no tripods. If there exists a $u \in A$ for which $A \cap \widetilde{\Delta}_u^{\mathrm{II}} \neq \varnothing$, then, as shown in Cases 1–3, we have

$$A \setminus (l_u \cup m_u) \subset \widetilde{\Delta}_u^{\mathrm{II}} \cup \widetilde{\Delta}_u^{\mathrm{IV}}.$$

It turns out, a similar condition holds for any point $v \in A$:

$$A \setminus (l_v \cup m_v) \subset \widetilde{\Delta}_v^{\mathrm{II}} \cup \widetilde{\Delta}_v^{\mathrm{IV}}.$$

Indeed, first consider the situation where $v \in A \cap \widetilde{\Delta}_u^{\mathrm{II}}$. Suppose that

$$(A \setminus (l_v \cup m_v)) \cap (\Delta \setminus (\widetilde{\Delta}_v^{\mathrm{II}} \cup \widetilde{\Delta}_v^{\mathrm{IV}})) \neq \varnothing.$$

Again, according to considerations in Cases 1–3, this is possible only if

$$A \setminus (l_v \cup m_v) \subset \widetilde{\Delta}_v^{\mathrm{I}} \cup \widetilde{\Delta}_v^{\mathrm{III}},$$

i.e., $A \cap (\widetilde{\Delta}_v^{\mathrm{II}} \cup \widetilde{\Delta}_v^{\mathrm{IV}}) = \varnothing$; however, by construction, we have $u \in \widetilde{\Delta}_v^{\mathrm{IV}}$, which is a contradiction. Next, suppose that

$$v \in (l_u \cup m_u) \cap A \qquad \text{and} \qquad (A \setminus (l_v \cup m_v)) \cap (\Delta \setminus (\widetilde{\Delta}_v^{\mathrm{II}} \cup \widetilde{\Delta}_v^{\mathrm{IV}})) \neq \varnothing.$$

Without loss of generality, we assume that $v \in l_u$ and $v_x < u_x$. We have $l_u = l_v$, $\widetilde{\Delta}_v^{\mathrm{III}} \subset \widetilde{\Delta}_u^{\mathrm{III}}$, and $A \cap \widetilde{\Delta}_u^{\mathrm{III}} = \varnothing$; therefore, $A \setminus (l_v \cup m_v) \subset \widetilde{\Delta}_v^{\mathrm{I}}$. The relations $A \cap \widetilde{\Delta}_u^{\mathrm{I}} = \varnothing$ and $A \cap \widetilde{\Delta}_v^{\mathrm{II}} = \varnothing$ and the absence of tripods imply the inclusions

$$A \setminus (l_v \cup m_v) \subset \widetilde{\Delta}_v^{\mathrm{I}} \cap m_u, \qquad A \setminus (l_u \cup m_u) \subset \widetilde{\Delta}_u^{\mathrm{II}} \cap m_v.$$

Take points $u' \in A \setminus (l_v \cup m_v)$ and $v' \in A \setminus (l_u \cup m_u)$ on the same horizontal line. Since A is ortho-convex, it follows that the rectangle with vertices u, u', v, and v' is entirely contained in A and has nonempty interior, which contradicts the assumption $\operatorname{int} A = \varnothing$. Thus, for all $u \in A$, we have

$$A \setminus (l_u \cup m_u) \subset \widetilde{\Delta}_u^{\mathrm{II}} \cup \widetilde{\Delta}_u^{\mathrm{IV}},$$

which implies, in particular, that the only vertices of Δ contained in A are the left upper vertex $(x(0), y(0))$ and the right lower vertex $(x(1), y(1))$. The construction of the required curve (i) is trivial: any circle of radius at most $d = \|(x(0), y(0)) - (x(1), y(1))\|$ centered at $(x(0), y(0))$ contains only one point of A; thus, for the parameter t we take r/d, where r is the distance from $(x(0), y(0))$ to $(x(t), y(t))$. The case where $A \cap \widetilde{\Delta}_u^{\mathrm{II}} = \varnothing$ for all $u \in A$ is handled in precisely the same way (we have $A \setminus (l_u \cup m_u) \subset \widetilde{\Delta}_u^{\mathrm{I}} \cup \widetilde{\Delta}_u^{\mathrm{III}}$ for any $u \in A$). This completes the proof of the proposition. $\square$

Before stating Proposition 4, we give the following definition. A continuous monotone curve $\gamma\colon C \to \mathbb{R}^2$ is said to be *unbounded in norm* if

$$C = (\alpha, \beta), \quad -\infty \le \alpha < \beta \le +\infty, \qquad \text{and} \qquad \|(x(t), y(t))\| \xrightarrow[\substack{t \to \alpha+0 \\ t \to \beta-0}]{} \infty.$$

Proposition 4. *If $\Gamma \neq \varnothing$ is a proper ortho-hyperplane, then Γ is a continuous monotone curve unbounded in norm.*

Proof. In the case where Γ is a horizontal or vertical line, the proof is trivial.

Let us show by contradiction that Γ contains no tripods. Let $u \in \Gamma$ be the node of a tripod, e.g.,

$$\gamma^{\mathrm{I}} \cup (s{\circ}\!\!-\!\!\bullet u) \cup (t{\circ}|\!\bullet u),$$

where $\gamma^{\mathrm{I}} \subset \Delta_u^{\mathrm{I}}$, $s = (-\infty, u_y)$, and $t = (u_x, -\infty)$. Then Δ_u^{III} is entirely contained in a closed ortho-half-space $H \in \mathbf{G^c}$ determining generating Γ. Clearly,

$$\Delta_u^{\mathrm{II}} \cap H \setminus m_u \subset l_u, \qquad \Delta_u^{\mathrm{IV}} \cap H \setminus l_u \subset m_u;$$

otherwise, there exist, e.g., $v \in \Delta_u^{\mathrm{III}} \setminus (l_u \cup m_u)$ and $w \in \Delta_u^{\mathrm{II}} \setminus (l_u \cup m_u)$ for which $v\bullet|\bullet w \neq \varnothing$, $v, w, \in \operatorname{int} H$. By Proposition 1, we have $v\bullet|\bullet w \subset \operatorname{int} H$, and hence $(v\bullet|\bullet w) \cap \Gamma \subset \operatorname{int} H$, which contradicts the definition of Γ. Therefore, we have

$$(\Delta_u^{\mathrm{II}} \cup \Delta_u^{\mathrm{IV}}) \setminus (l_u \cup m_u) \subset \mathbb{R}^2 \setminus H.$$

If $(\Delta_u^{\mathrm{I}} \setminus (l_u \cup m_u)) \cap (\mathbb{R}^2 \setminus H) \neq \varnothing$, then, taking $v \in (\Delta_u^{\mathrm{I}} \setminus (l_u \cup m_u)) \cap (\mathbb{R}^2 \setminus H)$, we obtain

$$\Delta_u^{\mathrm{I}} \cap \Delta_u^{\mathrm{III}} \setminus \{u\} \subset \mathbb{R}^2 \setminus H.$$

However, the curve $\gamma^{\mathrm{I}} \setminus \{u\} \subset \Delta_u^{\mathrm{I}}$ contains point arbitrarily close to u, so that

$$\varnothing \neq (\gamma^{\mathrm{I}} \setminus \{u\}) \cap (\Delta_u^{\mathrm{I}} \cap \Delta_v^{\mathrm{III}} \setminus \{u\}) \subset \mathbb{R}^2 \setminus H;$$

since H is closed, this curve cannot be contained in the boundary of the ortho-half-space $\mathbb{R}^2 \setminus H$, which contradicts the assumption. Therefore,

$$(\Delta_u^{\mathrm{I}} \setminus (l_u \cup m_u)) \cap (\mathbb{R}^2 \setminus H) = \varnothing,$$

or, equivalently, $\Delta_u^{\mathrm{I}} \setminus (l_u \cup m_u) \subset H$. The closedness of H allows us to strengthen this inclusion to $\Delta_u^{\mathrm{I}} \subset H$. Thus, we have

$$\Delta_u^{\mathrm{I}} \cup \Delta_u^{\mathrm{III}} = H, \qquad \mathbb{R}^2 \setminus H = (\Delta_u^{\mathrm{II}} \cup \Delta_u^{\mathrm{IV}}) \setminus (l_u \cup m_u);$$

as is easy to see, the second ortho-half-space is not connected, which again contradicts the definition of Γ. Thus, Γ does not contain the tripod specified above. A similar argument shows that tripods of other types ($s \neq (-\infty, u_y)$ and $t \neq (u_x, -\infty)$) are not contained in Γ either.

Thus, a proper ortho-hyperplane contains no tripods. The ortho-hyperplane Γ is nondegenerate and does not coincide with a horizontal or vertical line; hence Γ has a limit point $u \in \Gamma$, for which precisely one of the conditions

$$\Gamma \cap \Delta_u^{\mathrm{II}} \setminus (l_u \cup m_u) \neq \varnothing \qquad \text{and} \qquad \Gamma \cap \Delta_u^{\mathrm{III}} \setminus (l_u \cup m_u) \neq \varnothing$$

holds. To be specific, suppose that $\Gamma \cap \Delta_u^{\mathrm{II}} \setminus (l_u \cup m_u) \neq \varnothing$. Following the scheme of the proof of Proposition 3, we can show that

$$\Gamma \setminus (l_v \cup m_v) \subset (\Delta_v^{\mathrm{II}} \cup \Delta_v^{\mathrm{IV}}) \setminus (l_v \cup m_v)$$

for all $v \in \Gamma$ and construct a continuous monotone curve $\gamma\colon C \to \mathbb{R}^2$ determining Γ, e.g., by fixing a point $u \in \Gamma$, setting $\gamma(0) = u$, and defining the parameter t to equal the distance from u to $v \in \Gamma$ with positive (negative) sign if $v \in \Gamma \cap \Delta_u^{\mathrm{IV}} \setminus \{u\}$ (respectively, $v \in \Gamma \cap \Delta_u^{\mathrm{II}} \setminus \{u\}$).

Now, let us show that the curve γ is unbounded in norm. We argue by contradiction. First, suppose that the set C is not open, i.e., C is a half-open or closed interval. Let $C = [\alpha, \beta)$ (the case $C = (\alpha, \beta]$ is similar). In this case, the curve γ is entirely contained in one of the closed quarter-planes $\Delta_{\gamma(\alpha)}^{\mathrm{IV}}$ and $\Delta_{\gamma(\alpha)}^{\mathrm{II}}$; suppose that it is contained in $\Delta_{\gamma(\alpha)}^{\mathrm{IV}}$. If $\|(x(t), y(t))\| \xrightarrow[t \to \beta - 0]{} \infty$, then γ divides $\Delta_{\gamma(\alpha)}^{\mathrm{IV}}$ into two closed subsets A and B:

$$A \cup B = \Delta_{\gamma(\alpha)}^{\mathrm{IV}}, \qquad A \cap B = \Gamma.$$

The point $\gamma(\alpha) \in \Gamma$ is a vertex of the quarter-plane $\Delta_{\gamma(\alpha)}^{\mathrm{IV}}$; therefore, the ray $\Delta_{\gamma(\alpha)}^{\mathrm{IV}} \cap m_{\gamma(\alpha)}$ is entirely contained in one of the sets A and B. Let this set be A; then the ray $\Delta_{\gamma(\alpha)}^{\mathrm{IV}} \cap l_{\gamma(\alpha)}$ is entirely contained in B. Since Γ intersects only the quarter-planes II and IV with respect to any of its points, it follows that one of the rays mentioned above, say $\Delta_{\gamma(\alpha)}^{\mathrm{IV}} \cap m_{\gamma(\alpha)}$, contains a point $v \notin \Gamma$.

Consider any point $u \in \Delta_{\gamma(\alpha)}^{\mathrm{IV}} \cap m_{\gamma(\alpha)}$. Obviously, we have $u \in H$, where $H \in \mathbf{G^c}$ is a closed ortho-half-space determining Γ. It follows from the closedness of H that $l_u \cap A \subset H$ ($l_u \cap A$ and $m_u \cap A$ coincide with the intervals $u\bullet\!-\!\bullet(l_u \cap \Gamma)$ and $u\bullet|\bullet(m_u \cap \Gamma)$, respectively) and $(l_u \cap A)\bullet|\bullet\Gamma \subset H$. At the same time, we have

$$A = \bigcup \{(l_u \cap A)\bullet|\bullet\Gamma : u \in \Delta_{\gamma(\alpha)}^{\mathrm{IV}} \cap m_{\gamma(\alpha)}\};$$

therefore, $A \subset H$. Taking into account the relations $v \in A \setminus \Gamma$ and $m_v = m_{\gamma(\alpha)}$, we can write the boundary of A in explicit form and use a well-known property of the boundary:

$$v \in (m_v \cap A) \cup \Gamma = \operatorname{fr} A = \operatorname{fr}(A \cap H) \subset \operatorname{fr} A \cap \operatorname{fr} H \subset \operatorname{fr} H = \Gamma.$$

This contradicts the assumption $v \notin \Gamma$.

Suppose that $C = [\alpha, \beta)$ and $\|(x(t), y(t))\| \underset{t\to\beta-0}{\not\longrightarrow} \infty$. The functions $x(t)$ and $y(t)$ are monotone and bounded; hence there exists an $\widetilde{x} = \lim_{t\to\beta-0} x(t)$ and a $\widetilde{y} = \lim_{t\to\beta-0} y(t)$. Therefore, the point $(\widetilde{x}, \widetilde{y})$ is limit for the ortho-hyperplane Γ. At the same time, we have $(\widetilde{x}, \widetilde{y}) \notin \Gamma$, which is impossible, because any ortho-hyperplane is closed.

In the case $C = [\alpha, \beta]$, we again obtain a contradiction. This case is considered in the same way as the case where $C = [\alpha, \beta)$ and $\|(x(t), y(t))\| \underset{t\to\beta-0}{\longrightarrow} \infty$, but instead of the quarter-plane $\Delta^{\mathrm{IV}}_{\gamma(\alpha)}$, the compact rectangle

$$\Delta = \Delta^{\mathrm{IV}}_{\gamma(\alpha)} \cap \Delta^{\mathrm{II}}_{\gamma(\beta)} \supset \Gamma$$

is considered, and the rays $\Delta^{\mathrm{IV}}_{\gamma(\alpha)} \cap m_{\gamma(\alpha)}$ and $\Delta^{\mathrm{IV}}_{\gamma(\alpha)} \cap l_{\gamma(\alpha)}$ are replaced by the intervals $\Delta \cap m_{\gamma(\alpha)}$ and $\Delta \cap l_{\gamma(\alpha)}$, respectively.

It is easy to see that the remaining cases, in which $C = (\alpha, \beta)$ and

$$\|(x(t), y(t))\| \underset{t\to\alpha+0}{\not\longrightarrow} \infty \qquad \text{or} \qquad \|(x(t), y(t))\| \underset{t\to\beta-0}{\not\longrightarrow} \infty,$$

do not essentially differ from the case where $C = [\alpha, \beta)$ and $\|(x(t), y(t))\| \underset{t\to\beta-0}{\not\longrightarrow} \infty$, which has already been considered.

Thus, the continuous monotone curve γ determining the ortho-hyperplane Γ is unbounded in norm. This completes the proof of the proposition. $\square$

Proposition 5. *If a set $A \in \mathbf{G}^{\mathbf{c}}$ is closed, then any two points in this set can be joined by a compact continuous monotone curve entirely contained in A.*

Proof. Clearly, it suffices to consider the case of two points $u, v \in A$ not lying on the same horizontal or vertical line. The intersection $\Delta \cap A$, where Δ is the closed rectangle formed by the lines l_u, m_u, l_v, and m_v, is an ortho-convex compact set, because A is closed and ortho-convex. Let us show that $\Delta \cap A$, is also connected, i.e., this is an ortho-convex continuum.

Let π_{m_u} be the closed half-plane determined by the line m_u and containing the point v. Consider the sets $A_1 = A \setminus \operatorname{int} \pi_{m_u}$ and $A_2 = A \cap \pi_{m_u}$. These sets are closed; their union is connected, because this is A; and their intersection $A_1 \cap A_2 \subset \operatorname{fr} \pi_{m_u}$ is connected as well, because A is ortho-convex. According to a well-known theorem on connected sets [5], each of the sets A_1 and A_2 is connected. In a similar way, constructing a half-plane π_{l_u}, we prove that $A_2 \cap \pi_{l_u}$ is connected. Finally, having constructed the half-planes π_{m_v} and π_{l_v} containing u, we conclude that $\Delta \cap A$ is connected.

If $\operatorname{int}(\Delta \cap A) = \varnothing$, then the required curve is obtained by applying Proposition 3. Suppose that $\operatorname{int}(\Delta \cap A) \neq \varnothing$. Take any point w in $\operatorname{int}(\Delta \cap A)$. There exists a closed rectangle $\Delta_w \subset \operatorname{int} A$ for which $w \in \operatorname{int} \Delta_w \neq \varnothing$. Choose two closed half-planes π_1 and π_2 determined by two adjacent sides of Δ_w so that $w \in \pi_1 \cap \pi_2$ and $u, v \notin \pi_1 \cap \pi_2$. Consider the sets

$$A_1' = \Delta \cap A \setminus \operatorname{int}(\pi_1 \cap \pi_2), \qquad A_1'' = \Delta \cap A \cap \pi_1 \cap \pi_2.$$

These sets are connected, because they are closed, their union $A_1' \cup A_1'' = \Delta \cap A$ is connected (as shown above), and their intersection $A_1' \cap A_1'' = \Delta \cap A \cap \operatorname{fr}(\pi_1 \cap \pi_2)$ is connected as well (this follows from the ortho-convexity of $\Delta \cap A$ and the inclusion $\operatorname{fr} \pi_1 \cap \operatorname{fr} \pi_2 \subset \Delta \cap A$). Thus, each of the sets A_1' and A_1'' is an ortho-convex continuum. Moreover, the set A_1' satisfies the conditions $w \notin A_1'$ and $u, v \in A_1'$. If A_1' has an interior point, then we apply the same procedure to this set, and so on. As a result, we obtain a sequence

$$(\Delta \cap A) \supset A_1' \supset A_2' \supset \cdots$$

of embedded continua, in which each member contains u and v. We can assume this sequence to be at most countable, because the Euclidean plane is second countable. According to well-known theorems of topology, it has a limit, and this limit is a continuum. It contains the points u and v, has no interior points by construction, and is ortho-convex. Therefore, we can apply Proposition 3, which gives the required curve. This completes the proof of the proposition. $\square$

The results presented below describe several separation-type properties of ortho-convex sets, which are similar, in a certain respect, to the corresponding properties of convex sets in the plane. We mention at once that, in this paper, we give only the geometric statements of these results.

Proposition 6. *Any closed set A in $\mathbf{G^c}$ can be represented as the intersection of at most four closed proper ortho-half-spaces in $\mathbf{G^c}$ containing A.*

Proof. Consider the possible cases.

(1) Suppose that A is not bounded by any vertical or horizontal line. Without loss of generality, we can assume that A does not coincide with $\mathbb{R}^2$. Consider the set of horizontal lines intersecting A. On each of such lines l, we fix the point

$$u^l = (\sup\{u_x \mid u \in l,\, u_x \le (A \cap l)_x\},\, l_y)$$

if it exists; otherwise, we set $u^l = (-\infty, l_y)$. Obviously, the latter point exists when the intersection of l with A contains a left-unbounded ray. Each of the points u^l and $u^l_x \ne -\infty$ determines two rays, the open left ray $L^-_u = s{\circ\!-\!\circ}u^l$, $s = (-\infty, u^l_y)$, and the closed right ray $L^+_u = u^l{\bullet\!-\!\circ}t$, $t = (+\infty, u^l_y)$. As we shall see later on, each of the sets $\cup L^-_u$ and $\cup L^+_u$ has at most two connected components, which are, in turn, ortho-convex sets; moreover, these connected components turn out to be proper ortho-half-spaces, and their complements give some of the required ortho-half-spaces.

If there are more than one points of type u^l with $u^l_x = -\infty$, then the set $\{u^l_y \mid u^l_x = -\infty\}$ is closed and convex. Indeed, if there are more than one such points, then we can choose two points u^{l_1} and u^{l_2} with $u^{l_1}_x = u^{l_2}_x = -\infty$, for which $u^{l_1}_y \ge u^{l_2}_y$. Since A is ortho-convex, it follows that, for any points v^{l_1} and v^{l_2} on the open parts of left-unbounded rays contained in the sets $l_1 \cap A$ and $l_2 \cap A$, respectively, and satisfying the condition $v^{l_1}{\bullet\!|\!\bullet}v^{l_2} \ne \varnothing$, we have $v^{l_1}{\bullet\!|\!\bullet}v^{l_2} \subset A$. Therefore, for any $w \in v^{l_1}{\bullet\!|\!\bullet}v^{l_2}$, the ray $s{\circ\!-\!\circ}w$, $s = (-\infty, w_y)$, is contained in A, and hence the set $\{u^l_y \mid u^l_x = -\infty\}$ is convex. Let show that it is closed. Suppose that there exists a point

$$z' \in \operatorname{cl}\{u^l_y \mid u^l_x = -\infty\} \setminus \{u^l_y \mid u^l_x = -\infty\}.$$

For the line l' with $l'_y = z'$, we have $u^{l'} \in A$ and $u^{l'}_x \ne -\infty$, because the set A is unbounded and closed. Take a point $z'' \in \{u^l_y \mid u^l_x = -\infty\}$; without loss of generality, we can assume that $z'' < z'$. Since the set $\{u^l_y \mid u^l_x = -\infty\}$ is convex, it follows that $[z'', z') \subset \{u^l_y \mid u^l_x = -\infty\}$. Obviously, there exists a point $v \in A$ for which $v_x < u^{l'}_x$ and $v_y = z''$. By Proposition 5, v and $u^{l'}$ can be joined by a compact continuous monotone curve $\gamma_{vu^{l'}} \subset A$. For each point $w \in \gamma_{vu^{l'}}$ with $w_y \in [z'', z')$, we have

$$v_x \le w_x \le u^{l'}_x, \qquad s{\circ\!-\!\bullet}w \subset A, \quad s = (-\infty, w_y).$$

Take a point $v' \in s{\circ\!-\!\circ}u^{l'}$ with $v'_x = v_x$ and $v'_y = z'$ for $s = (-\infty, z')$ and choose any sequence $\{z_n \in [z'', z']\}$ converging to z'. This sequence determines the sequence $\{v_n \mid (v_n)_x = v_x,\, (v_n)_y = z_n\}$, which converges to v', and a sequence $\{w_n \mid w_n \in \gamma_{vu^{l'}},\, (w_n)_y = z_n\}$. Since $v_n \in s{\circ\!-\!\bullet}w_n \subset A$, it follows that $v_n \in A$; therefore, $v' \in A$, because A is closed. Thus, we have found a point $v' \in A$ for which $v'_x < u^{l'}_x$; this contradicts the definition of the point $u^{l'}_x$. Hence the set $\{u^l_y \mid u^l_x = -\infty\}$ is closed.

If there exists at least one (i.e., one or infinitely many) point of type u^l with $u^l_x = -\infty$, then the set $\cup L^-_u$ can be partitioned into two disjoint parts, the open upper part L^{II} above the upper line in $\{l \mid u^l_x = -\infty\}$ and the open lower part L^{III} below the lower line in $\{l \mid u^l_x = -\infty\}$. Possibly, $L^{\mathrm{II}} = \varnothing$ or $L^{\mathrm{III}} = \varnothing$. Clearly, in the absence of points u^l with $u^l_x = -\infty$, the open set $\cup L^-_u$ is not separated by straight lines. First, consider the situation in which there exists at least one point of type u^l with $u^l_x = -\infty$ in more detail.

Clearly, it suffices to determine the structure of one of the sets L^{II} and L^{III}. Consider, e.g., the set L^{III}. Let us show that it is ortho-convex. By construction, all horizontal intervals are contained in L^{III}. Take $u \in \operatorname{fr} L^{\mathrm{III}}$; clearly, $u \in \operatorname{fr} A$. Let us show that

$$A \cap \Delta^{\mathrm{III}}_u \setminus (l_u \cup m_u) = \varnothing.$$

Suppose that, on the contrary, there exists a $w \in A \cap \Delta^{\mathrm{III}}_u \setminus (l_u \cup m_u)$. Since the set A is unbounded, we can choose a point $v \in \operatorname{fr} L^{\mathrm{III}}$: $v_x < w_x$ with $v_y \ge u_y$ (moreover, it can be chosen in the set $\operatorname{fr} L^{\mathrm{III}} \cap \{l \mid u^l_x = -\infty\}$). By construction, we have $u, v, w \in A$, $v_x < w_x < u_x$, and $w_y < u_y \le v_y$. Since A is closed, it follows by Proposition 5 that the points v and w can be joined by a compact continuous monotone curve $\gamma_{vw} \subset A$; moreover, $(\gamma_{vw})_x < u_x$ and $u_y \in (\gamma_{vw})_y$. Since $u \in \operatorname{fr} L^{\mathrm{III}}$, it follows that, for $r \le \min\{|u_x - w_x|, |u_y - w_y|\}$, we have

$$(u + B_r) \cap L^{\mathrm{III}} \cap \pi^- \ne \varnothing, \qquad (u + B_r) \cap \gamma_{vw} = \varnothing,$$

where π^- is the open half-plane under the line l_u. Let $q \in (u + B_r) \cap L^{\mathrm{III}} \cap \pi^-$. Then the corresponding point $q^l \in l_q$ satisfies the conditions

$$(q^l)_x > q_x > w_x \ge (l_q \cap \gamma_{vw})_x, \qquad u_y > (q^l)_y > w_y.$$

Therefore, we have

$$l_q \cap \gamma_{vw} \in A, \qquad (q^l)_x > (l_q \cap \gamma_{vw})_x,$$

which contradicts the definition of the ray L^-_q. Thus, for all $u \in \operatorname{fr} L^{\mathrm{III}}$, we have

$$A \cap \Delta^{\mathrm{III}}_u \setminus (l_u \cup m_u) = \varnothing.$$

Next, take any two points u and v with $u_y > v_y$ in L^{III} on the same vertical line. By the definition of L^{III}, we have $u \in s{\circ}\!-\!{\circ}u^l$ and $v \in t{\circ}\!-\!{\circ}v^l$, where $s = (-\infty, u_y)$, $t = (-\infty, v_y)$, and $u^l, v^l \in \operatorname{fr} L^{\mathrm{III}}$; the relations readily imply the inclusion

$$(u \circ | \bullet v) \subset \Delta^{\mathrm{III}}_{u^l} \setminus (l_{u^l} \cup m_{u^l}).$$

Moreover, since $A \cap \Delta^{\mathrm{III}}_{u^l} \setminus (l_{u^l} \cup m_{u^l}) = \varnothing$, it follows that $(u\bullet|\bullet v) \cap A = \varnothing$. We see that, for any point $w \in (u\bullet|\bullet v)$, we have $w^l_x > w_x$, which means that $w \in L^{\mathrm{III}}$. Thus, $L^{\mathrm{III}} \in \mathbf{G}$. The proof of the connectedness of L^{III} is simple: any two points in L^{III} can be joined by the sides of a right angle entirely contained in L^{III}. As a consequence, for all $u \in \operatorname{fr} L^{\mathrm{III}}$, we have $L^-_u \subset L^{\mathrm{III}}$ and $L^+_u \subset \mathbb{R}^2 \setminus L^{\mathrm{III}}$.

Repeating the same considerations for the vertical lines intersecting A, we see that, for all $u \in \operatorname{fr} L^{\mathrm{III}}$, we have

$$M^-_u \subset L^{\mathrm{III}}, \qquad M^+_u \subset \mathbb{R}^2 \setminus L^{\mathrm{III}},$$

where

$$M^-_u = s{\circ}\!-\!{\circ}u^m, \quad s = (u_x, -\infty), \qquad M^+_u = u^m{\bullet}\!-\!{\circ}t, \quad t = (u_x, +\infty),$$
$$u^m = (m_x, \sup\{u_y \mid u \in m,\ u_y \le (A \cap m)_y\}).$$

These inclusions readily imply that the set $\mathbb{R}^2 \setminus L^{\mathrm{III}}$ is closed and belongs to the class $\mathbf{G^c}$. In other words, this is a closed proper ortho-half-space; by construction, $A \subset \mathbb{R}^2 \setminus L^{\mathrm{III}}$. The set $\mathbb{R}^2 \setminus L^{\mathrm{II}}$ has the same properties. As a result, we obtain

$$A \subset (\mathbb{R}^2 \setminus L^{\mathrm{II}}) \cap (\mathbb{R}^2 \setminus L^{\mathrm{III}}).$$

The situation in which there are no points of type u^l with $u^l_x = -\infty$ is considered in a similar way. Thus, $\mathbb{R}^2 \setminus \cup L^-_u$ is a closed proper ortho-half-space containing A.

If we chose a point

$$v^l = (\inf\{v_x \mid v \in l,\, v_x \geq (A \cap l)_x\},\, l_y)$$

instead of u^l at the beginning of (1), then, instead of the set $\cup L_u^-$, we would have to consider the set $\cup L_v^+$, for which assertions similar to those proved above hold: either

$$A \subset (\mathbb{R}^2 \setminus L^{\mathrm{I}}) \cap (\mathbb{R}^2 \setminus L^{\mathrm{IV}}),$$

where $\cup L_v^+ = L^{\mathrm{I}} \cup L^{\mathrm{IV}}$, or

$$A \subset \mathbb{R}^2 \setminus \cup L_v^+,$$

where $\mathbb{R}^2 \setminus L^{\mathrm{I,IV}}$ and $\mathbb{R}^2 \setminus \cup L_v^+$ are closed proper ortho-half-spaces.

(2) The set A is bounded by a vertical direct on only one side, say on the left. Then $\mu_0 = \sup\{\mu \in \mathbb{R} \mid \mu \leq A_x\}$ exists, and the vertical line m, $m_x = \mu_0$, intersects A in a closed convex subset of m, because A is closed and ortho-convex. Take $u \in A \cap m$. Let us construct a horizontal ray $s\circ\!\!-\!\!\bullet u$, $s = (-\infty, u_y)$, and add it to the set A. Applying the considerations of (1) to the resulting set $A' = A \cup (s\circ\!\!-\!\!\bullet u)$, we find $\mathbb{R}^2 \setminus (L^{\mathrm{II}})'$ and $\mathbb{R}^2 \setminus (L^{\mathrm{III}})'$ for this set. Augmenting $(L^{\mathrm{II}})'$ and $(L^{\mathrm{III}})'$ by the open half-plane π_m^- on the left of m and leaving the set $\mathbb{R}^2 \setminus \cup L_v^+$ unchanged, as in case (1) for A, we obtain the required proper ortho-half-spaces $\mathbb{R}^2 \setminus ((L^{\mathrm{II}})' \cup \pi_m^-)$, $\mathbb{R}^2 \setminus ((L^{\mathrm{III}})' \cup \pi_m^-)$, and either $\mathbb{R}^2 \setminus L^{\mathrm{I,IV}}$ or $\mathbb{R}^2 \setminus \cup L_v^+$. (The fact that $\mathbb{R}^2 \setminus ((L^{\mathrm{II}})' \cup \pi_m^-)$ and $\mathbb{R}^2 \setminus ((L^{\mathrm{III}})' \cup \pi_m^-)$ are ortho-half-spaces again follows from the considerations in case (1)).

(3) The set A bounded by a horizontal line from only one side, say from above. Then, by analogy with the preceding case, there exists a horizontal line l for which

$$l_y = \nu_0, \qquad \nu_0 = \inf\{\nu \in \mathbb{R} \mid \nu \geq A_y\}.$$

Therefore, the sets $\cup L_u^-$ and $\cup L_v^+$ are bounded by this line. We add the open half-plane π_l^+ above l to L^{I} and L^{II}. Again applying the technique of (1), we obtain the required proper ortho-half-spaces, which are either $\mathbb{R}^2 \setminus (L^{\mathrm{II}} \cup \pi_l^+)$ and $\mathbb{R}^2 \setminus L^{\mathrm{III}}$ or $\mathbb{R}^2 \setminus ((\cup L_u^-) \cup \pi_l^+)$ and either $\mathbb{R}^2 \setminus (L^{\mathrm{I}} \cup \pi_l^+)$ and $\mathbb{R}^2 \setminus L^{\mathrm{IV}}$ or $\mathbb{R}^2 \setminus ((\cup L_v^+) \cup \pi_l^+)$.

(4) If the set A is bounded by vertical or horizontal lines from more than one side, then we appropriately combine results obtained in cases (2) and (3), assuming that the sets L^i, $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, in (3) are obtained on the basis of (2), and again obtain the required ortho-half-spaces.

The proof is completed by the observation that, by virtue of the constructions performed in (1)–(4), the intersection of the obtained closed ortho-half-spaces containing A is A. Indeed, let us denote these ortho-half-spaces by H^k, $k \in K \subset \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, so that

$$\{H^k \mid k \in K\} \subset \{\mathbb{R}^2 \setminus L^{\mathrm{I}},\, \mathbb{R}^2 \setminus \cup L_u^-,\, \mathbb{R}^2 \setminus ((\cup L_u^-) \cup \pi_l^+),\, \dots\}.$$

Take any horizontal line l and any point $w \in l$. If $l \cap A \neq \varnothing$, then, according to (1)–(4), either $w \in A$ or there exists an $i \in K$ for which $w \in \mathbb{R}^2 \setminus H^i$. If $l \cap A = \varnothing$, then the set A is bounded, say above, by a horizontal line l' with $l'_y < l_y$; hence, according to (3), there exists a $j \in K$ for which $w \in \pi_{l'}^+ \subset \mathbb{R}^2 \setminus H^j$. Thus,

$$(\cup\{\mathbb{R}^2 \setminus H^k \mid k \in K\}) \cup A = \mathbb{R}^2.$$

Noting that $A \cap (\mathbb{R}^2 \setminus H^k) = \varnothing$ for all $k \in K$, we obtain $A = \cap\{H^k \mid k \in K\}$. This completes the proof of the proposition. □

Remark. The boundaries of the closed ortho-half-spaces H^i constructed in the proof are essentially supporting ortho-hyperplanes, i.e.,

$$\operatorname{fr} H^i \cap A \neq \varnothing = (\mathbb{R}^2 \setminus H^i) \cap A.$$

Corollary 6.1. *For any closed set $A \in \mathbf{G^c}$ and any point $u \notin A$, there exists a closed proper ortho-half-space H strongly separating u and A, i.e., such that $A \subset \operatorname{int} H$ and $u \in \mathbb{R}^2 \setminus H$.*

Proof. According to Proposition 6, the point u is contained in an open ortho-half-space supporting for A, e.g., in $\mathbb{R}^2 \setminus H^{\mathrm{IV}}$. Since $H^{\mathrm{IV}} \in \mathbf{G^c}$ and the point u is interior for $\mathbb{R}^2 \setminus H^{\mathrm{IV}}$, it follows that

$$\mathrm{fr}\, H^{\mathrm{IV}} \cap (\Delta_u^{\mathrm{II}} \setminus (l_u \cup m_u)) \neq \varnothing.$$

Take $v \in \mathrm{fr}\, H^{\mathrm{IV}} \cap (\Delta_u^{\mathrm{II}} \setminus (l_u \cup m_u))$. Let us show that the required ortho-half-space can be defined, e.g., by $H = H^{\mathrm{IV}} + w$, where $w = (u - v)/2$. First, note that the ortho-half-spaces H^{IV} and H are above the hyperplanes Γ^{IV} and Γ which they determine. Therefore, the inequality $w_y < 0$ directly implies the inclusion $H^{\mathrm{IV}} \subset H$, whence $A \subset H$. To prove the inclusion $H^{\mathrm{IV}} \subset \mathrm{int}\, H$, it suffices to show that $\Gamma^{\mathrm{IV}} \cap \Gamma = \varnothing$, which readily follows from Proposition 4 and the inequalities $w_x > 0$ and $w_y < 0$. Thus, $A \subset \mathrm{int}\, H$. The fact that $u \in \mathbb{R}^2 \setminus H$ follows from the relations

$$H^{\mathrm{IV}} \cap \Delta_v^{\mathrm{IV}} \setminus (l_v \cup m_v) = \varnothing, \qquad u \in \Delta_v^{\mathrm{IV}} \setminus (l_v \cup m_v) = \varnothing.$$

This proves the corollary. □

Corollary 6.2. *Any continuous monotone curve* Γ *unbounded in norm determines a closed ortho-half-space* $H \in \mathbf{G^c}$ *for which* $\Gamma = \mathrm{fr}\, H$.

Proof. Obviously, the curve Γ is a closed set in $\mathbf{G^c}$ located in opposite quarter-planes, say II and IV. This makes it possible to construct the required ortho-half-space H by using Proposition 6. Without loss of generality, we give the result of such a construction for only one special case of Γ. Suppose, e.g., that Γ is bounded by a line m on the left and by a line l from below; by analogy with the proof of Proposition 6, we assume these lines to be limit. We have $H = \mathrm{cl}((L^{\mathrm{III}})' \cup \pi_l^- \cup \pi_m^-)$, which proves the corollary. □

Proposition 7. *For any two disjoint closed sets* $A, B \in \mathbf{G^c}$*, there exists a closed proper ortho-half-space* H *such that* $A \subset H$ *and* $B \subset \mathrm{cl}(\mathbb{R}^2 \setminus H)$*. In other words, the proper ortho-hyperplane determined by the ortho-half-space* H *separates the sets* A *and* B.

Proof. Suppose that A and B cannot be separated by any horizontal direct (otherwise, there is nothing to prove). Since $A \cap B = \varnothing$ and A and B are ortho-convex, it follows that, for any horizontal line l intersecting these sets, we have either $(A \cap l)_x > (B \cap l)_x$ or $(A \cap l)_x < (B \cap l)_x$. This directly implies, in particular, that either $u_x^{l,A} \geq v_x^{l,B}$ or $u_x^{l,B} \geq v_x^{l,A}$, respectively.

Suppose, e.g., that some line l_1 satisfies the first condition. Then so does any line l_2 intersecting A and B. Indeed, if $(A \cap l_2)_x < (B \cap l_2)_x$, then choosing any $a_i \in A \cap l_i$ and $b_i \in B \cap l_i$ for each $i \in \{1, 2\}$ and applying Proposition 5, we see that, first, a_1 and a_2 can be joined by a compact continuous monotone curve $\gamma_{a_1 a_2} \subset A$ and b_1 and b_2 can be joined by a curve $\gamma_{b_1 b_2} \subset B$. Second, the curves $\gamma_{a_1 a_2}$ and $\gamma_{b_1 b_2}$ are monotone, and hence both of them are entirely enclosed by a minimal compact rectangle Δ with sides contained in l_1 and l_2. Continuing these curves by horizontal intervals to vertices of this rectangle Δ so that monotonicity is preserved and taking into account the inequalities $(a_1)_x > (b_1)_x$ and $(a_2)_x < (b_2)_x$, we obtain continuous monotone curves $\gamma_1 \supset \gamma_{a_1 a_2}$ and $\gamma_2 \supset \gamma_{b_1 b_2}$ entirely contained in Δ and joining different pairs of opposite vertices of Δ. By the theorem "on passing customs" [6], the curves γ_1 and γ_2 must have at least one common point. Moreover, by virtue of the inequalities $(a_1)_x > (b_1)_x$ and $(a_2)_x < (b_2)_x$, at least one of these common points belongs to both curves $\gamma_{a_1 a_2} \subset A$ and $\gamma_{b_1 b_2} \subset B$. However, this contradicts the assumption $A \cap B = \varnothing$. Therefore, for l_2, only the inequality $(A \cap l_2)_x > (B \cap l_2)_x$ can hold. In other words, the set A is on the right of B.

Let us show that one of the ortho-half-spaces determined by the set $\cup L_{u,A}^-$ constructed in Proposition 6 is as required. (In what follows, we omit the subscript indicating the set, assuming that the corresponding notation refers to A, unless otherwise specified; we also use some notation from the proof of Proposition 6 without mention.) Consider two cases.

(1) If the set A is not bounded on the left by a vertical line, then, obviously, either $B \cap \mathrm{cl}\, L^{\mathrm{II}} \neq \varnothing$ or $B \cap \mathrm{cl}\, L^{\mathrm{III}} \neq \varnothing$. Suppose, e.g., that $B \cap \mathrm{cl}\, L^{\mathrm{III}} \neq \varnothing$. The construction of the ortho-half-space H^{III} allow us to write the inclusion

$$B \cap \mathrm{cl}\, L^{\mathrm{III}} \subset \mathbb{R}^2 \setminus \mathrm{int}\, H^{\mathrm{III}}.$$

If follows from the considerations in case (3) of the proof of Proposition 6 and the fact that A is on the right of B that the inequality $B \setminus \operatorname{cl} L^{\mathrm{III}} \neq \varnothing$ holds if and only if A is bounded below by a horizontal line l, which can be assumed to be highest, and $B \setminus \operatorname{cl} L^{\mathrm{III}} \subset \operatorname{cl} \pi_l^-$, where π_l^- is the open half-plane below l. According to Proposition 6, these two inclusions give the required ortho-half-space $H = H^{\mathrm{III}}$.

(2) The set A is bounded on the left by a vertical line m, which can be assumed to be rightmost, as in case (2) of the proof of Proposition 6. If A and B are separated by this line, then there is nothing to prove. Thus, suppose that, on the contrary, $B \cap \pi_m^+ \neq \varnothing$, where π_m^+ is the open half-plane on the right of m. Take $u \in m \cap A$. Without loss of generality, we assume that

$$B \cap \operatorname{cl}(L^{\mathrm{III}})' \cap \pi_m^+ \neq \varnothing.$$

Let us show that $H = H^{\mathrm{III}}$. (Recall the expression for H^{III} derived in case (2) in the proof of Proposition 6: $H^{\mathrm{III}} = \mathbb{R}^2 \setminus ((L^{\mathrm{III}})' \cup \pi_m^+)$.) Since A is on the right of B, it follows from remarks made in case (1) that

$$B \cap \pi_{l_u}^- \cap \pi_m^+ \subset \operatorname{cl}(\mathbb{R}^2 \setminus H^{\mathrm{III}}).$$

Take any point v in $B \cap \operatorname{cl}(L^{\mathrm{III}})' \cap \pi_m^+$. Supposing that the set $B \cap \operatorname{cl} \pi_{l_u}^+ \cap \pi_m^+$ is nonempty, choose any point w in this set. According to Proposition 5, the points v and w can be joined by a compact continuous monotone curve $\gamma_{vw} \subset B$, which entirely contained π_m^+ as well, because it is monotone. The inequalities $w_y \geq u_y$ and $v_y \leq u_y$ imply the existence of an $s \in \gamma_{vw} \cap l_u \subset B$, and $s_x > u_x$. At the same time, as mentioned above, we have $(A \cap l_u)_x \geq (B \cap l_u)_x$, which implies $s_x \leq u_x$. This contradiction shows that $B \cap \operatorname{cl} \pi_{l_u}^+ \cap \pi_m^+ = \varnothing$. This equality, together with the expression for H^{III} and the inclusion obtained above, gives the final relation $B \subset \operatorname{cl}(\mathbb{R}^2 \setminus H^{\mathrm{III}})$, which proves the proposition. $\square$

It is clear, without any doubt, that all results obtained in this paper remain valid when the horizontal and vertical lines in the definition of an ortho-convex set are replaced by any two nonparallel lines l' and l'': it suffices to replace all sets in the above statements by those obtained from the initial ones by applying an appropriate linear transformation. In fact, ortho-convex sets are a special case of so-called biconvex sets [7], [8]. Therefore, hopefully, some properties of ortho-convex sets can be generalized to biconvex sets.

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