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Justification of the Averaging Method for Differential Equations with Large Rapidly Oscillating Summands and Boundary Conditions

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Abstract—The averaging method is justified for normal systems of differential equations with rapidly oscillating summands proportional to the square root of the oscillation frequency in the case of the boundary-value problem on a finite interval and for the problem of bounded solutions on the positive semiaxis with boundary condition at its left endpoint.

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1. INTRODUCTION

The averaging method, usually associated with the names of Krylov, Bogolyubov, and Mitropol'skii, originated in works dealing with celestial mechanics. In rigorous mathematical form, this method was described in the classical monographs [1], [2] for normal systems of differential equations in the case of the Cauchy problem on a finite interval and for some problems on the whole time axis (problems of periodic, almost periodic, and generally bounded solutions). However, it seems that the averaging method has not been developed for problems with boundary conditions. In the present paper, this method is justified for boundary-value problems on a finite interval and for the problem of bounded solutions on the positive semiaxis with boundary condition at its left endpoint. In the case of linear normal systems of differential equations with constant coefficients, such problems (not related to the averaging method) were posed, for example, in [3]. In addition, note that, in contrast to the classical systems of the averaging method [1], [2], the normal systems under consideration involve rapidly oscillating high-frequency summands proportional to the square root of the oscillation frequency. Earlier, we already considered various systems with large rapidly oscillating summands in the case of the Cauchy problem and problems on the whole axis in [4]–[7]. Note that the papers [4]–[7] were stimulated by Yudovich's important works (see, for example, [8], [9]), in which such systems were first considered, but this was done on a "physical" level without proper mathematical justification. Finally, note that, in the absence of large summands, Theorems 1 and 2, were published in [10], [11].

2. MAIN RESULTS

2.1. Problem on the Closed Interval

Let Ω be a domain in $\mathbb{R}^n$, and let

$$\Sigma = \{(x,t,\tau) : x \in \Omega,\ t \in [0,1],\ \tau \in [0,\infty)\}.$$

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On the interval $t \in [0, 1]$, we consider the boundary-value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x, t, \omega t) + \sqrt{\omega}\varphi(x, t, \omega t), \qquad \omega \gg 1, \\ Lx(0) = l, \qquad Rx(1) = r. \end{cases} \tag{1}$$

Here $f(x, t, \tau)$ and $\varphi(x, t, \tau)$ are vector functions continuous on Σ and taking values in $\mathbb{R}^n$, L and R are matrices having n columns and n_1 and n_2 rows, respectively, and l and r are column vectors of the corresponding dimensions; the entries of the matrices and vectors are real. Also assume the following.

1) The derivatives $f'_x(x, t, \tau)$, $\varphi'_x(x, t, \tau)$, $\varphi'_t(x, t, \tau)$, $\varphi''_{xt}(x, t, \tau)$, and $\varphi''_{xx}(x, t, \tau)$ exist and are continuous on Σ; the latter derivative, for example, is the matrix composed of all possible second-order derivatives of the components of the vector function $\varphi(x, t, \tau)$ with respect to x.

2) The vector functions $f(x, t, \tau)$, $\varphi(x, t, \tau)$, $\varphi'_t(x, t, \tau)$ and the matrix functions $f'_x(x, t, \tau)$, $\varphi'_x(x, t, \tau)$, $\varphi''_{xt}(x, t, \tau)$, and $\varphi''_{xx}(x, t, \tau)$ are uniformly bounded on Σ.

3) The matrix functions $f'_x(x, t, \tau)$, $\varphi''_{xt}(x, t, \tau)$, and $\varphi''_{xx}(x, t, \tau)$ satisfy, on Σ, a uniform Lipschitz condition for the variable x, i.e., there exists a constant $\lambda > 0$ such that, for all $(x_1, t, \tau), (x_2, t, \tau) \in \Sigma$,

$$|r(x_1, t, \tau) - r(x_2, t, \tau)| \le \lambda |x_1 - x_2|,$$

where $r = f'_x, \varphi''_{xt}, \varphi''_{xx}$ and $|u|$ is the norm of the vector $u \in \mathbb{R}^n$.

4) The vector function $f(x, t, \tau)$ and the matrix functions $f'_x(x, t, \tau)$, $\varphi''_{xt}(x, t, \tau)$, and $\varphi''_{xx}(x, t, \tau)$ satisfy Hölder's condition for the variable t on the set Σ, i.e., there exist constants $C > 0$ and $\gamma \in (0, 1)$ such that, for all $(x, t_1, \tau), (x, t_2, \tau) \in \Sigma$,

$$|r(x, t_2, \tau) - r(x, t_1, \tau)| \le C|t_2 - t_1|^\gamma,$$

where $r = f, f'_x, \varphi''_{xt}, \varphi''_{xx}$.

5) The limits

$$F(x, t) \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T (f(x, t, \tau) + \chi(x, t, \tau))\, d\tau,$$
$$F'_x(x, t) \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T (f'_x(x, t, \tau) + \chi'_x(x, t, \tau))\, d\tau,$$

where

$$\chi(x, t, \tau) = \varphi'_x(x, t, \tau) \int_0^\tau \varphi(x, t, s)\, ds,$$

exist uniformly with respect to the variable $(x, t) \in \Omega \times [0, 1]$.

6) The averaged problem

$$\begin{cases} \dfrac{dy}{dt} = F(y, t), \\ Ly(0) = l, \qquad Ry(1) = r \end{cases} \tag{2}$$

has a solution $\mathring{y}(t)$ (with values in Ω).

7) The limits

$$\lim_{N\to\infty} \frac{1}{\sqrt[4]{N}} \int_0^N r(x, t, \tau)\, d\tau = 0$$

for $r = \varphi, \varphi'_x, \varphi''_{xx}$ and

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \int_0^N s(x, t, \tau)\, d\tau = 0$$

for $s = \varphi'_t, \varphi''_{xt}$ exist uniformly with respect to $(x, t) \in \Omega \times [0, 1]$.

8) It is assumed that $n_1+n_2=n$ and also that

$$\begin{vmatrix} L \\ RV(1) \end{vmatrix} \neq 0,$$

where $V(t)$ is the matriciant of the system

$$\frac{dz}{dt}=F'_z(\mathring{y},t)z, \tag{3}$$

i.e., $V(t)$ is the fundamental system of solutions satisfying the condition $V(0)=E$.

In what follows, the symbol $C^\gamma(J)$, where $\gamma\in(0,1)$ and $J=[0,1]$ or $J=[0,\infty)$, will denote the Banach space of vector functions $x\colon J\to\mathbb{R}^n$ satisfying Hölder's condition with exponent γ and equipped with the norm

$$\|x\|_{C^\gamma(J)}=\sup_{t\in J}|x(t)|+\sup_{\substack{t_1,t_2\in J\\ t_1\neq t_2}}\frac{|x(t_2)-x(t_1)|}{|t_2-t_1|^\gamma}.$$

The following statement holds.

Theorem 1. *For each $\mu\in(0,1/2)$, there exists a number $\omega_0>0$ such that, in some $C^\mu([0;1])$-neighborhood of the vector function $\mathring{y}$, problem* (1) *has a unique solution x_ω for $\omega>\omega_0$, and the following limit equality holds*:

$$\lim_{\omega\to\infty}\|x_\omega-\mathring{y}\|_{C^\mu([0;1])}=0.$$

2.2. Problem on the Semiaxis

Let Ω be a domain in $\mathbb{R}^n$, and let

$$\Sigma=\{(x,t,\tau): x\in\Omega,\ t\in[0,\infty),\ \tau\in[0,\infty)\}.$$

On the semiaxis $t\in[0,\infty)$, we consider the boundary-value problem

$$\begin{cases}\dfrac{dx}{dt}=f(x,t,\omega t)+\sqrt{\omega}\varphi(x,t,\omega t), \qquad \omega\gg 1,\\ Mx(0)=\eta,\end{cases} \tag{4}$$

involving bounded solutions. Here $f(x,t,\tau)$ and $\varphi(x,t,\tau)$ are vector functions continuous on Σ and taking values in $\mathbb{R}^n$, M is a matrix, η is a vector whose dimension coincides with the number of rows in the matrix M; further, the elements of the matrix and the vector are real. Also assume the following.

1) The derivatives $f'_x(x,t,\tau)$, $\varphi'_x(x,t,\tau)$, $\varphi'_t(x,t,\tau)$, $\varphi''_{xt}(x,t,\tau)$, and $\varphi''_{xx}(x,t,\tau)$ exist and are continuous on Σ.

2) The vector functions $f(x,t,\tau)$, $\varphi(x,t,\tau)$, $\varphi'_t(x,t,\tau)$ and the matrix functions $f'_x(x,t,\tau)$, $\varphi'_x(x,t,\tau)$, $\varphi''_{xt}(x,t,\tau)$, and $\varphi''_{xx}(x,t,\tau)$ are uniformly bounded on Σ.

3) The matrix functions $f'_x(x,t,\tau)$, $\varphi''_{xt}(x,t,\tau)$, and $\varphi''_{xx}(x,t,\tau)$ on Σ satisfy the uniform Lipschitz condition in the variable x, i.e., there exists a constant $\lambda>0$ such that, for all $(x_1,t,\tau),(x_2,t,\tau)\in\Sigma$,

$$|r(x_1,t,\tau)-r(x_2,t,\tau)|\le\lambda|x_1-x_2|,$$

where $r=f'_x,\varphi''_{xt},\varphi''_{xx}$.

4) The vector function $f(x,t,\tau)$ and the matrix functions $f'_x(x,t,\tau)$, $\varphi''_{xt}(x,t,\tau)$, $\varphi''_{xx}(x,t,\tau)$ on the set Σ satisfy Hölder's condition in the variable t, i.e., there exist constants $C>0$ and $\gamma\in(0,1)$ such that, for all $(x,t_1,\tau),(x,t_2,\tau)\in\Sigma$,

$$|r(x,t_2,\tau)-r(x,t_1,\tau)|\le C|t_2-t_1|^\gamma,$$

where $r=f,f'_x,\varphi''_{xt},\varphi''_{xx}$.

5) The limits

$$F(x,t) \equiv \lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} (f(x,t,\tau) + \chi(x,t,\tau))\,d\tau,$$
$$F'_x(x,t) \equiv \lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} (f'_x(x,t,\tau) + \chi'_x(x,t,\tau))\,d\tau,$$

where

$$\chi(x,t,\tau) = \varphi'_x(x,t,\tau) \int_0^{\tau} \varphi(x,t,s)\,ds,$$

exist uniformly with respect to $a \in \mathbb{R}$ and $(x,t) \in \Omega \times [0,\infty)$.

6) The averaged problem

$$\begin{cases} \dfrac{dy}{dt} = F(y,t), \\ My(0) = \eta \end{cases} \tag{5}$$

has a bounded solution $\mathring{y}(t)$ (with values in Ω) on the semiaxis $t \in [0,\infty)$.

7) One of the following conditions hold:

(a) the limits

$$\lim_{N\to\infty} \frac{1}{\sqrt[4]{N}} \int_0^N t^{1/4} r(x,t,\tau)\,d\tau = 0 \qquad \text{for} \quad r = \varphi, \varphi'_x, \varphi''_{xx},$$
$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \int_0^N t^{1/2} s(x,t,\tau)\,d\tau = 0 \qquad \text{for} \quad s = \varphi'_t, \varphi''_{xt}$$

exist uniformly with respect to $(x,t) \in \Omega \times [1,\infty)$;

(b) the vector function $\varphi(x,t,\tau)$ is T-periodic ($T > 0$) in τ with zero mean

$$\frac{1}{T} \int_0^T \varphi(x,t,\tau)\,d\tau = 0.$$

8) The equation

$$\frac{du}{dt} = A(t)u, \tag{6}$$

where $A(t) = F'_x(\mathring{y}(t),t)$, is exponentially dichotomous on the semiaxis $t \in [0,\infty)$, and B_1, B_2 are subspaces of $\mathbb{R}^n$ ensuring this dichotomy (see, for example, [12, Chap. IV, Sec. 3] or Sec. 4.2 below). Denote by $\{e_k\}_1^r$ the basis in the subspace B_1 in which the solutions bounded on the right semiaxis begin and by S the matrix whose columns are the vectors columns e_k, $k = 1,\dots,r$. It is assumed that the matrix MS is invertible.

The following statement holds.

Theorem 2. *For each $\mu \in (0,1/2)$, there exists a number $\omega_0 > 0$ such that, in some $C^\mu([0;\infty))$-neighborhood of the vector function $\mathring{y}$, problem* (4) *has a unique bounded solution x_ω on the semiaxis $t \in [0,\infty)$ for $\omega > \omega_0$, and the following limit equality holds*:

$$\lim_{\omega\to\infty} \|x_\omega - \mathring{y}\|_{C^\mu([0;\infty))} = 0.$$

3. EXAMPLES

In this section, Theorems 1 and 2 are illustrated using simple examples.

1. On the interval $t \in [0, 1]$, consider the boundary-value problem

$$\begin{gathered} \ddot{x} + x + f(x,t)\sin(3\omega t) + \sqrt{\omega}\varphi_(x,t)\cos(\omega t) = 0, \\ x(0) = 0, \qquad x(1) = \cos(1), \qquad \omega \gg 1. \end{gathered} \tag{7}$$

Replacing $\dot{x} = -y$, we obtain the normal system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = x + f(x,t)\sin(3\omega t) + \sqrt{\omega}\varphi_(x,t)\cos(\omega t), \qquad \omega \gg 1, \\ x(0) = 1, \qquad x(1) = \cos(1). \end{cases} \tag{8}$$

Here[1] $(x(t), y(t))^T$ is the unknown vector function, and

$$L = (1, 0), \qquad R = (1, 0)$$

are the matrices of the boundary conditions, and we assume that the functions f and φ satisfy the conditions given in Sec. 2.1 (it suffices that, on the set $\{(x,t) \in \mathbb{R}^2 : |x| < 2, t \in [0, 1]\}$, the functions f, f'_x, φ, φ'_x, φ'_t, φ''_{xt}, φ''_{xx} be continuous and satisfy the uniform Lipschitz condition in x, t). It is easy to verify that the averaged problem is of the form

$$\begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = x, \\ x(0) = 1, \qquad x(1) = \cos(1). \end{cases} \tag{9}$$

Its solution is the vector function

$$\mathring{y}(t) = (\cos(t), \sin(t))^T;$$

hence

$$F'_x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad V(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

where $V(t)$ is the matriciant of the system of differential equations (9) (see (3)). Further,

$$RV(1) = \begin{pmatrix} \cos(1) & -\sin(1) \end{pmatrix},$$

so that

$$\begin{vmatrix} L \\ RV(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \cos(1) & -\sin(1) \end{vmatrix} = -\sin(1) \neq 0.$$

By Theorem 1, for each $\mu \in (0, 1/2)$, there exists an $\omega_0 > 0$ such that, in some $C^{1,\mu}([0;1])$-neighborhood of the function $\cos(t)$, problem (7) for $\omega > \omega_0$ has a unique solution $x_\omega(t)$, and the following limit equality holds:

$$\lim_{\omega \to \infty} \|x_\omega(t) - \cos(t)\|_{C^{1,\mu}([0;1])} = 0.$$

Here $\|u\|_{C^{1,\mu}} = \|u\|_{C^\mu} + \|(d/dt)u\|_{C^\mu}$.

[1] *Translator's note.* Here and below, to save space, column vectors with coordinates $x_1, \dots, x_n$ are denoted by $(x_1, \dots, x_n)^T$.

2. On the interval $t \in [0,1]$, consider the boundary-value problem

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + f_1(x_1,x_2,x_3,t)\sin(5\omega t) + \sqrt{\omega}\varphi_1(x_1,t)\cos(\omega t), \\ \dfrac{dx_2}{dt} = f_2(x_1,x_2,x_3,t)\cos(2\omega t) + \sqrt{\omega}\varphi_2(x_2,t)\sin(3\omega t), \\ \dfrac{dx_3}{dt} = x_3 + f_3(x_1,x_2,x_3,t)\cos(\omega t) + \sqrt{\omega}\varphi_3(x_3,t)\sin(7\omega t), \qquad \omega \gg 1, \\ x_1(0) = 1, \qquad x_2(0) = 1, \qquad 2x_2(1) + x_3(1) = 2 + \exp(1). \end{cases} \tag{10}$$

Here

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 2 & 1 \end{pmatrix}$$

are the matrices of the boundary conditions and the functions f_i, φ_i, $i = 1,2,3$, satisfy the conditions given in Sec. 2.1. In the previous example, these conditions were concretized. In this and subsequent examples, we do not concretize these conditions, because this is trivial. We can easily verify that the averaged problem is of the form

$$\begin{cases} \dfrac{dy_1}{dt} = -2y_1, \\ \dfrac{dy_2}{dt} = 0, \\ \dfrac{dy_3}{dt} = y_3, \\ y_1(0) = 1, \qquad y_2(0) = 1, \qquad 2y_2(1) + y_3(1) = 2 + \exp(1). \end{cases} \tag{11}$$

Its solution is the vector function $\mathring{y}(t) = (\exp(-2t), 1, \exp(t))^T$. Therefore,

$$F'_x = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad V(t) = \begin{pmatrix} \exp(-2t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(t) \end{pmatrix}.$$

Hence $RV(1) = \begin{pmatrix} 0 & 2 & \exp(1) \end{pmatrix}$, so that

$$\begin{vmatrix} L \\ RV(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & \exp(1) \end{vmatrix} = \exp(1) \neq 0.$$

By Theorem 1, for each $\mu \in (0, 1/2)$, there exists an $\omega_0 > 0$ such that, in some $C^\mu([0;1])$-neighborhood of the vector function $\mathring{y}(t)$, problem (10) has a unique solution $x_\omega(t)$, for $\omega > \omega_0$ and the following limit equality holds:

$$\lim_{\omega \to \infty} \|x_\omega(t) - \mathring{y}(t)\|_{C^\mu([0;1])} = 0.$$

3. Consider the following problem for the bounded (on the positive semiaxis) solutions of the system:

$$\begin{cases} \dfrac{dx_1}{dt} = 2x_1 + 3x_2 + f_1(x_1,x_2,t)\sin(2\omega t) + \sqrt{\omega}\varphi_1(x_1,t)\cos(\omega t), \\ \dfrac{dx_2}{dt} = 4x_1 + x_2 + f_2(x_1,x_2,t)\cos(3\omega t) + \sqrt{\omega}\varphi_2(x_2,t)\sin(7\omega t), \qquad \omega \gg 1, \\ x_1(0) = 3. \end{cases} \tag{12}$$

Here $M = \begin{pmatrix} 1 & 0 \end{pmatrix}$ is the boundary condition matrix and the functions f_i, φ_i, $i = 1, 2$ satisfy the conditions given in Sec. 2.2.

It is easy to see that the averaged problem is of the form

$$\begin{cases} \dfrac{dy_1}{dt} = 2y_1 + 3y_2, \\ \dfrac{dy_2}{dt} = 4y_1 + y_2, \\ y_1(0) = 3. \end{cases}$$

Its bounded solution is the vector function

$$\mathring{y}(t) = (3\exp(-2t), -4\exp(-2t))^T.$$

The matrix

$$F'_x = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

has two eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -2$, so that the exponential dichotomy condition holds. To the eigenvalue λ_2 corresponds the eigenvector $e_1 = (3, -4)^T$. Hence

$$|MS| = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} \right| = 3 \neq 0.$$

By Theorem 2, for each $\mu \in (0, 1/2)$, there exists an $\omega_0 > 0$ such that, in some $C^\mu([0;\infty))$-neighborhood of the vector function $\mathring{y}(t)$, problem (12) has a unique bounded solution x_ω on the semiaxis $t \in [0, \infty)$ for $\omega > \omega_0$, and the following limit equality holds:

$$\lim_{\omega\to\infty} \|x_\omega - \mathring{y}\|_{C^\mu([0;\infty))} = 0.$$

4. Consider another example of the boundary-value problem for bounded solutions on the positive semiaxis:

$$\begin{cases} \dfrac{dx_1}{dt} = 2x_1 + x_2 + f_1(x_1, x_2, x_3, t)\sin(2\omega t) + \sqrt{\omega}\varphi_1(x_1, t)\cos(5\omega t), \\ \dfrac{dx_2}{dt} = 2x_2 + f_2(x_1, x_2, x_3, t)\cos(3\omega t) + \sqrt{\omega}\varphi_2(x_2, t)\sin(\omega t), \\ \dfrac{dx_3}{dt} = -x_3 + f_3(x_1, x_2, x_3, t)\cos(\omega t) + \sqrt{\omega}\varphi_3(x_3, t)\sin(7\omega t), \qquad \omega \gg 1, \\ x_1(0) + 2x_2(0) + x_3(0) = 1. \end{cases} \tag{13}$$

Here $M = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$ is the boundary condition matrix and the functions f_i, φ_i, $i = 1, 2, 3$ satisfy the smoothness conditions given in Sec. 2.2.

The averaged problem

$$\begin{cases} \dfrac{dy_1}{dt} = 2y_1 + y_2, \\ \dfrac{dy_2}{dt} = 2y_2, \\ \dfrac{dy_3}{dt} = -y_3, \\ y_1(0) + 2y_2(0) + y_3(0) = 1 \end{cases}$$

has a bounded solution

$$\mathring{y}(t) = (0, 0, \exp(-t))^T.$$

The matrix

$$F'_x = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

has eigenvalues $\lambda_{1,2} = 2$ and $\lambda_3 = -1$, which are real, so that the exponential dichotomy condition holds. To the eigenvalue λ_3 corresponds the eigenvector $e_1 = (0, 0, 1)^T$, so that

$$|MS| = \left| \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| = 1 \neq 0.$$

By Theorem 2, for each $\mu \in (0, 1/2)$, there exists an $\omega_0 > 0$ such that, in some $C^\mu([0; \infty))$-neighborhood of the vector function $\mathring{y}(t)$, problem (13) has a unique bounded solution x_ω on the semiaxis $t \in [0, \infty)$ for $\omega > \omega_0$, and the following limit equality holds:

$$\lim_{\omega \to \infty} \|x_\omega - \mathring{y}\|_{C^\mu([0;\infty))} = 0.$$

4. FRAGMENTS OF THE PROOFS OF THEOREMS 1 AND 2

4.1. Passage to the Integral Equation

4.1.1. Problem on the closed interval. Let us begin with some well-known auxiliary results concerning the construction of the Green matrices for the boundary-value problem

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in [0, 1], \tag{14}$$

$$Lx(0) = l, \qquad Rx(1) = r, \tag{15}$$

where $A(t)$ is a square matrix of order n with continuous elements, $f(t)$ is a continuous n-dimensional vector function, L and R are rectangular matrices of dimensions $n_1 \times n$ and $n_2 \times n$, respectively, and l and r are vectors of the corresponding size. All the given problems are assumed real. By the symbol (C) we denote the following conditions:

$$\text{(C)}: \qquad n_1 + n_2 = n, \qquad \begin{vmatrix} L \\ RV(1) \end{vmatrix} \neq 0.$$

These conditions are necessary and sufficient for the unique solvability of the boundary-value problem (14), (15) for any f, l, and r. In the case of a constant matrix $A(t) \equiv A$, this assertion was proved in [3, Chap. I, Sec. 11]; in the case of a variable matrix $A(t)$ that proof, essentially, remains the same.

Under conditions (C), consider the Green matrices of problem (14), (15). These constitute the following triplet of matrix functions: $G(t, s)$ of order $n \times n$, $G_L(t)$ and $G_R(t)$ of sizes $n_1 \times n$ and $n_2 \times n$ satisfying the conditions

$$\begin{cases} G(s+0, s) - G(s-0, s) = E_n, \\ \dfrac{d}{dt} G(t, s) = A(t) G(t, s), \qquad t, s \in [0, 1], \quad t \neq s, \\ LG_L(0, s) = 0, \qquad RG_R(1, s) = 0, \end{cases}$$

$$\begin{cases} \dfrac{d}{dt}G_L(t) = A(t)G_L(t), & t \in [0,1], \\ LG_L(0) = E_{n_1}, \qquad RG_L(1) = 0; \end{cases} \qquad \begin{cases} \dfrac{d}{dt}G_R(t) = A(t)G_R(t), & t \in [0,1], \\ LG_R(0) = 0, \qquad RG_R(1) = E_{n_2}. \end{cases}$$

Problem (14) (15) is uniquely solvable, and its solution is expressed by the formula

$$x(t) = G_L(t)l + \int_0^1 G(t,s)f(s)\,ds + G_R(t)r.$$

This statement for a constant matrix $A(t) \equiv A$ was proved in [3, Chap. I, Sec. 11]; In the case of a variable matrix $A(t)$, the proof, essentially, remains valid.

Let us pass to the study of our nonlinear problem (1). Making the Krylov–Bogolyubov change of variables [2]

$$x(t) = v(t) + \frac{1}{\sqrt{\omega}} \int_0^{\omega t} \varphi(v(t), t, \tau)\,d\tau, \tag{16}$$

we obtain the problem

$$\begin{cases} \dfrac{dv}{dt} = f(v,t,\omega t) + \chi(v,t,\omega t) + r(v,t,\omega), & \omega \gg 1, \\ Lv(0) = l, \qquad Rv(1) = r - \dfrac{1}{\sqrt{\omega}} R \displaystyle\int_0^{\omega} \varphi(v(1), 1, \tau)\,d\tau. \end{cases} \tag{17}$$

The expression for the vector function $\chi(v,t,\tau)$ is the same as in Sec. 2.1, while that for the vector function $r(v,t,\omega)$ is fairly awkward. To simplify the latter function, we denote

$$K_\omega(t) \equiv \frac{1}{\sqrt{\omega}} \int_0^{\omega t} \varphi(v(t), t, \tau)\,d\tau, \qquad M_\omega(t) \equiv \frac{1}{\sqrt{\omega}} \int_0^{\omega t} \varphi'_v(v(t), t, \tau)\,d\tau.$$

Then

$$\begin{aligned} r(v,t,\omega) = {} & (E + M_\omega)^{-1} f(v(t) + K_\omega, t, \omega t) - f(v(t), t, \omega t) \\ & + \sqrt{\omega}(E + M_\omega)^{-1} \Big[\varphi(v(t) + K_\omega, t, \omega t) - \varphi(v(t), t, \omega t) \\ & \qquad - \omega^{-1} \int_0^{\omega t} \varphi'_s(v(t), s, \tau)|_{s=t}\,d\tau \Big] - \chi(v(t), t, \omega t). \end{aligned}$$

In problem (17), we replace $v = \mathring{y} + u$, where $\mathring{y}$ is the solution of problem (2), which was mentioned in Condition 6 of Sec. 2.1, obtaining

$$\begin{cases} \dfrac{du}{dt} - A(t)u = f(\mathring{y} + u, t, \omega t) + \chi(\mathring{y} + u, t, \omega t) + r(\mathring{y} + u, t, \omega) - F(\mathring{y}, t) - F'_u(\mathring{y}, t)u \\ \qquad\qquad \equiv \psi(u, t, \omega t), \\ Lu(0) = 0, \qquad Ru(1) = -\dfrac{1}{\sqrt{\omega}} R \displaystyle\int_0^{\omega} \varphi(\mathring{y}(1) + u(1), 1, \tau)\,d\tau, \qquad \omega \gg 1, \end{cases} \tag{18}$$

where $A(t) = F'_u(\mathring{y}, t)$. In (18), we pass to the equivalent integral equation

$$u(t) = \int_0^1 G(t,s)\psi(u(s), s, \omega s)\,ds - G_R(t)\frac{1}{\sqrt{\omega}} R \int_0^{\omega} \varphi(\mathring{y}(1) + u(1), 1, \tau)\,d\tau.$$

Here $G(t,s)$, $G_R(t)$ are the Green matrices that were mentioned above.

For $\gamma \in (0,1)$, we define the operator $N\colon C^\gamma([0;1]) \times [1;\infty] \to C^\gamma([0;1])$ as follows:

$$N(u,\omega) = \begin{cases} u(t) - \displaystyle\int_0^1 G(t,s)\psi(u(s),s,\omega s)\,ds \\ \qquad -G_R(t)\dfrac{1}{\sqrt{\omega}}R\displaystyle\int_0^\omega \varphi(\mathring{y}(1)+u(1),1,\tau)\,d\tau & \text{if} \quad \omega \in [1;\infty); \\ u(t) - \displaystyle\int_0^1 G(t,s)[F(\mathring{y}+u,s) - F(\mathring{y},s) - F'_u(\mathring{y},s)u]\,ds & \text{if} \quad \omega = \infty. \end{cases}$$

4.1.2. Problem on the semiaxis. Let us begin with some well-known auxiliary results.

On the semiaxis $J\colon t \in [0,\infty)$, we consider the differential equation

$$\frac{dx}{dt} = A(t)x, \tag{19}$$

where $A(t)$ is a square matrix of order n with elements continuous on J. We say that Eq. (19) is *exponentially dichotomous* on J if the space $\mathbb{R}^n$ splits into in the direct sum of its subspaces B_1 and B_2:

$$\mathbb{R}^n = B_1 \oplus B_2, \tag{20}$$

and, for positive numbers N_1, N_2, ν_1, and ν_2, the following conditions hold:

a) the solution $x_1(t)$ of Eq. (19) with initial condition $x_1(0) \in B_1$ for all $t, s \in J$, $t \geq s$ satisfies the estimate

$$|x_1(t)| \leq N_1 \exp(-\nu_1(t-s))|x_1(s)|,$$

b) the solution $x_2(t)$ of Eq. (19) with initial condition $x_2(0) \in B_2$ for all $t, s \in J$, $t \leq s$ satisfies the estimate

$$|x_2(t)| \leq N_2 \exp(\nu_2(t-s))|x_2(s)|.$$

Denote by P and Q the projections corresponding to the decomposition (20) and by $V(t)$ is the matriciant of system (19). Let us define the vector function

$$G(t,\tau) = \begin{cases} V(t)PV^{-1}(\tau) & \text{for } t > \tau, \\ -V(t)QV^{-1}(\tau) & \text{for } t < \tau, \end{cases} \tag{21}$$

which is called the *Green function* of Eq. (19) on the semiaxis J. For it, the following relations and important estimates hold (see [12, Chap. IV, Sec. 3]):

$$\frac{dG(t,\tau)}{dt} = A(t)G(t,\tau), \qquad t \neq \tau, \tag{22}$$

$$\frac{dG(t,\tau)}{d\tau} = -G(t,\tau)A(\tau), \qquad t \neq \tau, \tag{23}$$

and also

$$\begin{aligned} \|V(t)PV^{-1}(s)\| &\leq c_1 \exp(-\nu_1(t-s)), \qquad t \geq s, \\ \|V(t)QV^{-1}(s)\| &\leq c_2 \exp(-\nu_2(s-t)), \qquad s \geq t, \end{aligned} \tag{24}$$

where c_1 and c_2 are positive constants.

Now consider the inhomogeneous equation

$$\frac{dx}{dt} = A(t)x + f(t), \qquad f \in C(J). \tag{25}$$

The corresponding homogeneous equation ($f = 0$) is assumed exponentially dichotomous. As is well known [12, Chap. IV, Theorem 3.2], for any continuous vector function f bounded on J, the vector function

$$x_0(t) = \int_0^\infty G(t,s)f(s)\,ds, \qquad t \in J \tag{26}$$

is a bounded (on J) solution of Eq. (25). Let M and S be the same matrices as in Sec. 2.2. Any bounded (on J) solution $x(t)$ of Eq. (25) with boundary condition

$$Mx(0) = \eta, \tag{27}$$

is, obviously, of the form

$$x(t) = x_0(t) + V(t)v_0, \tag{28}$$

where $v_0 \in B_1$ and $Mv_0 = \eta - Mx_0(0)$. This implies

$$v_0 = Sa, \qquad (MS)a = \eta - \int_0^\infty MG(0,s)f(s)\,ds, \qquad a \in \mathbb{R}^r. \tag{29}$$

From (26), (28), (29), we obtain the following unique bounded (on J) solution of problem (25), (27):

$$\begin{aligned} x(t) &= \int_0^\infty [G(t,s) + V(t)S(MS)^{-1}MQV^{-1}(s)]f(s)\,ds + V(t)S(MS)^{-1}\eta \\ &\equiv \int_0^\infty G_1(t,s)f(s)\,ds + G_2(t)\eta. \end{aligned} \tag{30}$$

The matrix functions G_1 and G_2 are called (see [3, Chap. I, Sec. 14, where $A(t) \equiv A = \text{const}$]), the *Green functions of problem* (25), (27). We can easily verify that $G_1(t,\tau)$ satisfies relations (22), (23). Estimates (24) also imply the inequalities

$$\|G_1(t,s)\| \le c_1 \exp(-\nu(t-s)) + c_2 \exp(-\nu_1 t)\exp(-\nu_2 s),$$

where $t, s \in J$, $\nu = \min(\nu_1, \nu_2)$, and $c_1, c_2 = \text{const} > 0$.

Let us now turn to problem (4). In it, making the Krylov–Bogolyubov change of variables [2],

$$x(t) = v(t) + \frac{1}{\sqrt{\omega}}\int_0^{\omega t} \varphi(v(t),t,\tau)\,d\tau,$$

we obtain the problem

$$\begin{cases} \dfrac{dv}{dt} = f(v,t,\omega t) + \chi(v,t,\omega t) + r(v,t,\omega), \\ Mv(0) = \eta, \qquad \omega \gg 1. \end{cases} \tag{31}$$

Here the expressions for the vector functions $\chi(v,t,\tau)$ and $r(v,t,\omega)$ are of the same form as for these vector functions in Sec. 4.1.4.1.1.

In problem (31), we replace $v = \mathring{y} + u$, where $\mathring{y}$ is the solution of problem (5) mentioned above, obtaining

$$\begin{cases} \dfrac{du}{dt} - A(t)u = f(\mathring{y}+u,t,\omega t) + \chi(\mathring{y}+u,t,\omega t) + r(\mathring{y}+u,t,\omega) - F(\mathring{y},t) - F'_u(\mathring{y},t)u \\ \qquad\qquad\quad \equiv \psi(u,t,\omega t), \\ Mu(0) = 0, \qquad \omega \gg 1, \end{cases} \tag{32}$$

where $A(t) = F'_u(\mathring{y},t)$. In (32), we pass to the equivalent integral equation

$$u(t) = \int_0^\infty G_1(t,s)\psi(u(s),s,\omega s)\,ds.$$

Here $G_1(t,s)$ is the Green matrix introduced above.

For $\gamma \in (0,1)$, we define the operator $N\colon C^\gamma([0;\infty)) \times [1;\infty] \longrightarrow C^\gamma([0;\infty))$ as follows:

$$N(u,\omega) = \begin{cases} u(t) - \displaystyle\int_0^\infty G_1(t,s)\psi(u(s),s,\omega s)\,ds & \text{if } \omega \in [1,\infty); \\ u(t) - \displaystyle\int_0^\infty G_1(t,s)[F(\mathring{y}+u,s) - F(\mathring{y},s) - F'_u(\mathring{y},s)u]\,ds & \text{if } \omega = \infty. \end{cases}$$

4.2. Scheme of the Principal Part of the Proof

Theorems 1 and 2 follow, essentially, from the classical implicit-operator theorem if we use the following statement.

For the operators N defined in Sec. 4.1.4.1.1 and Sec. 4.1.4.1.2, respectively, the following lemma holds.

Lemma 1. *The operator N is continuous and is continuously Fréchet differentiable at the point $(0,\infty)$. Here $N(0,\infty) = 0$, $(\mathfrak{D}_u N)(0,\infty) = I$.*

The equalities given in the lemma are obvious. The proof of the continuity of the operator N and that of the existence and the continuity of its Fréchet differential $(\mathfrak{D}_u N)$ at the point $(0,\infty)$ are, essentially, simple, but cumbersome. Moreover, the techniques used the greater part of these proofs goes back to the classical theory of the averaging method (in which there are no large rapidly oscillating summands) and, therefore, are fairly well known. In this connection, we omit the proofs of the continuity of N and $(\mathfrak{D}_u N)$ and note only a simple technical lemma related to the large summand in problem (4).

Lemma 2. *Under the assumptions of Theorem 2, the following asymptotic relations uniform with respect to $(x,t) \in \Omega \times [0,\infty)$ hold:*

$$\frac{1}{\sqrt{\omega}}\int_0^{\omega t} r(x,t,\tau)\,d\tau = o(\omega^{-1/4}), \quad \omega \to \infty, \qquad \textit{for} \quad r = \varphi, \varphi'_x, \varphi''_{xx};$$

$$\frac{1}{\sqrt{\omega}}\int_0^{\omega t} s(x,t,\tau)\,d\tau = o(1), \quad \omega \to \infty, \qquad \textit{for} \quad s = \varphi'_t, \varphi''_{xt}.$$

Let us now clarify what we mean by the word "essentially" at the beginning of the subsection. The point is that, on the strength of the implicit-operator theorem, Lemma 1 implies the existence, relative uniqueness, and asymptotic proximity to zero in the norm of C^γ of the solution u_ω, $\omega \gg 1$, of problem (17), just as in the case of problem (32). After this, the two preceding changes of variables must also be taken into account. Namely, it is at this step that one must pass (and this is necessary!) from the spaces C^γ, $\gamma \in (0,1)$, to the spaces C^μ, $\mu \in (0,1/2)$.

Note, in conclusion, that the idea of applying the implicit-operator theorem to the theory of the averaging method was used earlier in [13], [14].

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