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Inequalities between Best Polynomial Approximations and Some Smoothness Characteristics in the Space L_2 and Widths of Classes of Functions

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Abstract—We obtain exact constants in Jackson-type inequalities for smoothness characteristics $\Lambda_k(f)$, $k \in \mathbb{N}$, defined by averaging the kth-order finite differences of functions $f \in L_2$. On the basis of this, for differentiable functions in the classes L_2^r, $r \in \mathbb{N}$, we refine the constants in Jackson-type inequalities containing the kth-order modulus of continuity ω_k. For classes of functions defined by their smoothness characteristics $\Lambda_k(f)$ and majorants Φ satisfying a number of conditions, we calculate the exact values of certain n-widths.

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1. INTRODUCTION

In solving a number of problems dealing with the approximation of functions of a real variable, one often uses various modifications of the classical smoothness characteristic of a function, its modulus of continuity [1], [2], because, in many cases, this is motivated by the specific features of the problems under consideration and leads to new meaningful results (see, for example, [2], [3]). So, to define the effective smoothness characteristics of functions, the papers of Tregub, Runovskii, Pustovoitov, Abilov, and others [3]–[8] dealt with various methods for averaging finite differences as well as with methods for their modification based on the application of smoothing operators, such as the Steklov operator [9]–[13], instead of the shift operator $T_h(f,x) := f(x+h)$. The subject matter of the present paper is similar and it involves the use of a specific smoothness characteristic of functions examined earlier in the paper of Runovskii [4].

Let $L_2 \equiv L_2([0,2\pi])$ be the space of Lebesgue measurable 2π-periodic functions whose norm is

$$\|f\| := \left\{\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2\,dx\right\}^{1/2} < \infty.$$

By the symbol $\Delta_h^k(f,x)$ we denote the kth finite difference of a function $f \in L_2$ at a point x with step width h, i.e.,

$$\Delta_h^k(f,x) := \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} f(x+jh).$$

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On the basis of [4], by the *averaged smoothness characteristic* of a function $f \in L_2$ we shall mean the quantity

$$\Lambda_k(f,t) := \left\{ \frac{1}{t} \int_0^t \|\Delta_h^k(f)\|^2\,dh \right\}^{1/2}, \tag{1.1}$$

where $t > 0$. Let us define, as usual, the kth modulus of continuity of a function $f \in L_2$ by the equality

$$\omega_k(f,t) := \sup_{|h|\le t} \|\Delta_h^k(f)\|. \tag{1.2}$$

It follows from formulas (1.1) and (1.2) that, for any $t > 0$, the following inequality holds:

$$\Lambda_k(f,t) \le \omega_k(f,t). \tag{1.3}$$

Let us point out a number of properties of the quantity (1.1), assuming that the functions f, f_1, f_2 are elements of the space L_2.

Property 1. *The function $\Lambda_k(f,t)$ is a continuous function for $t > 0$.*

Property 2. *The equality $\lim_{t\to 0+0} \Lambda_k(f,t) = 0$ is valid.*

Property 3. *The estimate $\Lambda_k(f,t) \le 2^k\|f\|$ holds for any $t > 0$.*

Property 4. *The following estimate holds: $\Lambda_k(f,nt) \le n^k \Lambda_k(f,t)$, where $n \in \mathbb{N}$ and $t > 0$ are arbitrary numbers.*

Proof. Making the change of variable $h = n\tau$ in formula (1.1), we obtain the relation

$$\Lambda_k(f,nt) = \left\{ \frac{1}{nt} \int_0^{nt} \|\Delta_h^k(f)\|^2\,dh \right\}^{1/2} = \left\{ \frac{1}{t} \int_0^{t} \|\Delta_{n\tau}^k(f)\|^2\,d\tau \right\}^{1/2}. \tag{1.4}$$

Further, let us consider the equality

$$\Delta_{n\tau}^k(f,x) = \sum_{j_1=0}^{n-1} \cdots \sum_{j_k=0}^{n-1} \Delta_\tau^k(f, x + (j_1 + \cdots + j_k)\tau).$$

Using this equality and taking into account the periodicity of the function f, we can write

$$\|\Delta_{n\tau}^k(f)\| \le n^k \|\Delta_\tau^k(f)\|. \tag{1.5}$$

From formulas (1.1), (1.4), and (1.5), we obtain property 4. $\square$

Property 5. *The following inequality holds:*

$$\Lambda_k(f_1 + f_2, t) \le \sqrt{2}\,(\Lambda_k(f_1,t) + \Lambda_k(f_2,t)).$$

Property 6. *The function $\Lambda_k(f,t)$ is almost increasing, i.e., there exists a constant c independent of t such that, for all $0 < t_1 < t_2$, the inequality $\Lambda_k(f,t_1) \le c\Lambda_k(f,t_2)$ holds.*

Proof. Using the sketch of proof for Theorem 3.1 from the paper of Runovskii [4], we can show that, for an arbitrary positive number t, the following inequality holds:

$$\omega_k(f,t) \le c\Lambda_k(f,t), \tag{1.6}$$

where the constant c is independent of $f \in L_2$ and t. From formulas (1.3) and (1.6), for $0 < t_1 < t_2$ and an arbitrary function $f \in L_2$, we obtain

$$\Lambda_k(f,t_1) \le \omega_k(f,t_1) \le \omega_k(f,t_2) \le c\Lambda_k(f,t_2),$$

i.e., property 6 holds. $\square$

2. JACKSON-TYPE INEQUALITIES IN THE CASE OF BEST APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS IN L_2

Let the symbol L_2^r, where $r \in \mathbb{N}$, denote the class of functions $f \in L_2$ whose $(r-1)$th derivatives are absolutely continuous and whose rth derivatives belong to the space L_2. By $\mathcal{T}_{n-1}$ we denote the subspace of trigonometric polynomials of degree $n-1$. For an arbitrary function $f \in L_2$ that has the Fourier series expansion

$$f(x) = \sum_{j=0}^{\infty} \lambda_j (a_j(f)\cos jx + b_j(f)\sin jx),$$

where

$$\lambda_j := \begin{cases} \dfrac{1}{2} & \text{if } j = 0, \\ 1 & \text{if } j \in \mathbb{N}, \end{cases}$$

and the equality is regarded in the sense of convergence in the metric of the space L_2, the value of its best approximation by elements of the subspace $\mathcal{T}_{n-1}$ is

$$E_{n-1}(f) := \inf_{T_{n-1}\in\mathcal{T}_{n-1}} \|f - T_{n-1}\| = \|f - S_{n-1}(f)\| = \left\{\sum_{j=n}^{\infty} \rho_j^2(f)\right\}^{1/2}, \qquad n \in \mathbb{N}. \tag{2.1}$$

Here

$$S_{n-1}(f,x) := \sum_{j=0}^{n-1} \lambda_j (a_j(f)\cos jx + b_j(f)\sin jx)$$

is the $(n-1)$th partial sum of the Fourier series of the function f and $\rho_j^2(f) := a_j^2(f) + b_j^2(f)$.

Recall that by an *inequality of Jackson type* we mean an inequality in which the value of the best approximation of a function by a finite-dimensional subspace is estimated from above by its smoothness characteristic.

In what follows, the ratio $0/0$ will be regarded as zero. Denote

$$\psi_{j,k}(x) := \left\{\frac{1}{x}\int_0^x (1-\cos jh)^k \, dh\right\}^{1/2}, \tag{2.2}$$

where $j, k \in \mathbb{N}$ and $x > 0$. Obviously,

$$\psi_{j,k}(x) = \psi_{1,k}(jx). \tag{2.3}$$

Theorem 1. *Let* $0 < t \le 3\pi/4$. *Then the following relation holds*:

$$\sup_{\substack{f\in L_2 \\ f\not\equiv\text{const}}} \frac{E_{n-1}(f)}{\Lambda_1(f, t/n)} = \frac{1}{\sqrt{2(1-\operatorname{sinc} t)}}, \tag{2.4}$$

where

$$\operatorname{sinc} t := \begin{cases} \dfrac{\sin(t)}{t} & \text{if } t \ne 0, \\ 1 & \text{if } t = 0. \end{cases}$$

Proof. For an arbitrary function $f \in L_2$, we have

$$\|\Delta_h^1(f)\|^2 = 2\sum_{j=1}^{\infty} \rho_j^2(f)(1-\cos jh).$$

Using this equality and formulas (1.1) and (2.2), we obtain

$$\Lambda_1(f,\tau)=\left\{2\sum_{j=1}^{\infty}\rho_j^2(f)\psi_{1,1}^2(j\tau)\right\}^{1/2}\ge\left\{2\sum_{j=n}^{\infty}\rho_j^2(f)(1-\operatorname{sinc} j\tau)\right\}^{1/2}. \tag{2.5}$$

Consider the function

$$(1-\operatorname{sinc}\tau)_0:=\begin{cases}1-\operatorname{sinc}\tau & \text{if } 0<\tau\le\dfrac{3\pi}{4},\\ 1-\dfrac{2\sqrt{2}}{3\pi} & \text{if } \dfrac{3\pi}{4}\le\tau<\infty.\end{cases}$$

From geometric considerations related to the behavior of the function $\operatorname{sinc}\tau$, we obtain the inequality

$$(1-\operatorname{sinc} a\tau)_0\ge(1-\operatorname{sinc} b\tau)_0, \tag{2.6}$$

where $0<\tau<\infty$, $1\le b<a<\infty$ are arbitrary numbers. In view of formulas (2.1) and (2.6), using inequality (2.5), we can write

$$\Lambda_1(f,\tau)\ge\sqrt{2(1-\operatorname{sinc} n\tau)_0}\,E_{n-1}(f).$$

Hence, for $0<\tau\le 3\pi/4$, we have the upper bound

$$\sup_{\substack{f\in L_2\\ f\not\equiv\text{const}}}\frac{E_{n-1}(f)}{\Lambda_1(f,\tau)}\le\frac{1}{\sqrt{2(1-\operatorname{sinc} n\tau)}}. \tag{2.7}$$

To obtain a lower bound for the extremal characteristic located on the left-hand side of equality (2.7), we consider the function $f_0(x):=\cos nx$ belonging to the space L_2. In view of relation (2.5), we have $\Lambda_1(f_0,\tau)=\sqrt{2(1-\operatorname{sinc} n\tau)}$; since $E_{n-1}(f_0)=1$, we have

$$\sup_{\substack{f\in L_2\\ f\not\equiv\text{const}}}\frac{E_{n-1}(f)}{\Lambda_1(f,\tau)}\ge\frac{E_{n-1}(f_0)}{\Lambda_1(f_0,\tau)}=\frac{1}{\sqrt{2(1-\operatorname{sinc} n\tau)}}. \tag{2.8}$$

Setting $\tau:=t/n$, where $0<t\le 3\pi/4$, from inequalities (2.7) and (2.8), we obtain the required equality (2.4). Theorem 1 is proved. □

Theorem 2. *Suppose that $n,r,k\in\mathbb{N}$ and $0<t\le 2\pi$. Then the following equality holds*:

$$\sup_{\substack{f\in L_2^r\\ f\not\equiv\text{const}}}\frac{n^rE_{n-1}(f)}{\Lambda_k(f^{(r)},t/n)}=\frac{1}{2^{k/2}\psi_{1,k}(t)}. \tag{2.9}$$

Proof. Let f be an arbitrary function from the class L_2^r. Since

$$\|\Delta_h^k(f^{(r)})\|^2=2^k\sum_{j=1}^{\infty}j^{2r}\rho_j^2(f)(1-\cos jh)^k,$$

it follows that, for any positive number τ, using formulas (1.1) and (2.2), (2.3), we can write

$$\Lambda_k(f^{(r)},\tau)=\left\{\frac{2^k}{\tau}\int_0^{\tau}\sum_{j=1}^{\infty}j^{2r}\rho_j^2(f)(1-\cos jh)^k\,dh\right\}^{1/2}\ge\left\{2^k\sum_{j=n}^{\infty}j^{2r}\rho_j^2(f)\psi_{1,k}^2(j\tau)\right\}^{1/2}. \tag{2.10}$$

Consider the auxiliary function

$$F_\tau(x):=x^{2r}\psi_{1,k}^2(x\tau)=\frac{x^{2r-1}}{\tau}\int_0^{x\tau}(1-\cos v)^k\,dv, \tag{2.11}$$

where $n \le x < \infty$. It follows from formula (2.11) that F_τ is an increasing function of x for an arbitrary fixed value of $\tau > 0$, i.e.,

$$\inf_{n\le x<\infty} F_\tau(x) = F_\tau(n). \tag{2.12}$$

Using formulas (2.1) and (2.10)–(2.12), we can write

$$\Lambda_k(f^{(r)},\tau) \ge \left\{2^k \sum_{j=n}^{\infty} F_\tau(j)\rho_j^2(f)\right\}^{1/2} \ge 2^{k/2}\, n^r\, \psi_{1,k}(n\tau)E_{n-1}(f).$$

Hence, setting $\tau := t/n$, we obtain the upper bound

$$\sup_{\substack{f\in L_2^r\\ f\not\equiv\mathrm{const}}} \frac{n^r E_{n-1}(f)}{\Lambda_k(f^{(r)},t/n)} \le \frac{1}{2^{k/2}\psi_{1,k}(t)}\,. \tag{2.13}$$

Let us obtain a lower bound for the extremal characteristic under consideration. To do this, we set $f_1(x) := \sin nx$. For $f_1 \in L_2$, using relations (2.1) and (2.10) we can write

$$E_{n-1}(f_1) = 1, \qquad \Lambda_k\left(f_1^{(r)}, \frac{t}{n}\right) = 2^{k/2} n^r \psi_{1,k}(t).$$

Therefore,

$$\sup_{\substack{f\in L_2^r\\ f\not\equiv\mathrm{const}}} \frac{n^r E_{n-1}(f)}{\Lambda_k(f^{(r)},t/n)} \ge \frac{n^r E_{n-1}(f_1)}{\Lambda_k(f_1^{(r)},t/n)} = \frac{1}{2^{k/2}\psi_{1,k}(t)}\,. \tag{2.14}$$

Comparing inequalities (2.13) and (2.14), we obtain the required relation (2.9). Theorem 2 is proved. □

It should be noted that $\lim_{t\to 0+0}\psi_{1,k}(t) = 0$ and, for $0 < t \le \pi$, $\psi_{1,k}$ is an increasing function. Indeed, calculating the first derivative of the function $\psi_{1,k}^2$ and estimating it from below for $0 < t \le \pi$, we obtain

$$\frac{d\psi_{1,k}^2}{dt} = \frac{d}{dt}\left(\frac{1}{t}\int_0^t (1-\cos v)^k\,dv\right) = \frac{1}{t}\left\{(1-\cos t)^k - \frac{1}{t}\int_0^t (1-\cos v)^k\,dv\right\}$$
$$> \frac{1}{t}\{(1-\cos t)^k - (1-\cos t)^k\} = 0.$$

Therefore, $0 < \psi_{1,k}(t_1) < \psi_{1,k}(t_2)$ if $0 < t_1 < t_2 \le \pi$.

3. RELATIONSHIP BETWEEN THEOREMS 1 AND 2 AND THE BEHAVIOR OF THE EXACT CONSTANTS IN JACKSON-TYPE INEQUALITIES FOR THE ORDINARY MODULUS OF CONTINUITY FOR THE CLASSES L_2^r, $r \in \mathbb{N}$

Consider the relationship between the result of Theorem 1 and the behavior of the constants in Jackson-type inequalities for 2π-periodic functions. To do this, we set

$$\mathcal{K}_{n,r}(\omega_k,t) := \sup_{\substack{f\in L_2^r\\ f\not\equiv\mathrm{const}}} \frac{n^r E_{n-1}(f)}{\omega_k(f^{(r)},t/n)}\,, \tag{3.1}$$

where $k,n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $f^{(0)} \equiv f$, $L_2^0 \equiv L_2$, $t > 0$. For $r = 0$, we assume $\mathcal{K}_n(\omega_k,t) := \mathcal{K}_{n,0}(\omega_k,t)$. Recall that, in the case $r = 0$, extremal characteristics of the form (3.1) were studied earlier in the papers of Chernykh, Arestov, Babenko, Vasil'ev, Kozko, and Rozhdestvenskii (see, for example, [14]–[19]). Thus, Chernykh showed [14] that[1]

$$\mathcal{K}_n(\omega_1,t) = \frac{1}{\sqrt{2}} \quad \text{for} \quad t \ge \pi, \qquad \mathcal{K}_n(\omega_1,t) > \frac{1}{\sqrt{2}} \quad \text{for} \quad 0 < t < \pi,$$

[1] *Translator's note*. Here and elsewhere, C_n^m stands for the binomial coefficient $\binom{n}{m}$.

$$\mathcal{K}_n(\omega_k, t) = \frac{1}{\sqrt{C_{2k}^k}} \quad \text{for} \quad n > k \geq 2, \quad 2\pi \leq t < \frac{2\pi n}{k}.$$

As to the behavior of the quantity $\mathcal{K}_n(\omega_1, t)$, Arestov and Chernykh noted [15] that, in the case $0 < t < \pi$, it is nonincreasing.

At present, the following equality is known:

$$\mathcal{K}_n(\omega_k, t) = \frac{1}{\sqrt{C_{2k}^k}} \qquad \text{for} \quad n \in \mathbb{N}, \quad t \geq \frac{7}{5}\pi.$$

Recall that the upper bound for $\mathcal{K}_n(\omega_k, t)$ was established by Vasil'ev [17] and the lower bound was obtained by Kozko and Rozhdestvenskii [18].

For $0 < t \leq 3\pi/4$, using inequalities (1.3) and Theorem 1, we derive the upper bound

$$\mathcal{K}_n(\omega_1, t) \leq \frac{1}{\sqrt{2(1 - \operatorname{sinc} t)}}. \tag{3.2}$$

From inequality (1.6), we obtain the lower bound

$$\frac{1}{c\sqrt{2(1 - \operatorname{sinc} t)}} \leq \mathcal{K}_n(\omega_1, t), \tag{3.3}$$

where the constant $c > 0$ is independent of t and $f \in L_2$. Relations (3.2), (3.3) imply the order-sharp estimate

$$\mathcal{K}_n(\omega_1, t) \asymp \frac{1}{\sqrt{2(1 - \operatorname{sinc} t)}}, \tag{3.4}$$

where $0 < t \leq 3\pi/4$, $n \in \mathbb{N}$. From our point of view, the order-sharp estimate (3.4) can be regarded as an extension of the corresponding result from [15] related to the study of the behavior of $\mathcal{K}_n(\omega_1, t)$.

Using inequality (1.3), from Theorem 2, we obtain

$$E_{n-1}(f) \leq \frac{\omega_k(f^{(r)}, t/n)}{2^{k/2} n^r \psi_{1,k}(t)},$$

where $f \in L_2^r$, $r, k, n \in \mathbb{N}$, $0 < t \leq 2\pi$; hence we have the following upper bound for $\mathcal{K}_{n,r}(\omega_k, t)$ (3.1):

$$\mathcal{K}_{n,r}(\omega_k, t) \leq \frac{1}{2^{k/2}\psi_{1,k}(t)}. \tag{3.5}$$

Taking into account inequality (1.6), formula (3.1), and Theorem 2, we can write

$$\frac{1}{c 2^{k/2}\psi_{1,k}(t)} \leq \mathcal{K}_{n,r}(\omega_k, t).$$

Therefore, the following relation holds in the sense of weak equivalence for the extremal characteristic (3.1):

$$\mathcal{K}_{n,r}(\omega_k, t) \asymp \frac{1}{2^{k/2}\psi_{1,k}(t)}, \tag{3.6}$$

where $n, r, k \in \mathbb{N}$; $0 < t \leq 2\pi$. In the case $n, r, k \in \mathbb{N}$ and $t = 2\pi$, Chernykh's result (mentioned above; see [14]) implies the upper bound

$$\mathcal{K}_{n,r}(\omega_k, 2\pi) \leq \frac{1}{\sqrt{C_{2k}^k}}. \tag{3.7}$$

Using the formula

$$\left(2\sin\frac{v}{2}\right)^{2k} = C_{2k}^k - 2\sum_{j=1}^{k}(-1)^{j+1} C_{2k}^{k-j} \cos jv \tag{3.8}$$

and relation (2.2), we can show the validity of the formula

$$2^{k/2}\psi_{1,k}(m\pi)=\sqrt{C_{2k}^k}, \qquad m\in\mathbb{N}.$$

Using this equality and relation (3.5), we obtain the upper bound for $\mathcal{K}_{n,r}(\omega_k)$ at the point $t=\pi$, i.e.,

$$\mathcal{K}_{n,r}(\omega_k,\pi)\le\frac{1}{\sqrt{C_{2k}^k}}. \tag{3.9}$$

Naturally, in the case $r\in\mathbb{N}$, the question of whether inequalities (3.7), (3.9) are sharp remains open; however, the fact (noted above) that, in view of formula (3.6), the function $\psi_{1,k}(t)$ is monotone increasing on the set $0<t\le\pi$ allows us to draw certain conclusions about the behavior of $\mathcal{K}_{n,r}(\omega_k,t)$ (3.1) for $t\in(0,\pi]$, for example, that this quantity is monotone decreasing on the point set $0<t\le\pi$ at the rate given by the right-hand side of relation (3.6).

4. JACKSON-TYPE INEQUALITIES FOR THE AVERAGED WEIGHTED SMOOTHNESS CHARACTERISTIC Λ_k FOR THE CLASSES L_2^r, $r\in\mathbb{N}$

Let

$$\eta_{j,k,r,p}(\varphi,\tau):=\biggl\{j^{rp}\int_0^\tau\psi_{1,k}^p(jt)\varphi(t)\,dt\biggr\}^{1/p}. \tag{4.1}$$

The following theorem can be regarded as an extension of a result of Ligun (see, for example, [20, Theorem 1]) to the case of the smoothness characteristic (1.1) under consideration in this paper.

Theorem 3. *Suppose that $n,r,k\in\mathbb{N}$, $0<p\le2$, $\tau\in(0,2\pi/n]$ is an arbitrary number, and φ is a nonnegative summable (on $[0,\tau]$) function not equivalent to zero. Then the following inequalities hold*:

$$\frac{1}{2^{k/2}\eta_{n,k,r,p}(\varphi,\tau)}\le\sup_{\substack{f\in L_2^r\\ f\not\equiv\mathrm{const}}}\frac{E_{n-1}(f)}{\{\int_0^\tau\Lambda_k^p(f^{(r)},t)\varphi(t)\,dt\}^{1/p}}\le\frac{1}{2^{k/2}\inf_{n\le j<\infty}\eta_{j,k,r,p}(\varphi,\tau)}. \tag{4.2}$$

Proof. Let us use the following version of Minkowski's inequality given in the monograph of Pinkus [21, p. 104];

$$\biggl\{\int_0^\tau\biggl(\sum_{j=n}^\infty|\widetilde{f}_j(t)|^2\biggr)^{p/2}dt\biggr\}^{1/p}\ge\biggl\{\sum_{j=n}^\infty\biggl(\int_0^\tau|\widetilde{f}_j(t)|^p\,dt\biggr)^{2/p}\biggr\}^{1/2}.$$

Setting $\widetilde{f}_j:=f_j\varphi^{1/p}$, where $j=n,n+1,\dots$, we obtain

$$\biggl\{\int_0^\tau\biggl(\sum_{j=n}^\infty|f_j(t)|^2\biggr)^{p/2}\varphi(t)dt\biggr\}^{1/p}\ge\biggl\{\sum_{j=n}^\infty\biggl(\int_0^\tau|f_j(t)|^p\varphi(t)\,dt\biggr)^{2/p}\biggr\}^{1/2}. \tag{4.3}$$

Using formulas (2.10), (4.3), (2.1), and (2.3), for an arbitrary function $f\in L_2^r$, $f\not\equiv\mathrm{const}$, we can write

$$\begin{aligned}\biggl\{\int_0^\tau\Lambda_k^p(f^{(r)},t)\varphi(t)\,dt\biggr\}^{1/p}&\ge\biggl\{2^{kp/2}\int_0^\tau\biggl(\sum_{j=n}^\infty j^{2r}\rho_j^2(t)\psi_{j,k}^2(t)\biggr)^{p/2}\varphi(t)\,dt\biggr\}^{1/p}\\ &\ge2^{k/2}\biggl\{\sum_{j=n}^\infty\biggl(j^{rp}\rho_j^p(f)\int_0^\tau\psi_{j,k}^p(t)\varphi(t)\,dt\biggr)^{2/p}\biggr\}^{1/2}\\ &=2^{k/2}\biggl\{\sum_{j=n}^\infty\rho_j^2(f)\eta_{j,k,r,p}^2(\varphi,\tau)\biggr\}^{1/2}\ge2^{k/2}E_{n-1}(f)\inf_{n\le j<\infty}\eta_{j,k,r,p}(\varphi,\tau).\end{aligned} \tag{4.4}$$

Hence we have the upper bound

$$\sup_{\substack{f\in L_2^r\\ f\not\equiv\mathrm{const}}} \frac{E_{n-1}(f)}{\{\int_0^\tau \Lambda_k^p(f^{(r)},t)\varphi(t)\,dt\}^{1/p}} \le \frac{1}{2^{k/2}\inf_{n\le j<\infty}\eta_{j,k,r,p}(\varphi,\tau)}. \tag{4.5}$$

To obtain a lower bound for the extremal characteristic on the left-hand side of equality (4.5), we consider the function $f_1(x) := \sin nx$ belonging to the class L_2^r. Using formulas (2.1) and (2.10) for f_1, we obtain

$$E_{n-1}(f_1) = 1, \qquad \Lambda_k(f_1^r, t) = 2^{k/2} n^r \psi_{1,k}(nt).$$

Therefore,

$$\sup_{\substack{f\in L_2^r\\ f\not\equiv\mathrm{const}}} \frac{E_{n-1}(f)}{\{\int_0^\tau \Lambda_k^p(f^{(r)},t)\varphi(t)\,dt\}^{1/p}} \ge \frac{E_{n-1}(f_1)}{\{\int_0^\tau \Lambda_k^p(f_1^{(r)},t)\varphi(t)\,dt\}^{1/p}}$$
$$= \frac{1}{2^{k/2}\{\int_0^\tau n^{rp}\psi_{1,k}^p(nt)\varphi(t)\,dt\}^{1/p}} = \frac{1}{2^{k/2}\eta_{n,k,r,p}(\varphi,\tau)}. \tag{4.6}$$

The double inequality (4.2) follows from relations (4.5), (4.6). This concludes the proof of Theorem 3. □

5. COROLLARIES OF THEOREM 3

Let us consider a few corollaries of Theorem 3.

Corollary 1. *Suppose that $n, r \in \mathbb{N}$, $0 < p \le 2$, $0 < \tau \le 3\pi/(4n)$, φ is a nonnegative summable (on $[0,\tau]$) function not equivalent to zero. Then the following equality holds*:

$$\sup_{\substack{f\in L_2^r\\ f\not\equiv\mathrm{const}}} \frac{n^r E_{n-1}(f)}{\{\int_0^\tau \Lambda_1^p(f^{(r)},t)\varphi(t)\,dt\}^{1/p}} = \frac{1}{\sqrt{2}\,\{\int_0^\tau (1-\operatorname{sinc} nt)^{p/2}\varphi(t)\,dt\}^{1/p}}. \tag{5.1}$$

Proof. In view of formula (4.2), to obtain relation (5.1), it suffices to verify the relation

$$\inf_{n\le j<\infty} \eta_{j,1,r,p}(\varphi,\tau) = \eta_{n,1,r,p}(\varphi,\tau) \tag{5.2}$$

if $0 < \tau \le 3\pi/(4n)$. Using formulas (2.2), (2.3), and (4.1), we can write

$$\eta_{j,1,r,p}^p(\varphi,\tau) = j^{rp}\int_0^\tau (1-\operatorname{sinc} jt)^{p/2}\varphi(t)\,dt. \tag{5.3}$$

Taking into account the fact that the function $\operatorname{sinc} y$ is monotone decreasing on the set $(0, 3\pi/4]$ and the relation $\operatorname{sinc}(3\pi/4) = \sup\{\operatorname{sinc} y : 3\pi/4 < y < \infty\}$, which can be verified by studying the behavior of the function under consideration, we obtain the inequality $\operatorname{sinc} y \ge \operatorname{sinc} xy$, valid for $0 < y \le 3\pi/4$ and $1 \le x < \infty$. Using this inequality, we obtain

$$x^\nu (1-\operatorname{sinc} xy)^\alpha \ge (1-\operatorname{sinc} y)^\alpha, \tag{5.4}$$

where ν and α are arbitrary positive numbers. Setting $x := j/n$, where $j, n \in \mathbb{N}$ and $j \ge n$, $y := nt$ $(0 < t \le \tau)$, $\nu := rp$, $\alpha := p/2$, and using inequality (5.4) we can write

$$j^{rp}(1-\operatorname{sinc} jt)^{p/2} \ge n^{rp}(1-\operatorname{sinc} nt)^{p/2}. \tag{5.5}$$

Multiplying both sides of inequality (5.5) by the function $\varphi(t)$, then integrating them over the variable t from 0 to τ, and using formula (5.3), we obtain

$$\eta_{j,1,r,p}(\varphi,\tau) \ge \eta_{n,1,r,p}(\varphi,\tau)$$

for any natural number $j \ge n$. Thus, relation (5.2) holds, and Corollary 1 is proved. □

Corollary 2. *Suppose that* $n, r, k \in \mathbb{N}$, $1/r \le p \le 2$, $\tau \in (0, 2\pi/n]$ *is an arbitrary number, and* $\varphi \equiv 1$. *Then the following equality holds*:

$$\sup_{\substack{f \in L_2^r \\ f \not\equiv \mathrm{const}}} \frac{E_{n-1}(f)}{\{\int_0^\tau \Lambda_k^p(f^{(r)}, t)\,dt\}^{1/p}} = \frac{1}{2^{k/2}\eta_{n,k,r,p}(1,\tau)}. \tag{5.6}$$

Proof. In view of the form of the quantity $\eta_{j,k,r,p}(1,\tau)$ (see formula (4.1)), to prove the equality

$$\inf_{n \le j < \infty} \eta_{j,k,r,p}(1,\tau) = \eta_{n,k,r,p}(1,\tau), \tag{5.7}$$

it suffices to show that the function

$$\gamma(x) := x^{rp} \int_0^\tau \psi_{1,k}^p(xt)\,dt$$

is nondecreasing for $x \ge n$. To do this, let us calculate its first derivative

$$\gamma'(x) := rpx^{rp-1} \int_0^\tau \psi_{1,k}^p(xt)\,dt + x^{rp} \int_0^\tau \frac{\partial}{\partial x}(\psi_{1,k}^p(xt))\,dt. \tag{5.8}$$

Using formulas (2.2), (2.3), we can readily verify the equality

$$\frac{\partial}{\partial x}(\psi_{1,k}^p(xt)) = \frac{t}{x}\frac{\partial}{\partial t}(\psi_{1,k}^p(xt)) \tag{5.9}$$

where t, x are nonzero numbers. From formula (5.8), with relation (5.9) taken into account, we see that

$$\gamma'(x) := x^{rp-1}\biggl\{rp \int_0^\tau \psi_{1,k}^p(xt)\,dt + \int_0^\tau t\frac{\partial}{\partial t}(\psi_{1,k}^p(xt))\,dt\biggr\}. \tag{5.10}$$

Integrating by parts the second integral on the right-hand side of relation (5.10), we obtain

$$\gamma'(x) := x^{rp-1}\biggl\{\tau\psi_{1,k}^p(x\tau) + (rp-1) \int_0^\tau \psi_{1,k}^p(xt)\,dt\biggr\}. \tag{5.11}$$

Since, in view of formulas (2.2), (2.3), the function $\psi_{1,k}$ is positive on the set $(0,\infty)$ and also $p \in [1/r, 2]$, it follows from relation (5.11) that $\gamma'(x) \ge 0$. Therefore,

$$\inf\{\gamma^{1/p}(x) : n \le x < \infty\} = \gamma^{1/p}(n),$$

which implies that relation (5.7) holds. From Theorem 3 and formula (5.7), we obtain the required equality (5.6). This concludes the proof of Corollary 2. □

Let $\tau_* := \beta/n$, where $\beta \in (0, 2\pi]$ is an arbitrary number, and let $\varphi_*(t) := g(nt)$. Then, for any $j \ge n$, we can write

$$\eta_{j,k,r,p}(\varphi_*, \tau_*) = \biggl\{j^{rp} \int_0^{\beta/n} \psi_{1,k}^p(jt) g(nt)\,dt\biggr\}^{1/p} = n^{r-1/p}\biggl\{\biggl(\frac{j}{n}\biggr)^{rp} \int_0^\beta \psi_{1,k}^p(jt/n) g(t)\,dt\biggr\}^{1/p}. \tag{5.12}$$

Hence we have

$$\inf_{n \le j < \infty} \eta_{j,k,r,p}\biggl(\varphi_*, \frac{\beta}{n}\biggr) \ge n^{r-1/p} \inf_{1 \le x < \infty} x^{rp} \int_0^\beta \psi_{1,k}^p(xt) g(t)\,dt. \tag{5.13}$$

Using formulas (5.12), (5.13), and Theorem 3, we obtain the following corollary.

Corollary 3. *Suppose that* $n, r, k \in \mathbb{N}$, $0 < p \le 2$, $\beta \in (0, 2\pi]$ *is an arbitrary number and* g *is a nonnegative summable (on* $[0,\beta]$*) function not equivalent to zero. Then the following inequalities hold*:

$$\frac{1}{2^{k/2}\{\mu_{k,r,p}(\beta, g, 1)\}^{1/p}} \le \sup_{\substack{f \in L_2^r \\ f \not\equiv \mathrm{const}}} \frac{n^r E_{n-1}(f)}{\{\int_0^\beta \Lambda_k^p(f^{(r)}, t/n) g(t)\,dt\}^{1/p}} \le \frac{1}{2^{k/2}\{\inf_{1 \le x < \infty} \mu_{k,r,p}(\beta, g, x)\}^{1/p}},$$

where

$$\mu_{k,r,p}(\beta,g,x) := x^{rp}\int_0^\beta \psi_{1,k}^p(xt)g(t)\,dt. \tag{5.14}$$

If the function g satisfies the condition

$$\inf_{1\le x<\infty}\mu_{k,r,p}(\beta,g,x)=\mu_{k,r,p}(\beta,g,1), \tag{5.15}$$

then the following relation holds:

$$\sup_{\substack{f\in L_2^r\\ f\not\equiv\text{const}}}\frac{n^rE_{n-1}(f)}{\{\int_0^\beta \Lambda_k^p(f^{(r)},t/n)g(t)\,dt\}^{1/p}}=\frac{1}{2^{k/2}\{\mu_{k,r,p}(\beta,g,1)\}^{1/p}}. \tag{5.16}$$

Further, let us find in what cases condition (5.15) holds.

Corollary 4. *Suppose that $n,r,k\in\mathbb{N}$, $0<p\le 2$, $\beta\in(0,2\pi]$, $g(t):=t^{rp-1}g_1(t)$, where g_1 is a nonincreasing nonnegative summable (on $[0,\beta]$) function not equivalent to zero. Then, for the function g, condition* (5.15) *holds and equality* (5.16) *is valid.*

Proof. To obtain the required result, we shall need the auxiliary function

$$g_2(t):=\begin{cases} g_1(t) & \text{if } 0\le t\le\beta,\\ g_1(\beta) & \text{if } \beta\le t<\infty.\end{cases}$$

For any values of $x\in[1,\infty)$, in view of formula (5.14), we have

$$\begin{aligned}\mu_{k,r,p}(\beta,t^{rp-1}g_1(t),x)&=x^{rp}\int_0^\beta \psi_{1,k}^p(xt)t^{rp-1}g_1(t)\,dt=\int_0^{\beta x}\psi_{1,k}^p(t)t^{rp-1}g_2\Big(\frac{t}{x}\Big)\,dt\\ &\ge\int_0^{\beta x}\psi_{1,k}^p(t)t^{rp-1}g_2(t)\,dt\ge\int_0^{\beta}\psi_{1,k}^p(t)t^{rp-1}g_1(t)\,dt=\mu_{k,r,p}(\beta,t^{rp-1}g_1(t),1),\end{aligned}$$

i.e., condition (5.15) holds, and hence the following relation is valid:

$$\sup_{\substack{f\in L_2^r\\ f\not\equiv\text{const}}}\frac{n^rE_{n-1}(f)}{\{\int_0^\beta \Lambda_k^p(f^{(r)},t/n)t^{rp-1}g_1(t)\,dt\}^{1/p}}=\frac{1}{2^{k/2}\{\mu_{k,r,p}(\beta,t^{rp-1}g_1(t),1)\}^{1/p}}.$$

Corollary 4 is proved. □

Corollary 5. *Suppose that $n,k\in\mathbb{N}$, $\beta\in(0,2\pi]$, g is a function differentiable at each point of the interval $(0,\beta)$, nonnegative, and summable on $[0,\beta]$, and not equivalent to zero, satisfying the condition*

$$(\widetilde r\widetilde p-1)g(t)\ge g'(t)t \tag{5.17}$$

for some $\widetilde r\in\mathbb{N}$ and $0<\widetilde p\le 2$ and for any $t\in(0,\beta)$ such that $\lim_{t\to0+0}g(t)t=0$. Then, for the given values of $\widetilde r$, $\widetilde p$, and the function g relation (5.15) *holds, and hence also so does relation* (5.16).

Proof. Let us consider the function

$$F(x):=\mu_{k,\widetilde r,\widetilde p}(\beta,g,x),\qquad\text{where}\quad 1\le x<\infty,$$

and, taking into account formula (5.14), study the behavior of its first derivative

$$\frac{dF}{dx}=\widetilde r\widetilde p x^{\widetilde r\widetilde p-1}\int_0^\beta\psi_{1,k}^{\widetilde p}(xt)g(t)\,dt+x^{\widetilde r\widetilde p}\int_0^\beta g(t)\frac{\partial}{\partial x}(\psi_{1,k}^{\widetilde p}(xt))\,dt. \tag{5.18}$$

Using formula (5.9), from relation (5.18) we obtain

$$\frac{dF}{dx}=x^{\widetilde r\widetilde p-1}\bigg\{\widetilde r\widetilde p\int_0^\beta\psi_{1,k}^{\widetilde p}(xt)g(t)\,dt+\int_0^\beta g(t)t\frac{\partial}{\partial t}(\psi_{1,k}^{\widetilde p}(xt))\,dt\bigg\}. \tag{5.19}$$

In view of formulas (2.2), (2.3), we have $\lim_{t\to 0+0}\psi_{1,k}(xt)=0$. It follows from the conditions for Corollary 5 that $\lim_{t\to 0+0} g(t)t=0$; hence, obviously, we have

$$\lim_{t\to 0+0} g(t)t\psi_{1,k}^{\widetilde{p}}(xt)=0.$$

In view of this relation, after integrating by parts the second integral on the right-hand side of (5.19), we obtain

$$\frac{dF}{dx}=x^{\widetilde{r}\widetilde{p}-1}\biggl\{\psi_{1,k}^{\widetilde{p}}(\beta x)g(\beta)\beta+\int_0^\beta \psi_{1,k}^{\widetilde{p}}(xt)((\widetilde{r}\widetilde{p}-1)g(t)-g'(t)t)\,dt\biggr\}.$$

It follows from inequality (5.17) and the fact that the functions $\psi_{1,k}$ and g are nonnegative that $dF/dx\ge 0$, i.e., the function F is nondecreasing on the set $1\le x<\infty$ for the values of $\widetilde{r}$, $\widetilde{p}$, and g indicated above. This implies that condition (5.15) holds and formula (5.16) is valid. Corollary 5 is proved. □

Let us show how this result can be used. Let $0<\beta\le\pi/n$. For the weight function, we consider the function $g_*(t):=\sin(nt/2)+0.5\sin nt$, used by Chernykh (see, for example, [14]). Set $r,k\in\mathbb{N}$, $2/r\le p\le 2$, and $l(x):=\operatorname{sinc}x-\cos x$, where $0<x\le\pi$. It is easy to show that, for any $x\in[0,\pi]$, the inequality $l(x)\ge 0$ holds. Then, for an arbitrary value of $t\in[0,\beta]$, we can write

$$\begin{aligned}(rp-1)g_*(t)-g_*'(t)t&=(rp-1)\biggl(\sin\frac{nt}{2}+\frac{1}{2}\sin nt\biggr)-\frac{nt}{2}\biggl(\cos\frac{nt}{2}+\cos nt\biggr)\\&\ge\frac{nt}{2}\biggl(l\biggl(\frac{nt}{2}\biggr)+l(nt)\biggr)\ge 0.\end{aligned}$$

Since the function g_* satisfies condition (5.17) for $r\in\mathbb{N}$ and $2/r\le p\le 2$ and for any $t\in(0,\beta)$, where $0<\beta\le\pi/n$, $n\in\mathbb{N}$, it follows from formula (5.16) that

$$\begin{aligned}&\sup_{\substack{f\in L_2^r\\ f\not\equiv\text{const}}}\frac{n^rE_{n-1}(f)}{\{\int_0^\beta\Lambda_k^p(f^{(r)},t/n)(\sin(nt/2)+(1/2)\sin nt)\,dt\}^{1/p}}\\&\qquad=\frac{1}{2^{k/2}\{\int_0^\beta\psi_{1,k}^p(t)(\sin(nt/2)+(1/2)\sin nt)\,dt\}^{1/p}}.\end{aligned}$$

6. EXACT VALUES OF THE n-WIDTHS OF THE CLASSES OF FUNCTIONS $W(\Lambda_1,\Phi)$ FROM L_2

Before stating other results, let us recall some necessary notions and definitions. Let $\mathbb{B}$ be the unit ball in L_2, let $\mathcal{M}$ be a convex centrally symmetric set from L_2, let $\mathcal{L}_n\subset L_2$ be an n-dimensional subspace, let $\mathcal{L}^n\subset L_2$ be a subspace of codimension n, let $V\colon L_2\to\mathcal{L}_n$ be a continuous linear operator, and let $V^\perp\colon L_2\to\mathcal{L}_n$ be a continuous linear projection operator. The quantities

$$\begin{aligned}b_n(\mathcal{M};L_2)&=\sup\{\sup\{\varepsilon>0:\varepsilon\mathbb{B}\cap\mathcal{L}_{n+1}\subset\mathcal{M}\}:\mathcal{L}_{n+1}\subset L_2\},\\d_n(\mathcal{M};L_2)&=\inf\bigl\{\sup\{\inf\{\|f-\varphi\|:\varphi\in\mathcal{L}_n\}:f\in\mathcal{M}\}:\mathcal{L}_n\subset L_2\bigr\},\\\delta_n(\mathcal{M};L_2)&=\inf\bigl\{\inf\{\sup\{\|f-Vf\|:f\in\mathcal{M}\}:VL_2\subset\mathcal{L}_n\}:\mathcal{L}_n\subset L_2\bigr\},\\d^n(\mathcal{M};L_2)&=\inf\{\sup\{\|f\|:f\in\mathcal{M}\cap\mathcal{L}^n\}:\mathcal{L}^n\subset L_2\},\\\Pi_n(\mathcal{M};L_2)&=\inf\bigl\{\inf\{\sup\{\|f-V^\perp f\|:f\in\mathcal{M}\}:V^\perp L_2\subset\mathcal{L}_n\}:\mathcal{L}_n\subset L_2\bigr\}\end{aligned}$$

are called, respectively, the *Bernstein, Kolmogorov, linear, Gelfand, and projection n-widths of $\mathcal{M}$* in L_2. Since L_2 is a Hilbert space, it follows that the following relations between these n-widths hold:

$$b_n(\mathcal{M};L_2)\le d^n(\mathcal{M};L_2)\le d_n(\mathcal{M};L_2)=\delta_n(\mathcal{M};L_2)=\Pi_n(\mathcal{M};L_2).\qquad(6.1)$$

Recall that, in the space L_2, the exact values of the n-widths of the classes of differentiable 2π-periodic functions defined by their moduli of continuity and other smoothness characteristics were calculated in the papers of Taikov [22], Shalaev [23], Esmaganbetov [24], and others (see, for example, [6]–[8], [11]–[13]).

Using the definition of the smoothness characteristic (1.1), we consider the following class of functions. Let $\Phi(\tau)$, where $0 \le \tau \le 2\pi$, be a continuous increasing function such that $\Phi(0) = 0$. In what follows, it will be called a *majorant*. By the symbol $W(\Lambda_1, \Phi)$ we denote the class of functions $f \in L_2$ for which the inequality $\Lambda_1(f, \tau) \le \Phi(\tau)$ holds for any $0 < \tau \le 2\pi$.

Let t_* denote the value of the argument of the function $\operatorname{sinc}\tau$ for which it attains its minimum on the set $(0, 2\pi]$. Obviously, t_* is the least positive root of the equation $\tau = \tan\tau$ ($4.49 < t_* < 4.51$) (see, for example, [7], [13]).

Set [7], [8]

$$(1-\operatorname{sinc}\tau)_* := \begin{cases} 1-\operatorname{sinc}\tau & \text{if } 0 \le \tau \le t_*, \\ 1-\operatorname{sinc}t_* & \text{if } t_* \le \tau < \infty. \end{cases}$$

For a set $\mathcal{M} \subset L_2$, we denote $E_{n-1}(\mathcal{M}) := \sup\{E_{n-1}(f) : f \in \mathcal{M}\}$. The following theorem is valid.

Theorem 4. *Suppose that, for all numbers $0 < \tau \le 2\pi$ and $n \in \mathbb{N}$, the majorant Φ satisfies the condition*

$$\frac{\Phi(\tau)}{\Phi(\pi/(2n))} \ge \left\{\frac{\pi(1-\operatorname{sinc}n\tau)_*}{\pi-2}\right\}^{1/2}. \tag{6.2}$$

Then the following relations hold:

$$q_{2n-1}(W(\Lambda_1,\Phi);L_2) = q_{2n}(W(\Lambda_1,\Phi);L_2) = E_{n-1}(W(\Lambda_1,\Phi)) = \sqrt{\frac{\pi}{2(\pi-2)}}\,\Phi\left(\frac{\pi}{2n}\right), \tag{6.3}$$

where $q_n(\,\cdot\,)$ is any one of the n-widths indicated above and the set of majorants Φ satisfying inequality (6.2)*, is nonempty.*

Proof. Using relation (2.4) in which we put $t := \pi/2$, we obtain the following upper bound for the value of the best polynomial approximation of an arbitrary function $f \in L_2$:

$$E_{n-1}(f) \le \sqrt{\frac{\pi}{2(\pi-2)}}\,\Lambda_1\left(f, \frac{\pi}{2n}\right).$$

In view of the definition of the class of functions $W(\Lambda_1, \Phi)$, using the last inequality and formula (6.1), we obtain the upper bounds

$$\begin{aligned} q_{2n}(W(\Lambda_1,\Phi);L_2) &\le q_{2n-1}(W(\Lambda_1,\Phi);L_2) \le d_{2n-1}(W(\Lambda_1,\Phi);L_2) \\ &\le E_{n-1}(W(\Lambda_1,\Phi)) \le \sqrt{\frac{\pi}{2(\pi-2)}}\,\Phi\left(\frac{\pi}{2n}\right). \end{aligned} \tag{6.4}$$

By relation (6.1), in order to obtain lower bounds for the n-widths of the class $W(\Lambda_1, \Phi)$, it suffices to find a lower bound for its Bernstein n-width. To do this, in the subspace of trigonometric polynomials $\mathcal{T}_n$ of degree n, we consider the ball

$$\mathbb{B}_{2n+1} := \left\{T_n \in \mathcal{T}_n : \|T_n\| \le \sqrt{\frac{\pi}{2(\pi-2)}}\,\Phi\left(\frac{\pi}{2n}\right)\right\}.$$

Using formula (2.5), for an arbitrary polynomial $T_n \in \mathcal{T}_n$, we obtain

$$\Lambda_1(T_n,\tau) = \left\{2\sum_{j=1}^{\infty}\rho_j^2(T_n)(1-\operatorname{sinc}j\tau)\right\}^{1/2} \le \left\{2(1-\operatorname{sinc}n\tau)_*\right\}^{1/2}\|T_n\|. \tag{6.5}$$

Using inequalities (6.5) and conditions (6.2), for an arbitrary polynomial $T_n \in \mathbb{B}_{2n+1}$, for any $\tau \in (0, 2\pi]$, we can write

$$\Lambda_1(T_n,\tau) \le \left\{\frac{\pi(1-\operatorname{sinc}n\tau)_*}{\pi-2}\right\}^{1/2}\Phi\left(\frac{\pi}{2n}\right) \le \Phi(\tau).$$

Thus, the inclusion $\mathbb{B}_{2n+1} \subset W(\Lambda_1, \Phi)$ holds. Using the definition of the Bernstein n-width and relation (6.1), we obtain

$$q_{2n}(W(\Lambda_1,\Phi);L_2) \geq b_{2n}(W(\Lambda_1,\Phi);L_2) \geq b_{2n}(\mathbb{B}_{2n+1};L_2) \geq \sqrt{\frac{\pi}{2(\pi-2)}}\,\Phi\left(\frac{\pi}{2n}\right). \tag{6.6}$$

The required equalities (6.3) are derived by comparing the upper bounds (6.4) with the lower bounds (6.6).

Let us show that the set of majorants satisfying condition (6.2), is nonempty. To do this, we consider, for example, the function $\Phi_*(\tau) = \tau^{\alpha/2}$ and specify a value of the constant α for which inequality (6.2) will hold for Φ_*. Substituting into formula (6.2) the value of Φ_* instead of Φ, we obtain the inequality

$$(\tau n)^{\alpha} \geq \frac{\pi^{1+\alpha}}{2^{\alpha}(\pi-2)}(1-\operatorname{sinc} n\tau)_*, \tag{6.7}$$

where $\tau \in (0, 2\pi]$. In the authors' paper [25], it was shown in the proof of the main theorem that inequality (6.7) holds for $\alpha := 2/(\pi-2)$. Therefore, the function $\Phi_*(\tau) = \tau^{1/(\pi-2)}$ satisfies relation (6.2). Theorem 4 is proved. □

7. EXACT VALUES OF THE n-WIDTHS OF THE CLASSES OF FUNCTIONS $W^r(\Lambda_k, \Phi)$, $r, k \in \mathbb{N}$, IN L_2

Let $r, k \in \mathbb{N}$, and let Φ be an arbitrary majorant. By the symbol $W^r(\Lambda_k, \Phi)$ we denote the class of functions $f \in L_2^r$ whose rth derivatives satisfy the condition $\Lambda_k(f^{(r)}, \tau) \leq \Phi(\tau)$, where $0 < \tau \leq 2\pi$.

Theorem 5. *Suppose that $n, k, r \in \mathbb{N}$, the function $\psi_{1,k}$ is defined by formulas* (2.2), (2.3)*, and $q_n(\,\cdot\,)$ is any one of the n-widths considered in Sec. 6. If, for an arbitrary $0 < \tau \leq 2\pi$ and $n \in \mathbb{N}$, the majorant Φ satisfies the condition*

$$\frac{\Phi(\tau)}{\Phi(\pi/n)} \geq \sqrt{\frac{2^k}{C_{2k}^k}}\,\psi_{1,k}(n\tau), \tag{7.1}$$

then the following relations hold:

$$q_{2n-1}(W^r(\Lambda_k,\Phi);L_2) = q_{2n}(W^r(\Lambda_k,\Phi);L_2) = E_{n-1}(W^r(\Lambda_k,\Phi)) = \frac{1}{\sqrt{C_{2k}^k}\,n^r}\,\Phi\left(\frac{\pi}{n}\right). \tag{7.2}$$

Here the set of majorants satisfying inequality (7.1)*, is nonempty.*

Proof. For an arbitrary function $f \in L_2^r$ and for $t := \pi$, using formula (2.9), we can write

$$E_{n-1}(f) \leq \frac{1}{n^r 2^{k/2}\psi_{1,k}(\pi)}\Lambda_k\left(f^{(r)}, \frac{\pi}{n}\right) = \frac{1}{\sqrt{C_{2k}^k}\,n^r}\Lambda_k\left(f^{(r)}, \frac{\pi}{n}\right).$$

Using the definition of the class $W^r(\Lambda_k, \Phi)$ and formula (6.1), we obtain the upper bounds

$$\begin{aligned} q_{2n}(W^r(\Lambda_k,\Phi);L_2) &\leq q_{2n-1}(W^r(\Lambda_k,\Phi);L_2) \leq d_{2n-1}(W^r(\Lambda_k,\Phi);L_2) \\ &\leq E_{n-1}(W^r(\Lambda_k,\Phi)) \leq \frac{1}{\sqrt{C_{2k}^k}\,n^r}\,\Phi\left(\frac{\pi}{n}\right). \end{aligned} \tag{7.3}$$

To derive the lower bounds for the same n-widths of the class $W^r(\Lambda_k, \Phi)$, it suffices, by virtue of formula (6.1), to find the lower bound for its Bernstein n-width. To do this, in the space $\mathcal{T}_n$, consider the ball

$$\mathbb{B}_{2n+1}^* := \left\{T_n \in \mathcal{T}_n : \|T_n\| \leq \frac{1}{\sqrt{C_{2k}^k}\,n^r}\,\Phi\left(\frac{\pi}{n}\right)\right\}.$$

Let T_n be an arbitrary polynomial belonging to $\mathcal{T}_n$. Using formulas (2.10), (2.11), we obtain

$$\Lambda_k(T_n^{(r)},\tau)=\left\{2^k\sum_{j=1}^{n} j^{2r}\rho_j^2(f)\psi_{1,k}^2(j\tau)\right\}^{1/2}=\left\{2^k\sum_{j=1}^{n}\rho_j^2(f)F_\tau(j)\right\}^{1/2},$$

where the function F_τ is defined by formula (2.11). Taking into account the fact that F_τ is an increasing function of its argument, we can write

$$\Lambda_k(T_n^{(r)},\tau)\le 2^{k/2}F_\tau^{1/2}(n)\|T_n\|=2^{k/2}n^r\psi_{1,k}(n\tau)\|T_n\|. \tag{7.4}$$

For an arbitrary polynomial $T_n\in\mathbb{B}_{2n+1}^*$, from formula (7.4) and condition (7.1), we obtain the inequality

$$\Lambda_k(T_n^{(r)},\tau)\le\frac{2^{k/2}}{\sqrt{C_{2k}^k}}\psi_{1,k}(n\tau)\Phi\left(\frac{\pi}{n}\right)\le\Phi(\tau),$$

where $0<\tau\le 2\pi$. Therefore, the ball $\mathbb{B}_{2n+1}^*$ belongs to the class $W^r(\Lambda_k,\Phi)$. Using relation (6.1) and the definition of the Bernstein n-width, we obtain the lower bounds

$$q_{2n}(W^r(\Lambda_k,\Phi);L_2)\ge b_{2n}(W^r(\Lambda_k,\Phi);L_2)\ge b_{2n}(\mathbb{B}_{2n+1}^*;L_2)\ge\frac{1}{\sqrt{C_{2k}^k}\,n^r}\,\Phi\left(\frac{\pi}{n}\right). \tag{7.5}$$

Comparing inequalities (7.3) and (7.5), we obtain the required equalities (7.2).

Further, let us show that the set of majorants satisfying condition (7.1), is nonempty. To do this, consider, for example, the function $\widetilde{\Phi}(\tau):=\tau^{\beta/2}$, where

$$\beta:=\frac{2^{2k}}{C_{2k}^k}-1. \tag{7.6}$$

It is known that, for a fixed n and $k\le[n/2]$ (where $[a]$ is the integer part of a number $a\in\mathbb{R}$), the quantity C_n^k increases and, for $k\ge[n/2]$, decreases and $2^n=\sum_{j=0}^n C_n^j$. Setting $n=2k$ and using this formula from combinatorics, in view of (7.6), we can write

$$\beta=\sum_{j=0}^{2k}\frac{C_{2k}^j}{C_{2k}^k}-1=2\sum_{j=0}^{k-1}\frac{C_{2k}^j}{C_{2k}^k}<2k,$$

i.e.,

$$0<\beta<2k, \tag{7.7}$$

In view of formulas (2.2), (2.3), condition (7.1) for the function $\widetilde{\Phi}$ takes the following form:

$$(n\tau)^{\beta+1}\ge\pi^\beta\frac{2^k}{C_{2k}^k}\int_0^{n\tau}(1-\cos h)^k\,dh.$$

Setting $t:=n\tau$, we obtain the inequality

$$t^{\beta+1}\ge\pi^\beta\frac{2^k}{C_{2k}^k}\int_0^{t}(1-\cos h)^k\,dh, \tag{7.8}$$

where $0\le t<\infty$. Let us prove that relation (7.8) holds; to do this, we introduce the auxiliary function

$$H(t):=t^{\beta+1}-\frac{\pi^\beta}{C_{2k}^k}\int_0^t\left(2\sin\frac{h}{2}\right)^{2k}dh. \tag{7.9}$$

As $t\to 0+0$, from formula (7.9), we obtain

$$H(t):=t^{\beta+1}\left(1-\frac{\pi^\beta}{C_{2k}^k(2k+1)}t^{2k-\beta}\right). \tag{7.10}$$

In view of inequality (7.7), it follows from relation (7.10) that, in an infinitely small neighborhood of zero $(0,\varepsilon)$, the function H takes positive values. Let us show that, on the whole interval $(0,\infty)$, it will be function of constant sign.

Let $0\le t\le\pi$. Arguing by contradiction, we assume that there exists a point $\xi\in(0,\pi)$ at which the function H changes sign. Since, in view of formulas (3.8), (7.6), and (7.9), we have $H(0)=H(\pi)=0$, it follows that, by Rolle's theorem, the first derivative

$$H'(t)=(\beta+1)t^{\beta}-\frac{\pi^{\beta}}{C_{2k}^{k}}\left(2\sin\frac{t}{2}\right)^{2k}$$

must have at least two different zeros on the interval $(0,\pi)$. In view of formulas (7.6), we can write

$$H'(t)=\frac{2^{2k}}{C_{2k}^{k}}\pi^{\beta}\left\{\left(\frac{t}{\pi}\right)^{\beta}-\left(\sin\frac{t}{2}\right)^{2k}\right\}=\frac{2^{2k}}{C_{2k}^{k}}\pi^{\beta}G(t).$$

Obviously, the function

$$G(t):=\left(\frac{t}{\pi}\right)^{\beta}-\left(\sin\frac{t}{2}\right)^{2k}$$

vanishes on the interval $(0,\pi)$ at the same points as the derivative H'. It is readily verified that this also applies to the functions

$$G_*(t):=\left(\frac{t}{\pi}\right)^{\beta/(2k)}-\sin\frac{t}{2},$$

i.e., G_* must have at least two different zeros on the interval $(0,\pi)$. Since $G_*(0)=G_*(\pi)=0$, it follows that, by Rolle's theorem, the first derivative

$$G_*'(t)=\frac{\beta}{2k\pi^{\beta/(2k)}}t^{\beta/(2k)-1}-\frac{1}{2}\cos\frac{t}{2} \tag{7.11}$$

must vanish at at least three different points on $(0,\pi)$. However, in view of formulas (7.7) and (7.11), G_*' is the difference of a convex (downward) monotone decreasing function and a monotone decreasing convex (upward) function. It follows from geometric considerations that, on the interval $(0,\pi)$, the function G_* has at most two different zeros. The resulting contradiction proves that, for any $t\in(0,\pi)$, the function $H(t)$ takes positive values.

Now let $\pi\le t\le 2\pi$. It follows from formula (7.11) that, on the closed interval $[\pi,2\pi]$, the derivative $G_*'(t)$ is positive. Hence, for $\pi\le t\le2\pi$, the function G_* is monotone increasing. Hence, in view of the equality $G_*(\pi)=0$, we have

$$G_*(t)>0\qquad\text{if}\quad t\in(\pi,2\pi].$$

It follows from the above that the derivative H' will also be nonnegative on the set $[\pi,2\pi]$, i.e., H is a nondecreasing function. Using the equality $H(\pi)=0$, we obtain the inequality $H(t)\ge0$, which is valid for any $t\in[\pi,2\pi]$.

Further, let $2\pi\le t<\infty$. Taking into account the form of the derivative H', we can conclude that, on this set, it takes only positive values, i.e., $H(t)$ will be a monotone increasing function for $2\pi\le t<\infty$. Taking into account relation (3.8), from formula (7.9) we obtain

$$H(2\pi):=(2\pi)^{\beta+1}-\frac{\pi^{\beta}}{C_{2k}^{k}}\int_0^{2\pi}\left(C_{2k}^{k}-2\sum_{j=1}^{k}(-1)^{j+1}C_{2k}^{k-j}\cos jh\right)dh=2\pi^{\beta+1}(2^{\beta}-1)>0,$$

i.e., $H(t)>0$ for any $t\in[2\pi,\infty)$. Summarizing the obtained results, we see that

$$H(t)\ge0\qquad\text{for}\quad 0\le t<\infty,$$

which implies that condition (7.1) for the majorant $\widetilde{\Phi}$ holds. Theorem 5 is proved. □

8. EXACT VALUES OF THE n-WIDTHS OF THE CLASSES OF FUNCTIONS $\widetilde{W}_p^r(\Lambda_1,\Phi)$, $r\in\mathbb{N}$, $0<p\le 2$, FROM L_2

Let the symbol $\widetilde{W}_p^r(\Lambda_1,\Phi)$, where $r\in\mathbb{N}$, $0<p\le 2$, and Φ is a majorant, denote the class of functions $f\in L_2^r$ whose rth derivatives $f^{(r)}$ satisfy the condition

$$\int_0^\tau \Lambda_1^p(f^{(r)},t)\,dt\le \Phi^p(\tau)\qquad \text{for any}\quad 0<\tau\le 2\pi.$$

Theorem 6. *Let $r\in\mathbb{N}$; $1/r\le p\le 2$; $q_n(\,\cdot\,)$ be any n-width considered in Sec. 6. If the majorant Φ satisfies the condition*

$$\frac{\Phi^p(\tau)}{\Phi^p(\pi/n)}\ge\frac{\int_0^{n\tau}(1-\operatorname{sinc}t)^{p/2}\,dt}{\int_0^{\pi}(1-\operatorname{sinc}t)^{p/2}\,dt}\tag{8.1}$$

for all values of $\tau\in(0,2\pi]$ and $n\in\mathbb{N}$, then the following equalities hold:

$$\begin{aligned}q_{2n}(\widetilde{W}_p^r(\Lambda_1,\Phi);L_2)&=q_{2n-1}(\widetilde{W}_p^r(\Lambda_1,\Phi);L_2)=E_{n-1}(\widetilde{W}_p^r(\Lambda_1,\Phi))\\&=\frac{n^{1/p-r}}{\sqrt2}\biggl\{\int_0^\pi(1-\operatorname{sinc}t)^{p/2}\,dt\biggr\}^{-1/p}\Phi\biggl(\frac{\pi}{n}\biggr).\end{aligned}\tag{8.2}$$

Here the set of majorants Φ satisfying inequality (8.1)*, is nonempty.*

Proof. Setting $\tau:=\pi/n$ in formula (5.6) and using relations (4.1) and (2.2), (2.3), for an arbitrary function $f\in L_2^r$, we obtain the following upper bounds for the value of its best polynomial approximation:

$$E_{n-1}(f)\le\frac{n^{1/p-r}}{\sqrt2}\biggl\{\int_0^\pi(1-\operatorname{sinc}t)^{p/2}\,dt\biggr\}^{-1/p}\biggl\{\int_0^{\pi/n}\Lambda_1^p(f^{(r)},t)\,dt\biggr\}^{1/p}.$$

By the definition of the class $\widetilde{W}^r(\Lambda_1,\Phi)$ and formula (6.1), we then have

$$\begin{aligned}q_{2n}(\widetilde{W}_p^r(\Lambda_1,\Phi);L_2)&\le q_{2n-1}(\widetilde{W}_p^r(\Lambda_1,\Phi);L_2)\le E_{n-1}(\widetilde{W}_p^r(\Lambda_1,\Phi))\\&\le\frac{n^{1/p-r}}{\sqrt2}\biggl\{\int_0^\pi(1-\operatorname{sinc}t)^{p/2}\,dt\biggr\}^{-1/p}\Phi\biggl(\frac{\pi}{n}\biggr).\end{aligned}\tag{8.3}$$

To obtain lower bounds for the same n-widths in the subspace of trigonometric polynomials $\mathcal{T}_n$, we consider the ball

$$\widetilde{\mathbb{B}}_{2n+1}:=\biggl\{T_n\in\mathcal{T}_n:\|T_n\|\le\frac{n^{1/p-r}}{\sqrt2}\biggl(\int_0^\pi(1-\operatorname{sinc}t)^{p/2}\,dt\biggr)^{-1/p}\Phi\biggl(\frac{\pi}{n}\biggr)\biggr\};$$

let us show the validity of the inclusion $\widetilde{\mathbb{B}}_{2n+1}\subset\widetilde{W}_p^r(\Lambda_1,\Phi)$. Using formula (7.4), for an arbitrary polynomial $T_n\in\mathcal{T}_n$ and for any $\tau\in(0,2\pi]$, we obtain

$$\int_0^\tau\Lambda_1^p(T_n^{(r)},t)\,dt\le 2^{p/2}n^{rp}\int_0^\tau(1-\operatorname{sinc}nt)^{p/2}\,dt\|T_n\|^p.$$

Then, for an arbitrary polynomial $T_n\in\widetilde{\mathbb{B}}_{2n+1}$, using conditions (8.1), we can write

$$\int_0^\tau\Lambda_1^p(T_n^{(r)},t)\,dt\le\frac{\int_0^{n\tau}(1-\operatorname{sinc}t)^{p/2}\,dt}{\int_0^{\pi}(1-\operatorname{sinc}t)^{p/2}\,dt}\,\Phi^p\biggl(\frac{\pi}{n}\biggr)\le\Phi^p(\tau),$$

where $0<\tau\le 2\pi$. Therefore, the ball $\widetilde{\mathbb{B}}_{2n+1}$ belongs to the class $\widetilde{W}_p^r(\Lambda_1,\Phi)$. Using formula (6.1) and the definition of the Bernstein n-width, we obtain the following lower bounds for the extremal characteristics (under consideration) of the class $\widetilde{W}_p^r(\Lambda_1,\Phi)$:

$$q_{2n}(\widetilde{W}_p^r(\Lambda_1,\Phi);L_2)\ge b_{2n}(\widetilde{W}_p^r(\Lambda_1,\Phi);L_2)\ge b_{2n}(\widetilde{\mathbb{B}}_{2n+1};L_2)$$

$$\geq \frac{n^{1/p-r}}{\sqrt{2}} \left\{ \int_0^\pi (1-\operatorname{sinc} t)^{p/2}\, dt \right\}^{-1/p} \Phi\left(\frac{\pi}{n}\right). \tag{8.4}$$

Comparing the upper bounds (8.3) with the lower bounds (8.4), we obtain the required equalities (8.2).

Further, let us show that the set of majorants satisfying condition (8.1), is nonempty. To do this, let us consider, for example, the function $\widetilde{\Phi}(\tau) := \tau^{\gamma/p}$, where

$$\gamma := \frac{\pi}{\int_0^\pi (1-\operatorname{sinc} t)^{p/2}\, dt}. \tag{8.5}$$

For it, condition (8.1) takes the following form:

$$\left(\frac{\tau n}{\pi}\right)^\gamma \geq \frac{\int_0^{n\tau} (1-\operatorname{sinc} t)^{p/2}\, dt}{\int_0^\pi (1-\operatorname{sinc} t)^{p/2}\, dt}.$$

Setting $v := \tau n$, where $0 \leq v < \infty$, and using formula (8.5), we obtain the inequality

$$v^\gamma \geq \gamma \pi^{\gamma-1} \int_0^v (1-\operatorname{sinc} t)^{p/2}\, dt, \tag{8.6}$$

which we must prove. As a preliminary let us calculate upper and lower bounds for γ. To derive an upper bound, we need the inequality

$$1-\operatorname{sinc} t > \left(\frac{t}{\pi}\right)^2, \qquad \text{where} \quad t \in (0,\pi).$$

To establish this inequality, we set

$$\theta(t) := 1-\operatorname{sinc} t - \left(\frac{t}{\pi}\right)^2 = \frac{t-\sin t - t^3/\pi^2}{t} =: \frac{\theta_1(t)}{t}.$$

As $t \to 0+0$, we have

$$\theta_1(t) = t^3\left(\frac{1}{3!} - \frac{1}{\pi^2} + O(t^2)\right),$$

i.e., θ_1 is a positive function in an infinitely small neighborhood on the right of zero. Let us show that θ_1 is a function of constant sign on the interval $(0,\pi)$. To do this, arguing by contradiction, we assume that, on the interval $(0,\pi)$, there exists a point at which θ_1 changes sign. Since $\theta_1(0) = \theta_1(\pi) = 0$, it follows that, by Rolle's theorem, the first derivative

$$\theta_1'(t) = 1-\cos t - \frac{3t^2}{\pi^2}$$

must have at least two different zeros on $(0,\pi)$. Taking into account the equality $\theta_1'(0) = 0$ and using a similar considerations, we see that the second derivative

$$\theta_1''(t) = \sin t - \frac{6t}{\pi^2}$$

must also have at least two different zeros on the interval $(0,\pi)$. However, from geometric considerations, it is obvious that θ_1'' has only one zero on $(0,\pi)$. The resulting contradiction proves the validity of the desired inequality, using which, together with formula (8.5), we obtain

$$\gamma < \frac{\pi}{\int_0^\pi (t/\pi)^p\, dt} = 1+p. \tag{8.7}$$

It follows from geometric considerations that, for any $t \in (0,\pi)$, the inequality $\operatorname{sinc} t > 1 - t/\pi$ holds. In this connection, from relation (8.5), we obtain

$$\gamma > \frac{\pi}{\int_0^\pi (t/\pi)^{p/2}\, dt} = 1+\frac{p}{2}. \tag{8.8}$$

Taking into account formula (8.6), let us consider the auxiliary function

$$Q(v) := v^\gamma - \gamma\pi^{\gamma-1}\int_0^v (1-\operatorname{sinc} t)^{p/2}\,dt \tag{8.9}$$

and show that it is nonnegative on the set $0 \le v < \infty$. The arguments are carried out in three stages, depending on the values of the variable v:

a) $0 \le v \le \pi$;

b) $\pi \le v \le 2\pi$;

c) $2\pi \le v < \infty$.

a) Let $0 \le v \le \pi$. For $0 \le t \le \pi$, the inequality $\sin t \ge t - t^3/6$ holds; hence, from relation (8.9), we obtain

$$Q(v) \ge v^\gamma\left(1 - \frac{\gamma\pi^{\gamma-1}}{6^{p/2}(1+p)}v^{p+1-\gamma}\right). \tag{8.10}$$

It follows from formulas (8.7) and (8.10) that, as $v \to 0+0$, the function Q takes only positive values. Let us show that Q is a function of constant sign on the interval $(0,\pi)$. To do this, we argue by contradiction, assuming that there exists a point $\xi \in (0,\pi)$ such that the function Q changes sign as its argument v passes through it. Using formula (8.5), from relation (8.9), we obtain $Q(0) = Q(\pi) = 0$. Hence, in view of Rolle's theorem, we see that the first derivative

$$Q'(v) = \gamma\pi^{\gamma-1}\left(\left(\frac{v}{\pi}\right)^{\gamma-1} - (1-\operatorname{sinc} v)^{p/2}\right) \tag{8.11}$$

must have at least two different zeros on the interval $(0,\pi)$. It follows from formula (8.11) that the function

$$Q_*(v) := \left(\frac{v}{\pi}\right)^{(\gamma-1)2/p} - 1 + \operatorname{sinc} v \tag{8.12}$$

must have a similar number of different zeros at of the same points on $(0,\pi)$. Using formula (8.12), we obtain $Q_*(0) = Q_*(\pi) = 0$. Hence the function Q_* must have at least four different zeros on the closed interval $[0,\pi]$. Taking into account the expression

$$Q_*(v) = \frac{\widetilde{Q}_*(v)}{v}, \tag{8.13}$$

where

$$\widetilde{Q}_*(v) := \pi^{(1-\gamma)2/p}v^{1+(\gamma-1)2/p} - v + \sin v, \tag{8.14}$$

we see that, in view of expression (8.14), the function $\widetilde{Q}_*$ must also have at least four different zeros on the closed interval $[0,\pi]$. Using Rolle's theorem, we see that the first derivative

$$\widetilde{Q}'_*(v) = \pi^{(1-\gamma)2/p}\left(1 + \frac{2}{p}(\gamma-1)\right)v^{(\gamma-1)2/p} - 1 + \cos v, \tag{8.15}$$

must vanish at at least three different points on the interval $(0,\pi)$. Since $\widetilde{Q}'_*(0) = 0$, using similar considerations, we see that the second derivative

$$\widetilde{Q}''_*(v) = \pi^{(1-\gamma)2/p}\left(1 + \frac{2}{p}(\gamma-1)\right)(\gamma-1)\frac{2}{p}v^{(\gamma-1)2/p-1} - \sin v \tag{8.16}$$

must have at least three different zeros on $(0,\pi)$. Taking into account inequality (8.8), from relation (8.16), we obtain $\widetilde{Q}''_*(0) = 0$. Hence the third derivative

$$\widetilde{Q}'''_*(v) = \pi^{(1-\gamma)2/p}\left(1 + \frac{2}{p}(\gamma-1)\right)(\gamma-1)\frac{2}{p}\left((\gamma-1)\frac{2}{p} - 1\right)v^{(\gamma-1)2/p-2} - \cos v \tag{8.17}$$

must vanish at at least three different points on the interval $(0,\pi)$. Using inequalities (8.7), we see that the function $v^{(\gamma-1)2/p-2}$ is positive convex (downward) and monotone decreasing on $(0,\pi)$. Taking into account the behavior of the function $\cos v$ on $(0,\pi)$, we conclude that, in view of formula (8.17), the function $\widetilde{Q}_*'''$ must have at most two different zeros on the interval $(0,\pi)$. The resulting contradiction proves the validity of inequality (8.6) in case a).

Further, consider case b). It follows from formula (8.16) that the function $\widetilde{Q}_*''$ takes positive values on the closed interval $[\pi,2\pi]$, i.e., $\widetilde{Q}_*'$ is a monotone increasing function on $[\pi,2\pi]$. Using formula (8.8), from relation (8.15), we obtain

$$\widetilde{Q}_*'(\pi)=\frac{2}{p}(\gamma-1)-1>0.$$

Therefore, $\widetilde{Q}_*'(v)>0$ for any $v\in[\pi,2\pi]$, which means that the function $\widetilde{Q}_*$ is monotone increasing on this point set. Since $\widetilde{Q}_*(\pi)=0$, it follows that, for $\pi<v\le 2\pi$, the inequality $\widetilde{Q}_*(v)>0$ holds. In view of formulas (8.11)–(8.13), we see that the inequality $Q'(v)>0$ holds for $\pi<v\le 2\pi$. Since $Q(\pi)=0$, it follows that Q is a positive monotone increasing function on the set $(\pi,2\pi]$ and, in view of relation (8.9), this means that, in case b), inequality (8.6) holds.

Let us pass to the study of case c). Analyzing the function (8.15) and taking into account inequality (8.8), for any $2\pi\le v<\infty$, we obtain the following lower bound of the derivative $\widetilde{Q}_*'$:

$$\begin{aligned}\widetilde{Q}_*'(v)&\ge \pi^{(1-\gamma)2/p}\biggl(1+\frac{2}{p}(\gamma-1)\biggr)\min_{2\pi\le v<\infty} v^{(\gamma-1)2/p}-1+\min_{2\pi\le v<\infty}\cos v\\ &=2\biggl\{2^{(\gamma-1)2/p-1}\biggl(1+\frac{2}{p}(\gamma-1)\biggr)-1\biggr\}>2.\end{aligned}$$

This means that $\widetilde{Q}_*'$ takes positive values on $[2\pi,\infty)$, i.e., the function $\widetilde{Q}_*$ is monotone increasing. It follows from formulas (8.8) and (8.14) that

$$\widetilde{Q}_*(2\pi)=2\pi(2^{(\gamma-1)2/p}-1)>0;$$

hence we see that, in view of formulas (8.11)–(8.13), the derivative Q' takes only positive values on the set $[2\pi,\infty)$, i.e., the function Q is monotone increasing. It follows from case b) that $Q(2\pi)>0$, i.e., $Q(v)>0$ for an arbitrary v, $2\pi\le v<\infty$, and inequality (8.6) also holds in case c). Theorem 6 is proved. □

For the classes of functions considered above, the calculation of the exact values of the Fourier cosine and sine coefficients is also of significant interest. Without loss of generality, we shall obtain one of such results following from Theorem 6, because the corresponding results following from Theorems 4 and 5 can be stated and proved in a similar way.

Corollary 6. *Let all the assumptions of Theorem* 6 *hold. Then, for an arbitrary* $n\in\mathbb{N}$, *the following equalities hold*:

$$\sup_{f\in\widetilde{W}_p^r(\Lambda_1,\Phi)}|a_n(f)|=\sup_{f\in\widetilde{W}_p^r(\Lambda_1,\Phi)}|b_n(f)|=\frac{n^{1/p-r}}{\sqrt{2}}\biggl\{\int_0^\pi(1-\operatorname{sinc}t)^{p/2}\,dt\biggr\}^{-1/p}\Phi\biggl(\frac{\pi}{n}\biggr), \qquad (8.18)$$

where $a_n(\,\cdot\,)$ *and* $b_n(\,\cdot\,)$ *are the Fourier cosine and sine coefficients, respectively.*

Proof. Without loss of generality, we obtain the required result for the Fourier cosine coefficient $a_n(\,\cdot\,)$. Since

$$a_n(f)=\frac{1}{\pi}\int_0^{2\pi}(f(t)-S_{n-1}(t))\cos(nt)\,dt,$$

where $S_{n-1}(f)$ is a partial sum of the Fourier series of the function $f \in \widetilde{W}_p^r(\Lambda_1, \Phi)$, using the Cauchy–Bunyakovskii inequality and formula (2.1), we can write

$$|a_n(f)| \le \|f - S_{n-1}(f)\| = E_{n-1}(f).$$

Hence, using relation (8.2), we obtain

$$\sup_{f \in \widetilde{W}_p^r(\Lambda_1,\Phi)} |a_n(f)| \le E_{n-1}(\widetilde{W}_p^r(\Lambda_1, \Phi)) = \frac{n^{1/p-r}}{\sqrt{2}} \left\{ \int_0^\pi (1 - \operatorname{sinc} t)^{p/2}\, dt \right\}^{-1/p} \Phi\left(\frac{\pi}{n}\right). \tag{8.19}$$

To obtain the lower bound, consider the function

$$f_2(x) := \frac{n^{1/p-r}}{\sqrt{2}} \left\{ \int_0^\pi (1 - \operatorname{sinc} t)^{p/2}\, dt \right\}^{-1/p} \Phi\left(\frac{\pi}{n}\right) \cos nx,$$

which, as is readily verified, belongs to the ball $\widetilde{\mathbb{B}}_{2n+1}$, which was introduced in the proof of Theorem 6. Since $\widetilde{\mathbb{B}}_{2n+1} \subset \widetilde{W}_p^r(\Lambda_1, \Phi)$, it follows that f_2 is an element of the class $\widetilde{W}_p^r(\Lambda_1, \Phi)$. Then

$$\sup_{f \in \widetilde{W}_p^r(\Lambda_1,\Phi)} |a_n(f)| \ge |a_n(f_2)| = \frac{n^{1/p-r}}{\sqrt{2}} \left\{ \int_0^\pi (1 - \operatorname{sinc} t)^{p/2}\, dt \right\}^{-1/p} \Phi\left(\frac{\pi}{n}\right). \tag{8.20}$$

The required equality (8.18) is obtained from relations (8.19), (8.20). This concludes the proof of Corollary 6. □

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