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On Optimal Banach Spaces Containing a Weight Cone of Monotone or Quasiconcave Functions

V. D. Stepanov*

Peoples' Friendship University of Russia, Moscow, Russia
Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia
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Abstract—Optimal (minimal) Banach spaces containing given cones of monotone or quasiconcave functions on the semiaxis from weighted Lebesgue spaces are described. Exact formulas for the norm of the optimal space are presented. All cases of the summation parameter are studied.

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1. INTRODUCTION

Let $\mathfrak{M}$ denote the class of all Lebesgue measurable functions on $\mathbb{R}_+ := [0, \infty)$, and let

$$\mathfrak{M}^+ := \{f \in \mathfrak{M} : f \geq 0\}.$$

Let $K \subset \mathfrak{M}^+$ be a cone with a positive homogeneous functional $\sigma\colon K \to \mathbb{R}_+$, i.e., if $\alpha > 0$ and $f \in K$, then $\alpha f \in K$, $\sigma(\alpha f) = \alpha\sigma(f)$; furthermore, $\sigma(f) = 0$ if and only if $f = 0$ almost everywhere.

The following problem is well known: For a given cone K, construct an optimal Banach space $X_0 \subset \mathfrak{M}$ such that $K \hookrightarrow X_0$ and if $K \hookrightarrow X$ for a Banach space $X \subset \mathfrak{M}$ as well, then $X \subset X_0$. Here $K \hookrightarrow X$ denotes a continuous embedding. This problem was considered in the paper of Gol'dman and Zabreiko [1], where its close association with similar problems on optimal spaces in the embedding theory for Sobolev, Besov, and Calderón spaces, and Bessel and Riesz potentials, etc. (see, for example, [2]–[7]), as well as with applications to estimates of approximation numbers, was noted.

Let $\mathfrak{M}^{\downarrow} \subset \mathfrak{M}^+$ be the subset of all nonincreasing functions, and let $\mathfrak{M}^{\uparrow} \subset \mathfrak{M}^+$ be the subset of all nondecreasing functions. Let $0 < p \leq \infty$, and let $v \in \mathfrak{M}^+$, be a given weight function (a weight). We consider the weight cone of monotone functions $K = \mathcal{L}_{p,v} \subset \mathfrak{M}^{\downarrow}$, that, for $0 < p < \infty$, is of the form

$$\mathcal{L}_{p,v} := \left\{ f \in \mathfrak{M}^{\downarrow} : \|f\|_{p,v} := \left(\int_0^\infty \left(\frac{1}{t} \int_0^t f \right)^p v(t)\, dt \right)^{1/p} < \infty \right\}$$

and, for $p = \infty$,

$$\mathcal{L}_{\infty,v} := \left\{ f \in \mathfrak{M}^{\downarrow} : \|f\|_{\infty,v} := \operatorname*{ess\,sup}_{t \geq 0} \left(\frac{1}{t} \int_0^t f \right) v(t) < \infty \right\},$$

and obtain the optimal Banach space X_0 containing this cone. Explicit formulas for the norm of the space $X_0 \supset K$ will be presented in Sec. 3 (Theorem 2).

Section 2 deals with the auxiliary problem of finding the Banach space associated with the cone K, a problem that can be reduced to the characterization of the weighted inequalities of the embedding $\mathcal{L}_{p,v}$ in a weighted Lebesgue space on the semiaxis. The solution of this problem was first given in [8] and [9], but more explicit formulas were obtained by Gogatishvili and Pick in [10] and [11]. However, the methods

*E-mail: stepanov@mi.ras.ru

of discretization [9] and antidiscretization used in the proofs in [10] and [11] are fairly complicated; therefore, in Sec. 2, we give simple proofs of the assertions that we need; these proofs differ from those in [9]–[12], while the techniques developed there will be used in our arguments in Secs. 3 and 4. In conclusion, in Sec. 4, we solve a similar problem for the weight cone of quasiconcave functions

$$\Omega_{0,1} := \biggl\{ f \in \mathfrak{M}^+ : f \in \mathfrak{M}^{\uparrow},\ \frac{f(t)}{t} \in \mathfrak{M}^{\downarrow} \biggr\}.$$

For $0 < p \le \infty$ and a weight function $v \in \mathfrak{M}^+$, we consider the weight cone of functions $K = \mathfrak{L}_{p,v}$ of the form

$$\mathfrak{L}_{p,v} := \biggl\{ f \in \Omega_{0,1} : \|f\|_{\mathfrak{L}_{p,v}} := \biggl(\int_0^\infty [f]^p v \biggr)^{1/p} < \infty \biggr\}, \qquad 0 < p < \infty,$$

or, for $p = \infty$,

$$\mathfrak{L}_{\infty,v} := \Bigl\{ f \in \Omega_{0,1} : \|f\|_{\mathfrak{L}_{\infty,v}} := \operatorname*{ess\,sup}_{t \ge 0} f(t)v(t) < \infty \Bigr\},$$

and obtain the optimal Banach space X_0 containing this cone. Note that the cone $\mathcal{L}_{p,v}$ was first studied by Sawyer [13] and then this cone, as well as the cone $\mathfrak{L}_{p,v}$, was studied intensively in connection with problems involving Lorentz spaces and related problems in analysis [8]–[12], [14]–[26], etc.

Throughout the paper, products of the form $0 \cdot \infty$ will be assumed equal to 0. The relation $A \ll B$ means $A \le cB$ with constant c depending only on p; $A \approx B$ is equivalent to $A \ll B \ll A$. We use the notation := and =: to define new quantities. If $0 < p \le \infty$, then $p \ne 1, \infty$, $p = 1$ $p' := 1$ $p = \infty$.

$$p' := \begin{cases} \dfrac{p}{p-1} & \text{if } p \ne 1, \infty, \\ \infty & \text{if } p = 1, \\ 1 & \text{if } p = \infty. \end{cases}$$

2. WEIGHTED INEQUALITIES ON THE CONE OF MONOTONE FUNCTIONS

2.1. Let $0 < p < \infty$. For a fixed $g \in \mathfrak{M}^{\downarrow}$, we consider the characteristic problem for the inequality

$$\int_0^\infty fg \le C \biggl(\int_0^\infty \biggl(\frac{1}{t} \int_0^t f \biggr)^p v(t)\,dt \biggr)^{1/p}, \qquad f \in \mathfrak{M}^{\downarrow}, \tag{2.1}$$

where $0 < p < \infty$, the constant C is assumed least possible, and the problem is to find a sharp estimate of the constant C in terms of the weight v and the parameter p. Further, we assume that the nondegeneracy conditions hold:

$$0 < \int \frac{v(z)\,dz}{(z+t)^p} < \infty, \qquad \int_0^1 z^{-p} v(z)\,dz = \int_1^\infty v = \infty. \tag{2.2}$$

If these conditions do not hold, then it is necessary to analyze a number of pathological cases.

For $p = \infty$, a similar problem is considered for the inequality

$$\int_0^\infty fg \le C \operatorname*{ess\,sup}_{t \ge 0} \biggl(\frac{1}{t} \int_0^t f \biggr) v(t), \qquad f \in \mathfrak{M}^{\downarrow}.$$

Set

$$Pv(t) := \frac{1}{t} \int_0^t v, \qquad Q_p v(t) := p t^{p-1} \int_t^\infty s^{-p} v(s)\,ds.$$

Then, by a trivial verification procedure, we can establish that

$$PQ_p = Q_p P = P + \frac{1}{p} Q_p.$$

The case $p = 1$**.** In this case, for $g \in \mathfrak{M}^+$, inequality (2.1) is of the form

$$\int_0^\infty fg \le C \int_0^\infty f(s)\,ds \int_s^\infty \frac{v(t)}{t}\,dt, \qquad f \in \mathfrak{M}^\downarrow,$$

and its characterization is known [14, Proposition 1]:

$$C = \sup_{s>0} \frac{\int_0^s g}{\int_0^s \bigl(\int_z^\infty (v(t)/t)\,dt\bigr)\,dz} = \sup_{s>0} \frac{Pg(s)}{PQ_1 v(s)}\,.$$

The case $p \ne 1, \infty$**.** Let

$$F(t) := \biggl(\frac{1}{t}\int_0^t f\biggr)^p, \qquad f \in \mathfrak{M}^\downarrow,$$

and, without loss of generality, suppose that

$$g(t) = \int_t^\infty \frac{h(s)\,ds}{s}\,, \qquad h \in \mathfrak{M}^+.$$

Then (2.1) is equivalent to

$$\biggl(\int_0^\infty F^{1/p} h\biggr)^p \le C^p \int_0^\infty Fv. \tag{2.3}$$

Since

$$F(t) \in \mathfrak{M}^\downarrow, \qquad t^p F(t) \in \mathfrak{M}^\uparrow,$$

it follows from Sinnamon's lemma [20, Lemma 2.3] that there exists an $\overline{F}(t) \approx F(t)$ and a sequence of functions $\{w_n\} \subset \mathfrak{M}^+$ such that

$$\int_0^\infty \frac{w_n(z)\,dz}{(z+t)^p} \uparrow \overline{F}(t);$$

therefore, (2.3) is equivalent to

$$\biggl(\int_0^\infty h(t)\biggl(\int_0^\infty \frac{w(z)\,dz}{(z+t)^p}\biggr)^{1/p} dt\biggr)^p \ll C^p \int_0^\infty Vw, \qquad w \in \mathfrak{M}^+, \tag{2.4}$$

where

$$V(z) := \int_0^\infty \frac{v(t)\,dt}{(z+t)^p} \approx t^{1-p} PQ_p v(t).$$

Set

$$S_p w(t) := \int_0^\infty \frac{w(z)\,dz}{(z+t)^p}\,.$$

Then (2.4) is equivalent to

$$\biggl(\int_0^\infty [S_p w]^{1/p} h\biggr)^p \ll C_0^p \int_0^\infty Vw, \qquad w \in \mathfrak{M}^+, \tag{2.5}$$

where $C_0 \approx C$.

The case $0 < p < 1$**.** For the best constant C_0 in inequality (2.5), using Theorem [27, Chap. XI, Sec. 1.5, Theorem 4], we obtain

$$C_0^p = \sup_{z>0} \frac{1}{V(z)} \biggl(\int_0^\infty \frac{h(t)\,dt}{t+z}\biggr)^p.$$

Since

$$\int_0^\infty \frac{h(t)\,dt}{t+z} \approx \frac{1}{z}\int_0^z h + \int_z^\infty \frac{h(t)\,dt}{t} = \frac{1}{z}\int_0^z \int_s^\infty \frac{h(t)\,dt}{t}\,ds = \frac{1}{z}\int_0^z g, \tag{2.6}$$

it follows that, for $0 < p < 1$ and $g \in \mathfrak{M}^{\downarrow}$, the best constant in (2.1) satisfies the two-sided estimate

$$C \approx \sup_{z>0}\biggl(\frac{1}{z}\int_0^z g\biggr)\frac{1}{V^{1/p}(z)} \approx \sup_{z>0}\biggl(\frac{1}{z}\int_0^z g\biggr)\frac{z^{1/p'}}{(PQ_pv(z))^{1/p}}.$$

The case $1 < p < \infty$**.** It is easy to see that (2.5), and hence also (2.1), is equivalent to the simultaneous validity of two inequalities

$$\biggl(\int_0^\infty \frac{h(t)}{t}\biggl(\int_0^t w\biggr)^{1/p} dt\biggr)^p \le C_1^p \int_0^\infty wV_1,$$
$$\biggl(\int_0^\infty h(t)\biggl(\int_t^\infty w\biggr)^{1/p} dt\biggr)^p \le C_2^p \int_0^\infty wV_2,$$

where

$$V_1(z) := \frac{1}{z^p}\int_0^z v + \int_z^\infty t^{-p}v(t)\,dt \approx z^{1-p}PQ_pv(z), \qquad V_2(z) := z^pV_1(z);$$

further,

$$C \approx C_1 + C_2.$$

It is known [28, Theorem 3.3] (see also [29, formula (1.18)] that, for the best constants C_1 and C_2, the following estimates hold:

$$C_1^{p'} \approx \int_0^\infty [V_1(x)]^{1/(1-p)}\biggl(\int_x^\infty \frac{h(s)\,ds}{s}\biggr)^{1/(p-1)}\frac{h(x)}{x}\,dx,$$
$$C_2^{p'} \approx \int_0^\infty [V_2(x)]^{1/(1-p)}\biggl(\int_0^x h(t)\,dt\biggr)^{1/(p-1)} h(x)\,dx.$$

Integrating by parts, we obtain

$$C_1^{p'} \approx \int_0^\infty \biggl(\int_x^\infty \frac{h(s)\,ds}{s}\biggr)^{p'}\frac{x^{1/(p-1)}\int_0^x v}{V_2^{p'}(x)}\,dx,$$
$$C_2^{p'} \approx \int_0^\infty \biggl(\int_0^x h\biggr)^{p'}\frac{x^{p-1}\int_x^\infty s^{-p}v(s)\,ds}{V_2^{p'}(x)}\,dx,$$

Then

$$C_1^{p'} \approx \int_0^\infty \biggl(\int_x^\infty \frac{h(s)\,ds}{s}\biggr)^{p'}\frac{Pv(x)}{(PQ_pv(x))^{p'}}\,dx,$$
$$C_2^{p'} \approx \int_0^\infty \biggl(\frac{1}{x}\int_0^x h\biggr)^{p'}\frac{Q_pv(x)}{(PQ_pv(x))^{p'}}\,dx.$$

Since $PQ_pv \ge Pv$, $PQ_pv \ge Q_pv$, it follows that

$$\frac{Pv(x)}{(PQ_pv(x))^{p'}} = \frac{Pv(x)PQ_pv(x)}{(PQ_pv(x))^{p'+1}} \ge \frac{Pv(x)Q_pv(x)}{(PQ_pv(x))^{p'+1}},$$
$$\frac{Q_pv(x)}{(PQ_pv(x))^{p'}} = \frac{Q_pv(x)PQ_pv(x)}{(PQ_pv(x))^{p'+1}} \ge \frac{Pv(x)Q_pv(x)}{(PQ_pv(x))^{p'+1}}$$

and, using (2.6), we can write

$$C^{p'} \approx C_1^{p'} + C_2^{p'} \gg \int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\,dt\right)^{p'} \frac{Pv(x)Q_pv(x)}{(PQ_pv(x))^{p'+1}}\,dx.$$

To prove of the reverse inequality

$$C^{p'} \approx C_1^{p'} + C_2^{p'} \ll \int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\,dt\right)^{p'} \frac{Pv(x)Q_pv(x)}{(PQ_pv(x))^{p'+1}}\,dx, \tag{2.7}$$

we note that (2.7) is equivalent to two inequalities

$$C_1^{p'} \approx \int_0^\infty \left(\int_x^\infty \frac{h(s)\,ds}{s}\right)^{p'} \frac{Pv(x)dx}{(PQ_pv(x))^{p'}} \ll \int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\,dt\right)^{p'} \frac{Pv(x)Q_pv(x)}{(PQ_pv(x))^{p'+1}}\,dx, \tag{2.8}$$

$$C_2^{p'} \approx \int_0^\infty \left(\frac{1}{x}\int_0^x h\right)^{p'} \frac{Q_pv(x)\,dx}{(PQ_pv(x))^{p'}} \ll \int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\,dt\right)^{p'} \frac{Pv(x)Q_pv(x)}{(PQ_pv(x))^{p'+1}}\,dx. \tag{2.9}$$

Recall that here $g(t) = \int_t^\infty (h(s)/s)\,ds$.

Let us show that inequalities (2.8) and (2.9), and hence also (2.7), hold. For (2.8), we have

$$\int_0^\infty [g]^{p'} \frac{Pv}{(PQ_pv)^{p'}} \ll \int_0^\infty \left(\frac{1}{x}\int_0^x g\right)^{p'} \psi(x)\,dx, \qquad g \in \mathfrak{M}^{\downarrow}, \tag{2.10}$$

where

$$\psi := \frac{PvQ_pv}{(PQ_pv)^{p'+1}}.$$

By Theorem [8, Theorem 4.2(a)], (2.10) is valid if and only if

$$\int_0^t \frac{Pv}{(PQ_pv)^{p'}} \ll \int_0^t s^{p'-1}\,ds \int_s^\infty z^{-p'}\psi(z)\,dz. \tag{2.11}$$

Integrating by parts under condition (2.2), we obtain

$$\int_s^\infty z^{-p'}\psi(z)\,dz = \int_s^\infty z^{-p'} \frac{Pv(z)Q_pv(z)}{(PQ_pv)^{p'+1}}\,dz = \int_s^\infty \left(\int_0^z v\right) \frac{Q_pv(z)}{(\int_0^z Q_pv)^{p'+1}}\,dz$$
$$\approx \int_s^\infty \left(\int_0^z v\right) d\left(-\left(\int_0^z Q_pv\right)^{-p'}\right) \ge \frac{\int_0^s v}{(\int_0^s Q_pv)^{p'}}.$$

Thus,

$$\int_0^t s^{p'-1}\,ds \int_s^\infty z^{-p'}\psi(z)\,dz \gg \int_0^t s^{p'-1} \frac{\int_0^s v}{(\int_0^s Q_pv)^{p'}}\,ds = \int_0^t \frac{Pv}{(PQ_pv)^{p'}},$$

and, therefore, inequality (2.11) is proved.

Let us now verify (2.9). By the change

$$u(x) := \int_0^x h,$$

we reduce (2.9) to the inequality

$$\int_0^\infty [u(x)]^{p'} \frac{Q_pv(x)}{(\int_0^x Q_pv)^{p'}}\,dx \ll \int_0^\infty \left(\int_x^\infty s^{-2}u(s)\,ds\right)^{p'} \psi(x)\,dx, \tag{2.12}$$

where $u \in \mathfrak{M}^{\uparrow}$. By Theorem [8, Theorem 4.2(b)], (2.12) is valid if and only if

$$\int_t^\infty \frac{Q_pv(x)}{(\int_0^x Q_pv)^{p'}}\,dx \approx \frac{1}{(\int_0^t Q_pv)^{p'-1}} = \frac{1}{t^{p'-1}(PQ_pv(t))^{p'-1}} \ll \frac{1}{t^{p'}}\int_0^t \psi + \int_t^\infty \frac{\psi(z)\,dz}{z^{p'}}. \tag{2.13}$$

To prove the last inequality, we write

$$\int_0^t \psi = \int_0^t \frac{Pv(z)Q_pv(z)}{(PQ_pv)^{p'+1}}\,dz = \int_0^t \frac{Pv(z)Q_pv(z)}{(Q_pPv)^{p'+1}}\,dz = \int_0^t \frac{z^{-p-1}(\int_0^z v)\int_z^\infty s^{-p}v(s)\,ds}{(\int_z^\infty s^{-p-1}(\int_0^s v)\,ds)^{p'+1}}\,dz$$

$$\approx \int_0^t \biggl(\int_z^\infty s^{-p}v(s)\,ds\biggr)\,d\biggl(\biggl(\int_z^\infty s^{-p-1}\biggl(\int_0^s v\biggr)\,ds\biggr)^{-p'}\biggr) \gg \frac{\int_t^\infty s^{-p}v(s)\,ds}{(\int_z^\infty s^{-p-1}(\int_0^s v)\,ds)^{p'}}\,.$$

Then

$$\frac{1}{t^{p'}}\int_0^t \psi \gg \frac{Q_pv(t)}{t^{p'-1}(PQ_pv(t))^{p'}}\,. \tag{2.14}$$

For the second summand on the right-hand side of (2.13), we write

$$\int_t^\infty \frac{\psi(z)\,dz}{z^{p'}} = \int_t^\infty \frac{(\int_0^z v)Q_pv(z)}{(\int_0^z Q_pv)^{p'+1}}\,dz = \int_t^\infty \biggl(\int_0^z v\biggr)\,d\biggl(-\biggl(\int_0^z Q_pv\biggr)^{-p'}\biggr)$$

$$\gg \frac{\int_0^t v}{(\int_0^t Q_pv)^{p'}} = \frac{Pv(t)}{t^{p'-1}(PQ_pv(t))^{p'}}\,. \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\frac{1}{t^{p'}}\int_0^t \psi + \int_t^\infty \frac{\psi(z)\,dz}{z^{p'}} \gg \frac{Pv(t)+Q_pv(t)}{t^{p'-1}(PQ_pv(t))^{p'}} = \frac{1}{t^{p'-1}(PQ_pv(t))^{p'-1}}$$

and, therefore, (2.13) is proved.

Thus, for $1 < p < \infty$ and $g \in \mathfrak{M}^{\downarrow}$, the best constant in (2.1) satisfies the relation

$$C \approx \biggl(\int_0^\infty \biggl(\frac{1}{x}\int_0^x g(t)\,dt\biggr)^{p'} \frac{Pv(x)Q_pv(x)}{(PQ_pv(x))^{p'+1}}\,dx\biggr)^{1/p'}.$$

2.2. We shall now discard the monotonicity condition for the function g in inequality (2.1).

Let $w(x) \in \mathfrak{M}^+$. By the change $f(x) = \int_x^\infty \rho$, the problem of characterizing inequality (2.1) for $g = w$ can be reduced to the two-sided estimate of the functional

$$J := \sup_{f\in\mathfrak{M}^{\downarrow}} \frac{\int_0^\infty fw}{(\int_0^\infty((1/t)\int_0^t f)^p v(t)\,dt)^{1/p}} = \sup_{\rho\in\mathfrak{M}^+} \frac{\int_0^\infty \rho W}{(\int_0^\infty((1/t)\int_0^t ds\int_s^\infty \rho)^p v(t)\,dt)^{1/p}}\,, \tag{2.16}$$

where

$$W(t) := \int_0^t w.$$

Applying [30, Theorem 4.1], we obtain

$$J = \sup_{\rho\in\mathfrak{M}^+} \frac{\int_0^\infty \rho\overline{W}}{(\int_0^\infty((1/t)\int_0^t ds\int_s^\infty \rho)^p v(t)\,dt)^{1/p}}\,, \tag{2.17}$$

where

$$\overline{W}(x) := x\sup_{t\ge x} Pw(t) =: x\mathcal{W}(x). \tag{2.18}$$

Since $\overline{W}(x) \in \mathfrak{M}^{\uparrow}$, $\overline{W}(x)/x \in \mathfrak{M}^{\downarrow}$, it follows that [10, Lemma 2.8] there exists a Borel measure $d\eta$ such that

$$\overline{W}(x) \approx x\int_{[0,\infty)} \frac{d\eta(s)}{x+s}\,. \tag{2.19}$$

Changing the order of integration, we obtain

$$J \approx \sup_{\rho\in\mathfrak{M}^+} \frac{\displaystyle\int_{[0,\infty)}\biggl(\int_0^\infty \frac{x\rho(x)}{x+s}\,dx\biggr)\,d\eta(s)}{\displaystyle\biggl(\int_0^\infty\biggl(\int_0^\infty \frac{x\rho(x)}{x+s}\,dx\biggr)^p v(t)\,dt\biggr)^{1/p}}.$$

Just as in the argument in the first part, we can assume without loss of generality that

$$\biggl(\int_0^\infty \frac{x\rho(x)\,dx}{x+s}\biggr)^p \approx \int_0^\infty \frac{\nu(z)\,dz}{(z+s)^p}, \qquad J^p \approx \sup_{\nu\in\mathfrak{M}^+} \frac{\displaystyle\biggl(\int_{[0,\infty)}\biggl(\int_0^\infty \frac{\nu(z)}{(z+s)^p}\,dz\biggr)^{1/p} d\eta(s)\biggr)^p}{\int_0^\infty \nu V}. \tag{2.20}$$

Just as in the characterization of inequality (2.4) for the case $1<p<\infty$, we can write

$$J \approx \biggl(\int_0^\infty \biggl(\sup_{t\ge x}\frac{1}{t}\int_0^t w\biggr)^{p'} \frac{(Pv(x))Q_pv(x)}{(PQ_pv(x))^{p'+1}}\,dx\biggr)^{1/p'}.$$

For $0<p<1$, applying [27, Chap. XI, Sec. 1.5, Theorem 4] and using (2.20), we obtain

$$J \approx \sup_{t\ge 0} \frac{t^{1/p'}Pw(t)}{[PQ_pv(t)]^{1/p}}, \qquad 0<p<1,$$

and, similarly, for $p=1$,

$$J = \sup_{t\ge 0}\frac{Pw(t)}{PQ_1v(t)}, \qquad p=1.$$

2.3. The case $p=\infty$. Here the problem is to characterize the inequality

$$\int_0^\infty fw \le C \operatorname*{ess\,sup}_{t\ge 0}\biggl(\frac{1}{t}\int_0^t f\biggr)v(t) =: \|vPf\|_\infty, \qquad f\in\mathfrak{M}^\downarrow, \tag{2.21}$$

for a fixed function $w\in\mathfrak{M}^+$. First, note [11] that as $f\in\mathfrak{M}^\downarrow$,

$$\|vPf\|_\infty = \|\mathcal{V}Pf\|_\infty, \qquad \text{where} \quad \mathcal{V}(t) := t\sup_{s\ge t}\frac{\overline{v}(s)}{s}, \quad \overline{v}(s) := \operatorname*{ess\,sup}_{\tau\in[0,s]} v(\tau). \tag{2.22}$$

Just as for (2.16) and (2.17), we see that (2.21) is equivalent to the two-sided estimate of the functional

$$\mathcal{J} := \sup_{\rho\in\mathfrak{M}^+}\frac{\int_0^\infty \rho\overline{W}}{\operatorname{ess\,sup}_{t\ge 0}\mathcal{V}(t)((1/t)\int_0^t ds\int_s^\infty \rho)}.$$

Note that

$$\frac{1}{t}\int_0^t ds\int_s^\infty \rho \approx \int_0^\infty \frac{s\rho(s)\,ds}{t+s} =: S\rho(t);$$

therefore, in view of (2.19), we have

$$\mathcal{J} \approx \sup_{\rho\in\mathfrak{M}^+}\frac{\int_{[0,\infty)} S\rho\,d\eta}{\|\mathcal{V}S\rho\|_\infty}. \tag{2.23}$$

Obviously,

$$\mathcal{J} \ll \int_0^\infty \frac{d\eta}{\mathcal{V}}.$$

Since

$$\frac{1}{\mathcal{V}} \in \mathfrak{M}^{\downarrow}, \qquad \frac{t}{\mathcal{V}(t)} \in \mathfrak{M}^{\uparrow},$$

it follows that there exists a Borel measure $d\lambda$ for which

$$\frac{1}{\mathcal{V}(t)} \approx \int_{[0,\infty)} \frac{s\,d\lambda(s)}{t+s}, \tag{2.24}$$

and a sequence $\{\lambda_n\} \subset \mathfrak{M}^+$ such that

$$\int_0^\infty \frac{s\lambda_n(s)\,ds}{t+s} \uparrow \int_{[0,\infty)} \frac{s\,d\lambda(s)}{t+s} \approx \frac{1}{\mathcal{V}(t)}.$$

If, in (2.23), we take $\rho = \lambda_n$, then

$$\mathcal{J} \gg \int_{[0,\infty)} \frac{d\eta}{\mathcal{V}};$$

therefore,

$$\mathcal{J} \approx \int_{[0,\infty)} \frac{d\eta}{\mathcal{V}} \approx \int_{[0,\infty)} \overline{W}\,d\lambda.$$

Thus, we have proved the following statement.

Theorem 1. *Let $0 < p \le \infty$, let $w \in \mathfrak{M}^+$, and let $v \in \mathfrak{M}^+$ be a weight function satisfying condition* (2.2). *If*

$$J := \sup_{f \in \mathfrak{M}^{\downarrow}} \frac{\int_0^\infty fw}{(\int_0^\infty [Pf]^p v)^{1/p}},$$

then

$$J \approx \left(\int_0^\infty \left(\sup_{t \ge x} \frac{1}{t}\int_0^t w\right)^{p'} \frac{(Pv(x)Q_pv(x)}{(PQ_pv(x))^{p'+1}}\,dx\right)^{1/p'}, \qquad 1 < p < \infty, \tag{2.25}$$

$$J \approx \sup_{t \ge 0} \frac{t^{1/p'}Pw(t)}{[PQ_pv(t)]^{1/p}}, \qquad 0 < p < 1, \tag{2.26}$$

$$J = \sup_{t \ge 0} \frac{Pw(t)}{PQ_1v(t)}, \qquad p = 1, \tag{2.27}$$

and

$$\sup_{f \in \mathfrak{M}^{\downarrow}} \frac{\int_0^\infty fw}{\operatorname{ess\,sup}_{t \ge 0} v(t)Pf(t)} \approx \int_{[0,\infty)} \overline{W}\,d\lambda, \tag{2.28}$$

where the measure $d\lambda$ is related to the weight function v by the formulas (2.22) *and* (2.24) *and $\overline{W}$ is given by w in* (2.18).

3. OPTIMAL BANACH SPACES CONTAINING A CONE OF MONOTONE FUNCTIONS

For a given cone $K \subset \mathfrak{M}^+$ with positive homogeneous functional σ, by an *associated* Banach space K' we mean the space with the norm

$$\|g\|_{K'} := \sup_{0 \ne h \in K} \frac{\int_0^\infty |g|h}{\sigma(h)}.$$

Under sufficiently general assumptions, Gol'dman and Zabreiko [1, Theorem 2.2] proved that the Banach space X_0 associated with K' is optimal, i.e.,

$$\|f\|_{X_0} = \sup_{0 \ne g \in K'} \frac{\int_0^\infty |fg|}{\|g\|_{K'}}.$$

Theorem 2. *Let $0 < p \le \infty$, let $v \in \mathfrak{M}^+$ be a weight function satisfying condition* (2.2)*, and let $K \subset \mathfrak{M}^+$ be a cone coinciding with $\mathcal{L}_{p,v}$ or $\mathcal{L}_{\infty,v}$. For $f \in \mathfrak{M}^+$, denote*

$$f^{\downarrow}(t) := \operatorname*{ess\,sup}_{s \ge t} |f(s)|.$$

Then, the norms of the optimal Banach spaces X_0 for the cones K satisfy the following relations:

$$1) \quad \|f\|_{X_0} \approx \left(\int_0^\infty [Pf^{\downarrow}]^p \frac{P\psi Q_{p'}\psi}{(PQ_{p'}\psi)^{p+1}}\right)^{1/p}, \qquad 1 < p < \infty, \tag{3.1}$$

where ψ is given by the formula $\psi := PvQ_pv/(PQ_pv)^{p'+1}$;

$$2) \quad \|f\|_{X_0} = \int_0^\infty f^{\downarrow} Q_1 v, \qquad p = 1; \tag{3.2}$$

$$3) \quad \|f\|_{X_0} \approx \int_0^\infty f^{\downarrow}(t) Q_p v(t) \left(\int_0^t Q_p v\right)^{1/p-1} dt, \qquad 0 < p < 1; \tag{3.3}$$

$$4) \quad \|f\|_{X_0} \approx \sup_{t \ge 0} \frac{f^{\downarrow}(t)}{\int_{[t,\infty)} d\lambda}, \qquad p = \infty, \tag{3.4}$$

where the measure $d\lambda$ is related to the weight function v by formulas (2.22) *and* (2.24).

Proof. Let $1 < p < \infty$. Then, by Theorem 1 (see (2.25)),

$$\|g\|_{K'} := \sup_{0 \ne h \in \mathcal{L}_{p,v}} \frac{\int_0^\infty |g|h}{\|h\|_{\mathcal{L}_{p,v}}} \approx \left(\int_0^\infty \left(\sup_{t \ge x} \frac{1}{t}\int_0^t |g|\right)^{p'} \frac{(Pv(x))Q_pv(x)}{(PQ_pv(x))^{p'+1}}\, dx\right)^{1/p'}.$$

Thus,

$$\|f\|_{X_0} \approx \sup_{g \in \mathfrak{M}^+} \frac{\int_0^\infty |f|g}{\left(\int_0^\infty \left(\sup_{t \ge x} \frac{1}{t}\int_0^t g\right)^{p'} \left(\frac{(Pv(x))Q_pv(x)}{(PQ_pv(x))^{p'+1}}\right) dx\right)^{1/p'}}.$$

Further, using an argument similar to that in the proof of Theorem 3.3 from [30], we obtain

$$\|f\|_{X_0} \approx \sup_{g \in \mathfrak{M}^+} \frac{\int_0^\infty f^{\downarrow} g}{(\int_0^\infty (\sup_{t \ge x} Pg(t))^{p'} \psi(x)\, dx)^{1/p'}} =: \sup_{g \in \mathfrak{M}^+} \frac{\int_0^\infty f^{\downarrow} g}{(\int_0^\infty [G(t)]^{p'} \psi(x)\, dx)^{1/p'}} =: F.$$

Applying (2.25), we can write

$$\begin{aligned} F &= \sup_{g \in \mathfrak{M}^+} \frac{\int_{[0,\infty)} Pg(t) t\, d(-f^{\downarrow}(t))}{(\int_0^\infty [G(x)]^{p'} \psi(x)\, dx)^{1/p'}} \ge \sup_{g \in \mathfrak{M}^{\downarrow}} \frac{\int_{[0,\infty)} Pg(t) t\, d(-f^{\downarrow}(t))}{(\int_0^\infty [Pg(x)]^{p'} \psi(x)\, dx)^{1/p'}} \\ &= \sup_{g \in \mathfrak{M}^{\downarrow}} \frac{\int_0^\infty f^{\downarrow} g}{(\int_0^\infty [Pg(x)]^{p'} \psi(x)\, dx)^{1/p'}} \approx \left(\int_0^\infty [Pf^{\downarrow}]^p \frac{P\psi Q_{p'}\psi}{(PQ_{p'}\psi)^{p+1}}\right)^{1/p} =: \mathcal{F} \end{aligned}$$

and, therefore, the lower bound in (3.1) is proved.

Further, note that $G(t) \in \mathfrak{M}^{\downarrow}$, $tG(t) \in \mathfrak{M}^{\uparrow}$; therefore, by Sinnamon's lemma [20, Lemma 2.3], there exists a $\overline{G}(t) \approx G(t)$ and a sequence of functions $\{g_n\} \subset \mathfrak{M}^{\downarrow}$ such that

$$Pg_n(t) \uparrow \overline{G}(t).$$

Again applying (2.25) and the monotone convergence theorem, we obtain

$$\int_{[0,\infty)} Pg(t) t\, d(-f^{\downarrow}(t)) \le \int_{[0,\infty)} G(t) t\, d(-f^{\downarrow}(t)) \approx \int_{[0,\infty)} \overline{G}(t) t\, d(-f^{\downarrow}(t))$$

$$= \lim_{n\to\infty}\int_{[0,\infty)} Pg_n(t)t\,d(-f^{\downarrow}(t)) = \lim_{n\to\infty}\int_0^\infty f^{\downarrow}g_n \ll \mathcal{F}\lim_{n\to\infty}\biggl(\int_0^\infty [Pg_n]^{p'}\psi\biggr)^{1/p'}$$
$$= \mathcal{F}\biggl(\int_0^\infty [\overline{G}]^{p'}\psi\biggr)^{1/p'} \approx \mathcal{F}\biggl(\int_0^\infty [G]^{p'}\psi\biggr)^{1/p'},$$

and the upper bound in (3.1) is established.

Further, let $p=1$, and let $K=\mathcal{L}_{1,v}$. Applying (2.27), we obtain

$$\|g\|_{K'} := \sup_{0\neq h\in\mathcal{L}_{1,v}} \frac{\int_0^\infty |g|h}{\|h\|_{\mathcal{L}_{1,v}}} = \sup_{t>0}\frac{P|g|(t)}{PQ_1v(t)} =: \biggl\|\frac{P|g|(t)}{PQ_1v(t)}\biggr\|_\infty.$$

Thus,

$$\|f\|_{X_0} = \sup_{g\in\mathfrak{M}^+}\frac{\int_0^\infty |f|g}{\|(Pg)/(PQ_1v)\|_\infty} = \sup_{g\in\mathfrak{M}^+}\frac{\int_0^\infty f^{\downarrow}g}{\|(Pg)/(PQ_1v)\|_\infty} \ge \int_0^\infty f^{\downarrow}Q_1v.$$

Since

$$\int_0^\infty f^{\downarrow}g = \int_{[0,\infty)} Pg(t)t\,d(-f^{\downarrow}(t))$$
$$\le \biggl\|\frac{Pg}{PQ_1v}\biggr\|_\infty \int_{[0,\infty)} PQ_1v(t)t\,d(-f^{\downarrow}(t)) = \biggl\|\frac{Pg}{PQ_1v}\biggr\|_\infty \int_0^\infty f^{\downarrow}Q_1v,$$

it follows that

$$\|f\|_{X_0} \le \int_0^\infty f^{\downarrow}Q_1v$$

and, therefore, (3.2) is proved.

In the same way, we consider the case $0<p<1$. By Theorem 1 (see (2.26)), we have

$$\|g\|_{K'} := \sup_{0\neq h\in\mathcal{L}_{p,v}} \frac{\int_0^\infty |g|h}{\|h\|_{\mathcal{L}_{p,v}}} \approx \sup_{t>0}\frac{t^{1/p'}P|g|(t)}{[PQ_pv(t)]^{1/p}}.$$

Hence

$$\|f\|_{X_0} \approx \sup_{g\in\mathfrak{M}^+}\frac{\int_0^\infty |f|g}{\sup_{t>0}(t^{1/p'}Pg(t)/[PQ_pv(t)]^{1/p})} = \sup_{g\in\mathfrak{M}^+}\frac{\int_0^\infty f^{\downarrow}g}{\sup_{t>0}(t^{1/p'}Pg(t)/[PQ_pv(t)]^{1/p})}$$
$$= \sup_{g\in\mathfrak{M}^+}\frac{\int_{[0,\infty)}(\int_0^t g)\,d(-f^{\downarrow}(t))}{\sup_{t>0}(\int_0^t g/[\int_0^t Q_pv]^{1/p})} = \frac{1}{p}\int_0^\infty f^{\downarrow}(t)Q_pv(t)\biggl(\int_0^t Q_pv\biggr)^{1/p-1}dt$$

and, therefore, (3.3) is proved.

In the case $p=\infty$, $K=\mathcal{L}_{\infty,v}$, using (2.28), we obtain

$$\|g\|_{K'} := \sup_{0\neq h\in\mathcal{L}_{\infty,v}} \frac{\int_0^\infty |g|h}{\|h\|_{\mathcal{L}_{\infty,v}}} \approx \int_{[0,\infty)} G\,d\lambda,$$
$$\|f\|_{X_0} \approx \sup_{g\in\mathfrak{M}^+}\frac{\int_0^\infty |f|g}{\int_{[0,\infty)} G\,d\lambda} = \sup_{g\in\mathfrak{M}^+}\frac{\int_0^\infty f^{\downarrow}g}{\int_{[0,\infty)} G\,d\lambda} = \sup_{g\in\mathfrak{M}^+}\frac{\int_{[0,\infty)} Pg(t)t\,d(-f^{\downarrow}(t))}{\int_{[0,\infty)}(\sup_{s\ge t}Pg(s))t\,d\lambda(t)}$$
$$= \sup_{g\in\mathfrak{M}^+}\frac{\int_0^\infty f^{\downarrow}g}{\int_{[0,\infty)} Pg(t)t\,d\lambda(t)} = \sup_{g\in\mathfrak{M}^+}\frac{\int_{[0,\infty)}(\int_0^t g)\,d(-f^{\downarrow}(t))}{\int_{[0,\infty)}(\int_0^t g)\,d\lambda(t)} = \sup_{t\ge 0}\frac{f^{\downarrow}(t)}{\int_{[t,\infty)} d\lambda}$$

and, therefore, (3.4) is proved. □

4. OPTIMAL BANACH SPACES CONTAINING A CONE OF QUASICONCAVE FUNCTIONS

Here we study a similar problem for the cone generated by the set of quasiconcave functions

$$\Omega_{0,1} := \left\{ f \in \mathfrak{M}^+ : f \in \mathfrak{M}^{\uparrow},\ \frac{f(t)}{t} \in \mathfrak{M}^{\downarrow} \right\}.$$

Let $0 < p \le \infty$, and let $v \in \mathfrak{M}^+$ be a given weight function satisfying condition (2.2). We consider the weight cone of quasiconcave functions $K = \mathfrak{L}_{p,v}$, whose form depends on the parameter p:

$$\mathfrak{L}_{p,v} := \left\{ f \in \Omega_{0,1} : \|f\|_{\mathfrak{L}_{p,v}} := \left(\int_0^\infty [f]^p v \right)^{1/p} < \infty \right\}, \qquad 0 < p < \infty,$$

$$\mathfrak{L}_{\infty,v} := \left\{ f \in \Omega_{0,1} : \|f\|_{\mathfrak{L}_{\infty,v}} := \operatorname*{ess\,sup}_{t \ge 0} f(t) v(t) < \infty \right\},$$

and obtain the optimal Banach space X_0 containing this cone.

First, we need an analog of Theorem 1. Denote

$$\widetilde{w}(t) := \int_t^\infty w, \qquad v_p(t) := t^p v(t), \qquad \overline{v}_1(s) := \operatorname*{ess\,sup}_{\tau \in [0,s]} v_1(\tau), \qquad \mathcal{V}_1(t) := t \sup_{s \ge t} \frac{\overline{v}_1(s)}{s}.$$

Theorem 3. *Let $0 < p \le \infty$, let $w \in \mathfrak{M}^+$, and let $v \in \mathfrak{M}^+$ be a weight function satisfying condition* (2.2). *If*

$$J := \sup_{f \in \Omega_{0,1}} \frac{\int_0^\infty fw}{\|f\|_{\mathfrak{L}_{p,v}}},$$

then

$$J \approx \left(\int_0^\infty (P\widetilde{w})^{p'} \frac{(Pv_p Q_p v_p}{(PQ_p v_p)^{p'+1}} \right)^{1/p'}, \qquad 1 < p < \infty, \tag{4.1}$$

$$J \approx \sup_{t \ge 0} \frac{t^{1/p'} P\widetilde{w}(t)}{[PQ_p v_p(t)]^{1/p}}, \qquad 0 < p \le 1, \tag{4.2}$$

and

$$\sup_{f \in \Omega_{0,1}} \frac{\int_0^\infty fw}{\|f\|_{\mathfrak{L}_{\infty,v}}} \approx \int_0^\infty \frac{sw(s)\,ds}{\mathcal{V}_1(s)}. \tag{4.3}$$

Proof. Let $0 < p < \infty$. Applying Sinnamon's lemma (see [20, Lemma 2.3]), it is easy to see that

$$J \approx \sup_{\nu \in \mathfrak{M}^{\downarrow}} \frac{\int_0^\infty (\int_0^t \nu) w(t)\,dt}{(\int_0^\infty (P\nu)^p v_p)^{1/p}} = \sup_{\nu \in \mathfrak{M}^{\downarrow}} \frac{\int_0^\infty \nu \widetilde{w}}{(\int_0^\infty (P\nu)^p v_p)^{1/p}}.$$

In view of the inclusion $\widetilde{w} \in \mathfrak{M}^{\downarrow}$, applying Theorem 1, we obtain (4.1) and (4.2). Let the measure $d\lambda_1$ be related to $\mathcal{V}_1$ in the same way as $d\lambda$ is related to $\mathcal{V}$ in Theorem 1. Then, applying (2.28), we obtain

$$\sup_{f \in \Omega_{0,1}} \frac{\int_0^\infty fw}{\|f\|_{\mathfrak{L}_{\infty,v}}} \approx \sup_{\nu \in \mathfrak{M}^{\downarrow}} \frac{\int_0^\infty \nu \widetilde{w}}{\operatorname{ess\,sup}_{t \ge 0} P\nu(t) v_1(t)} \approx \int_{[0,\infty)} \left(\int_0^t \widetilde{w} \right) d\lambda_1(t)$$
$$= \int_0^\infty \left(\int_t^\infty w \right) \int_{[t,\infty)} d\lambda_1 \approx \int_0^\infty \frac{sw(s)\,ds}{\mathcal{V}_1(s)}. \qquad \square$$

Let

$$\Phi_p(t) := \frac{t}{(\int_0^t Q_p v)^{1/p}}, \qquad \mathcal{F}_p^{-1}(t) := t \sup_{s\ge t} \frac{1}{s} \sup_{z\in[0,s]} \Phi_p(z).$$

Then, just as in the case of the measure $d\lambda$, there exists a Borel measure $d\tau$ for which

$$\mathcal{F}_p(t) \approx \int_{[0,\infty)} \frac{s\,d\tau(s)}{s+t}. \tag{4.4}$$

(This relation is similar to (2.24).)

Theorem 4. *Let $0 < p \le \infty$, let $v \in \mathfrak{M}^+$ be a weight function satisfying condition* (2.2)*, and let $K \subset \mathfrak{M}^+$ be a cone coinciding with $\mathfrak{L}_{p,v}$ or $\mathfrak{L}_{\infty,v}$. For $f \in \mathfrak{M}^+$, denote*

$$f^{\uparrow}(t) := \operatorname*{ess\,sup}_{s\le t} |f(s)|.$$

Then, the norms of the optimal Banach spaces X_0 for the cones K satisfy the relations

$$1) \quad \|f\|_{X_0} \approx \left(\int_0^\infty \left(\frac{f^{\uparrow}(t)}{t}\right)^p \frac{P\psi_p Q_{p'}\psi_p}{(PQ_{p'}\psi_p)^{p+1}}\right)^{1/p}, \qquad 1 < p < \infty,$$

where ψ_p is given by the formula $\psi_p := Pv_pQ_pv_p/(PQ_pv_p)^{p'+1}$;

$$2) \quad \|f\|_{X_0} \approx \int_{[0,\infty)} t \sup_{s\ge t} \frac{f^{\uparrow}(s)}{s}\,d\tau(t), \qquad 0 < p \le 1,$$

where the measure $d\tau$ is given by relation (4.4)*;*

$$3) \quad \|f\|_{X_0} \approx \sup_{t\ge 0} \frac{f^{\uparrow}(t)}{\int_{[t,\infty)} d\lambda}, \qquad p = \infty,$$

where the measure $d\lambda$ is related to the weight function v by the formulas (2.22) *and* (2.24).

Proof. Let $1 < p < \infty$. Then, by Theorem 3 (see (4.1)),

$$\|g\|_{K'} := \sup_{0\ne h\in\mathfrak{L}_{p,v}} \frac{\int_0^\infty |g|h}{\|h\|_{\mathfrak{L}_{p,v}}} \approx \left(\int_0^\infty \left(\frac{1}{t}\int_0^t ds \int_s^\infty |g|\right)^{p'} \psi_p(t)\,dt\right)^{1/p'}.$$

Thus,

$$\|f\|_{X_0} \approx \sup_{g\in\mathfrak{M}^+} \frac{\int_0^\infty |f|g}{(\int_0^\infty((1/t)\int_0^t ds\int_s^\infty g)^{p'}\psi_p(t)\,dt)^{1/p'}}.$$

Applying [30, Corollary 3.4] and, further, Theorem 1, we obtain

$$\begin{aligned}
\|f\|_{X_0} &\approx \sup_{g\in\mathfrak{M}^+} \frac{\int_0^\infty f^{\uparrow}g}{(\int_0^\infty((1/t)\int_0^t ds\int_s^\infty g)^{p'}\psi_p(t)\,dt)^{1/p'}} \\
&= \sup_{g\in\mathfrak{M}^+} \frac{\int_{[0,\infty)}(\int_s^\infty g)\,df^{\uparrow}(s)}{(\int_0^\infty((1/t)\int_0^t ds\int_s^\infty g)^{p'}\psi_p(t)\,dt)^{1/p'}} = \sup_{\nu\in\mathfrak{M}^{\downarrow}} \frac{\int_{[0,\infty)}\nu\,df^{\uparrow}(s)}{(\int_0^\infty(P\nu)^{p'}\psi_p)^{1/p'}} \\
&\approx \left(\int_0^\infty \left(\frac{f^{\uparrow}(t)}{t}\right)^p \frac{P\psi_pQ_{p'}\psi_p}{(PQ_{p'}\psi_p)^{p+1}}\right)^{1/p}.
\end{aligned}$$

Let $0 < p \le 1$. Then, by Theorem 3 (see (4.2)),

$$\|g\|_{K'} \approx \sup_{t\ge 0} \frac{t^{1/p'}P\widetilde{|g|}(t)}{[PQ_pv_p(t)]^{1/p}} =: \sup_{t\ge 0} \Phi_p(t)P\widetilde{|g|}(t).$$

Therefore,

$$\|f\|_{X_0} \approx \sup_{g\in\mathfrak{M}^+} \frac{\int_0^\infty |f|g}{\sup_{t\ge 0}\Phi_p(t)P\widetilde{|g|}(t)}.$$

Again, applying [30, Corollary 3.4] and, further, Theorem 1 (see formula (2.28)), we obtain

$$\|f\|_{X_0} \approx \sup_{g\in\mathfrak{M}^+} \frac{\int_0^\infty f^\uparrow g}{\sup_{t\ge 0}\Phi_p(t)P\widetilde{g}(t)} = \sup_{g\in\mathfrak{M}^+} \frac{\int_{[0,\infty)} \widetilde{g}\,df^\uparrow}{\sup_{t\ge 0}\Phi_p(t)P\widetilde{g}(t)} = \sup_{\nu\in\mathfrak{M}^\downarrow} \frac{\int_{[0,\infty)} \nu\,df^\uparrow}{\sup_{t\ge 0}\Phi_p(t)P\nu(t)}$$
$$\approx \int_{[0,\infty)} t\sup_{s\ge t}\frac{f^\uparrow(s)}{s}\,d\tau(t).$$

For $p=\infty$, applying (4.3), we can write

$$\|g\|_{K'} \approx \int_0^\infty \frac{s|g(s)|(s)\,ds}{\mathcal{V}_1(s)};$$

therefore,

$$\|f\|_{X_0} \approx \sup_{g\in\mathfrak{M}^+} \frac{\int_0^\infty |f|g}{\int_0^\infty (s|g(s)|/\mathcal{V}_1(s))\,ds} = \sup_{t\ge 0}\frac{|f(t)|\mathcal{V}_1(t)}{t}.$$

$\square$

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