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Homotopy Properties of ∞-Simplicial Coalgebras and Homotopy Unital Supplemented A_∞-Algebras

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Abstract—The homotopy theory of ∞-simplicial coalgebras is developed; in terms of this theory, an additional structure on the tensor bigraded coalgebra of a graded module is described such that endowing the coalgebra with this structure is equivalent to endowing the given graded module with the structure of a homotopy unital A_∞-algebra.

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It is well known [1] that endowing a graded module with the structure of an A_∞-algebra is equivalent to endowing the tensor bigraded coalgebra of this module with the structure of a differential bigraded coalgebra. The bigrading of this tensor coalgebra is usually convolved, i.e., the tensor algebra of the suspension of the graded module is considered. The equivalence mentioned above is a very useful tool for investigating the homotopy and category properties of A_∞-algebras, because it reduces studying these properties to examining the corresponding properties of differential free coalgebras.

On the other hand, in [2], the notion of a homotopy unital A_∞-algebra was introduced, which is the homotopy counterpart of the notion of a unital (i.e., having a unit) associative differential algebra. As well as in [1], there arises the important and interesting question of describing an additional structure on the tensor bigraded coalgebra of a graded module such that endowing the coalgebra with this structure is equivalent to endowing the given module with the structure of a homotopy unital A_∞-algebra.

This paper is devoted to the development of the homotopy theory of ∞-simplicial coalgebras; in terms of this theory, an answer to the question posed above is given. The paper consists of three sections. In the first section, we recall the necessary definitions, constructions, and assertions from [3] related to the notion of a differential ∞-simplicial module, which is the homotopy invariant counterpart of the notion of a differential simplicial module. In the second section, we describe the construction of a tensor product of ∞-simplicial modules and introduce the notion of an ∞-simplicial coalgebra. We also prove the homotopy invariance of the structure of an ∞-simplicial coalgebra under homotopy equivalences of the type of SDR-data (strong deformation retractions of special form) of differential coalgebras. In the third section, we introduce the notion of a homotopy unital supplemented A_∞-algebra, which is a homotopy generalization of the notion of an supplemented associative algebra with unit. In the case of connected graded modules, i.e., nonnegatively graded modules for which the module of elements of grade zero is the base ring, the notions of a homotopy unital supplemented A_∞-algebra and a homotopy unital A_∞-algebra coincide. It is proved that endowing a graded module with the structure of a homotopy unital supplemented A_∞-algebra is equivalent to endowing the tensor bigraded coalgebra of this module with the structure of an ∞-simplicial coalgebra. This statement, as applied to connected graded modules, answers the above-posed question about an additional structure on a tensor bigraded coalgebra. On the basis of this equivalence, we obtain a simplicial method for calculating structural relations for homotopy unital supplemented A_∞-algebras and, in particular, for homotopy unital A_∞-algebras, which is simpler than the method for calculating structural relations proposed in [4].

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We also apply this equivalence to prove the homotopy invariance of the structure of a homotopy unital supplemented A_∞-algebra; in particular, for the case of connected graded modules, we obtain a new, in comparison with [4], proof of the homotopy invariance of the structure of a homotopy unital A_∞-algebra.

All modules and maps of modules considered in this paper are, respectively, K-modules and K-linear maps of modules, where K is any commutative ring with unit.

1. COLORED ALGEBRAS OF SIMPLICIAL FACES AND DEGENERACIES AND ∞-SIMPLICIAL MODULES

In what follows, by a *colored graded module* X we mean any family of graded modules $X = \{X(s,t)_m\}$, $m \in \mathbb{Z}$, indexed by all pairs of elements $(s,t) \in I \times I$, where I is a set of nonnegative integers. A *map* $f\colon X \to Y$ of colored graded modules is any family of maps

$$f = \{f(s,t) : X(s,t) \to Y(s,t)\}_{s,t\in I}$$

of graded modules.

The *tensor product* of colored graded modules X and Y is defined as the colored graded module $X \otimes Y$ for which

$$(X \otimes Y)(s,t)_m = \bigoplus_{k\in I} \bigoplus_{p+q=m} X(s,k)_p \otimes Y(k,t)_q.$$

A *colored graded algebra* (A,π) is any colored graded module A endowed with a multiplication $\pi\colon A \otimes A \to A$, which is a map of colored graded modules satisfying the associativity condition $\pi(\pi \otimes 1) = \pi(1 \otimes \pi)$.

The *unit* of a colored algebra (A,π) is a family $1_* = \{1_k\}_{k\in I}$ of elements $1_k \in A(k,k)_0$ such that $\pi(1_s \otimes a) = a = \pi(a \otimes 1_t)$ for each element $a \in A(s,t)_m$, where $s,t \in I$ and $m \in \mathbb{Z}$.

In what follows, by K_I we denote the graded module defined by the relations $K_I(s,s)_m = K$ for $m = 0$ and $s \in I$, $K_I(s,s)_m = 0$ for $m \neq 0$ and $s \in I$, and $K_I(s,t)_m = 0$ for $s \neq t$ and $m \in \mathbb{Z}$ and colored by colors from I. It is easy to see that, using multiplication in the ring K, we can consider K_I as a colored graded algebra (K_I,π).

The base colored graded algebra in this paper is the colored algebra (S,π) of simplicial faces and degeneracies considered in [3]. The colored algebra (S,π) is generated by elements $\partial_i^n \in S(n-1,n)_0$ with $n-1 \in I$ and $s_i^n \in S(n+1,n)_0$ with $n \in I$ and $i \in \mathbb{Z}$, $0 \le i \le n$, subject to the simplicial commutation relations

$$\partial_i^{n-1}\partial_j^n = \partial_{j-1}^{n-1}\partial_i^n, \qquad i<j, \quad n-1 \in I, \tag{1.1}$$

$$s_i^{n+1}s_j^n = s_{j+1}^{n+1}s_i^n, \qquad i \le j, \quad n \in I, \tag{1.2}$$

$$\partial_i^{n+1}s_j^n = \begin{cases} s_{j-1}^{n-1}\partial_i^n, & i<j, \quad n-1 \in I,\\ 1_n, & i=j, \quad i=j+1, \quad n \in I,\\ s_j^{n-1}\partial_{i-1}^n, & i>j+1, \quad n-1 \in I, \end{cases} \tag{1.3}$$

where $1_* = \{1_n\}_{n\in I}$ is the unit of the colored algebra (S,π) and $ab = \pi(a \otimes b)$, $a,b \in (S,\pi)$.

A colored graded coalgebra (C,∇) is defined as the colored graded module $C = \{C(s,t)_m\}_{s,t\in I}$, $m \in \mathbb{Z}$, $m \ge 0$, together with a comultiplication $\nabla\colon C \to C \otimes C$, which is a map of colored graded modules satisfying the condition $(\nabla \otimes 1)\nabla = (1 \otimes \nabla)\nabla$. The notions of a counit $\varepsilon\colon C \to K_I$ and a cosupplementation $\nu\colon K_I \to C$ for a colored graded coalgebra (C,∇) are defined in the standard way.

A *curved colored coalgebra* (C,∇,ϑ) or, briefly, a *colored ϑ-coalgebra* is a graded colored coalgebra (C,∇) together with a map $\vartheta\colon C_\bullet \to (K_I)_{\bullet-2}$ of colored graded modules which has degree (-2) and satisfies the condition

$$\vartheta(c_2')c_{n-2}'' = c_{n-2}'\vartheta(c_2'')$$

for all $c_n \in C(s,t)_n$ with $n \geq 2$ and $s, t \in I$; here the elements $c_2' \in C(s,s)_2$, $c_{n-2}'' \in C(s,t)_{n-2}$, $c_{n-2}' \in C(s,t)_{n-2}$, and $c_2'' \in C(t,t)_2$ are determined from c_n by the relation

$$\nabla(c_n) = \cdots + c_2' \otimes c_{n-2}'' + \cdots + c_{n-2}' \otimes c_2'' + \cdots \in (C \otimes C)(s,t)_n.$$

The map ϑ is called the *curvature* of the colored coalgebra (C, ∇).

In what follows, by the *counit* and the *cosupplementation* of a colored ϑ-coalgebra (C, ∇, ϑ) we understand those of the graded color coalgebra (C, ∇).

The base colored ϑ-coalgebra in this paper is the colored ϑ-coalgebra $(S^!, \nabla, \vartheta)$ considered in [3], which is Koszul dual to the quadratic-scalar colored algebra (S, π).

Let us describe $(S^!, \nabla, \vartheta)$. First, we recall that the *suspension* of a colored graded module X is the colored graded module SX defined by $(SX)(s,t)_{m+1} = X(s,t)_m$ for any $s, t \in I$. The elements of SX are traditionally denoted by $[x]$, where $x \in X$.

Let M denote the colored graded module of the generators of the colored algebra (S, π). Thus, M is determined by the following conditions:

(1) $M(s,t)_m = 0$ for $s, t \in I$ and $m > 0$;

(2) $M(s,t)_0 = 0$ for $(s,t) \neq (n-1, n)$, $n-1 \in I$, and $(s,t) \neq (n+1, n)$, $n \in I$;

(3) $M(n-1, n)_0$ is the free K-module with generators ∂_i^n, where $n-1 \in I$ and $0 \leq i \leq n$;

(4) $M(n+1, n)_0$ is the free K-module with generators s_i^n, where $n \in I$ and $0 \leq i \leq n$.

For this module M and its suspension SM, consider the rearrangement map

$$T \colon SM \otimes SM \to SM \otimes SM$$

of colored graded modules defined at the generators of the colored graded module $SM \otimes SM$ by

$$T([\partial_i^{n-1}] \otimes [\partial_j^n]) = \begin{cases} [\partial_{j-1}^{n-1}] \otimes [\partial_i^n], & i < j, \quad n-2 \in I, \\ [\partial_j^{n-1}] \otimes [\partial_{i+1}^n], & i \geq j, \quad n-2 \in I, \end{cases}$$

$$T([s_i^{n+1}] \otimes [s_j^n]) = \begin{cases} [s_{j+1}^{n+1}] \otimes [s_i^n], & i \leq j, \quad n \in I, \\ [s_j^{n+1}] \otimes [s_{i-1}^n], & i > j, \quad n \in I, \end{cases}$$

$$T([\partial_i^{n+1}] \otimes [s_j^n]) = \begin{cases} [s_{j-1}^{n-1}] \otimes [\partial_i^n], & i < j, \quad n-1 \in I, \\ [s_j^{n-1}] \otimes [\partial_{i-1}^n], & i > j+1, \quad n-1 \in I, \end{cases}$$

$$T([\partial_{i+1}^{n+1}] \otimes [s_i^n]) = [\partial_{i+1}^{n+1}] \otimes [s_i^n], \qquad i \geq 0, \quad n \in I,$$

$$T([\partial_i^{n+1}] \otimes [s_i^n]) = [\partial_i^{n+1}] \otimes [s_i^n], \qquad i \geq 0, \quad n \in I,$$

$$T([s_i^{n-1}] \otimes [\partial_j^n]) = \begin{cases} [\partial_{j+1}^{n+1}] \otimes [s_i^n], & i < j, \quad n-1 \in I, \\ [\partial_j^{n+1}] \otimes [s_{i+1}^n], & i \geq j, \quad n-1 \in I. \end{cases}$$

It is easy to see that the map T satisfies the condition $T^2 = \mathrm{id}$.

Let Σ_n be the symmetric group of permutations on $1, 2, \ldots, n$. We define the action of each transposition $\tau_k = (k+1, k) \in \Sigma_n$, $1 \leq k \leq n-1$, on the colored graded module $(SM)^{\otimes n}$ by

$$\tau_k([a_1] \otimes \cdots \otimes [a_n]) = [a_1] \otimes \cdots \otimes T([a_k] \otimes [a_{k+1}]) \otimes \cdots \otimes [a_n],$$

where $[a_1], \ldots, [a_n]$ are any generators of SM. A straightforward calculation by the formulas for the rearrangement map T shows that the actions of the transpositions τ_k on $(SM)^{\otimes n}$ satisfy the relations

$$\tau_k^2 = \mathrm{id}, \quad 1 \leq k \leq n, \qquad \tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_{k+1}, \quad 1 \leq k \leq n-2,$$
$$\tau_k \tau_m = \tau_m \tau_k, \qquad 1 \leq k \leq n-1, \quad 1 \leq m \leq n-1, \quad |k - m| \geq 2.$$

It follows that the standard procedure for decomposing any permutation $\sigma \in \Sigma_n$ into a product $\tau_{k_q} \cdots \tau_{k_1}$ of transpositions of neighboring numbers determines a left action

$$\nu \colon \Sigma_n \times (SM)^{\otimes n} \to (SM)^{\otimes n}$$

of the group Σ_n on the colored graded module $(SM)^{\otimes n}$ by the rule

$$\nu(\sigma, [a_1] \otimes \cdots \otimes [a_n]) = \tau_{k_q}(\cdots(\tau_{k_1}([a_1] \otimes \cdots \otimes [a_n]))\cdots).$$

The above relations for the actions of τ_k imply that the action ν does not depend on the choice of the decomposition of σ into a product of transpositions of neighboring numbers. It is easy to see that $(SM)^{\otimes n}$ contains elements whose isotropy groups with respect to this action of Σ_n on $(SM)^{\otimes n}$ are not trivial. Given any element $[a_1] \otimes \cdots \otimes [a_n] \in (SM)^{\otimes n}$, let

$$\mathcal{O}([a_1] \otimes \cdots \otimes [a_n]) \subset (SM)^{\otimes n}$$

denote its orbit under the action of Σ_n on $(SM)^{\otimes n}$ specified above.

It follows from the definition of the rearrangement map T that the orbit $\mathcal{O}([a_1] \otimes \cdots \otimes [a_k])$ of any element $[a_1] \otimes \cdots \otimes [a_k] \in (SM)^{\otimes k}(m,n)$, $m,n \in I$, contains precisely one element of the form

$$[\partial^{n+q-p+1}_{i_1}] \otimes \cdots \otimes [\partial^{n+q}_{i_p}] \otimes [s^{n+q-1}_{j_q}] \otimes \cdots \otimes [s^n_{j_1}],$$

where $p \geq 0$, $q \geq 0$, $p+q=k \geq 1$, $m=n+q-p$, $i_1 < \cdots < i_p$, and $j_q > \cdots > j_1$. In what follows, we refer to elements of $(SM)^{\otimes k}$, $k \geq 1$, of this form as *ordered elements*.

Recall the description of the colored ϑ-coalgebra $(S^!, \nabla, \vartheta)$ given in [3]. The colored graded module $S^!$ is defined by the conditions

(1) $(S^!)^{(k)}(m,n)_l = 0$ for $l \neq k$, $k \geq 1$, and $n,m \in I$;

(2) $(S^!)^{(k)}(m,n)_k$ with $k \geq 1$ and $n,m \in I$ is the free K-module with generators

$$[\partial^{n+q-p+1}_{i_1}] \mathbin{\widehat{\wedge}} \cdots \mathbin{\widehat{\wedge}} [\partial^{n+q}_{i_p}] \mathbin{\widehat{\wedge}} [s^{n+q-1}_{j_q}] \mathbin{\widehat{\wedge}} \cdots \mathbin{\widehat{\wedge}} [s^n_{j_1}] = \sum_{\mathcal{O}(\alpha)} (-1)^{\varepsilon}[a_1] \otimes \cdots \otimes [a_{p+q}], \tag{1.4}$$

where $\alpha = [\partial^{n+q-p+1}_{i_1}] \otimes \cdots \otimes [\partial^{n+q}_{i_p}] \otimes [s^{n+q-1}_{j_q}] \otimes \cdots \otimes [s^n_{j_1}]$ is any ordered element, $k=p+q$, $m=n+q-p$, and $\mathcal{O}(\alpha) = \{[a_1] \otimes \cdots \otimes [a_{p+q}]\}$ is the orbit of α; the exponent ε in (1.4) is defined by

$$\varepsilon = \operatorname{sign}([a_1] \otimes \cdots \otimes [a_{p+q}]) = i_1 + \cdots + i_p + j_1 + \cdots + j_q + l_1 + \cdots + l_{p+q},$$

where the numbers $l_1, \dots, l_{p+q}$ are determined by the relation

$$[a_1] \otimes \cdots \otimes [a_{p+q}] = [\nu^{k_1}_{l_1}] \otimes \cdots \otimes [\nu^{k_{p+q}}_{l_{p+q}}]$$

in which $\nu^{k_i}_{l_i}$ is $\partial^{k_i}_{l_i}$ or $s^{k_i}_{l_i}$ for $1 \leq i \leq p+q$, $k_{p+q}=n$, and $\nu^{k_1}_{l_1}$ is $\partial^{m+1}_{l_1}$ or $s^{m-1}_{l_1}$.

For example, it follows from (1.4) that

$$[\partial^n_1] \mathbin{\widehat{\wedge}} [\partial^{n+1}_2] \mathbin{\widehat{\wedge}} [s^n_2] = [\partial^n_1] \otimes [\partial^{n+1}_2] \otimes [s^n_2] - [\partial^n_1] \otimes [\partial^{n+1}_1] \otimes [s^n_2] + [\partial^n_1] \otimes [s^{n-1}_1] \otimes [\partial^n_1],$$
$$[\partial^{n+1}_{i+1}] \mathbin{\widehat{\wedge}} [s^n_i] = [\partial^{n+1}_{i+1}] \otimes [s^n_i], \quad [\partial^{n+1}_i] \mathbin{\widehat{\wedge}} [s^n_i] = [\partial^{n+1}_i] \otimes [s^n_i], \qquad i \geq 0.$$

Consider the comultiplication of the colored graded coalgebra $(S^!, \nabla)$. Let

$$\alpha = [\partial^{n+q-p+1}_{i_1}] \otimes \cdots \otimes [\partial^{n+q}_{i_p}] \otimes [s^{n+q-1}_{j_q}] \otimes \cdots \otimes [s^n_{j_1}]$$

be any ordered element. For each element $\gamma = [a_1] \otimes \cdots \otimes [a_{p+q}] \in \mathcal{O}(\alpha)$, by $P(\gamma)$ we denote the set of all representations of γ in the form

$$\gamma = ([a_1] \otimes \cdots \otimes [a_z]) \otimes ([a_{z+1}] \otimes \cdots \otimes [a_{p+q}]), \qquad 1 \leq z \leq p+q-1,$$

where $[a_1]\otimes\dots\otimes[a_z]$ and $[a_{z+1}]\otimes\dots\otimes[a_{p+q}]$ are ordered elements. The values of the comultiplication ∇ of the coalgebra $S^!$ at the generators $\beta=[\partial^{n+q-p+1}_{i_1}]\mathbin{\widehat{\wedge}}\dots\mathbin{\widehat{\wedge}}[\partial^{n+q}_{i_p}]\mathbin{\widehat{\wedge}}[s^{n+q-1}_{j_q}]\mathbin{\widehat{\wedge}}\dots\mathbin{\widehat{\wedge}}[s^{n}_{j_1}]$ of the module $(S^!)^{(k)}(m,n)_k$, where $m,n\in I$, $k\geq 1$, $p\geq 0$, $q\geq 0$, $p+q=k$, and $m=n+q-p$, are

$$\nabla(\beta)=1_{n+q-p}\otimes\beta + \sum_{P(\gamma\in\mathcal{O}(\alpha))}(-1)^{\operatorname{sign}(\gamma)}([a_1]\mathbin{\widehat{\wedge}}\dots\mathbin{\widehat{\wedge}}[a_z])\otimes([a_{z+1}]\mathbin{\widehat{\wedge}}\dots\mathbin{\widehat{\wedge}}[a_{p+q}])+\beta\otimes 1_n,$$

where $\alpha=[\partial^{n+q-p+1}_{i_1}]\otimes\dots\otimes[\partial^{n+q}_{i_p}]\otimes[s^{n+q-1}_{j_q}]\otimes\dots\otimes[s^{n}_{j_1}]$ and $\gamma=[a_1]\otimes\dots\otimes[a_{p+q}]$.

The *curvature* $\vartheta\colon S^!_\bullet\to(K_I)_{\bullet-2}$ of the colored graded coalgebra $(S^!,\nabla)$ is defined at the generators of $S^!$ specified above by

$$\vartheta([\partial^{n+q-p+1}_{i_1}]\mathbin{\widehat{\wedge}}\dots\mathbin{\widehat{\wedge}}[\partial^{n+q}_{i_p}]\mathbin{\widehat{\wedge}}[s^{n+q-1}_{j_q}]\mathbin{\widehat{\wedge}}\dots\mathbin{\widehat{\wedge}}[s^{n}_{j_1}])=0,\qquad (p,q)\neq(1,1),$$

$$\vartheta([\partial^{n+1}_{i}]\mathbin{\widehat{\wedge}}[s^{n}_{j}])=\begin{cases}1, & i=j,\quad i=j+1,\\ 0 & \text{otherwise.}\end{cases}$$

We proceed to the necessary constructions and facts related to the notion of an ∞-simplicial module [3].

In what follows, by a *differential bigraded module* we mean any differential bigraded module (X,d) of the form $X=\{X_{n,m}\}$, where $n,m\in\mathbb{Z}$, $n\geq 0$, and $d\colon X_{*,\bullet}\to X_{*,\bullet-1}$.

The tensor product of a colored graded module X and a differential bigraded module (Y,d) is defined as the differential bigraded module $(X\otimes Y,d)$, where

$$(X\otimes Y)_{n,m}=\bigoplus_{s\in I}\bigoplus_{p+q=m}X(n,s)_p\otimes Y_{s,q},\qquad n\in I,\quad m\in\mathbb{Z},$$

and the value of the differential at each element $x\otimes y\in X(n,s)_p\otimes Y_{s,q}$ equals

$$d(x\otimes y)=(-1)^{n-s+p}x\otimes d(y).$$

Given any differential bigraded modules (X,d) and (Y,d), consider the differential bigraded module $(\operatorname{Hom}(X;Y),d)$. The elements of each module $\operatorname{Hom}(X;Y)_{n,m}$ are arbitrary maps $f\colon X_{*,\bullet}\to Y_{*+n,\bullet+m}$ of bigraded modules which have bidegree (n,m); at the elements $f\in\operatorname{Hom}(X;Y)_{n,m}$, the differential is given by

$$d(f)=df+(-1)^{n+m+1}fd\colon X_{*,\bullet}\to Y_{*+n,\bullet+m-1}.$$

Given the colored graded coalgebra $(S^!,\nabla)$ and any differential bigraded modules (X,d), (Y,d), and (Z,d), we define a map

$$\cup\colon(\operatorname{Hom}(S^!\otimes Y;Z)\otimes\operatorname{Hom}(S^!\otimes X;Y))_{*,\bullet}\to\operatorname{Hom}(S^!\otimes X;Z)_{*,\bullet}$$

of bigraded modules by setting

$$g\cup f=g(1\otimes f)(\nabla\otimes 1)\quad\text{for } g\in\operatorname{Hom}(S^!\otimes Y;Z)\qquad\text{and}\qquad f\in\operatorname{Hom}(S^!\otimes X;Y).$$

It is easy to show that $\cup$ is a map of differential bigraded modules which has the associativity property, i.e.,

$$d(g\cup f)=d(g)\cup f+(-1)^{n+m}g\cup d(f),\qquad (g\cup f)\cup l=g\cup(f\cup l),$$

where $g\in\operatorname{Hom}(S^!\otimes Y,Z)_{n,m}$.

In what follows, given any $f\in\operatorname{Hom}(X;Y)_{n,m}$, by $\widehat{f}$ we denote the map $\widehat{f}\in\operatorname{Hom}(S^!\otimes X;Y)_{n,m}$ defined by

$$\widehat{f}=(\varepsilon\otimes f)\colon S^!\otimes X\to K_I\otimes Y=Y,$$

where $\varepsilon\colon S^!\to K_I$ is the counit of the colored graded coalgebra $(S^!,\nabla)$.

Now let us consider the notion of differential ∞-simplicial module [3], which is the homotopy invariant counterpart of the notion of a differential simplicial module.

A *differential* ∞-*simplicial module*, or, briefly, an ∞-*simplicial module*, is any differential bigraded module (X, d) together with a map $\psi\colon (S^! \otimes X)_{*,\bullet} \to X_{*,\bullet-1}$ of bigraded modules which has bidegree $(0, -1)$ and satisfies the conditions

(1) $\psi(\nu \otimes 1) = d\colon (K_I \otimes X)_{*,\bullet} = X_{*,\bullet} \to X_{*,\bullet-1}$, where $\nu\colon K_I \to S^!$ is the cosupplementation of the colored ϑ-coalgebra $(S^!, \nabla, \vartheta)$;

(2) $\psi \cup \psi = -\widetilde{\vartheta}$, where the map $\widetilde{\vartheta}\colon (S^! \otimes X)_{*,\bullet} \to X_{*,\bullet-2}$ is defined by

$$\widetilde{\vartheta} = \vartheta \otimes 1\colon (S^! \otimes X)_{*,\bullet} \to (K_I \otimes X)_{*,\bullet-2} = X_{*,\bullet-2}.$$

Representing the structure map $\psi\colon (S^! \otimes X)_{*,\bullet} \to X_{*,\bullet-1}$ of any ∞-simplicial module (X, d, ψ) in the form $\psi = \widehat{d} + \psi'$, where $\widehat{d}$ is defined as $\widehat{f}$ (see above) for $f = d \in \mathrm{Hom}(X; X)_{0,-1}$, we see that the map $\psi' = \psi - \widehat{d}\colon (S^! \otimes X)_{*,\bullet} \to X_{*,\bullet-1}$ satisfies the condition $\psi'(\nu \otimes 1) = 0$, and $\psi \cup \psi = -\widetilde{\vartheta}$ if and only if

$$\widehat{d} \cup \widehat{d} = 0, \quad d(\psi') + \psi' \cup \psi' + \widetilde{\vartheta} = 0.$$

Since the condition $\widehat{d} \cup \widehat{d} = 0$ is equivalent to $d^2 = 0$, it follows that specifying an ∞-simplicial module (X, d, ψ) is equivalent to specifying a triple (X, d, ψ'), where ψ' is a map satisfying the conditions $\psi'(\nu \otimes 1) = 0$ and $d(\psi') + \psi' \cup \psi' + \widetilde{\vartheta} = 0$.

By a *morphism* $f\colon (X, d, \psi) \to (Y, d, \psi)$ of ∞-simplicial modules we mean a map

$$f\colon (S^! \otimes X)_{*,\bullet} \to Y_{*,\bullet}$$

of bigraded modules which has degree $(0, 0)$ and satisfies the condition $\psi \cup f = f \cup \psi$.

It follows from $\psi \cup f = f \cup \psi$ that the map $f_\nu = f(\nu \otimes 1)\colon (K_I \otimes X)_{*,\bullet} = X_{*,\bullet} \to Y_{*,\bullet}$ of bigraded modules satisfies the condition $df_\nu = f_\nu d$, i.e., is a map of differential bigraded modules. Representing a morphism $f\colon (X, d, \psi) \to (Y, d, \psi)$ of ∞-simplicial modules in the form $f = \widehat{f_\nu} + f'$, we see that the map $f' = f - \widehat{f_\nu}\colon (S^! \otimes X)_{*,\bullet} \to Y_{*,\bullet}$ satisfies the condition $f'(\nu \otimes 1) = 0$, and $\psi \cup f = f \cup \psi$ if and only if

$$\widehat{d} \cup \widehat{f_\nu} = \widehat{f_\nu} \cup \widehat{d}, \qquad d(f') - f' \cup \psi' + \psi' \cup f' - \widehat{f_\nu} \cup \psi' + \psi' \cup \widehat{f_\nu} = 0.$$

Moreover, it is clear that $\psi \cup f = f \cup \psi$ implies $\widetilde{\vartheta} \cup f = f \cup \widetilde{\vartheta}$.

The composition $g \circ f$ of morphisms $f\colon (X, d, \psi) \to (Y, d, \psi)$ and $g\colon (Y, d, \psi) \to (Z, d, \psi)$ of ∞-simplicial modules is defined as the morphism $g \cup f\colon (X, d, \psi) \to (Z, d, \psi)$ of ∞-simplicial modules. Clearly, the operation of taking the composition of morphisms is associative; moreover, for each ∞-simplicial module (X, d, ψ), the identity morphism $\widehat{1}_X\colon (X, d, \psi) \to (X, d, \psi)$ is defined, where 1_X is the identity map of the module X. Thus, ∞-simplicial modules and their morphisms form a category.

A *homotopy* $h\colon (X, d, \psi) \to (Y, d, \psi)$ between morphisms

$$f\colon (X, d, \psi) \to (Y, d, \psi) \qquad \text{and} \qquad g\colon (X, d, \psi) \to (Y, d, \psi)$$

of ∞-simplicial modules is defined as a map $h\colon (S^! \otimes X)_{*,\bullet} \to Y_{*,\bullet+1}$ satisfying the condition $\psi \cup h + h \cup \psi = f - g$.

Since $\psi \cup h + h \cup \psi = f - g$, it follows that the map

$$h_\nu = h(\nu \otimes 1)\colon (K_I \otimes X)_{*,\bullet} = X_{*,\bullet} \to Y_{*,\bullet+1}$$

of bigraded modules satisfies the condition $dh_\nu + h_\nu d = f_\nu - g_\nu$, i.e., is a homotopy between the maps f_ν and g_ν of differential bigraded modules. Representing any homotopy h between morphisms $f, g\colon (X, d, \psi) \to (Y, d, \psi)$ of ∞-simplicial modules in the form $h = \widehat{h_\nu} + h'$, we see that the map

$h' = h - \widehat{h}_\nu\colon (S^! \otimes X)_{*,\bullet} \to Y_{*,\bullet+1}$ satisfies the condition $h'(\nu \otimes 1) = 0$, and $\psi \cup h + h \cup \psi = f - g$ if and only if

$$\widehat{d} \cup \widehat{h}_\nu + \widehat{h}_\nu \cup \widehat{d} = \widehat{f}_\nu - \widehat{g}_\nu, \qquad d(h') + h' \cup \psi' + \psi' \cup h' + \widehat{h}_\nu \cup \psi' + \psi' \cup \widehat{h}_\nu = f' - g'.$$

Moreover, it is clear that $\psi \cup h + h \cup \psi = f - g$ implies $\widetilde{\vartheta} \cup h = h \cup \widetilde{\vartheta}$.

Let $\eta\colon (X, d, \psi) \rightleftarrows (Y, d, \psi) : \xi$ be any morphisms of ∞-simplicial modules such that $\eta \cup \xi = \widehat{1}_Y$, and let $h\colon (X, d, \psi) \to (X, d, \psi)$ be any homotopy between the morphisms $\xi \cup \eta$ and $\widehat{1}_X$ of ∞-simplicial modules which satisfies the conditions $\eta \cup h = 0$, $h \cup \xi = 0$, and $h \cup h = 0$. Any such triple $(\eta\colon (X, d, \psi) \rightleftarrows (Y, d, \psi) : \xi, h)$ is called SDR-*data for ∞-simplicial modules*.

Consider the homotopy properties of ∞-simplicial modules. Recall that SDR-*data for differential bigraded modules* is any triple $(\eta\colon (X, d) \rightleftarrows (Y, d) : \xi, h)$, where $\eta\colon X_{*,\bullet} \rightleftarrows Y_{*,\bullet} : \xi$ is a map of differential bigraded modules and $h : X_{*,\bullet} \to X_{*,\bullet+1}$ is a homotopy between $\xi\eta$ and 1_X satisfying the conditions $\eta h = 0$, $h\xi = 0$, and $hh = 0$.

It is worth mentioning that the conditions $\eta h = 0$, $h\xi = 0$, and $hh = 0$, which must hold for SDR-data $(\eta\colon (X, d) \rightleftarrows (Y, d) : \xi, h)$, are not restrictive. Indeed, as shown in [5], if these conditions do not hold, then, defining the new homotopy $h' = h'' d\, h''$, where $h'' = (\xi\eta - 1_X)h(\xi\eta - 1_X)$, we obtain SDR-data $(\eta\colon (X, d) \rightleftarrows (Y, d) : \xi, h')$.

The following theorem asserts the homotopy invariance of the structure of an ∞-simplicial module [3].

Theorem 1.1. *Suppose given any ∞-simplicial module (X, d, ψ) with $\psi = \widehat{d} + \psi'$ and any* SDR-*data $(\eta\colon (X, d) \rightleftarrows (Y, d) : \xi, h)$ for differential bigraded modules. Then there is an ∞-simplicial module structure $(Y, d, \overline{\psi})$ on (Y, d) for which the map $\overline{\psi} = \widehat{d} + \overline{\psi}'$ is defined by*

$$\overline{\psi}' = \sum_{n \geq 0} \widehat{\eta} \cup \psi' \cup \underbrace{(\widehat{h} \cup \psi') \cup \cdots \cup (\widehat{h} \cup \psi')}_{n} \cup \widehat{\xi}. \tag{1.5}$$

Moreover, the maps $\overline{\xi} = \widehat{\xi} + \overline{\xi}'$, $\overline{\eta} = \widehat{\eta} + \overline{\eta}'$, and $\overline{h} = \widehat{h} + \overline{h}'$ defined by

$$\overline{\xi}' = \sum_{n \geq 0} \widehat{h} \cup \psi' \cup \underbrace{(\widehat{h} \cup \psi') \cup \cdots \cup (\widehat{h} \cup \psi')}_{n} \cup \widehat{\xi}, \tag{1.6}$$

$$\overline{\eta}' = \sum_{n \geq 0} \widehat{\eta} \cup \psi' \cup \underbrace{(\widehat{h} \cup \psi') \cup \cdots \cup (\widehat{h} \cup \psi')}_{n} \cup \widehat{h}, \tag{1.7}$$

$$\overline{h}' = \sum_{n \geq 0} \widehat{h} \cup \psi' \cup \underbrace{(\widehat{h} \cup \psi') \cup \cdots \cup (\widehat{h} \cup \psi')}_{n} \cup \widehat{h} \tag{1.8}$$

determine the SDR-*data $(\overline{\eta}\colon (X, d, \psi) \rightleftarrows (Y, d, \overline{\psi}) : \overline{\xi}, \overline{h})$ for ∞-simplicial modules.*

Differential ∞-simplicial modules, as well as differential simplicial modules, can be considered from the functional point of view. Indeed, for any ∞-simplicial module (X, d, ψ) with $\psi = \widehat{d} + \psi'$, we can define the family of maps

$$\widetilde{(\partial s)} = \{(\partial s)_{(i_1,\dots,i_p|j_q,\dots,j_1)} : X_{n,\bullet} \to X_{n-p+q,\bullet+p+q-1}\}, \qquad p \geq 0, \quad q \geq 0, \quad p+q \geq 1,$$
$$0 \leq i_1 < \cdots < i_p \leq n+q, \qquad n+q-1 \geq j_q > \cdots > j_1 \geq 0,$$
$$(\partial s)_{(i_1,\dots,i_p|j_q,\dots,j_1)}(x) = \psi'(([\partial^{n+q-p+1}_{i_1}] \mathbin{\widehat{\wedge}} \cdots \mathbin{\widehat{\wedge}} [\partial^{n+q}_{i_p}] \mathbin{\widehat{\wedge}} [s^{n+q-1}_{j_q}] \mathbin{\widehat{\wedge}} \cdots \mathbin{\widehat{\wedge}} [s^n_{j_1}]) \otimes x).$$

We denote the maps $(\partial s)_{(i_1,\dots,i_p|j_q,\dots,j_1)}$ with $q = 0$ in the family $\widetilde{(\partial s)}$ by $\partial_{(i_1,\dots,i_p)}$ and the maps $(\partial s)_{(i_1,\dots,i_p|j_q,\dots,j_1)}$ with $p = 0$ by $s_{(j_q,\dots,j_1)}$. Since $d(\psi') + \psi' \cup \psi' + \widetilde{\vartheta} = 0$, it follows that the maps in the family $\widetilde{(\partial s)}$ satisfy the relations

$$d((\partial s)_{(i_1,\dots,i_p|j_q,\dots,j_1)}) = \sum_{P(\gamma \in \mathcal{O}(\alpha))} (-1)^{\operatorname{sign}(\gamma)+1} (\partial s)_{(l_1,\dots,l_t|m_1,\dots,m_k)} (\partial s)_{(x_1,\dots,x_c|y_1,\dots,y_d)}, \qquad (p,q) \neq (1,1), \tag{1.9}$$

$$d((\partial s)_{(i|j)}) = \begin{cases} s_{(j-1)}\partial_{(i)} - \partial_{(i)}s_{(j)}, & i < j, \\ 1 - \partial_{(i)}s_{(j)}, & i = j, \quad i = j+1, \\ s_{(j)}\partial_{(i-1)} - \partial_{(i)}s_{(j)}, & i > j+1, \end{cases} \tag{1.10}$$

where the set $P(\gamma \in \mathcal{O}(\alpha))$ is the same as in the above expression for the comultiplication ∇ of the coalgebra $(S^!, \nabla)$, $t + c = p$, $k + d = q$, $\gamma = [a_1] \otimes \cdots [a_{p+q}]$,

$$([a_1] \otimes \cdots \otimes [a_z]) \otimes ([a_{z+1}] \otimes \cdots \otimes [a_{p+q}]) \in P(\gamma \in \mathcal{O}(\alpha)), \qquad 1 \le z \le p+q-1,$$

and

$$(\partial s)_{(l_1,\dots,l_t|m_1,\dots,m_k)}(g) = \psi'(([a_1]\ \widehat{\wedge} \cdots \widehat{\wedge}\ [a_z]) \otimes g), \qquad z = t+k,$$
$$(\partial s)_{(x_1,\dots,x_c|y_1,\dots,y_d)}(r) = \psi'(([a_{t+k+1}]\ \widehat{\wedge} \cdots \widehat{\wedge}\ [a_{p+q}]) \otimes r), \qquad p+q-z = c+d.$$

For example, (1.9) implies the relations

$$d((\partial s)_{(2|2,1)}) = -\partial_{(2)}s_{(2,1)} - (\partial s)_{(2|2)}s_{(1)} + (\partial s)_{(2|1)}s_{(1)},$$
$$d((\partial s)_{(1,2|2)}) = -\partial_{(1)}(\partial s)_{(2|2)} - \partial_{(1,2)}s_{(2)} + \partial_{(1)}(\partial s)_{(1|2)} - (\partial s)_{(1|1)}\partial_{(1)}.$$

It is easy to see that it follows from the above formulas defining the maps $(\partial s)_{(i_1,\dots,i_p|j_q,\dots,j_1)}$ that these maps completely determine the structure map $\psi = \widehat{d} + \psi'$. Thus, the following lemma is valid.

Lemma 1.1. *Specifying an ∞-simplicial module (X, d, ψ) is equivalent to specifying a triple $(X, d, \widetilde{(\partial s)})$ defined above and satisfying relations* (1.9) *and* (1.10).

In what follows, we identify the triples (X, d, ψ) and $(X, d, \widetilde{(\partial s)})$ corresponding to each other and use the same term "∞-simplicial module" for both of them.

Note that a special case of (1.9) is given by the relations

$$d(\partial_{(i)}) = 0, \quad i \ge 0, \qquad d(\partial_{(i,j)}) = \partial_{(j-1)}\partial_{(i)} - \partial_{(i)}\partial_{(j)}, \quad i < j,$$
$$d(s_{(i)}) = 0, \quad i \ge 0, \qquad d(s_{(i,j)}) = s_{(j)}s_{(i-1)} - s_{(i)}s_{(j)}, \quad i > j.$$

These relations, together with (1.10), say that, for any ∞-simplicial module $(X, d, \widetilde{(\partial s)})$, the maps $\partial_{(i)} \colon X_{n,\bullet} \to X_{n-1,\bullet}$ and $s_{(j)} \colon X_{n,\bullet} \to X_{n+1,\bullet}$ of differential modules satisfy the simplicial commutation relations (1.1)–(1.3) up to homotopy. In other words, the quadruple $(X, d, \partial_{(i)}, s_{(j)})$ is a differential simplicial module up to homotopy.

2. TENSOR PRODUCT OF ∞-SIMPLICIAL MODULES AND ∞-SIMPLICIAL COALGEBRAS

For a colored ϑ-coalgebra $(S^!, \nabla, \vartheta)$ and any differential bigraded modules (X, d) and (Y, d), consider the map

$$L\colon ((S^! \otimes S^!) \otimes (X \otimes Y))_{*,\bullet} \to ((S^! \otimes X) \otimes (S^! \otimes Y))_{*,\bullet}$$

of bigraded modules whose values at the generators of the colored module $S^! \otimes S^!$ and any elements $x \otimes y \in X_{k,m} \otimes Y_{l,t} \subset (X \otimes Y)_{k+l,m+t}$ are defined by

(1) $L((1_{k+l} \otimes 1_{k+l}) \otimes (x \otimes y)) = (1_k \otimes x) \otimes (1_l \otimes y);$

(2) $L((((\partial s)^{k+l}_{(i_1,\dots,i_p|j_q,\dots,j_1)} \otimes 1_{k+l})) \otimes (x \otimes y))$

$$= \begin{cases} ((\partial s)^{k}_{(i_1,\dots,i_p|j_q,\dots,j_1)} \otimes x) \otimes (1_l \otimes y) & \text{if } j_q \le k+q-1,\ i_p \le k+q, \\ 0 & \text{otherwise;} \end{cases}$$

(3) $L((1_{k+l+q-p} \otimes (\partial s)^{k+l}_{(i_1,\dots,i_p|j_q,\dots,j_1)}) \otimes (x \otimes y))$

$$= \begin{cases} (-1)^{(p+q)k}(1_k \otimes x) \otimes ((\partial s)^l_{(i_1-k,\dots,i_p-k|j_q-k,\dots,j_1-k)} \otimes y) \\ \qquad \text{if } j_1 > k,\ i_1 > k, \\ 0 \qquad \text{otherwise;} \end{cases}$$

(4) $$L(((\partial s)^{k+l+b-a}_{(i_1,\dots,i_p|j_q,\dots,j_1)} \otimes (\partial s)^{k+l}_{(\mu_1,\dots,\mu_a|\nu_b,\dots,\nu_1)}) \otimes (x \otimes y))$$
$$= \begin{cases} (-1)^{(a+b)k}((\partial s)^k_{(i_1,\dots,i_p|j_q,\dots,j_1)} \otimes x) \otimes ((\partial s)^l_{(\mu_1-k,\dots,\mu_a-k|\nu_b-k,\dots,\nu_1-k)} \otimes y), \\ \qquad \text{if } j_q \le k+q-1,\ i_p \le k+q,\ \nu_1 > k,\ \mu_1 > k, \\ 0 \qquad \text{otherwise.} \end{cases}$$

Using the map L, we define a map

$$\overline{\otimes}\colon (\mathrm{Hom}(S^! \otimes X_1; Y_1) \otimes \mathrm{Hom}(S^! \otimes X_2; Y_2))_{*,\bullet} \to \mathrm{Hom}(S^! \otimes (X_1 \otimes X_2); Y_1 \otimes Y_2)_{*,\bullet}$$

of bigraded modules at any $f \in \mathrm{Hom}(S^! \otimes X_1; Y_1)$ and $g \in \mathrm{Hom}(S^! \otimes X_2; Y_2))$ by

$$f \mathbin{\overline{\otimes}} g = (f \otimes g)L(\nabla \otimes 1_{X_1 \otimes X_2})\colon S^! \otimes (X_1 \otimes X_2) \to Y_1 \otimes Y_2.$$

It is easy to see that the map $f \mathbin{\overline{\otimes}} g = \widehat{(f \mathbin{\overline{\otimes}} g)}_\nu + (f \mathbin{\overline{\otimes}} g)'$ satisfies the relations

$$(f \mathbin{\overline{\otimes}} g)_\nu = f_\nu \otimes g_\nu, \widehat{(f \mathbin{\overline{\otimes}} g)}_\nu = \widehat{f}_\nu \mathbin{\overline{\otimes}} \widehat{g}_\nu, \qquad (f \mathbin{\overline{\otimes}} g)' = f' \mathbin{\overline{\otimes}} g' + \widehat{f}_\nu \mathbin{\overline{\otimes}} g' + f' \mathbin{\overline{\otimes}} \widehat{g}_\nu.$$

Moreover, a direct calculation shows that the maps $\cup$ and $\overline{\otimes}$ considered above are related to each other by the "sign permutation rule," i.e.,

$$(f_1 \mathbin{\overline{\otimes}} g_1) \cup (f_2 \mathbin{\overline{\otimes}} g_2) = (-1)^{(n+m)(s+t)}(f_1 \cup f_2) \mathbin{\overline{\otimes}} (g_1 \cup g_2), \tag{2.1}$$

provided that the bidegrees of the maps g_1 and f_2 are (n,m) and (s,t), respectively.

Definition 2.1. The *tensor product of differential ∞-simplicial modules* (X,d,ψ) and (Y,d,ψ) is the ∞-simplicial module $(X \otimes Y, d, \psi)$, where $(X \otimes Y, d)$ is the tensor product of the corresponding differential bigraded modules and the map $\psi\colon (S^! \otimes (X \otimes Y))_{*,\bullet} \to (X \otimes Y)_{*,\bullet-1}$ is defined by

$$\psi = \psi \mathbin{\overline{\otimes}} \widehat{1}_Y + \widehat{1}_X \mathbin{\overline{\otimes}} \psi. \tag{2.2}$$

It is easy to see that the tensor product of any ∞-simplicial modules is an ∞-simplicial module. Indeed, we have $\widetilde{\vartheta}_{X \otimes Y} = \widetilde{\vartheta}_X \mathbin{\overline{\otimes}} \widehat{1}_Y + \widehat{1}_X \mathbin{\overline{\otimes}} \widetilde{\vartheta}_Y$; thus, applying (2.1), we obtain

$$\begin{aligned} \psi \cup \psi &= (\psi \mathbin{\overline{\otimes}} \widehat{1} + \widehat{1} \mathbin{\overline{\otimes}} \psi) \cup (\psi \mathbin{\overline{\otimes}} \widehat{1} + \widehat{1} \mathbin{\overline{\otimes}} \psi) \\ &= (\psi \mathbin{\overline{\otimes}} \widehat{1}) \cup (\psi \mathbin{\overline{\otimes}} \widehat{1}) + (\psi \mathbin{\overline{\otimes}} \widehat{1}) \cup (\widehat{1} \mathbin{\overline{\otimes}} \psi) + (\widehat{1} \mathbin{\overline{\otimes}} \psi) \cup (\psi \mathbin{\overline{\otimes}} \widehat{1}) + (\widehat{1} \mathbin{\overline{\otimes}} \psi) \cup (\widehat{1} \mathbin{\overline{\otimes}} \psi) \\ &= (\psi \cup \psi) \mathbin{\overline{\otimes}} (\widehat{1} \cup \widehat{1}) + (\psi \cup \widehat{1}) \mathbin{\overline{\otimes}} (\widehat{1} \cup \psi) + (-1)^{(-1)(-1)}(\widehat{1} \cup \psi) \mathbin{\overline{\otimes}} (\psi \cup \widehat{1}) \\ &\quad + (\widehat{1} \cup \widehat{1}) \mathbin{\overline{\otimes}} (\psi \cup \psi) = -(\widetilde{\vartheta} \mathbin{\overline{\otimes}} \widehat{1} + \widehat{1} \mathbin{\overline{\otimes}} \widetilde{\vartheta}) = -\widetilde{\vartheta}. \end{aligned}$$

Now let us consider the tensor product of ∞-simplicial modules from the functional point of view. Let $(X \otimes Y, d, \widetilde{(\partial s)})$ be the tensor product of any ∞-simplicial modules $(X, d, \widetilde{(\partial s)})$ and $(Y, d, \widetilde{(\partial s)})$. It follows from relation (2.2) and the definition of the map $\overline{\otimes}$ that, for the ∞-simplicial module $(X \otimes Y, d, \widetilde{(\partial s)})$, the family of maps

$$\widetilde{(\partial s)} = \{(\partial s)_{(i_1,\dots,i_p|j_q,\dots,j_1)}\colon (X \otimes Y)_{n,\bullet} \to (X \otimes Y)_{n-p+q,\,\bullet+p+q-1}\}$$

is defined at each $x \otimes y \in X_{k,m} \otimes Y_{l,t}$ by the rule

$$(\partial s)_{(i_1,\dots,i_p|j_q,\dots,j_1)}(x \otimes y)$$

$$= \begin{cases} (\partial s)_{(i_1,\dots,i_p|j_q,\dots,j_1)}(x) \otimes y & \text{if } j_q \le k+q-1, \\ & \quad i_p \le k+q, \\ (-1)^{(p+q-1)k+m} x \otimes (\partial s)_{(i_1-k,\dots,i_p-k|j_q-k,\dots,j_1-k)}(y) & \text{if } j_1 > k,\ i_1 > k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

It is easy to see that if ∞-simplicial modules $(X, d, \widetilde{(\partial s)})$ and $(Y, d, \widetilde{(\partial s)})$ are differential simplicial modules (X, d, ∂_i, s_j) and (Y, d, ∂_i, s_j), respectively, then relation (2.3) defines the differential simplicial module $(X \otimes Y, d, \partial_i, s_j)$ for which the ∂_i and s_j take the following values at each $x \otimes y \in X_{k,m} \otimes Y_{l,t}$:

$$\partial_i(x \otimes y) = \begin{cases} \partial_i(x) \otimes y, & 0 \le i \le k, \\ (-1)^m x \otimes \partial_{i-k}(y), & k < i \le k+l, \end{cases}$$

$$s_j(x \otimes y) = \begin{cases} s_j(x) \otimes y, & 0 \le j \le k, \\ (-1)^m x \otimes s_{i-k}(y), & k < j \le k+l. \end{cases}$$

It is worth mentioning that the usual construction of the diagonal tensor product of differential simplicial modules is related to the new construction of the tensor product of differential simplicial modules described above by the Alexander–Whitney and Eilenberg–MacLane maps, which are maps of differential simplicial modules rather than only maps of the corresponding chain bicomplexes in the case under consideration.

Definition 2.2. A *differential ∞-simplicial coalgebra* or, briefly, an *∞-simplicial coalgebra* (X, d, ψ, ∇), is an ∞-simplicial module (X, d, ψ) together with an associative comultiplication $\nabla\colon X \to X \otimes X$ for which the map $\widehat{\nabla}\colon S^! \otimes X \to X \otimes X$ is a morphism $\widehat{\nabla}\colon (X, d, \psi) \to (X \otimes X, d, \psi)$ of ∞-simplicial modules, i.e., satisfies the condition

$$\widehat{\nabla} \cup \psi = (\widehat{1}_X \mathbin{\overline{\otimes}} \psi + \psi \mathbin{\overline{\otimes}} \widehat{1}_X) \cup \widehat{\nabla}. \tag{2.4}$$

Clearly, for a map ψ represented in the form $\psi = \widehat{d} + \psi'$, condition (2.4) is equivalent to the conditions

$$\widehat{\nabla} \cup \widehat{d} = (\widehat{1}_X \mathbin{\overline{\otimes}} \widehat{d} + \widehat{d} \mathbin{\overline{\otimes}} \widehat{1}_X) \cup \widehat{\nabla}, \qquad \widehat{\nabla} \cup \psi' = (\widehat{1}_X \mathbin{\overline{\otimes}} \psi' + \psi' \mathbin{\overline{\otimes}} \widehat{1}_X) \cup \widehat{\nabla}.$$

The former is equivalent to $\nabla d = (1 \otimes d + d \otimes 1)\nabla$; therefore, for any ∞-simplicial coalgebra (X, d, ψ, ∇), the triple (X, d, ∇) is a differential coalgebra.

Definition 2.3. A *morphism $f\colon (X, d, \psi, \nabla) \to (Y, d, \psi, \nabla)$ of differential ∞-simplicial coalgebras* is a morphism $f\colon (X, d, \psi) \to (Y, d, \psi)$ of ∞-simplicial modules which satisfies the condition

$$\widehat{\nabla} \cup f = (f \mathbin{\overline{\otimes}} f) \cup \widehat{\nabla}. \tag{2.5}$$

Clearly, for a map f represented in the form $f = \widehat{f_\nu} + f'$, condition (2.5) is equivalent to

$$\widehat{\nabla} \cup \widehat{f_\nu} = (\widehat{f_\nu} \mathbin{\overline{\otimes}} \widehat{f_\nu}) \cup \widehat{\nabla}, \qquad \widehat{\nabla} \cup f' = (f' \mathbin{\overline{\otimes}} f' + \widehat{f_\nu} \mathbin{\overline{\otimes}} f' + f' \mathbin{\overline{\otimes}} \widehat{f_\nu}) \cup \widehat{\nabla}.$$

The former condition is equivalent to $f_\nu \nabla = (f_\nu \otimes f_\nu)\nabla$; therefore, for any morphism

$$f\colon (X, d, \psi) \to (Y, d, \psi)$$

of ∞-simplicial coalgebras, the map $f_\nu\colon (X, d) \to (Y, d)$ of differential modules is a map

$$f_\nu\colon (X, d, \nabla) \to (Y, d, \nabla)$$

of differential coalgebras.

Definition 2.4. A *homotopy $h\colon (X, d, \psi, \nabla) \to (Y, d, \psi, \nabla)$ between morphisms*

$$f, g\colon (X, d, \psi, \nabla) \to (Y, d, \psi, \nabla)$$

of ∞-simplicial coalgebras is a homotopy $h\colon (X, d, \psi) \to (Y, d, \psi)$ between morphisms

$$f, g\colon (X, d, \psi) \to (Y, d, \psi)$$

of ∞-simplicial modules that satisfies the condition

$$\widehat{\nabla} \cup h = (h \mathbin{\overline{\otimes}} f + g \mathbin{\overline{\otimes}} h) \cup \widehat{\nabla}. \tag{2.6}$$

It is easy to see that, for a map h represented in the form $h = \widehat{h}_\nu + h'$, condition (2.6) is equivalent to

$$\widehat{\nabla} \cup \widehat{h}_\nu = (\widehat{h}_\nu \,\overline{\otimes}\, \widehat{f}_\nu + \widehat{g}_\nu \,\overline{\otimes}\, \widehat{h}_\nu) \cup \widehat{\nabla},$$
$$\widehat{\nabla} \cup h' = (h' \,\overline{\otimes}\, f' + h' \,\overline{\otimes}\, \widehat{f}_\nu + \widehat{h}_\nu \,\overline{\otimes}\, f' + g' \,\overline{\otimes}\, h' + g' \,\overline{\otimes}\, \widehat{h}_\nu + \widehat{g}_\nu \,\overline{\otimes}\, h') \cup \widehat{\nabla}.$$

The former condition is equivalent to $\nabla h_\nu = (h_\nu \otimes f_\nu + g_\nu \otimes h_\nu)\nabla$; therefore, for any homotopy $h\colon (X,d,\psi,\nabla) \to (Y,d,\psi,\nabla)$ between morphisms $f,g\colon (X,d,\psi,\nabla) \to (Y,d,\psi,\nabla)$ of ∞-simplicial coalgebras, a homotopy $h_\nu\colon X \to Y$ between maps $f_\nu, g_\nu\colon (X,d) \to (Y,d)$ of differential modules is a homotopy between the maps $f_\nu, g_\nu\colon (X,d,\nabla) \to (Y,d,\nabla)$ of differential coalgebras. Note that the notion of a homotopy between maps of differential coalgebras which we use here has become widely accepted at present.

Suppose given any morphisms $\eta\colon (X,d,\psi,\nabla) \rightleftarrows (Y,d,\psi,\nabla) : \xi$ of differential ∞-simplicial coalgebras such that $\eta \cup \xi = \widehat{1}_Y$, and let $h\colon (X,d,\psi,\nabla) \to (X,d,\psi,\nabla)$ be any homotopy between the morphisms $\xi \cup \eta$ and $\widehat{1}_X$ of ∞-simplicial coalgebras which satisfies the conditions $\eta \cup h = 0$, $h \cup \xi = 0$, and $h \cup h = 0$. Any such triple $(\eta\colon (X,d,\psi,\nabla) \rightleftarrows (Y,d,\psi,\nabla) : \xi, h)$ is called SDR-*data for ∞-simplicial coalgebras*.

It is easy to show that, given any SDR-data $(\eta\colon (X,d,\psi,\nabla) \rightleftarrows (Y,d,\psi,\nabla) : \xi, h)$ for ∞-simplicial coalgebras, SDR-data $(\eta_\nu\colon (X,d,\nabla) \rightleftarrows (Y,d,\nabla) : \xi_\nu, h_\nu)$ for differential coalgebras are defined. We say that SDR-data $(\eta\colon (X,d,\psi,\nabla) \rightleftarrows (Y,d,\psi,\nabla) : \xi, h)$ for ∞-simplicial coalgebras extend the SDR-data $(\eta\colon (X,d,\nabla) \rightleftarrows (Y,d,\nabla) : \xi, h)$ for differential coalgebras if $\eta = \eta_\nu$, $\xi = \xi_\nu$, and $h = h_\nu$.

Now, let us prove the homotopy invariance of the structure of an ∞-simplicial coalgebra under homotopy equivalences of the type of SDR-data for differential coalgebras.

Theorem 2.1. *Suppose given any ∞-simplicial coalgebra (X,d,ψ,∇) and* SDR-*data*

$$(\eta\colon (X,d,\nabla) \rightleftarrows (Y,d,\nabla) : \xi, h)$$

for differential coalgebras. Then relations (1.5)–(1.8) *define the structure of an ∞-simplicial coalgebra $(Y,d,\overline{\psi},\nabla)$ on (Y,d,∇) and, in addition, determine* SDR-*data*

$$(\overline{\eta}\colon (X,d,\psi,\nabla) \rightleftarrows (Y,d,\overline{\psi},\nabla) : \overline{\xi}, \overline{h}\,)$$

for ∞-simplicial coalgebras which extend SDR-*data $(\eta\colon (X,d,\nabla) \rightleftarrows (Y,d,\nabla) : \xi, h)$ for differential coalgebras.*

Proof. To any SDR-data $(\eta\colon (X,d,\nabla) \rightleftarrows (Y,d,\nabla) : \xi, h)$ for differential coalgebras there correspond SDR-data $(\widehat{\eta}\colon (X,d,\widehat{d},\nabla) \rightleftarrows (Y,d,\widehat{d},\nabla) : \widehat{\xi}, \widehat{h})$ for ∞-simplicial coalgebras; in particular, the following conditions hold:

$$\widehat{\nabla} \cup \widehat{\eta} = (\widehat{\eta} \,\overline{\otimes}\, \widehat{\eta}) \cup \widehat{\nabla}, \qquad \widehat{\nabla} \cup \widehat{\xi} = (\widehat{\xi} \,\overline{\otimes}\, \widehat{\xi}) \cup \widehat{\nabla}, \qquad \widehat{\nabla} \cup \widehat{h} = (\widehat{h} \,\overline{\otimes}\, (\widehat{\xi} \cup \widehat{\eta}) + \widehat{1} \,\overline{\otimes}\, \widehat{h}) \cup \widehat{\nabla}.$$

Using these conditions and relation (2.1), we obtain the following chain of equalities for each summand $\widehat{\eta} \cup \psi' \cup (\widehat{h} \cup \psi')^{\cup n} \cup \widehat{\xi}$ in (1.5):

$$\begin{aligned}
&\widehat{\nabla} \cup (\widehat{\eta} \cup \psi' \cup (\widehat{h} \cup \psi')^{\cup n} \cup \widehat{\xi}) = (\widehat{\eta} \,\overline{\otimes}\, \widehat{\eta}) \cup \widehat{\nabla} \cup \psi' \cup (\widehat{h} \cup \psi')^{\cup n} \cup \widehat{\xi} \\
&\quad = (\widehat{\eta} \,\overline{\otimes}\, \widehat{\eta}) \cup (\widehat{1} \,\overline{\otimes}\, \psi' + \psi' \,\overline{\otimes}\, \widehat{1}) \cup \widehat{\nabla} \cup (\widehat{h} \cup \psi')^{\cup n} \cup \widehat{\xi} \\
&\quad = (\widehat{\eta} \,\overline{\otimes}\, (\widehat{\eta} \cup \psi') + (\widehat{\eta} \cup \psi') \,\overline{\otimes}\, \widehat{\eta}) \cup (\widehat{h} \,\overline{\otimes}\, (\widehat{\xi} \cup \widehat{\eta}) + \widehat{1} \,\overline{\otimes}\, \widehat{h}) \cup \widehat{\nabla} \cup \psi' \cup (\widehat{h} \cup \psi')^{\cup(n-1)} \cup \widehat{\xi} \\
&\quad = (\widehat{\eta} \,\overline{\otimes}\, (\widehat{\eta} \cup \psi' \cup \widehat{h}) + (\widehat{\eta} \cup \psi' \cup \widehat{h}) \,\overline{\otimes}\, \widehat{\eta}) \cup \widehat{\nabla} \cup \psi' \cup (\widehat{h} \cup \psi')^{\cup(n-1)} \cup \widehat{\xi} = \cdots \\
&\quad = (\widehat{1} \,\overline{\otimes}\, (\widehat{\eta} \cup \psi' \cup (\widehat{h} \cup \psi')^{\cup n} \cup \widehat{\xi}) + (\widehat{\eta} \cup \psi' \cup (\widehat{h} \cup \psi')^{\cup n} \cup \widehat{\xi}) \,\overline{\otimes}\, \widehat{1}) \cup \widehat{\nabla}.
\end{aligned}$$

It follows that the map $\overline{\psi} = \widehat{d} + \overline{\psi}'$ defined by (1.5) satisfies condition (2.4). In a similar way, it can be shown that the morphisms $\overline{\xi} = \widehat{\xi} + \overline{\xi}'$ and $\overline{\eta} = \widehat{\eta} + \overline{\eta}'$ of ∞-simplicial modules satisfy condition (2.5) and the homotopy $\overline{h} = \widehat{h} + \overline{h}'$ between these morphisms of ∞-simplicial modules satisfies condition (2.6). ◻

3. HOMOTOPY UNITAL SUPPLEMENTED A_∞-ALGEBRAS AND TENSOR ∞-SIMPLICIAL COALGEBRAS

First, we recall the necessary definitions and constructions related to the notions of (asymmetric) operad and of algebra over an operad in the category of differential modules (see, e.g., [6]).

By a *differential family*, or, briefly, a *family* $\mathscr{E} = \{\mathscr{E}(j)\}_{j\geq 0}$ we mean any family of differential modules $(\mathscr{E}(j), d)$, $j \geq 0$. We define a *morphism* $f\colon \mathscr{E}' \to \mathscr{E}''$ *of families* to be any family of maps $\alpha = \{\alpha(j) : (\mathscr{E}'(j), d) \to (\mathscr{E}''(j), d)\}_{j\geq 0}$ of differential modules. Given any families $\mathscr{E}'$ and $\mathscr{E}''$, we define the family $\mathscr{E}' \times \mathscr{E}''$ by

$$(\mathscr{E}' \times \mathscr{E}'')(j) = \bigoplus_{j_1+\dots+j_k=j} \mathscr{E}'(k) \otimes \mathscr{E}''(j_1) \otimes \dots \otimes \mathscr{E}''(j_k), \qquad j \geq 0.$$

Clearly, the $\times$-product of families thus defined is associative; i.e., for any families $\mathscr{E}$, $\mathscr{E}'$, and $\mathscr{E}''$, we have the isomorphism of families $\mathscr{E} \times (\mathscr{E}' \times \mathscr{E}'') \approx (\mathscr{E} \times \mathscr{E}') \times \mathscr{E}''$.

An (asymmetric) *operad* $(\mathscr{E}, \gamma)$ is any family $\mathscr{E}$ together with a family morphism $\gamma\colon \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ satisfying the condition $\gamma(\gamma \times 1) = \gamma(1 \times \gamma)$. Moreover, there is an element $1 \in \mathscr{E}(1)_0$ such that, for each $e_j \in \mathscr{E}(j)$, $j \geq 0$, we have $\gamma(1 \otimes e_j) = e_j$ and, for each $e_j \in \mathscr{E}(j)$, $j \geq 1$, we have

$$\gamma(e_j \otimes 1 \otimes \dots \otimes 1) = e_j.$$

In what follows, we write elements of the form $\gamma(e_k \otimes e_{j_1} \otimes \dots \otimes e_{j_k})$ as $e_k(e_{j_1} \otimes \dots \otimes e_{j_k})$. An *operad morphism* $f(\mathscr{E}', \gamma) \to (\mathscr{E}'', \gamma)$ is defined as a family morphism $f\colon \mathscr{E}' \to \mathscr{E}''$ satisfying the condition $f\gamma = \gamma(f \times f)$.

A canonical example of an operad is the operad $(\mathscr{E}_X, \gamma)$ which is defined for any differential module (X, d) by

$$(\mathscr{E}_X(j), d) = (\mathrm{Hom}(X^{\otimes j}; X), d), \qquad \gamma(f_k \otimes f_{j_1} \otimes \dots \otimes f_{j_k}) = f_k(f_{j_1} \otimes \dots \otimes f_{j_k}).$$

An *algebra over an operad* $(\mathscr{E}, \gamma)$, or, briefly, an $\mathscr{E}$*-algebra* (X, d, α), is any differential module (X, d) together with a fixed operad morphism $\alpha\colon \mathscr{E} \to \mathscr{E}_X$. A *morphism* $f\colon (X, d, \alpha) \to (Y, d, \alpha)$ *of* $\mathscr{E}$*-algebras* is any map $f\colon (X, d) \to (Y, d)$ of differential modules for which $f_*\alpha = f^*\alpha\colon \mathscr{E} \to \mathscr{E}_{(X,Y)}$, where the family $\mathscr{E}_{(X,Y)}$ is defined by $(\mathscr{E}_{(X,Y)}(j), d) = (\mathrm{Hom}(X^{\otimes j}; Y), d)$, and $f_*\colon\mathscr{E}_X \to \mathscr{E}_{(X,Y)}$ and $f^*\colon \mathscr{E}_Y \to \mathscr{E}_{(X,Y)}$ are the family morphisms induced by f.

An important example of an operad is the Stasheff operad (A_∞, γ). As a graded operad, (A_∞, γ) is free with generators $\pi_n \in A_\infty(n + 2)_n$, $n \geq 0$, and at the generators π_{n+1}, $n \geq -1$, the differential takes the values

$$d(\pi_{n+1}) = \sum_{m=1}^{n+1} \sum_{t=1}^{m+1} (-1)^{t(n-m)+n+1} \pi_{m-1}(\underbrace{1 \otimes \dots \otimes 1}_{t-1} \otimes \pi_{n-m+1} \otimes \underbrace{1 \otimes \dots \otimes 1}_{m-t+1}). \tag{3.1}$$

For example, in the case of $n = -1; 0; 1$, relations (3.1) have the form

$$d(\pi_0) = 0, \qquad d(\pi_1) = \pi_0(\pi_0 \otimes 1) - \pi_0(1 \otimes \pi_0),$$
$$d(\pi_2) = \pi_0(\pi_1 \otimes 1 + 1 \otimes \pi_1) - \pi_1(\pi_0 \otimes 1 \otimes 1 - 1 \otimes \pi_0 \otimes 1 + 1 \otimes 1 \otimes \pi_0).$$

It is easy to see that endowing a differential module (A, d), where $A = \{A_n\}$, $n \in \mathbb{Z}$, $n \geq 0$, and $d\colon A_\bullet \to A_{\bullet-1}$, with the structure (A, d, α) of an A_∞-algebra is equivalent to specifying a family of maps $\{\pi_n = \alpha(\pi_n)\colon (A^{\otimes(n+2)})_\bullet \to A_{\bullet+n} \mid n \in \mathbb{Z},\ n \geq 0\}$ satisfying relations (3.1) for (A, d).

Recall that a *unital differential algebra* (A, d, π, ν) is defined as the differential algebra (A, d, π), where $A = \{A_n\}$, $n \in \mathbb{Z}$, $n \geq 0$, and $d\colon A_\bullet \to A_{\bullet-1}$, with associative multiplication $\pi\colon A \otimes A \to A$ together with a map $\nu\colon K \to A$ of graded K-modules, where $K_0 = K$ and $K_i = 0$ for $i \neq 0$, which is called the *unit of the differential algebra* (A, d, π) and satisfies the conditions

$$\pi(\nu \otimes 1) = 1\colon A = K \otimes A \to A, \qquad \pi(1 \otimes \nu) = 1\colon A = A \otimes K \to A. \tag{3.2}$$

Clearly, the unit $\nu\colon K \to A$ is completely determined by the element $\nu(1) \in A_0$, $1 \in K$, which is called the unit of the differential algebra (A, d, π) and denoted by $1 \in A_0$.

Following [4], we now recall the notion of a homotopy unital A_∞-algebra [2]. Consider the operad $(A_\infty^{su}\langle u,h\rangle,\gamma)$ introduced in [4]. As a graded operad, $(A_\infty^{su}\langle u,h\rangle,\gamma)$ is an operad with generators $\pi_n\in(A_\infty^{su}\langle u,h\rangle)(n+2)_n$, where $n\geq 0$, $1^{su}\in(A_\infty^{su}\langle u,h\rangle)(0)_0$, $u\in(A_\infty^{su}\langle u,h\rangle)(0)_0$, and $h\in(A_\infty^{su}\langle u,h\rangle)(0)_1$, satisfying the relations

$$\pi_0(1^{su}\otimes 1)=1,\qquad \pi_0(1\otimes 1^{su})=1,\qquad \pi_n(1^{\otimes k}\otimes 1^{su}\otimes 1^{\otimes(n-k+1)})=0,\quad n>0,\tag{3.3}$$

where $0\leq k\leq n+1$; the differential is defined at the generators specified above by relations (3.1), and

$$d(1^{su})=0,\qquad d(u)=0,\qquad d(h)=1^{su}-u.\tag{3.4}$$

In the operad $(A_\infty^{su}\langle u,h\rangle,\gamma)$, consider the suboperad (A_∞^{hu},γ) with generators

$$\tau_0^0=u\in(A_\infty^{su}\langle u,h\rangle)(0)_0,\qquad \tau_n^{\varnothing}=\pi_{n-1}\in(A_\infty^{su}\langle u,h\rangle)(n+1)_{n-1},\quad n\geq 1,$$

$$\tau_n^{j_q,\dots,j_1}=\pi_{n-1}(\underbrace{\overbrace{\underbrace{1^{\otimes n_1}}_{j_1}\otimes h\otimes 1^{\otimes n_2}}^{j_2}\otimes h\otimes 1^{\otimes n_3}\cdots\otimes 1^{n_k}\otimes h\otimes 1^{\otimes n_{k+1}}\otimes\cdots\otimes 1^{\otimes n_q}}_{j_q}\otimes h\otimes 1^{\otimes n_{q+1}})$$

$$\in(A_\infty^{su}\langle u,h\rangle)(n-q+1)_{n+q-1},\qquad n\geq 1,\quad q\geq 1,\quad n\geq j_q>\cdots>j_1\geq 0,\quad n_s\geq 0,$$

$$1\leq s\leq q+1,\quad n_1+\cdots+n_{q+1}=n-q+1,\quad j_k=n_1+\cdots+n_k+k-1,\quad 1\leq k\leq q.$$

It is worth mentioning for clarity that each j_k, $1\leq k\leq q$, is the number of all tensor multipliers on the left of the kth occurence of h counting from the origin of the tensor battery to the right. For example, using this rule, we readily obtain

$$\tau_4^{3,1}=\pi_3(1\otimes h\otimes 1\otimes h\otimes 1),\qquad \tau_3^{3,2}=\pi_2(1\otimes 1\otimes h\otimes h),$$
$$\tau_5^{5,1,0}=\pi_4(h\otimes h\otimes 1\otimes 1\otimes 1\otimes h).$$

In [4], the element $\tau_n^{j_q,\dots,j_1}$ is denoted by $m_{n_1,n_2,\dots,n_{q+1}}$, where the numbers $n_1,\dots,n_{q+1}$ are the same as in the above expression for $\tau_n^{j_q,\dots,j_1}$. The values of the differential at the generators $\tau_n^{j_q,\dots,j_1}$, where $n\geq 0$, $q\geq 0$, $n+q\geq 1$, and $\tau_n^{j_q,\dots,j_1}=\tau_n^{\varnothing}=\pi_{n-1}$ for $q=0$ and $n\geq 1$, are completely determined by (3.1), (3.4) and (3.3). It is easy to check that

$$\begin{aligned}&d(\tau_0^0)=0,\quad d(\tau_1^{\varnothing})=0,\qquad d(\tau_1^0)=1-\pi_0(\tau_0^0\otimes 1),\\ &d(\tau_1^1)=1-\pi_0(1\otimes\tau_0^0),\qquad d(\tau_1^{1,0})=\tau_1^0\tau_0^0-\tau_1^1\tau_0^0.\end{aligned}\tag{3.5}$$

Moreover, a straightforward calculation shows that, for any $n\geq 0$ and $q\geq 0$, we have

$$\begin{aligned}d(\tau_{n+2}^{j_q,\dots,j_1})&=\sum_{m=1}^{n+1}\sum_{t=1}^{m+1}(-1)^{\lambda}\tau_m^{j_q-(n-m+2),\dots,j_k-(n-m+2),j_l,\dots,j_1}\\&\qquad\times(\underbrace{1\otimes\cdots\otimes 1}_{t-1-l}\otimes\tau_{n-m+2}^{j_{k-1}-(t-1),\dots,j_{l+1}-(t-1)}\otimes\underbrace{1\otimes\cdots\otimes 1}_{m-t-q+k})\\&\quad+\sum_{i=1}^{q}(-1)^{n+i}\tau_{n+2}^{j_q,\dots,j_{i+1},j_{i-1},\dots,j_1}(\underbrace{1\otimes\cdots\otimes 1}_{j_i-(i-1)}\otimes\tau_0^0\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-j_i-q+i+2}),\end{aligned}\tag{3.6}$$

where $\lambda=t(n-m)+n+1+(n-m)(q-k+l+1)+q(k-l)+kl$ and the summation in the first term on the right-hand side is over all numbers $k\geq 1$ and $l\geq 0$ such that, for each fixed t,

$$\begin{aligned}0\leq j_1<\cdots<j_l\leq t-2<j_{l+1}<\cdots<j_{l+(k-l-1)}\\=j_{k-1}<t+n-m+2\leq j_k<\cdots<j_q\leq n+2.\end{aligned}$$

Obviously, for $q=0$, $k=1$, and $l=0$, relation (3.6) transforms into (3.1).

The algebras (A,d,α) over the operad (A_∞^{hu},γ), i.e., the A_∞^{hu}-algebras, are called *homotopy unital A_∞-algebras*.

It is easy to see that endowing a differential module (A, d), where $A = \{A_n\}$, $n \in \mathbb{Z}$, $n \geq 0$, and $d\colon A_\bullet \to A_{\bullet-1}$, with the structure of an A_∞^{hu}-algebra (A, d, α) is equivalent to specifying a family of maps

$$\{\tau_n^{j_q,\dots,j_1} = \alpha(\tau_n^{j_q,\dots,j_1})\colon (A^{\otimes(n-q+1)})_\bullet \to A_{\bullet+n+q-1} \mid n \in \mathbb{Z},\ n \geq 0,\ q \geq 0,\ n+q \geq 1\},$$
$$n-q+1 \geq 0, \qquad j_q, \dots, j_1 \in \mathbb{Z}, \qquad n \geq j_q > \dots > j_1 \geq 0,$$

satisfying relations (3.5) and (3.6) for (A, d). The left-hand sides of relations (3.5) and (3.6) for the maps $\tau_n^{j_q,\dots,j_1}\colon (A^{\otimes(n-q+1)})_\bullet \to A_{\bullet+n+q-1}$ are calculated by the usual formula

$$d(\tau_n^{j_q,\dots,j_1}) = d\tau_n^{j_q,\dots,j_1} + (-1)^{n+q}\tau_n^{j_q,\dots,j_1}d.$$

In [4], it was shown that relations (3.5) and (3.6) are equivalent to the structural relations for homotopy unital A_∞-algebras given in [2].

Note that the third and fourth equalities in (3.5) say that the map $\tau_0^0\colon K \to A$ satisfies (3.2) up to homotopy; i.e., up to homotopy, the map τ_0^0 is the unit of the differential homotopy associative algebra (A, d, π_0).

Of course, relations (3.6) are very cumbersome. However, later on (after the proof of Theorem 3.1), we describe a simple simplicial method for calculating these relations.

Definition 3.1. A *homotopy unital supplemented A_∞-algebra* or, briefly, an *supplemented A_∞^{hu}-algebra*, is defined as any A_∞^{hu}-algebra $(A, d, \tau_n^{j_q,\dots,j_1})$ together with maps $\varepsilon_1, \varepsilon_2\colon A \to K$ of graded modules satisfying the relations

$$\varepsilon_i d = 0, \quad \varepsilon_i\tau_0^0 = 1, \quad \varepsilon_i\pi_0 = \pi(\varepsilon_i \otimes \varepsilon_i), \qquad i = 1, 2, \tag{3.7}$$

where $\pi_0 = \tau_1^\varnothing$ and π is multiplication in the base ring K.

Note that the notion of an supplemented A_∞^{hu}-algebra generalizes the notion of an supplemented associative differential algebra with unit, i.e., an associative differential algebra with unit on which a map of differential algebras to the base ring K is defined. Indeed, if an supplemented A_∞^{hu}-algebra $(A, d, \varepsilon_i, \tau_n^{j_q,\dots,j_1})$ is such that $\varepsilon_1 = \varepsilon_2 = \varepsilon$, $\tau_0^0 = \nu \neq 0$, $\tau_1^\varnothing = \pi_0 \neq 0$, and $\tau_n^{j_q,\dots,j_1} = 0$ for all other n and $j_q, \dots, j_1$, then the quintuple $(A, d, \pi_0, \nu, \varepsilon)$ is an supplemented associative differential algebra with unit.

It is easy to see that, for any supplemented A_∞^{hu}-algebra $(A, d, \varepsilon_i, \tau_n^{j_q,\dots,j_1})$, we have $\varepsilon_i\tau_n^{j_q,\dots,j_1} = 0$, $n+q > 1$, from dimensional considerations. It is also easy to see that each connected A_∞^{hu}-algebra $(A, d, \tau_n^{j_q,\dots,j_1})$ ("connected" means that $A_0 = K$) is an supplemented A_∞^{hu}-algebra $(A, d, \varepsilon_i, \tau_n^{j_q,\dots,j_1})$, where $\varepsilon_1 = \varepsilon_2\colon A_0 = K \to K$ is the identity map of the base ring K and $\varepsilon : A_m \to K$ is the zero map for all $m > 0$.

We proceed to specifying a relationship between supplemented A_∞^{hu}-algebras and tensor ∞-simplicial coalgebras. Given an supplemented A_∞^{hu}-algebra $(A, d, \varepsilon_i, \tau_n^{j_q,\dots,j_1})$, consider the tensor differential bigraded coalgebra $(T(A), d, \nabla)$, where $T(A)_{n,m} = (A^{\otimes n})_m$, $n \geq 0$, $m \geq 0$, $d\colon T(A)_{n,\bullet} \to T(A)_{n,\bullet-1}$ is an ordinary differential in the tensor product, $A^{\otimes 0} = K$ is the base ring, and the comultiplication $\nabla\colon T(A)_{*,\bullet} \to (T(A) \otimes T(A))_{*,\bullet}$ is defined by

$$\nabla(a_1 \otimes \dots \otimes a_n) = \sum_{i=0}^{n} (-1)^{k_i}(a_1 \otimes \dots \otimes a_i) \otimes (a_{i+1} \otimes \dots \otimes a_n),$$

where $k_i = i(\deg(a_{i+1}) + \dots + \deg(a_n))$. On the bigraded module $T(A)$, we define a family of maps

$$\widetilde{(\partial s)} = \{(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)}\colon T(A)_{n,m} \to T(A)_{n-p+q,m+p+q-1}\},$$

where $p \geq 0$, $q \geq 0$, $p+q \geq 1$, $0 \leq i_1 < \dots < i_p \leq n+q$, and $n+q-1 \geq j_q > \dots > j_1 \geq 0$, by the following rules:

(1) for $p \geq 1$ and $q = 0$, we set

$$(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)} = (\partial s)^n_{(i_1,\dots,i_p|\varnothing)} = \partial^n_{(i_1,\dots,i_p)}$$
$$= \begin{cases} (-1)^{m-1}(\varepsilon_1 \cdot 1) \otimes 1^{\otimes(n-2)} & \text{if } p=1,\ i_1=0, \\ (-1)^{m-1} 1^{\otimes(n-2)} \otimes (1 \cdot \varepsilon_2) & \text{if } p=1,\ i_1=n, \\ (-1)^{p(m-1)} 1^{\otimes(k-1)} \otimes \tau_p^{\varnothing} \otimes 1^{\otimes(n-p-k)} & \\ \qquad \text{if } 1 \leq k \leq n-p \text{ and } (i_1,\dots,i_p)=(k,k+1,\dots,k+p-1), & \\ 0 \quad \text{otherwise,} & \end{cases} \tag{3.8}$$

where $(\varepsilon_1 \cdot 1)(a_1 \otimes a_2) = \varepsilon_1(a_1)a_2$, and $(1 \cdot \varepsilon_2)(a_1 \otimes a_2) = a_1\varepsilon_2(a_2)$;

(2) for $p = 0$ and $q \geq 1$, we set

$$(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)} = (\partial s)^n_{(\varnothing|j_q,\dots,j_1)} = s^n_{(j_q,\dots,j_1)}$$
$$= \begin{cases} (-1)^m 1^{\otimes j_1} \otimes \tau_0^0 \otimes 1^{\otimes(n-j_1)} & \text{if } \quad q=1, \\ 0 & \text{if } \quad q>1; \end{cases} \tag{3.9}$$

(3) for $p \geq 1$ and $q \geq 1$, we set

$$(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)}$$
$$= \begin{cases} (-1)^{(p+q)(m+q-1)} 1^{\otimes(k-1)} \otimes \tau_p^{j_q-(k-1),\dots,j_1-(k-1)} \otimes 1^{\otimes(n-p+q-k)} & \\ \qquad \text{if } 1 \leq k \leq n-p+q,\ (i_1,\dots,i_p)=(k,k+1,\dots,k+p-1),\ j_1 \geq k-1, & \\ 0 \quad \text{otherwise.} & \end{cases} \tag{3.10}$$

Theorem 3.1. *For any supplemented A_∞^{hu}-algebra $(A, d, \varepsilon_i, \tau_n^{j_q,\dots,j_1})$, the quadruple (specified above) $(T(A), d, \nabla, \widetilde{(\partial s)})$ is an ∞-simplicial coalgebra.*

Proof. First, we show that the triple $(T(A), d, \widetilde{(\partial s)})$ is an ∞-simplicial module. We must check relations (1.9) and (1.10) for the family of maps

$$\{(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)} \colon T(A)_{n,m} \to T(A)_{n-p+q,m+p+q-1}\}$$

defined by (3.8)–(3.10). For the maps

$$(\partial s)^{p-q+1}_{(i_1,\dots,i_p|j_q,\dots,j_1)} \colon (A^{\otimes(p-q+1)})_c \to A_{c+p+q-1}, \qquad 0 \leq p \leq 1, \quad p \geq j_q > \dots > j_1 \geq 0,$$

relations (1.9) and (1.10) follow in an obvious way from (3.5), (3.7), and (3.1) with $n = -1, 0$. Now, let us check (1.9) for the maps

$$(\partial s)^{n+3-q}_{(1,2,\dots,n+2|j_q,\dots,j_1)} = (-1)^{(n+2+q)(c+q-1)} \tau_{n+2}^{j_q,\dots,j_1} \colon (A^{\otimes(n+3-q)})_c \to A_{c+n+q+1},$$
$$n \geq 0, \quad q \geq 0, \quad n+2 \geq j_q > \dots > j_1 \geq 0.$$

It follows from (3.8)–(3.10) that, in the case under consideration, relation (1.9) can be written in the form

$$d((\partial s)^{n+3-q}_{(1,2,\dots,n+2|j_q,\dots,j_1)}) = \sum_{m=1}^{n+1}\sum_{t=1}^{m+1} (-1)^{\operatorname{sign}(\sigma_{m,t,k,l})+1} (\partial s)^{m-q+k-l}_{(1,2,\dots,m|j_q-(n-m+2),\dots,j_k-(n-m+2),j_l,\dots,j_1)}$$
$$\times (\partial s)^{n+3-q}_{(t-l,t-l+1\dots,t-l+n-m+1|j_{k-1}-l,\dots,j_{l+1}-l)}$$
$$+ \sum_{i=1}^{q} (-1)^{\operatorname{sign}(\sigma_i)+1} (\partial s)^{n+4-q}_{(1,2,\dots,n+2|j_q,\dots,j_{i+1},j_{i-1},\dots,j_1)} s^{n+3-q}_{(j_i-(i-1))}, \tag{3.11}$$

where

$$d((\partial s)^{n+3-q}_{(1,2,\dots,n+2|j_q,\dots,j_1)}) = d(\partial s)^{n+3-q}_{(1,2,\dots,n+2|j_q,\dots,j_1)} + (\partial s)^{n+3-q}_{(1,2,\dots,n+2|j_q,\dots,j_1)}d$$

and the summation in the first term on the right-hand side is over all numbers $k \geq 1$ and $l \geq 0$ satisfying, for each fixed t, the inequalities

$$0 \leq j_1 < \dots < j_l \leq t-2 < j_{l+1} < \dots < j_{k-1} < t+n-m+2 \leq j_k < \dots < j_q \leq n+2.$$

Each permutation $\sigma_i \in \Sigma_{n+2+q}$ in the second term on the right-hand side of (3.11) breaks every element $a_1 \otimes \dots \otimes a_{n+2} \otimes b_q \otimes \dots \otimes b_1 \in (SM)^{n+2+q}$ into three blocks as

$$(a_1 \otimes \dots \otimes a_{n+2} \otimes b_q \otimes \dots \otimes b_{i+1}) \otimes (b_i) \otimes (b_{i-1} \otimes \dots \otimes b_1)$$

and transposes (in the sense of the action of Σ_{n+2+q} on $(SM)^{n+2+q}$) the second and the third block. Each permutation $\sigma_{m,t,k,l} \in \Sigma_{n+2+q}$ in the first term on the right-hand side of (3.11) is the product of the permutations $(\nu_{m,k,l})(\varrho_{m,k})(\sigma_{m,t})$ acting on $(SM)^{n+2+q}$ by the following rules:

(1) $\sigma_{m,t} \in \Sigma_{n+2+q}$ breaks each element $a_1 \otimes \dots \otimes a_{n+2} \otimes b_q \otimes \dots \otimes b_1$ into four blocks as

$$(a_1 \otimes \dots \otimes a_{t-1}) \otimes (a_t \otimes \dots \otimes a_{t+n-m+1}) \otimes (a_{t+n-m+2} \otimes \dots \otimes a_{n+2}) \otimes (b_q \otimes \dots \otimes b_1)$$

and transposes (in the sense of the action of Σ_{n+2+q} on $(SM)^{n+2+q}$) the second and the third block;

(2) $\varrho_{m,k} \in \Sigma_{n+2+q}$ breaks each element $a_1 \otimes \dots \otimes a_{n+2} \otimes b_q \otimes \dots \otimes b_1$ into four blocks as

$$(a_1 \otimes \dots \otimes a_m) \otimes (a_{m+1} \otimes \dots \otimes a_{n+2}) \otimes (b_q \otimes \dots \otimes b_k) \otimes (b_{k-1} \otimes \dots \otimes b_1)$$

and transposes (in the sense of the action of Σ_{n+2+q} on $(SM)^{n+2+q}$) the second and the third block;

(3) $\nu_{m,k,l} \in \Sigma_{n+2+q}$ breaks each element $a_1 \otimes \dots \otimes a_{n+q-k+3} \otimes b_{k-1} \otimes b_1$ into three blocks as

$$(a_1 \otimes \dots \otimes a_{m+q-k+1}) \otimes (a_{m+q-k+2} \otimes \dots \otimes a_{n+q-k+3} \otimes b_{k-1} \otimes \dots \otimes b_{l+1}) \otimes (b_l \otimes \dots \otimes b_1)$$

and transposes (in the sense of the action of Σ_{n+2+q} on $(SM)^{n+2+q}$) the second and the third block.

It is easy to check that the result of the action of each permutation σ_i on any ordered element of the form $\alpha = [\partial_1] \otimes [\partial_2] \otimes \dots \otimes [\partial_{n+2}] \otimes [s_{j_q}] \otimes \dots \otimes [s_{j_1}]$ is the element

$$\beta = [\partial_1] \otimes [\partial_2] \otimes \dots \otimes [\partial_{n+2}] \otimes [s_{j_q}] \otimes \dots \otimes [s_{j_{i+1}}] \otimes [s_{j_{i-1}}] \otimes \dots \otimes [s_{j_1}] \otimes [s_{j_1-(i-1)}].$$

Clearly, $\operatorname{sign}(\beta) = i-1 \equiv \operatorname{sign}(\sigma_i) \pmod 2$. Direct calculations show also that the result of the action of each permutation $\sigma_{m,t,k,l}$ on any ordered element of the form

$$\alpha = [\partial_1] \otimes [\partial_2] \otimes \dots \otimes [\partial_{n+2}] \otimes [s_{j_q}] \otimes \dots \otimes [s_{j_1}]$$

for which $j_{k-1} < t+n-m+2 \leq j_k$ and $j_l \leq t-2 < j_{l+1}$ is the element

$$\gamma = [\partial_1] \otimes [\partial_2] \otimes \dots \otimes [\partial_m] \otimes [s_{j_q-(n-m+2)}] \otimes \dots \otimes [s_{j_k-(n-m+2)}] \otimes [s_{j_l}] \otimes \dots \otimes [s_{j_1}] \\ \otimes [\partial_{t-l}] \otimes [\partial_{t-l+1}] \otimes \dots \otimes [\partial_{t-l+n-m+1}] \otimes [s_{j_{k-1}-l}] \otimes \dots \otimes [s_{j_{l+1}-l}].$$

It is easy to show that, for γ, we have

$$\operatorname{sign}(\gamma) \equiv t(n-m) + n + (n-m)(q-k+l+1) + nm + kl \equiv \operatorname{sign}(\sigma_{m,t,k,l}) \pmod 2.$$

Multiplying both sides of (3.6) by $(-1)^{(n+2+q)(c+q-1)}$, taking into account the last congruence, and using (3.8)–(3.10), we obtain (3.11). Thus, we have checked (1.9) for the maps $(\partial s)^{n+3-q}_{(1,2,\dots,n+2|j_q,\dots,j_1)}$ with $n \geq 0$ and $n+2 \geq j_q > \dots > j_1 \geq 0$. In a similar way, relations (1.9) are verified for the maps $(\partial s)^{n}_{(i_1,\dots,i_p|j_q,\dots,j_1)}$ with $p \geq 2$ in the general case. Thus, for each supplemented A^{hu}_∞-algebra

$(A, d, \varepsilon_i, \tau_n^{j_q,\dots,j_1})$, the triple $(T(A), d, \widetilde{(\partial s)})$ is an ∞-simplicial module. Let us endow this ∞-simplicial module with the comultiplication $\nabla\colon T(A)_{*,\bullet} \to (T(A) \otimes T(A))_{*,\bullet}$ specified above. A straightforward calculation using (2.3) and (3.8)–(3.10) shows that, for each map $(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)} \in \widetilde{(\partial s)}$, we have

$$\nabla(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)} = (\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)}\nabla.$$

Since $(T(A), d, \nabla)$ is a differential coalgebra, it follows that

$$\nabla(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)} = (\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)}\nabla$$

if and only if condition (2.4) holds; therefore, the quadruple $(T(A), d, \nabla, \widetilde{(\partial s)})$ is an ∞-simplicial coalgebra. $\square$

The proof of Theorem 3.1 given above provides a convenient simplicial method for calculating (3.6). Indeed, as seen from this proof, $d(\tau_{n+2}^{j_q,\dots,j_1})$ with $n \geq 0$ and $q \geq 0$ is calculated as follows.

(1) Write the simplicial expression $\partial_1\partial_2 \dots \partial_{n+2}s_{j_q} \dots s_{j_1}$ from $\tau_{n+2}^{j_q,\dots,j_1}$.

(2) Write out the element $\partial_1\partial_2 \dots \partial_{n+2}s_{j_q} \dots s_{j_1}$ and all elements obtained from it by using the simplicial relations between faces and degeneracies, except those of the forms $\partial_i s_i = 1$ and $\partial_{i+1}s_i = 1$.

(3) For each element $\gamma = a_1 \cdots a_{n+q+2}$ obtained in (2), find all partitions (if they exist) of γ into two blocks $(a_1 \cdots a_z) \mid (a_{z+1} \cdots a_{n+q+2})$ of the forms $\partial_1\partial_2 \cdots \partial_m s_{k_\mu} \cdots s_{k_1}$, where $m \geq k_\mu > \dots > k_1 \geq 0$, $m \geq 1$, and $\mu \geq 0$, and $\partial_t\partial_{t+1} \cdots \partial_{t+p-1}s_{l_\lambda} \cdots s_{l_1}$, where $p+t-1 \geq l_\lambda > \dots > l_1 \geq t-1$, $t \geq 1$, $p \geq 1$, and $\lambda \geq 0$, respectively.

(4) For each partition $(\partial_1\partial_2 \cdots \partial_m s_{k_\mu} \cdots s_{k_1}) \mid (\partial_t\partial_{t+1} \cdots \partial_{t+p-1}s_{l_\lambda} \cdots s_{l_1})$ of $\gamma = a_1 \cdots a_{n+q+2}$ found in (3), write the corresponding element

$$(-1)^{\mathrm{sign}(\gamma)+1+q\mu+nm}\tau_m^{k_\mu,\dots,k_1}(\underbrace{1 \otimes \dots \otimes 1}_{t-1} \otimes \tau_p^{l_\lambda-(t-1),\dots,l_1-(t-1)} \otimes 1 \otimes \dots \otimes 1),$$

where $\mathrm{sign}(\gamma) = \mathrm{sign}([a_1] \otimes \dots \otimes [a_{n+q+2}])$. Summing all such elements over all partitions obtained in (3) for all elements obtained in (2), calculate the first sum on the right-hand side of (3.6).

(5) For each element $\gamma = a_1 \cdots a_{n+q+2}$ obtained in (2), find a partition (if it exists) of this element into two blocks $(a_1 \cdots a_{n+q+1}) \mid (a_{n+q+2})$, where the first block has the form $(\partial_1\partial_2 \cdots \partial_{n+2}s_{k_{q-1}} \cdots s_{k_1})$ with $n+2 \geq k_{q-1} > \dots > k_1 \geq 0$ and the second block has the form (s_i) with $i \geq 0$.

(6) For each partition $(\partial_1\partial_2 \cdots \partial_{n+2}s_{k_{q-1}} \cdots s_{k_1})|(s_i)$ found in (5) for $\gamma = a_1 \cdots a_{n+q+2}$, write the corresponding element

$$(-1)^{\mathrm{sign}(\gamma)+1+n}\tau_{n+2}^{k_{q-1},\dots,k_1}(\underbrace{1 \otimes \dots \otimes 1}_{i} \otimes \tau_0^0 \otimes 1 \otimes \dots \otimes 1),$$

where $\mathrm{sign}(\gamma) = \mathrm{sign}([a_1] \otimes \dots \otimes [a_{n+q+2}])$. Summing all such elements over all partitions obtained in (5) for all elements obtained in (2), calculate the second sum on the right-hand side of (3.6).

For example, using the simplicial method for writing relations (3.6) described above, we readily obtain

$$d(\tau_2^0) = \tau_1^0\pi_0 + \pi_0(\tau_1^0 \otimes 1) - \pi_1(\tau_0^0 \otimes 1 \otimes 1),$$
$$d(\tau_2^{2,0}) = -\tau_1^1\tau_1^0 - \tau_1^0\tau_1^1 - \tau_2^2(\tau_0^0 \otimes 1) + \tau_2^0(1 \otimes \tau_0^0).$$

Now let us consider the situation opposite to that considered above, where the tensor differential coalgebra $(T(A), d, \nabla)$ of any differential module (A, d) is endowed with the structure of an ∞-simplicial coalgebra $(T(A), d, \nabla, \widetilde{(\partial s)})$. It follows from (2.4) that all maps $(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)} \in \widetilde{(\partial s)}$ satisfy the condition

$$\nabla(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)} = (\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)}\nabla.$$

This condition and (2.3) imply that, to specify a family of maps $\widetilde{(\partial s)}$, it suffices to specify only maps

$$\partial^1_{(i)}\colon T(A)_{1,\bullet} = A_\bullet \to K_\bullet = T(A)_{0,\bullet}, \quad 0 \leq i \leq 1,$$

$$s^0_{(0)}\colon T(A)_{0,\bullet} = K_\bullet \to A_\bullet = T(A)_{1,\bullet},$$
$$(\partial s)^{p-q+1}_{(1,\dots,p|j_q,\dots,j_1)}\colon T(A)_{p-q+1,\bullet} = (A^{\otimes(p-q+1)})_\bullet \to A_{\bullet+p+q-1} = T(A)_{1,\bullet+p+q-1},$$
$$p \geq 1, \quad q \geq 0, \quad p \geq j_q > \dots > j_1 \geq 0,$$

because the remaining maps $(\partial s)^n_{(i_1,\dots,i_p|j_q,\dots,j_1)}\colon T(A)_{n,m} \to T(A)_{n-p+q,m+p+q-1}$ in the family (∂s) are uniquely determined by (3.8)–(3.10), provided that

$$\varepsilon_i = (-1)^{m-1}\partial^1_{(i)}, \qquad \tau^0_0 = s^0_{(0)}, \qquad \tau^{j_q,\dots,j_1}_p = (-1)^{(p+q)(m+q-1)}(\partial s)^{p-q+1}_{(1,\dots,p|j_q,\dots,j_1)}. \tag{3.12}$$

Let us show that the quadruple $(A, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_p)$, where ε_i and $\tau^{j_q,\dots,j_1}_p$ are specified by (3.12), is an supplemented A^{hu}_∞-algebra. It is required to check (3.5)–(3.7). Relations (3.5) are obtained from

$$\tau^{\varnothing}_1 = (-1)^{m-1}\partial^2_{(1)}, \qquad \tau^0_1 = (\partial s)^1_{(1|0)}, \qquad \tau^1_1 = (\partial s)^1_{(1|1)}, \qquad \tau^{1,0}_1 = (\partial s)^0_{(1|1,0)}$$

by using (1.9), (1.10), and (3.8)–(3.10). Relations (3.6) are obtained from (3.11) and (3.8)–(3.10) by using an argument similar to that in the proof of Theorem 3.1. Let us prove (3.7). Obviously, $d(\partial^1_{(0)}) = 0$ and $d(\partial^1_{(1)}) = 0$. Since $(\partial s)^0_{(0|0)} = (\partial s)^0_{(1|0)} = 0$, it follows that $\partial^1_{(0)}s^0_{(0)} = \partial^1_{(1)}s^0_{(0)} = 1$, and since $\partial^2_{(0,1)} = \partial^2_{(1,2)} = 0$, it follows that $\partial^1_{(0)}\partial^2_{(1)} = \partial^1_{(0)}\partial^2_{(0)}$ and $\partial^1_{(1)}\partial^2_{(2)} = \partial^1_{(1)}\partial^2_{(1)}$. These relations and (3.8)–(3.10) imply (3.7). Thus, the quadruple $(A, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_p)$ under consideration is an supplemented A^{hu}_∞-algebra. Applying Theorem 3.1, we obtain the following result.

Theorem 3.2. *Endowing a differential module (A, d) with the structure of an supplemented A^{hu}_∞-algebra $(A, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_p)$ is equivalent to endowing the corresponding tensor differential coalgebra $(T(A), d, \nabla)$ with the structure of an ∞-simplicial coalgebra $(T(A), d, \psi, \nabla)$.*

In what follows, by a connected differential module (A, d) we understand any nonnegatively graded differential module satisfying the condition $A_0 = K$, where K is the base ring.

Corollary 3.1. *Endowing a connected differential module (A, d) with the structure of an A^{hu}_∞-algebra $(A, d, \tau^{j_q,\dots,j_1}_p)$ is equivalent to endowing the corresponding tensor differential coalgebra $(T(A), d, \nabla)$ with the structure of an ∞-simplicial coalgebra $(T(A), d, \psi, \nabla)$.*

Definition 3.2. By a *morphism* $f\colon (X, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_n) \to (Y, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_n)$ *of supplemented A^{hu}_∞-algebras* we mean a morphism of the corresponding ∞-simplicial coalgebras

$$f\colon (T(X), d, \psi, \nabla) \to (T(Y), d, \psi, \nabla).$$

Definition 3.3. By a *homotopy* $h\colon (X, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_n) \to (Y, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_n)$ *between morphisms* $f, g\colon (X, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_n) \to (Y, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_n)$ *of supplemented A^{hu}_∞-algebras* we mean a homotopy $h\colon (T(X), d, \psi, \nabla) \to (T(Y), d, \psi, \nabla)$ between the corresponding morphisms

$$f, g\colon (T(X), d, \psi, \nabla) \to (T(Y), d, \psi, \nabla)$$

of ∞-simplicial coalgebras. SDR-*data for supplemented A^{hu}_∞-algebras* are the corresponding SDR-data for ∞-simplicial coalgebras.

The following theorem, which is a corollary of Theorems 2.1 and 3.2, asserts the homotopy invariance of the structure of an supplemented A^{hu}_∞-algebra under homotopy equivalences of the type of SDR-data for differential modules.

Theorem 3.3. *Suppose given an supplemented A^{hu}_∞-algebra $(X, d, \varepsilon_i, \tau^{j_q,\dots,j_1}_n)$,* SDR-*data*

$$(\eta\colon (X, d) \rightleftarrows (Y, d) : \xi, h)$$

for differential modules, and the SDR-*data*

$$(T(\eta)\colon (T(X), d, \nabla) \rightleftarrows (T(Y), d, \nabla) : T(\xi), T(h))$$

for tensor differential coalgebras corresponding to the SDR-*data* $(\eta\colon (X,d) \rightleftarrows (Y,d) : \xi, h)$ *for differential modules. Then relations* (1.5)–(1.8) *define the structure of an supplemented* A_∞^{hu}-*algebra* $(Y, d, \varepsilon_i, \overline{\tau}_n^{j_q,\dots,j_1})$ *on* (Y,d) *and* SDR-*data*

$$(\overline{T(\eta)}\colon (X, d, \varepsilon_i, \tau_n^{j_q,\dots,j_1}) \rightleftarrows (X, d, \varepsilon_i, \overline{\tau}_n^{j_q,\dots,j_1})) : \overline{T(\xi)}, \overline{T(h)}\,)$$

for supplemented A_∞^{hu}-*algebras which extend the* SDR-*data for tensor differential coalgebras specified above.*

Understanding morphisms, homotopies, and SDR-data for connected A_∞^{hu}-algebras as the corresponding morphisms, homotopies, and SDR-data for supplemented A_∞^{hu}-algebras, we obtain the following obvious corollary of Theorem 3.3.

Corollary 3.2. *Suppose given a connected* A_∞^{hu}-*algebra* $(X, d, , \tau_n^{j_q,\dots,j_1})$, SDR-*data*

$$(\eta\colon (X,d) \rightleftarrows (Y,d) : \xi, h)$$

for connected differential modules, and the SDR-*data*

$$(T(\eta)\colon (T(X), d, \nabla) \rightleftarrows (T(Y), d, \nabla) : T(\xi), T(h))$$

for tensor differential coalgebras corresponding to the given SDR-*data* $(\eta\colon (X,d) \rightleftarrows (Y,d) : \xi, h)$ *for differential modules. Then relations* (1.5)–(1.8) *define the structure of an* A_∞^{hu}-*algebra* $(Y, d, \overline{\tau}_n^{j_q,\dots,j_1})$ *on* (Y,d) *and* SDR-*data*

$$(\overline{T(\eta)}\colon (X, d, \tau_n^{j_q,\dots,j_1}) \rightleftarrows (X, d, \overline{\tau}_n^{j_q,\dots,j_1})) : \overline{T(\xi)}, \overline{T(h)}\,)$$

for A_∞^{hu}-*algebras extending the* SDR-*data for tensor differential coalgebras specified above.*

It is worth mentioning that Corollary 3.2, unlike the corresponding assertion in [4], provides SDR-data for A_∞^{hu}-algebras rather than only the quasi-isomorphism of A_∞^{hu}-algebras.

In conclusion, we mention that Corollary 3.2 can be proved without the connectedness assumption on the differential modules. To this end, it suffices to consider the colored algebra (S', π) obtained from the colored algebra (S, π) of faces and degeneracies by "forgetting" the generators ∂_0^n and ∂_n^n with $n \geq 0$ and all relations containing them. Replacing the colored algebra (S, π) by the colored algebra (S', π) in all of the above considerations, we obtain a proof of Corollary 3.2 without the connectedness assumption on the differential modules.

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