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On the Primality Property of Central Polynomials of Prime Varieties of Associative Algebras

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Abstract—In the paper, it is proved that, if $f(x_1,\dots,x_n)g(y_1,\dots,y_m)$ is a multilinear central polynomial for a verbally prime T-ideal Γ over a field of arbitrary characteristic, then both polynomials $f(x_1,\dots,x_n)$ and $g(y_1,\dots,y_m)$ are central for Γ.

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All algebras considered in the paper are assumed to be associative algebras over a field.

A multilinear polynomial $f = f(x_1,\dots,x_n)$ is referred to as a *central polynomial* for a T-ideal Γ if $f \notin \Gamma$ and $[f, y] \in \Gamma$. In other words, if A is an algebra with the ideal of identities Γ, then all values $f(a_1,\dots,a_n)$ $(a_1,\dots,a_n \in A)$ belong to the center of the algebra A, and here the polynomial f is not an identity of the algebra A.

Regev [1] proved that, if a multilinear polynomial $f(x_1,\dots,x_n)g(y_1,\dots,y_m)$ is a central polynomial for the algebra of matrices of order k, then both polynomials f and g are central. Diniz [2] proved a similar property for the ideals of identities of the algebras $M_k(E)$ (the algebras of matrices of order k over the Grassmann algebra) and $M_{k,k}$ (the matrix superalgebra $M_{k,k}$, which is a subalgebra of $M_{2k}(E)$) over a field of characteristic zero. Both ideals of identities are verbally prime.

The notion of verbally prime T-ideal, which is one of the main notions in the theory of identities of associative algebras, war introduced by Kemer in [3], namely, a T-ideal Γ is said to be *verbally prime* if, for arbitrary T-ideals Γ_1 and Γ_2, it follows from the inclusion $\Gamma_1 \cdot \Gamma_2 \subseteq \Gamma$ that either $\Gamma_1 \subseteq \Gamma$ or $\Gamma_2 \subseteq \Gamma$. A variety is said to be *prime* if the ideal of identities of this variety is verbally prime. Kemer [3] described all prime varieties over a field of characteristic zero. The problem of describing (and also of studying the properties) of prime varieties over a field of positive characteristic is at present one of the key problems in the associative PI-theory.

As is well known, for every nonzero verbally prime T-ideal Γ, there is a multilinear central polynomial. Over a field of characteristic zero, this follows from the results of the papers [3]–[4]; namely, Kemer has proved that every prime variety is either $\mathrm{Var}(M_{n,k})$ or $\mathrm{Var}(M_n(G))$, Razmyslov constructed central polynomials for varieties of the first kind, and Okhitin, for varieties of the second kind. The existence of central polynomials for prime varieties over a field of positive characteristic was proved by Belov in [6].

In the present paper, we generalize results of Regev and Diniz by proving the primality property for the central polynomials for arbitrary verbally prime T-ideals, both over a field of characteristic zero and over a field of positive characteristic.

Theorem 1. *Let Γ be a verbally prime T-ideal over a field of arbitrary characteristic. If the membership relation $[fg, y] \in \Gamma$ holds for a multilinear polynomial $[fg, y]$, then one of the following three cases occurs*:

1) $f \in \Gamma$;

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2) $g \in \Gamma$;

3) *both polynomials* f *and* g *are central for the* T*-ideal* Γ.

Theorem 1 is a consequence of Belov's results (see [6]) on stability, on the existence of weak identities, and on central polynomials for prime varieties.

Recall the definition of *stable* T-ideal. If $f(x_1, \dots, x_n)$ is a multilinear polynomial, write

$$f(x_1, \dots, x_n) = \sum_i u_i x_k v_i,$$

where u_i, v_i are some polynomials. Introduce the reflection operator A_{x_k} with respect to the variable x_k on the space P_n of multilinear polynomials in variables $x_1, \dots, x_n$,

$$A_{x_k}\left(\sum_i u_i x_k v_i\right) = \sum_i v_i x_k u_i.$$

A T-ideal Γ is said to be *stable* if, for any n, the spaces $P_n \cap \Gamma$ are closed with respect to all reflection operators $A_{x_1} \dots A_{x_n}$. The notion of stable T-ideal was introduced by Latyshev in [7] and [8]. The stability of verbally prime T-ideals over fields of characteristic zero was proved by Okhitin in [5]. Okhitin's proof uses the classification of prime varieties over fields of characteristic zero which was obtained by Kemer in [3] and, therefore, it cannot be carried over to the case of positive characteristic of the ground field.

Over fields of positive characteristic, the stability of an arbitrary verbally prime T-ideal was proved by Belov in [6]. It can readily be shown that, for a stable T-ideal, the existence of a central polynomial is equivalent to the existence of an identity which is weak with respect to some variable (a polynomial $f(x_1, \dots, x_n)$ is said to be an *identity of a* T*-ideal* Γ which is *weak with respect to a variable* x_1 if $f(x_1, \dots, x_n) \notin \Gamma$ and $f([x, y], x_2 \dots, x_n) \in \Gamma$). In the same paper, Belov proved that every verbally prime T-ideal admits weak identities, which is equivalent (due to the stability property) to the existence of central polynomials.

Proof of Theorem 1. We shall prove that, if $[fg, y] \in \Gamma$, then either $f \in \Gamma$ or $[g, x] \in \Gamma$. It can proved in a quite similar way that either $g \in \Gamma$ or $[f, x] \in \Gamma$. These two assertions, taken together, are equivalent to the theorem. Everywhere in the proof, all polynomials are multilinear, and all congruences are modulo Γ.

Note the equation

$$\sum_{i=1}^{n} g(x_1, \dots, [x_i, x], \dots, x_n) = [g(x_1, \dots, x_n), x].$$

This implies that, if $f(g(x_1, \dots, x_n), y_2, \dots, y_m) \in \Gamma$, then

$$f([g(x_1, \dots, x_n), x], y_2, \dots, y_m) \in \Gamma.$$

We shall say that, in this case, the polynomial $f([g, x], \dots)$ is obtained from the polynomial $f(g, \dots)$ by the substitution $g \to [g, x]$.

One can immediately obtain the following property of the reflection operators:

$$A_x([f(x, \dots)g, z]) = \widetilde{f}(g[z, x], \dots),$$

where $\widetilde{f} = \widetilde{f}(x, \dots) = A_x f(x, \dots)$. The assumption of Theorem 1 and this property imply the identity

$$\widetilde{f}(g[z, x]) \equiv 0. \tag{1}$$

In (1) and in what follows, in the representation of the polynomial $\widetilde{f}$, we omit all variables except for the first one.

From identity (1), using the substitution $g \to [a, g]$, we obtain the identity

$$\widetilde{f}([a, g][z, x]) \equiv 0. \tag{2}$$

Substituting $a = ab$ into (2), we have, using (2),

$$0 \equiv \widetilde{f}([ab, g][z, x]) = \widetilde{f}(a[b, g][z, x] + [a, g]b[z, x]) \equiv \widetilde{f}([b, g]a[z, x] + [a, g]b[z, x]). \quad (3)$$

Substitute $z = bz$ into (2),

$$0 \equiv \widetilde{f}([a, g][bz, x]) = \widetilde{f}([a, g]b[z, x] + [a, g][b, x]z). \quad (4)$$

Subtract identity (4) from identity (3):

$$0 \equiv \widetilde{f}([b, g]a[z, x] - [a, g][b, x]z) = \widetilde{f}([b, g][az, x] - [b, g][a, x]z - [a, g][b, x]z),$$

and hence, modulo identity (2),

$$\widetilde{f}([b, g][a, x]z + [a, g][b, x]z) \equiv 0. \quad (5)$$

Applying the operator A_z to (5), we obtain the identity

$$f(z, \dots) \cdot ([b, g][a, x] + [a, g][b, x]) \equiv 0.$$

Since the ideal Γ is verbally prime, it follows that either $f \equiv 0$ (in this case, Theorem 1 is proved) or Γ contains the identity

$$[b, g][a, x] + [a, g][b, x] \equiv 0. \quad (6)$$

Applying the operator A_x to this identity, we obtain

$$[x[b, g], a] + [x[a, g], b] \equiv 0. \quad (7)$$

Substituting a central polynomial $c(z_1, \dots, z_t)$ of a verbally prime T-ideal Γ instead of x into (7), we obtain the identity

$$c(z_1, \dots, z_t) \cdot ([[b, g], a] + [[a, g], b]) \equiv 0.$$

Since the T-ideal Γ is verbally prime and $c(z_1, \dots, z_t) \notin \Gamma$, it follows that

$$[[b, g], a] + [[a, g], b] \equiv 0. \quad (8)$$

Making the substitution $a = ax$ in (8), we obtain, modulo (8),

$$\begin{aligned} 0 &\equiv [[b, g], ax] + [[ax, g], b] = a[[b, g], x] + [[b, g], a]x + [a[x, g], b] + [[a, g]x, b] \\ &= a[[b, g], x] + [[b, g], a]x + a[[x, g], b] + [a, b][x, g] + [a, g][x, b] + [[a, g], b]x \\ &\equiv [a, b][x, g] + [a, g][x, b], \end{aligned}$$

i.e.,

$$[a, x][b, g] + [a, g][b, x] \equiv 0.$$

This identity, together with (6), implies the identity

$$[[u, v], [t, g]] \equiv 0. \quad (9)$$

Making the substitution $g \to [t, g]$ and $b \to [u, v]$ in (6) and using (9), we obtain

$$0 \equiv [[u, v], [t, g]][a, x] + [a, [t, g]][[u, v], x] \equiv [a, [t, g]][[u, v], x]. \quad (10)$$

We conclude from (10) and from the T-primality of the ideal Γ that either $[a, [t, g]] \in \Gamma$ or $[[u, v], x] \in \Gamma$. In any case, the following inclusion holds:

$$[a, [t, g]] \in \Gamma. \quad (11)$$

Substituting $a \to [z, t]$ into (6) and using the Jacobi identity and (11), we can write

$$\begin{aligned} 0 &\equiv [b, g][[z, t], x] + [[z, t], g][b, x] \\ &= [b, g][[z, t], x] - [[t, g], z][b, x] - [[g, z], t][b, x] \equiv [b, g][[z, t], x]. \end{aligned} \quad (12)$$

It follows from this identity and from the T-primality of the ideal Γ that either $[b, g] \in \Gamma$ (in this case, Theorem 1 is proved) or $[[z, t], x] \in \Gamma$. Consider the case $[[z, t], x] \in \Gamma$. The identity $[fg, x] \equiv 0$ can be represented in the form $f[g, x] + [f, x]g \equiv 0$; whence, by the substitution $g \to [g, y]$, we obtain the identity $[f, x][g, y] \equiv 0$. Since the ideal Γ is verbally prime, it follows that either $[g, y] \equiv 0$ (in this case, Theorem 1 is proved) or $[f, x] \equiv 0$. In the last case, it follows from the identity $f[g, x] + [f, x]g \equiv 0$ that $f[g, x] \equiv 0$; whence either $f \equiv 0$ or $[g, x] \equiv 0$, as was to be proved. □

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