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Existence of the Stationary Solution of a Rayleigh-Type Equation*

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Abstract—A fluid flow along a semi-infinite plate with small periodic irregularities on the surface is considered for large Reynolds numbers. The boundary layer has a double-deck structure: a thin boundary layer ("lower deck") and a classical Prandtl boundary layer ("upper deck"). The aim of this paper is to prove the existence and uniqueness of the stationary solution of a Rayleigh-type equation, which describes oscillations of the vertical velocity component in the classical boundary layer.

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1. INTRODUCTION

In this paper, we continue our study of the Rayleigh-type equation (1.8), which was started in [1]; see also [2]. It is known that the Rayleigh equation (see [3]) plays an important role in fluid mechanics problems; see [4]. In this paper, this equation is considered on a semi-infinite cylinder (see (1.8), (1.9)), and it describes oscillations of the vertical velocity component in the classical Prandtl boundary layer (in the "upper deck" of a boundary layer with a double-deck boundary layer structure, see region II in Fig. 2) in the problem of an incompressible viscosity fluid flow along a semi-infinite flat plate with small periodic perturbations on the surface (see Fig. 1) for large Reynolds number $\mathbf{Re}$; for more details, see below.

In [1], it was proved that the stationary solution of the Rayleigh-type equation (1.8) exists and is unique for all $x > \delta$ and $\delta > M$, where x is the distance from the edge of the plate and M is a constant; see (2.1). The aim of this paper is to prove that a stationary solution of the Rayleigh-type equation (1.8) exists and is unique for all $x > \delta$ and $\delta \in (0, M]$ (i.e., at the edge of the plate).

As will be shown in Section 2, the proof of the existence of the solution in this case is reduced to proving that the discrete spectrum of a Schrödinger-type operator on the half-space with a potential in the form of a well of a small depth (see Fig. 3) is empty, and the last statement is proved (see Lemma 1).

We note that the results of this paper (see Theorem 2) together with the results in [1] prove that the stationary solution of the Rayleigh-type equation (2.2) exists and is unique for all $x > \delta$ and $\delta > 0$ (i.e., in the entire region under study) and this fact actually means the existence of the double-deck structure (because all equations describing this structure are solvable).

In this section, we present the main results from [1, 2], which we need for further discussion.

We assume that the plate surface is described by the relation

$$y_s = \varepsilon^{4/3}\mu\big(x, x/\varepsilon\big), \tag{1.1}$$

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where $\varepsilon = 1/\sqrt{\mathbf{Re}}$ is a small parameter,

$$\mu(x, \xi + 2\pi) = \mu(x, \xi), \qquad \int_0^{2\pi} \mu(x, \xi)\, d\xi = 0.$$

We suppose that the upstream flow is a plane-parallel one with velocity $\mathbf{U_0} = (1, 0)$.

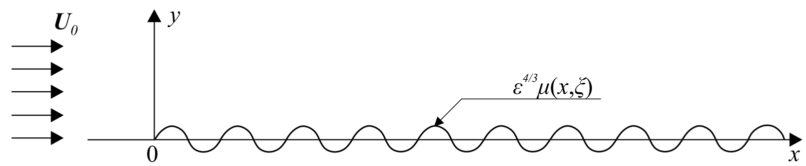

Fig. 1.

This problem is described by the system of Navier–Stokes and continuity equations

$$\begin{cases} \varepsilon^{-2/3} \dfrac{\partial \mathbf{U}}{\partial t} + \langle \mathbf{U}, \nabla \rangle \mathbf{U} = -\nabla p + \varepsilon^2 \Delta \mathbf{U}, \\ \langle \nabla, \mathbf{U} \rangle = 0, \end{cases} \tag{1.2}$$

where $\mathbf{U} = (u, v)$ is the velocity vector and p is the pressure. The boundary conditions are

$$\mathbf{U}\Big|_{\substack{y = \ y_s \\ x>0}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \frac{\partial u}{\partial y}\Big|_{\substack{y=0 \\ x<0}} = 0, \ v\Big|_{\substack{y=0 \\ x<0}} = 0, \ \mathbf{U}\Big|_{y \to \pm\infty} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{U}\Big|_{x \to -\infty} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.3}$$

According to the results of [2], the asymptotic solution of problem (1.2), (1.3) has a double-deck structure, which consists of a thin boundary layer and a classical Prandtl boundary layer; see Fig. 2. The coefficient $\varepsilon^{-2/3}$ at $\partial/\partial t$ in (1.2) is due to the fact that the double-deck structure generates a special hierarchy of times. The time scale for velocity fluctuations in the thin boundary layer is significantly smaller than that in the Prandtl boundary layer; see Theorem 1 below.

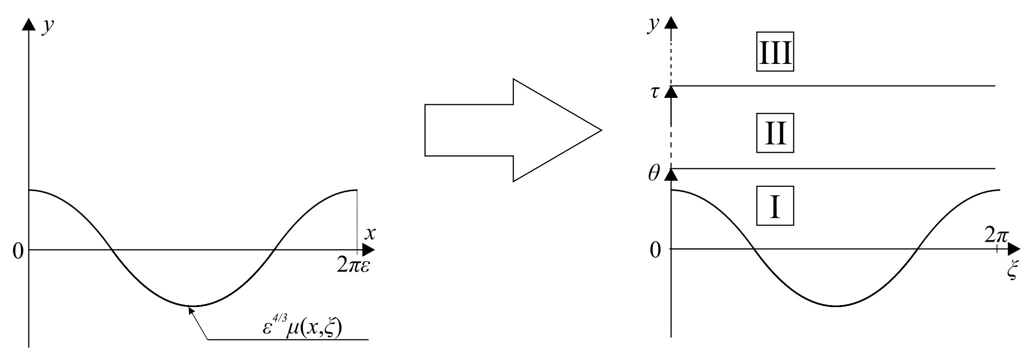

Fig. 2.

For further discussion, we introduce the following definitions.

Definition 1. For any 2π–periodic smooth the function $g(x, \xi)$ on $\mathbb{R} \times [0, 2\pi]$, we define

(i) the *mean value* by the formula

$$\overline{g}(x) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} g(x, \xi)\, d\xi,$$

(ii) the *oscillating part* by the formula

$$\widetilde{g}(x,\xi) \stackrel{\text{def}}{=} g(x,\xi) - \overline{g}(x).$$

Definition 2. For any 2π−periodic smooth the function $\widetilde{g}(x,\xi)$ on $\mathbb{R}\times[0,2\pi]$ such that $\overline{g}(x,\xi)=0$, we define the function

$$G(x,\xi) \stackrel{\text{def}}{=} \int^{\xi} \widetilde{g}(x,\xi)\,d\xi$$

so that $\overline{G}(x,\xi)=0$.

We introduce the scale

$$\theta = \frac{y}{\varepsilon^{4/3}}, \quad \tau = \frac{y}{\varepsilon}, \quad \xi = \frac{x}{\varepsilon}. \tag{1.4}$$

As indicated above, it follows from the results of [2] that our asymptotic solution of (1.2), (1.3) has a double-deck structure. We shall make use of the following notation. The superscript at the function stands for the number of the deck on which the function is defined (in the sense of Definition 3 below): I is the thin boundary layer, II is the classical Prandtl boundary layer, and III is the external region; see Fig. 2.

Definition 3. Let $N \in \mathbb{Z}_+$ be sufficiently large. Then

(i) a smooth 2π-periodic function decaying as $|\theta^{-N}|$, $\theta\to\infty$ is called the *boundary function in the thin boundary layer* I;

(ii) a smooth 2π-periodic function decaying as $|\tau^{-N}|$, $\tau\to\infty$ is called the *boundary function in the classical boundary layer* II.

We suppose that the initial conditions for problem (1.2), (1.3) are

$$\begin{aligned}
U(x,y) &= f'\big(\tau/\sqrt{x}\big) + \varepsilon^{1/3}\big(U_1^{\mathrm{I}}(x,\xi,\theta) + u_1^{\mathrm{II}}(x,\xi,\tau)\big) + O\big(\varepsilon^{2/3}\big),\\
V(x,y) &= \varepsilon^{2/3}\big(V_2^{\mathrm{I}}(x,\xi,\theta) + \widetilde{V}_2^{\mathrm{II}}(x,\xi,\tau)\big) + O(\varepsilon),\\
P(x,y) &= P_0 + \varepsilon^{2/3}\widetilde{P}_2^{\mathrm{II}}(x,\xi,\tau) + O(\varepsilon),
\end{aligned} \tag{1.5}$$

where the functions U_1^{I}, V_2^{I}, $\widetilde{V}_2^{\mathrm{II}}$, $\widetilde{P}_2^{\mathrm{II}}$ are boundary layer functions (see Definition 3 above) and $P_0=\text{const}$.

The main result of [2] (see also [1]) is the following theorem.

Theorem 1. *Let $x\ge\delta>0$. Then the formal asymptotic solution of problem* (1.2), (1.3) *has the form*

$$\begin{aligned}
u(t,x,y) &= u_0(x,\tau) + \varepsilon^{1/3}\big(u_1^{\mathrm{I}}(t,x,\xi,\theta) + \overline{u}_1^{\mathrm{II}}(t,x,\xi,\tau)\big) + O(\varepsilon^{2/3}),\\
v(t,x,y) &= \varepsilon^{2/3}\big(\widetilde{v}_2^{\mathrm{I}}(t,x,\xi,\theta) + \widetilde{v}_2^{\mathrm{II}}(t,x,\xi,\tau)\big) + O(\varepsilon),\\
p(t,x,y) &= p_0 + \varepsilon^{2/3}\widetilde{p}_2^{\mathrm{II}}(t,x,\xi,\tau) + O(\varepsilon),
\end{aligned} \tag{1.6}$$

where $\theta=y/\varepsilon^{4/3}$, $\tau=y/\varepsilon$, $\xi=x/\varepsilon$, p_0 is constant, $u_0=f'\big(\tau/\sqrt{x}\big)$. The the function $f(\gamma)$ is the Blasius function; see [5] *and Fig.* 4.

The functions u_1^{I} and $\widetilde{v}_2^{\mathrm{I}}$ are determined by the relations

$$u_1^{\mathrm{I}} = u_1^* - \overline{u}_1^{\mathrm{II}}\big|_{\tau=0} - \theta\frac{\partial u_0}{\partial\tau}\bigg|_{\tau=0}, \quad \widetilde{v}_2^{\mathrm{I}} = v_2^* - \widetilde{v}_2^{\mathrm{II}}\big|_{\tau=0},$$

where the functions u_1^* and v_2^* describing the flow in the thin boundary layer are the solution of the boundary-value problem

$$\begin{cases} \dfrac{\partial u_1^*}{\partial t} + u_1^* \dfrac{\partial u_1^*}{\partial \xi} + v_2^* \dfrac{\partial u_1^*}{\partial \theta} = \dfrac{\partial^2 u_1^*}{\partial \theta^2} + \dfrac{\partial \widetilde{p}_2^{\mathrm{II}}}{\partial \xi}\Big|_{\tau=0}, \\ \dfrac{\partial u_1^*}{\partial \xi} + \dfrac{\partial v_2^*}{\partial \theta} = 0, \end{cases} \tag{1.7}$$

$$u_1^*\big|_{\theta=\mu} = \mu \frac{\partial u_0}{\partial \tau}\Big|_{\tau=0}, \; v_2^*\big|_{\theta=\mu} = 0, \quad u_1^*\big|_{\xi} = u_1^*\big|_{\xi+2\pi}, \quad v_2^*\big|_{\xi} = v_2^*\big|_{\xi+2\pi},$$
$$\frac{\partial u_1^*}{\partial \theta}\Big|_{\theta\to\infty} \to \frac{\partial u_0}{\partial \tau}\Big|_{\tau=0}, \qquad \frac{\partial u_1^*}{\partial \xi}\Big|_{\theta\to\infty} \to 0$$

with the initial condition $u_1^*\big|_{t=0} = U_1^{\mathrm{I}} + u_1^{\mathrm{II}}\big|_{\tau=0} + \theta \dfrac{\partial u_0}{\partial \tau}\Big|_{\tau=0}$.

The the function $\widetilde{v}_2^{\mathrm{II}}$ describing the flow in the classical boundary layer is a solution of the following Rayleigh-type equation:

$$\varepsilon^{1/3} \frac{\partial}{\partial t} \Delta \int^{\xi} \widetilde{v}_2^{\mathrm{II}}\, d\xi + u_0 \Delta \widetilde{v}_2^{\mathrm{II}} - \widetilde{v}_2^{\mathrm{II}} \frac{\partial^2 u_0}{\partial \tau^2} = 0 \tag{1.8}$$

$$\widetilde{v}_2^{\mathrm{II}}\big|_{\tau=0} = \lim_{\theta\to\infty} \widetilde{v}_2^*, \quad \lim_{\tau\to\infty} \widetilde{v}_2^{\mathrm{II}} = 0, \quad \widetilde{v}_2^{\mathrm{II}}\big|_{\xi} = \widetilde{v}_2^{\mathrm{II}}\big|_{\xi+2\pi}. \tag{1.9}$$

with the initial condition $\widetilde{v}_2^{\mathrm{II}}\big|_{t=0} = \widetilde{V}_1^{\mathrm{II}}$.

The function $\overline{u}_1^{\mathrm{II}}$ is a solution of the linearized Prandtl equation

$$\begin{cases} \varepsilon^{-2/3} \dfrac{\partial \overline{u}_1^{\mathrm{II}}}{\partial t} + u_0 \dfrac{\partial \overline{u}_1^{\mathrm{II}}}{\partial x} + \overline{u}_1^{\mathrm{II}} \dfrac{\partial u_0}{\partial x} + \overline{v}_3^* \dfrac{\partial \overline{u}_1^{\mathrm{II}}}{\partial \tau} + \overline{v}_4^* \dfrac{\partial \overline{u}_1^{\mathrm{II}}}{\partial \tau} - \dfrac{\partial^2 \overline{u}_1^{\mathrm{II}}}{\partial \tau^2} = 0, \\ \dfrac{\partial \overline{u}_1^{\mathrm{II}}}{\partial x} + \dfrac{\partial \overline{v}_4^*}{\partial \tau} = 0 \end{cases} \tag{1.10}$$

with some boundary conditions, but this function plays no significant role if we are interested in the properties of the flow near the surface.

The pressure $\widetilde{p}_2^{\mathrm{II}}$ is determined by the expression

$$\frac{\partial \widetilde{p}_2^{\mathrm{II}}}{\partial \xi} = u_0 \frac{\partial \widetilde{v}_2^{\mathrm{II}}}{\partial \tau} - \widetilde{v}_2^{\mathrm{II}} \frac{\partial u_0}{\partial \tau} + \varepsilon^{1/3} \frac{\partial}{\partial t} \int^{\xi} \frac{\partial \widetilde{v}_2^{\mathrm{II}}}{\partial \tau}\, d\xi. \tag{1.11}$$

2. EXISTENCE OF THE STATIONARY SOLUTION OF A RAYLEIGH-TYPE EQUATION

In [1] we proved that the stationary solution of the Rayleigh-type equation (1.8), (1.9) exists for all $x > M$ (x is the distance from the edge of the plate), where

$$M = \max_{\gamma\in[0,\infty)} \left| \frac{f'''(\gamma)}{f'(\gamma)} \right|. \tag{2.1}$$

The aim of this paper is to prove that the stationary solution of the Rayleigh-type equation (1.8) exists for all $x \geq \delta$, where $\delta \in (0, M]$.

Theorem 2. *The stationary solution of the Rayleigh-type equation* (1.8), (1.9) *exists and is unique for all $x \geq \delta$, where $\delta \in (0, M]$, and M is defined above; see* (2.1).

Proof. We write the stationary equation corresponding to (1.8) in the classical boundary layer variable $\eta = \tau/\sqrt{x} = y/(\varepsilon\sqrt{x})$. Taking into account that $u_0 = f'(\tau/\sqrt{x}) = f'(\eta)$, we have

$$\begin{cases} f'(\eta)\left(\dfrac{1}{x}\dfrac{\partial^2\widetilde{v}}{\partial\eta^2} + \dfrac{\partial^2\widetilde{v}}{\partial\xi^2}\right) - \dfrac{1}{x}\widetilde{v}f'''(\eta) = 0, \\ \widetilde{v}\big|_{\eta=0} = \lim\limits_{\theta\to\infty} v_2^*, \quad \widetilde{v}\big|_{\eta\to\infty} \to 0, \quad \widetilde{v}\big|_{\xi} = \widetilde{v}\big|_{\xi+2\pi}. \end{cases} \tag{2.2}$$

We expand the function $\widetilde{v}$ into the Fourier series

$$\widetilde{v} = \sum_{k\neq 0} v_k(\eta)e^{ik\xi}. \tag{2.3}$$

By substituting (2.3) into (2.2), we obtain the equations for the coefficients v_k:

$$-v_k'' + Uv_k + xk^2v_k = 0, \quad k \neq 0, \quad k \in \mathbb{Z}, \quad v_k(0) = v_{0k}, \quad v_k\big|_{\eta\to\infty} \to 0, \tag{2.4}$$

where

$$U(\eta) = f'''(\eta)/f'(\eta), \tag{2.5}$$

and v_{0k} is a coefficient of the Fourier expansion of the function $\widetilde{v}_2^*\big|_{\theta\to\infty}$ (i.e., it is the boundary condition expansion; see (1.9)).

The potential $U(\eta)$ is a well of depth M and it has the following properties:

$$U(0) = 0, \qquad U\big|_{\eta\to\infty} = O\big(|\eta|^{-N}\big),$$

where N is any number; see Fig. 3 and also [6].

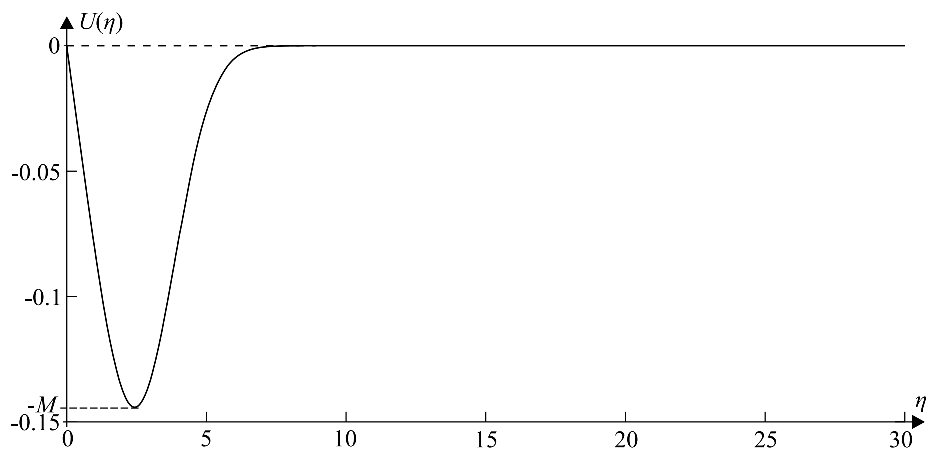

Fig. 3. The potential $U(\eta)$, $M \approx 0.1442$.

Let us reduce problem (2.4) to a problem with zero boundary condition at $\gamma = 0$. We put

$$v_k = \phi_k + g_k v_{0k},$$

where $g_k \in C^\infty[0,\infty)$ is a given function such that

$$g_k\big|_{\eta=0} = 1, \quad g_k\big|_{\eta\to\infty} = O\big(\eta^{-N}\big), \quad \forall\, N \in \mathbb{Z}_+. \tag{2.6}$$

Then we have

$$\begin{cases} (\hat{H} + xk^2)\phi_k = -v_{0k}(\hat{H} + xk^2)g_k, \\ \phi_k\big|_{\eta=0} = 0, \quad \phi_k\big|_{\eta\to\infty} \to 0, \end{cases} \tag{2.7}$$

where $\hat{H}$ is the differential expression

$$\hat{H} = -\frac{d^2}{d\eta^2} + U(\eta).$$

It was shown in [1] that if $x \geq \delta > M$, then problem (2.7) is uniquely solvable. Here we consider the case $\delta \in (0, M]$.

By H we denote the self-adjoint operator in $L_2(0, +\infty)$ with the differential expression $\hat{H}$ subject to the Dirichlet condition at $\eta = 0$. The domain of this operator is

$$D(H) = \{u \in W_2^2(0, +\infty) : u(0) = 0\}.$$

Then problem (2.7) can be rewritten in terms of the above introduced operator

$$(H + xk^2)\phi_k = -v_{0k}(\hat{H} + xk^2)g_k. \tag{2.8}$$

This equation is uniquely solvable if and only if $-xk^2$ is not in the spectrum of operator H. Since the potential U is fast decaying at infinity, it follows that the essential spectrum of operator H is $[0, +\infty)$. Since $k \neq 0$, $x > 0$, the number $-xk^2$ belongs to the spectrum of H only if it is a discrete eigenvalue of H. Our next lemma asserts the absence of such eigenvalues.

Lemma 1. *The discrete spectrum of operator H is empty.*

Proof. We argue by contradiction. Let $\lambda_0 < 0$ be the lowest eigenvalue of operator H and ψ_0 be the associated eigenfunction normalized in $L_2(0, +\infty)$. Then by the minimax principle we know that

$$\lambda_0 = \inf_{\substack{u \in W_2^1(0,+\infty),\\ u(0)=0,\ u \not\equiv 0}} \frac{\|u'\|^2_{L_2(0,+\infty)} + (Uu, u)_{L_2(0,+\infty)}}{\|u\|^2_{L_2(0,+\infty)}} = \|\psi_0'\|^2_{L_2(0,+\infty)} + (U\psi_0, \psi_0)_{L_2(0,+\infty)}.$$

Then the function $|\psi_0|$ also minimizes the above infimum and, therefore, it is an eigenfunction associated with λ_0. Thus, we can assume that the function ψ_0 is non-negative. We also note that since $\lambda_0 < 0$, the function ψ_0 decays exponentially at infinity: $\psi_0(\eta) = O\big(e^{-\sqrt{|\lambda_0|}\eta}\big)$, $\eta \to +\infty$.

Consider the function $\psi_1 = f'(\eta)$. The properties of Blasius function imply that $\psi_1 \in C^2[0, +\infty)$, $\psi_1(0) = 0$, $\psi_1(\eta) > 0$, $\eta > 0$ and $\psi_1(\eta) \to 1$ as $\eta \to +\infty$ (cf. Fig. 4).

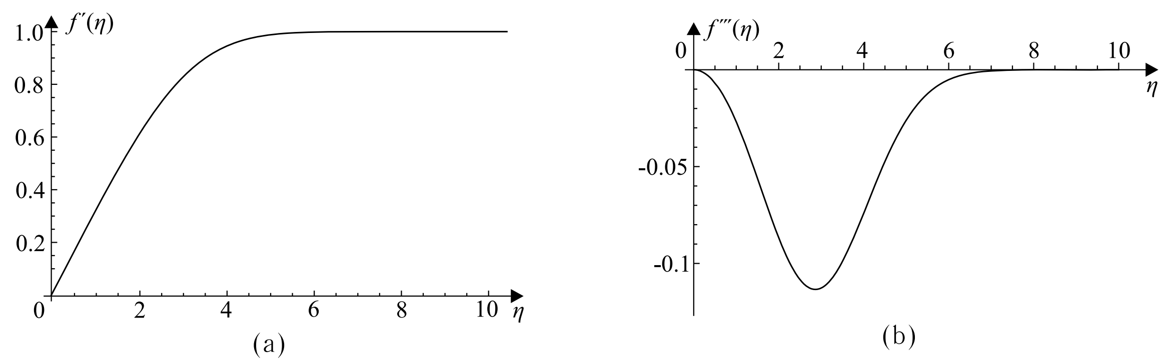

Fig. 4.

It is straightforward to check that the function ψ_1 solves the equation

$$-\psi_1'' + U\psi_1 = 0, \quad \eta > 0.$$

We multiply this equation by ψ_0 and integrate twice by parts over $(0, R)$, where $R > 0$ is a fixed constant:

$$0 = \int_0^R \psi_0(-\psi_1'' + U\psi_1)d\eta = \lambda_0 \int_0^R \psi_0\psi_1 d\eta - \psi_0(R)\psi_1'(R) + \psi_0'(R)\psi_1(R).$$

We pass to the limit as $R \to +\infty$ and employ the above described behavior of ψ_0, ψ_1 at infinity to obtain:

$$\int_0^{+\infty} \psi_0\psi_1 d\eta = 0.$$

This identity is impossible since both functions ψ_0, ψ_1 are non-negative. This completes the proof. $\square$

The proven lemma implies the unique solvability of equation (2.8), and therefore, of problem (2.7). Hence, the stationary solution of the Rayleigh-type equation (1.8) exists and is unique for all $x \geq \delta$ and for all $\delta \in (0, M]$. $\square$

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