ISSN 0001-4346, Mathematical Notes, 2015, Vol. 98, No. 3, pp. 503–514. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © E. V. Yushkov, M. O. Korpusov, 2015, published in Matematicheskie Zametki, 2015, Vol. 98, No. 3, pp. 448–462.

Global Unsolvability of One-Dimensional Problems for Burgers-Type Equations

E. V. Yushkov* **and M. O. Korpusov**

Lomonosov Moscow State University, Moscow, Russia

Received February 24, 2014

Abstract—In this paper, we study the global solvability of well-known equations used to describe nonlinear processes with dissipation, namely, the Burgers equation, the Korteweg–de Vries–Burgers equation, and the modified Korteweg–de Vries–Burgers equation. Using a method due to Pokhozhaev, we obtain necessary conditions for the blow-up of global solutions and estimates of the blow-up time and blow-up rate in bounded and unbounded domains. We also study the effect of linear and nonlinear viscosity on the occurrence of a gradient catastrophe in finite time.

DOI: 10.1134/S0001434615090163

Keywords: *Burgers equation, global unsolvability of Burgers-type equations, Korteweg–de Vries–Burgers equation, nonlinear process with dissipation, blow-up time, gradient catastrophe, maximum principle, method of test functions.*

This paper is dedicated to the memory
of the remarkable man and outstanding scientist,
Stanislav Ivanovich Pokhozhaev.

1. INTRODUCTION

In 1915, Bateman obtained the following partial differential equation describing the effects of both nonlinear propagation of waves and diffusion:

$$u_t + uu_x = \nu u_{xx}. \tag{1}$$

In the literature, Eq. (1) is now known as the *Burgers equation* and is considered as one of the standard equations of the theory of nonlinear waves. The rigorous derivation of Eq. (1) from the system of equations of gas dynamics involving viscosity and heat conduction, the behavior of self-similar solutions, the asymptotics as $\nu \to 0$, the questions of uniqueness and solvability were studied in great detail in the classical textbooks [1]–[3]. The most important fact is that, in 1950, Cole and Hopf managed to reduce, by the nonlinear change

$$u(x,t) = -2\nu(\ln \phi(x,t))_x,$$

the Burgers equation to the linear heat equation and obtain an exact smooth solution of the corresponding Cauchy problem.

The present paper deals with the singular solutions of Burgers-type equations, not existing globally in time. Using the method of test functions proposed and developed by Pokhozhaev and Mitidieri [5], [4], we shall show that the linear dissipative summand does not always ensure the boundedness of the solution $u(x,t)$ and its derivative $u_x(x,t)$. In other words, nonlinearity results in that, for certain initial and boundary conditions, there exists a finite time T such that the functional

$$J(t) = \int u(x,t)\varphi(x)\,dx \to \infty \qquad \text{as} \quad t \to T$$

*E-mail: yushkov.msu@mail.ru

for some smooth test function $\varphi(x)$.

To exclude a possible misunderstanding, we must note that, for the Burgers equation, the maximum principle holds. Therefore, the solution of the initial boundary-value problem for Eq. (1) with boundary conditions of the first kind at both endpoints of a closed interval is bounded for all instants of time. However, the maximum principle does not preclude the occurrence of a gradient catastrophe or the occurrence of a blow-up under nonclassical boundary conditions as well as the global unsolvability of initial boundary-value problems for the Korteweg–de Vries–Burgers equation

$$u_t + uu_x + \beta u_{xxx} = \nu u_{xx}, \tag{2}$$

describing collisionless waves in a plasma and for the modified equation Korteweg–de Vries–Burgers equation

$$u_t - u^2 u_x + \beta u_{xxx} = \nu u_{xx}, \tag{3}$$

which is the main equation for media with cubic nonlinearity describing magnetic sound and Alfven waves [6].

The present paper is a continuation of the series of papers devoted to the global unsolvability of one-dimensional problems of hydrodynamic type, which was initiated by Pokhozhaev's result for the Korteweg–de Vries equation [7], [8]:

$$u_t + uu_x + \beta u_{xxx} = 0. \tag{4}$$

The method of test functions has turned out to be convenient for the study of both the equations indicated above and systems of shallow-water equations [9]. The principal advantage of this method is the possibility of proving the blow-up result for initial boundary-value problems with classical and nonclassical boundary conditions that can be given at one endpoint or both endpoints. The greatest difficulty associated with this method is to choose the best test function and to prove the local (in time) solvability. Unfortunately, at present, there are no complete answers to these problems, but, despite this setback, the use of the method of test functions is a convenient practical approach to estimating the solvability time for problems studied in present-day mathematical physics.

2. BLOW-UP OF THE SOLUTION OF THE BURGERS EQUATION

Let us analyze the general scheme of the method of test functions, using the initial boundary-value problem for the Burgers equation

$$u_t + uu_x = \nu u_{xx}, \qquad x \in [0, L], \tag{5}$$

$$u|_{t=0} = u_0(x), \qquad \nu > 0, \tag{6}$$

as an example. The boundary conditions for problem (5) (6) will be given in what follows. Suppose that we can find the time $T > 0$ for which the classical solution of the problem under consideration exists and belongs to the class

$$u(x,t) \in C^{(1)}([0,T]; C^{(2)}([0,L])).$$

Using the method of test functions, we shall obtain sufficient conditions for the blow-up of the solution; to do this, we multiply (5) by the function $\varphi(x) \in C^{(2)}([0,L])$ and integrate by parts:

$$\frac{d}{dt}\int_0^L u(x,t)\varphi(x)\,dx = \frac{1}{2}\int_0^L u^2\varphi_x\,dx + \nu\int_0^L u\varphi_{xx}\,dx + G(u,\varphi)|_{x=L} - G(u,\varphi)|_{x=0}, \tag{7}$$

where

$$G(u,\varphi) = -\frac{u^2}{2}\varphi + \nu u_x\varphi - \nu u\varphi_x.$$

Let the function $\varphi(x)$ be monotone nondecreasing: $\varphi_x \ge 0$. Then the following equality holds:

$$\int_0^L (u^2\varphi_x + 2\nu u\varphi_{xx})\,dx = \int_0^L w^2\varphi_x\,dx - \nu^2\int_0^L \frac{\varphi_{xx}^2}{\varphi_x}\,dx, \qquad \text{where} \quad w = u + \nu\frac{\varphi_{xx}}{\varphi_x}. \tag{8}$$

In view of the Cauchy–Bunyakovskii inequality, we can find the following lower bound for the second summand on the right-hand side:

$$\left|\int_0^L w(x,t)\varphi(x)\,dx\right|^2 \le \int_0^L w^2(x,t)\varphi_x(x)\,dx \int_0^L \frac{\varphi^2(x)}{\varphi_x(x)}\,dx. \tag{9}$$

Naturally, the function $\varphi(x)$ is chosen so that all the resulting integrals are convergent. Now expression (7) can be rewritten as

$$\frac{dJ}{dt} \ge \frac{1}{2}\left|\int_0^L \frac{\varphi^2(x)}{\varphi_x(x)}\,dx\right|^{-1} J^2 + G(u,\varphi)|_0^L - \frac{\nu^2}{2}\int_0^L \frac{\varphi_{xx}^2(x)}{\varphi_x(x)}\,dx, \tag{10}$$

where

$$J = \int_0^L w(x,t)\varphi(x)\,dx.$$

Suppose that there exists a test function $\varphi(x)$ for which $G(u,\varphi)|_0^L$ is independent of time. If there is no such function, then $G(u,\varphi)|_0^L$ must be considered separately, for example, assuming that the constant independent of t is bounded above. For convenience, denote

$$k^2 = \frac{1}{2}\left|\int_0^L \frac{\varphi^2(x)}{\varphi_x(x)}\,dx\right|^{-1} < \infty, \qquad a^2 = \frac{\nu^2}{2}\int_0^L \frac{\varphi_{xx}^2(x)}{\varphi_x(x)}\,dx - G(u,\varphi)\big|_0^L < \infty,$$

and rewrite inequality (10) as

$$\frac{dJ}{dt} \ge k^2 J^2 - a^2. \tag{11}$$

Applying the theory of ordinary differential inequalities, we obtain the following statement.

Theorem 1. *Let the initial function $u_0(x) \in L^{(1)}([0,L])$ and the boundary conditions of problem* (5), (6) *be such that there exists a nondecreasing test function $\varphi(x) \in C^{(2)}([0,L])$ for which the following inequality holds*:

$$J_0 = J(0) = \int_0^L \left(u_0(x) + \nu\frac{\varphi_{xx}(x)}{\varphi_x(x)}\right)\varphi(x)\,dx > \frac{a}{k}. \tag{12}$$

Then there does not exist a global (*in time*) *classical solution and the following estimate holds*:

$$J(t) \ge \frac{a}{k}\frac{1 + c_0\exp(2akt)}{1 - c_0\exp(2akt)}, \qquad c_0 = \frac{kJ_0 - a}{kJ_0 + a},$$

and hence

$$\lim_{t\to T} J(t) = +\infty, \qquad T \le -\frac{1}{2ak}\ln\left(\frac{kJ_0 - a}{kJ_0 + a}\right).$$

Example 1. Note that, in practice, the method of test functions is very convenient, especially when the conditions at one of the endpoints are redefined. For example, suppose that, in problem (5), (6) on the closed interval $[0,1]$, the following conditions are imposed on the function $u(x,t)$ and its derivative on the right boundary:

$$u|_{x=1} = u_x|_{x=1} = 0.$$

Then choosing the test function as $\varphi(x) = x^3$, we obtain

$$k^2 = \frac{15}{2}, \qquad a^2 = 6\nu^2,$$

and Theorem 1 immediately implies the global (in time) unsolvability of the equation under the following additional conditions on the initial function:

$$\int_0^1 x^3 u_0(x)\,dx > \nu\frac{2(3-\sqrt{5}\,)}{3\sqrt{5}}. \tag{13}$$

Thus, if the boundary conditions are not conditions of the first kind at both endpoints of the closed interval, then the maximum principle does not prevent the validity of the blow-up result. However, we can encounter the situation in which a problem with boundary conditions can be ill-defined only at one boundary. Therefore, to preserve mathematical rigor in each particular case, it is necessary to solve the problem of local solvability, which can often be done by using the contraction mapping method [10].

Example 2 (Gradient catastrophe)**.** Now suppose that the boundary conditions are given so that the maximum principle ensures the global (in time) boundedness of the solution. Let us study the "soft blow-up" phenomenon for the initial boundary-value problem (5), (6) in a domain Ω under the classical boundary conditions. Let there exist a smooth bounded solution $u(x,t) \le \max_x\{u_0(x)\}$. Differentiating Eq. (5) with respect to x, we obtain

$$(u_x)_t + u_x^2 + uu_{xx} = \nu u_{xxx}. \tag{14}$$

Substituting the expression for u_{xx} from (5), we rewrite (14) as

$$(u_x)_t + u_x^2 + \frac{1}{\nu}uu_t + \frac{1}{\nu}u^2u_x = \nu u_{xxx}. \tag{15}$$

Let us multiply the equation by the test function $\varphi(x)$ and integrate by parts over the domain Ω, obtaining

$$-\frac{d}{dt}\int \varphi\biggl(u_x + \frac{u^2}{2\nu}\biggr)\,dx = \int \varphi\biggl(u_x^2 + \frac{1}{\nu}u^2u_x\biggr)\,dx + \nu\int \varphi_{xxx}u\,dx + Gr|_{\partial\Omega}, \tag{16}$$

where $Gr = \nu(-u_{xx}\varphi + u_x\varphi_x - u\varphi_{xx})$. Denote

$$w = -u_x - \frac{u^2}{2\nu}\,.$$

Using the maximum principle ($|u(x,t)| \le M$) and assuming, for simplicity, that the boundary conditions are independent of time, we obtain the inequality

$$\frac{d}{dt}\int \varphi w\,dx \ge \int \varphi w^2\,dx - M^4\int \frac{\varphi}{4\nu^2}\,dx - M\nu\int |\varphi_{xxx}|\,dx + Gr|_{\partial\Omega}. \tag{17}$$

Let the test function be nonnegative, and let us use the Cauchy–Bunyakovskii inequality

$$\biggl(\int \varphi w\,dx\biggr)^2 \le \int \varphi w^2\,dx \int \varphi\,dx. \tag{18}$$

By analogy with (9), we obtain the inequality

$$\frac{d}{dt}\int \varphi w\,dx \ge \biggl(\int \varphi\,dx\biggr)^{-1}\biggl[\int \varphi w\,dx\biggr]^2 - M^4\int \frac{\varphi}{4\nu^2}\,dx - M\nu\int |\varphi_{xxx}|\,dx + Gr|_{\partial\Omega}. \tag{19}$$

Let us make the change

$$J(t) = \int \varphi\biggl(-u_x - \frac{u^2}{2\nu}\biggr)\,dx,$$

$$k^2 = \biggl(\int \varphi\,dx\biggr)^{-1}, \qquad a^2 = M^4\int \frac{\varphi}{4\nu^2}\,dx + M\nu\int |\varphi_{xxx}|\,dx + Gr|_{\partial\Omega},$$

and rewrite inequality (19) as an ordinary differential inequality of the form (11) for which we can immediately write the blow-up result using the example of Theorem 1 in which condition (12) must be replaced by the following one:

$$J_0 = J(0) = \int \biggl(-u_x(0) - \frac{u^2(0)}{2\nu}\biggr)\varphi(x)\,dx > \frac{a}{k}\,. \tag{20}$$

Note that since it is not required that the function $\varphi(x)$ satisfy the monotonicity condition, but only that it be positive, it follows that result can be formulated for both the initial boundary-value problem and the Cauchy problem.

Further, there are two interesting aspects: first, it is obvious that by choosing a solitory initial profile with long leading front and decreasing the amplitude, we can always ensure that condition (20) will hold and wave breaking will occur. In other words, if the perturbation is given sufficient distance to travel, the wave will break even in the presence of dissipation. Second, decreasing or increasing ν for a fixed initial profile and a chosen test function, we can always ensure that condition (20) will not hold and, therefore, there will be no blow-up.

3. BLOW-UP OF THE SOLUTION OF THE KORTEWEG—DE VRIES—BURGERS EQUATION

It becomes clear from the examined problem that the important feature of the method of test functions is its universality in the application to equations differing by summands of the form $u_{x\dots xx}$. Since the main idea is to integrate by parts, it follows that the order of the derivative will define only the boundary conditions, not the form of the final inequality.

As an example, let us consider the initial boundary-value problem for the Korteweg—de Vries—Burgers equation describing processes in a medium with both dispersion and dissipation. Let us show that even the presence of two factors, restraining wave breaking will not lead to the global solvability (in time) of the problem for arbitrary initial data,

$$u_t + uu_x + \beta u_{xxx} = \nu u_{xx}, \qquad x \in [0, L], \tag{21}$$

$$u|_{t=0} = u_0(x), \qquad \nu > 0, \quad \beta > 0. \tag{22}$$

The processes described by such a model are observed, for example, in a plasma, where they are called collisionless, because dissipation is not determined by the collision of particles, but is caused by other factors. As examples of such processes, we can indicate, in particular, ion-sound and magnetic sound waves first studied by Sagdeev in the 1960s [6].

Suppose that there exists an $T > 0$ for which the classical solution of the problem under consideration belongs to the class

$$u(x,t) \in C^{(1)}([0,T]; C^{(3)}([0,L])).$$

Multiplying Eq. (21) by the test function $\varphi(x) \in C^{(3)}([0,L])$ and integrating by parts, we obtain

$$\frac{d}{dt}\int_0^L u(x,t)\varphi(x)\,dx = \frac{1}{2}\int_0^L u^2\varphi_x\,dx + \int_0^L u(\nu\varphi_{xx} + \beta\varphi_{xxx})\,dx + G(u,\varphi)|_{x=L} - G(u,\varphi)|_{x=0}, \tag{23}$$

where

$$G(u,\varphi) = -\biggl(\frac{u^2}{2} + \beta u_{xx}\biggr)\varphi + (\nu\varphi + \beta\varphi_x)u_x - (\nu\varphi_x + \beta\varphi_{xx})u.$$

Let us choose a nondecreasing test function $\varphi_x(x) \geq 0$; then, collecting a perfect square, we obtain the equation

$$\frac{dJ}{dt} = \frac{1}{2}\int_0^L w^2\varphi_x\,dx - \frac{1}{2}\int_0^L \frac{(\nu\varphi_{xx} + \beta\varphi_{xxx})^2}{\varphi_x}\,dx + G(u,\varphi)\big|_0^L, \tag{24}$$

where

$$J(t) = \int_0^L w(x,t)\varphi(x)\,dx, \qquad w = u + \frac{\nu\varphi_{xx} + \beta\varphi_{xxx}}{\varphi_x}.$$

Let us use the Cauchy—Bunyakovskii inequality to find estimate (9); then, from (24), we can obtain the inequality

$$\frac{dJ}{dt} \geq k^2 J(t) - a^2, \qquad k^2 = \frac{1}{2}\biggl|\int_0^L \frac{\varphi^2(x)}{\varphi_x(x)}\,dx\biggr|^{-1}, \qquad a^2 = \frac{1}{2}\int_0^L \frac{(\nu\varphi_{xx} + \beta\varphi_{xxx})^2}{\varphi_x}\,dx + G(u,\varphi)\big|_0^L; \tag{25}$$

here we have again assumed that the function $G(u,\varphi)|_0^L$ is independent of time or that this function is bounded above by a constant independent of time. Thus, we have proved the statement.

Theorem 2. *Let the initial function $u_0(x)$ belong to $L^{(1)}([0,L])$, and let the boundary conditions for problem* (21), (22) *be such that there exists a nondecreasing test function $\varphi(x)\in C^{(3)}([0,L])$ for which the following inequality holds*:

$$J_0 = J(0) = \int_0^L \left(\varphi u_0(x) + \frac{\varphi}{\varphi_x}(\nu\varphi_{xx} + \beta\varphi_{xxx})\right) dx > \frac{a}{k}; \tag{26}$$

then the result of Theorem 1 *is valid.*

Example 3. Consider problem (21), (22) on the closed interval $[0,1]$ with boundary conditions only at the right boundary:

$$u|_{x=1} = u_x|_{x=1} = u_{xx}|_{x=1} = 0. \tag{27}$$

Then we can chose the power-type test function $\varphi(x) = x^5$ for which we will have the equalities

$$k^2 = \frac{35}{2}, \qquad a^2 = \frac{40}{3}(\nu^2 + 9\beta\nu + 27\beta^2)$$

and, therefore, if the initial condition satisfies the inequality

$$\int_0^1 x^5 u_0(x)\,dx > 4\sqrt{\frac{\nu^2}{21} + \frac{3\beta\nu}{7} + \frac{9\beta^2}{7}} - 3\beta - \frac{4\nu}{5}, \tag{28}$$

then we will have the global unsolvability for the function

$$J(t) = \int_0^1 (x^5 u_0(x) + 4\nu x^4 + 12\beta x^3)\,dx.$$

Example 4. In the following example, we can use the absence of the maximum principle for the Korteweg–de Vries–Burgers equation. This allows us to assume that the blow-up of the solution of problem (21), (22) can also occur for boundary conditions of the first kind that are given at both endpoints of the closed interval.

Indeed, consider problem (21), (22) on the closed interval $[0,1]$ with boundary conditions

$$u|_{x=0} = u|_{x=1} = u_{xx}|_{x=0} = 0. \tag{29}$$

For convenience, suppose that $\alpha = \nu/\beta \geq 5$ and choose the test function $\varphi(x) = -(1-x)^\alpha$. Then the boundary conditions yield

$$G(u,\varphi)|_0^1 = [\beta u_{xx}\varphi - (\nu\varphi + \beta\varphi_x)u_x]_{x=0} = 0, \tag{30}$$

and the sufficient conditions for the blow-up of the functional

$$J(t) = \nu\frac{\alpha-1}{\alpha} - \beta(\alpha-2) - \int_0^1 (1-x)^\alpha u(x,t)\,dx,$$

$$k^2 = \frac{\alpha(\alpha+2)}{2}, \qquad a^2 = \frac{\alpha(\alpha-1)^2}{2}\left(\frac{\nu^2}{\alpha-2} - \frac{2\beta\nu(\alpha-2)}{\alpha-3} + \frac{\beta^2(\alpha-2)^2}{\alpha-4}\right)$$

immediately follow from Theorem 2.

Finally, if we assume that the solution $u(x,t)$ is bounded, then, in the same way as above, we can obtain sufficient conditions for a gradient catastrophe. Note that, in the given examples, the power-type test function $\varphi = x^\lambda(x-L)^\mu$ was chosen. This is the traditional choice of the function $\varphi(x)$ used to exclude the left or right boundary condition. However, to obtain the optimal estimate of the blow-up time, it is necessary to find the best test function, i.e., to solve the corresponding variational problem in each particular case.

4. BLOW-UP IN THE MODIFIED KORTEWEG–DE VRIES–BURGERS EQUATION

The successful application of the method of test functions to the study of the global solvability of various problems for the Korteweg–de Vries equation, Zakharov–Kuznetsov equation, Kadomtsev–Petviashvili equation, Khokhlov–Zabolotskaya equation, Ostrovskii equation, Burgers equation, Korteweg–de Vries–Burgers equation, and many others containing the gradient nonlinearity uu_x demonstrate the wide possibilities of the method [11]. However, the idea of the approach is also applicable to problems involving nonlinearities of other types.

Example 5. Consider the initial boundary-value problem for the widely-known modified Korteweg–de Vries equation with dissipation:

$$u_t - u^2 u_x + \beta u_{xxx} = \nu u_{xx}, \qquad x \in [0, L], \tag{31}$$

$$u|_{t=0} = u_0(x), \qquad u|_{x=0} = u_x|_{x=0} = u|_{x=L} = 0, \qquad \nu > 0, \quad \beta > 0. \tag{32}$$

Equation (31) is used, in particular, for modeling waves in nonlinear transmission lines [1]. The zero boundary conditions are chosen to reduce the amount of calculations, but we can also prove the global (in time) unsolvability for boundary conditions of general form by analogy with the previous problems.

Multiplying Eq. (31) by the test function $\varphi(x,t) = u(x,t)\phi(x)$ and integrating by parts, we can obtain the following expression:

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2\phi\,dx = -\frac{1}{4}\int_0^L u^4\phi_x,dx + \frac{1}{2}\int_0^L u^2(\nu\phi_{xx} + \beta\phi_{xxx})\,dx - \int_0^L u_x^2\left(\frac{3\beta}{2}\phi_x + \nu\phi\right)dx$$
$$+ \left[\frac{1}{4}\phi u^4 + (\nu\phi + \beta\phi_x)uu_x - \frac{1}{2}(\nu\phi_x + \beta\phi_{xx})u^2 + \frac{\beta}{2}u_x^2\phi - \beta u u_{xx}\phi\right]_0^L. \tag{33}$$

To obtain a lower bound for the left-hand side, it is necessary to choose the test function so that the third summand on the right is bounded. In the general case, the following condition must hold:

$$\left(\frac{3\beta}{2}\phi_x + \nu\phi\right) \le 0.$$

Consider the following particular case by choosing the test function as $\phi(x) = \exp(-2\nu x/3\beta)$; then, for the boundary conditions (32), expression (33) can be rewritten as

$$\frac{dJ}{dt} \ge \frac{\nu}{3\beta}\int_0^L u^4 \exp\left(-\frac{2\nu}{3\beta}x\right)dx + \frac{4\nu^3}{27\beta^2}\int_0^L u^2\exp\left(-\frac{2\nu}{3\beta}x\right)dx, \tag{34}$$

where

$$J(t) = \int_0^L u^2 \exp\left(-\frac{2\nu}{3\beta}x\right)dx.$$

Let us use a rough estimate to obtain the blow-up result: let us get rid of the second nonnegative summand on the right-hand side of (34) and bound the first summand by the Cauchy–Bunyakovskii inequality

$$\left(\int_0^L u^2\exp\left(-\frac{2\nu x}{3\beta}\right)dx\right)^2 \le \frac{3\beta}{2\nu}\int_0^L u^4\exp\left(-\frac{2\nu x}{3\beta}\right)dx. \tag{35}$$

Then (34) can be written as the ordinary differential inequality

$$\frac{dJ}{dt} \ge kJ^2, \qquad k = \frac{2\nu^2}{9\beta^2};$$

solving it, we obtain the following statement.

Theorem 3. *The classical nontrivial solution* $u(x,t)$ *of problem* (31), (32) *belonging to the class* $C^{(1)}([0,T];C^{(3)}([0,L]))$ *cannot exist on a time interval greater than* $[0,T]$ *and must satisfy the estimate*

$$J(t) \geq \frac{J(0)}{1-kJ(0)t}, \qquad T \leq \frac{1}{kJ(0)}. \tag{36}$$

Acting according to the proposed scheme, we can also apply the method of test functions to problems with nonlinearities of the type $|u|^n u_x$ and additional summands of the form $u_{x\dots xx}$ arising in the physics of solitons [13], [12]. Thus, linear dissipation is not the principal factor ensuring the global (in time) solvability of problems with gradient nonlinearity.

5. BLOW-UP IN UNBOUNDED DOMAINS

For the Burgers problem, we considered problems on a closed interval with the exception of the "soft blow-up." However, the method of test functions is also applicable in unbounded domains.

Example 6. Consider the following problem for the Korteweg–de Vries–Burgers equation on the half-line:

$$u_t + uu_x + \beta u_{xxx} = \nu u_{xx}, \qquad x \in [0,+\infty), \tag{37}$$

$$u|_{t=0} = u_0(x). \tag{38}$$

Suppose that there exists a classical solution of the problem belonging to the class

$$C^{(1)}([0,T];C^{(3)}([0,+\infty))).$$

For the test function we take $\varphi(x) = -\exp(-x)$ and choose the following special boundary conditions to reduce the amount of calculations:

$$u|_{x=0} = 0, \qquad u_x|_{x=0} = 1, \qquad u_{xx}|_{x=0} = \frac{\nu-\beta}{\beta}.$$

Let us multiply Eq. (37) by $\varphi(x)$ and then integrate by parts, obtaining

$$\frac{d}{dt}\int_0^{+\infty} u(x,t)(-e^{-x})\,dx = \frac{1}{2}\int_0^{+\infty} u^2 e^{-x}\,dx + (\beta-\nu)\int_0^{+\infty} ue^{-x}\,dx. \tag{39}$$

Denote

$$J(t) = \int_0^{+\infty} w(x,t)e^{-x}\,dx, \qquad w = -(u+\beta-\nu),$$

and rewrite equality (39) as

$$\frac{dJ}{dt} = \frac{1}{2}\int_0^{+\infty} w^2 e^{-x}\,dx - \frac{(\nu-\beta)^2}{2}. \tag{40}$$

Using the Cauchy–Bunyakovskii inequality

$$\left(\int_0^{+\infty} w(x,t)e^{-x}\,dx\right)^2 \leq \int_0^{+\infty} w^2 e^{-x}\,dx \int_0^{+\infty} e^{-x}\,dx,$$

we write (40) as the ordinary differential inequality

$$\frac{dJ}{dt} \geq k^2 J^2 - a^2, \qquad k^2 = \frac{1}{2}, \qquad a^2 = \frac{(\nu-\beta)^2}{2}. \tag{41}$$

Then we can formulate the following blow-up result.

Theorem 4. *Let the initial function $u_0(x) \in L^{(1)}([0, L])$ of problem (37), (38) satisfies the inequality*

$$\int_0^{+\infty} u_0(x)e^{-x}\,dx < \nu - \beta - |\nu - \beta|; \tag{42}$$

then the result of Theorem 1 holds.

Note that often the correct choice of test function can decrease the number of boundary conditions sufficient for global unsolvability. For example, for problem (37), (38), the blow-up result can also be obtained for the case

$$u|_{x=0} = C_1, \quad u_{xx}|_{x=0} = C_2, \qquad C_1, C_2 = \text{const}$$

if the test function $\varphi(x) = \exp(-\nu x/\beta)$ is taken.

Example 7. Similarly, the blow-up result for the modified Korteweg–de Vries–Burgers equation on the half-line can only be obtained with one boundary condition, although of special nonphysical form:

$$u_t - u^2 u_x + \beta u_{xxx} = \nu u_{xx}, \qquad x \in [0, +\infty), \tag{43}$$

$$u|_{t=0} = u_0(x), \qquad \frac{1}{2}u^4 + \beta u_x^2 - 2\beta u u_{xx}\bigg|_{x=0} < a^2. \tag{44}$$

To do this, it suffices to take the test function in the form

$$\varphi = u(x,t)\phi(x) = u(x,t)\exp\biggl(-\frac{\nu}{\beta}x\biggr).$$

Then, denoting $J(t) = \int u^2 \exp(-\nu x/\beta)\,dx$ and using Eq. (33), we obtain the inequality

$$\frac{dJ}{dt} \geq \frac{\nu}{2\beta}\int_0^L u^4 \exp\biggl(-\frac{\nu x}{\beta}\biggr)\,dx - \biggl[\frac{1}{2}u^4 + \beta u_x^2 - 2\beta u u_{xx}\biggr]_0 \geq k^2 J^2 - a^2, \tag{45}$$

where $k^2 = \nu^2/2\beta^2$, from which we find sufficient (for a blow-up) conditions on the initial function $u_0(x)$:

$$\int_0^{+\infty} u_0^2(x)\exp\biggl(-\frac{\nu x}{\beta}\biggr)\,dx > \frac{a}{k}. \tag{46}$$

Concluding the discussion on the subject of unbounded domains, it is impossible to avoid the paper [7] in which the method of test functions for the Korteweg–de Vries equation was used to obtain the blow-up result for the solution of the Cauchy problem. However, in the proof given in [7], special nonphysical constraints on the growth at infinity were imposed on the initial function; therefore, despite the fact that similar results for Burgers-type equations can also be obtained, we will not dwell on this subject.

6. THE MAXIMUM PRINCIPLE

The last section is devoted to several technical questions and some derivations from the results obtained. In studying the gradient catastrophe for the Burgers equation (5), we used the maximum principle, but, due to the lack of references on this subject, we shall give its proof.

Maximum Principle. *If the function $u(x,t)$ defined and continuous in the closed domain $0 \leq t \leq T$ and $0 \leq x \leq L$ satisfies the Burgers equation (5) at the points of the domain $0 < x < L$, $0 < t \leq T$, then the maximum and minimum values of the function $u(x,t)$ are attained either at the initial instant of time, or on the boundary.*

Proof. The proof of the principle is obtained by arguing by contradiction. The case $u(x,t) \equiv \text{const}$ is trivial. Denote by M the maximum value of the function for $t = 0$ or for $x = 0, L$. Assume that the function $u(x,t)$ attains its maximum value $M + \varepsilon$ at the interior point (x_0, t_0). Let us find the point (x_1, t_1) at which

$$\frac{\partial^2 u}{\partial x^2}(x_1, t_1) \le 0, \qquad \frac{\partial u}{\partial t}(x_1, t_1) > 0. \tag{47}$$

For some constant k, let us introduce the auxiliary function

$$v(x,t) = u(x,t) + k(t_0 - t). \tag{48}$$

Obviously,

$$v(x_0, t_0) = u(x_0, t_0) = M + \varepsilon \qquad \text{and} \qquad k(t_0 - t) \le (kT).$$

Choose $(kT) < \varepsilon/2$; then the maximum value of the function $v(x,t)$ for $t = 0$ or for $x = 0, L$ is at most $M + \varepsilon/2$. In view of the continuity of the function $v(x,t)$, it must, at some point (x_1, t_1), attain its maximum value:

$$v(x_1, t_1) \ge v(x_0, t_0) = M + \varepsilon.$$

Therefore, at the point (x_1, t_1): $t_1 > 0$, $0 < x_1 < L$, the following inequalities must hold:

$$v_{xx}(x_1, t_1) = u_{xx}(x_1, t_1) \le 0, \qquad v_t(x_1, t_1) = u_t(x_1, t_1) - k \ge 0, \qquad v_x(x_1, t_1) = u_x(x_1, t_1) = 0.$$

Thus, the function $u(x,t)$ does not satisfy the Burgers equation at the interior point (x_1, t_1). The second part of the theorem on the minimum value does not require a special proof, because the function $u_1(-x,t) = -u(x,t)$ has maximum value where u is minimal.

The theorem is proved. □

In view of the given scheme of proof, we can conclude that a similar principle of the boundedness of the solution will hold not only for Eq. (5), but also for equations with nonlinear viscosity on the right-hand side. For example, for the simple-wave equation regularized by smoothing on s small time interval and used for modeling hydrodynamic processes on shallow water,

$$u_t + uu_x = \nu(|u|^q u)_{xx}, \tag{49}$$

or for the equation with nonlinear viscosity of simplified form,

$$u_t + uu_x = \nu |u|^q u_{xx}. \tag{50}$$

As a consequence of the maximum principle, we can obtained the global unsolvability result, because after differentiation of the equation with respect to x, the left-hand side will contain only two summands u_{xt} and u_x^2 ensuring a blow-up, while all the other summands will be either bounded or of first order in u_x.

Example 8. Using the method of test functions, we shall prove the occurrence of the gradient catastrophe for Eq. (43) in the case $q = 2$. Let us rewrite (49) in the form

$$u_t + u_x(u - 6\nu u u_x) = 3\nu u^2 u_{xx}, \tag{51}$$

from which we see that the maximum principle holds. To prove this, it suffices to take into account the fact that the second summand on the left-hand side vanishes because of the equality $u_x = 0$ and the fact that the multiplier before u_{xx} is not negative. Thus, there exist constants m and M such that $m \le |u(x,t)| \le M$ for all (x,t). For the simplicity of calculations, we assume that $m > 0$ and $M \le 1$.

Let us differentiate (51) with respect to x,

$$(u_x)_t + u_x^2 + uu_{xx} = \nu(u^3)_{xxx}$$

and substitute the expression for uu_{xx} resulting from (51):

$$uu_{xx} = \frac{1}{3\nu}\frac{u_t}{u} + \frac{u_x}{3\nu} - 2u_x^2, \tag{52}$$

obtaining the equation

$$\frac{\partial}{\partial t}\left(\frac{1}{3\nu}\ln u + u_x\right) = \nu(u^3)_{xxx} - \frac{1}{3\nu}u_x + u_x^2. \tag{53}$$

Let us now multiply (53) by the test function $\varphi(x) \in C^{(3)}(\Omega)$ and integrate over the domain Ω. Let the function $\varphi(x) \geq 0$ have finite support on Ω; then, after integrating by parts, the conditions on the boundary $\partial\Omega$ will yield zero, and Eq. (53) becomes

$$\frac{d}{dt}\int \varphi w\,dx = \int \varphi u_x^2\,dx + \frac{1}{3\nu}\int \varphi_x u\,dx - \int \varphi_{xxx}u^3\,dx, \qquad \text{where} \quad w = \frac{1}{3\nu}\ln u + u_x. \tag{54}$$

Let us use the obvious estimate

$$w^2 \leq 2u_x^2 + \frac{2}{9\nu^2}(\ln u)^2.$$

Then, using the maximum principle, we can rewrite the expression (55) in the form

$$\frac{d}{dt}\int \varphi w\,dx \geq \frac{1}{2}\int \varphi w^2\,dx - \frac{1}{3\nu}\int |\varphi_x||u|\,dx - \int |\varphi_{xxx}||u|\,dx - \frac{1}{9\nu^2}\int |\varphi|(\ln u)^2\,dx. \tag{55}$$

Using inequality (18) and introducing the notation

$$J(t) = \int \varphi\left(\frac{1}{3\nu}\ln u + u_x\right)dx,$$

$$k^2 = \frac{1}{2}\left(\int \varphi\,dx\right)^{-1}, \qquad a^2 = \frac{1}{3\nu}|\varphi_x|\,dx + \int |\varphi_{xxx}|\,dx + \frac{1}{9\nu^2}\int |\varphi|(\ln m)^2\,dx,$$

we obtain the global (in time) unsolvability result from the ordinary differential inequality (11) under the condition $J(0) > a/k$. Note that if the assumption $M \leq 1$ did not hold, then a similar estimate for a^2 would contain the constant M and $(\ln m)$ could be replaced by an estimate of $(\ln M)$, which, certainly, would not change essentially the result obtained.

It is also of interest that the gradient catastrophe in the case considered above occurs not because of the gradient nonlinearity, but is caused by the nonlinearity $-2u_x^2$ on the right-hand side (see (52)). In particular, if, for Eq. (49) with $q = 1$, we carry out similar transformations, then, for the Cauchy problem with zero conditions, as $x \to \infty$, the gradient and dissipative nonlinearities cancel each other out and the gradient catastrophe will not occur and, conversely, the conservation law

$$\frac{d}{dt}\int_{-\infty}^{+\infty} \varphi(x)\left(u_x + \frac{u}{2\nu}\right)dx = 0$$

will hold.

Finally, all the equations considered in this paper are model equations for perturbations propagating in one direction. Thus, this must be taken into account in the setting of the initial boundary-value problem. In particular, in passing to a fixed reference system, the equations may contain an additional summand of the form c_0u_x, where c_0 is the velocity of the reference system in which the equation is of the form (1), (2), or (3). Nevertheless, this summand will not essentially affect either the proof of a “severe” blow-up involving the additional summand

$$c_0\int \varphi_x u\,dx,$$

or the study of the gradient catastrophe involving two additional summands, u_t on the left-hand side and $\varphi_x u^2/2$ on the right-hand side. Finally, in the study of processes involving perturbations propagating in both directions, problems must be posed for Boussinesq-type equations whose global solvability will be studied by the method of test functions in another paper.

7. CONCLUSIONS

The method of test functions used in the study of the global (in time) unsolvability of problems for Burgers-type equations has a very wide range of applications. Using this method, one can show that the presence of linear and nonlinear dissipation is not a sufficient condition for the existence of bounded smooth solutions. It is shown that, under certain boundary and initial conditions, both gradient nonlinearity and nonlinearity in the dissipative summand can lead to a blow-up of the solution or to a gradient catastrophe in finite time. Using the proposed method, we can obtain estimates of the blow-up rate and blow-up time depending on the initial and boundary data as well as on the choice of the test function.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research under grant 13-01-12460-ofi-m.

REFERENCES

1. N. M. Ryskin and D. I. Trubetskov, *Nonlinear Waves* (Nauka, Moscow, 2000) [in Russian].
2. G. B. Whitham, *Linear and Nonlinear Waves* (Wiley-Interscience, New York—London—Sydney, 1974; Mir, Moscow, 1977).
3. O. V. Rudenko and S. I. Soluyan, *Theoretical Foundations of Nonlinear Acoustics* (Nauka, Moscow, 1975) [in Russian].
4. S. I. Pokhozhaev, "On the blow-up of solutions to nonlinear initial-boundary value problems," in *Trudy Mat. Inst. Steklov*, Vol. 260: *Theory of Functions and Nonlinear Partial Differential Equations*, Collection of papers dedicated to the 70th anniversary of the Corresponding Member of the Russian Academy of Sciences S. I. Pokhozhaev (MAIK, Moscow, 2008), pp. 213—226 [Proc. Steklov Inst. Math. **260**, 204—217 (2008)].
5. E. Mitidieri and S. I. Pokhozhaev, "A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities," in *Trudy Mat. Inst. Steklov* (Nauka, Moscow, 2001), Vol. 234, pp. 3—383 [Proc. Steklov Inst. Math. **234**, 1—362 (2001)].
6. G. M. Zaslavskii and R. Z. Sagdeev, *Introduction to Nonlinear Physics*: *From the Pendulum to Turbulence and Chaos* (Nauka, Moscow, 1988) [in Russian].
7. S. I. Pokhozhaev, "On the nonexistence of global solutions of the Cauchy problem for the Korteweg—de Vries equation," Funktsional. Anal. Prilozhen. **46** (4), 51—60 (2012) [Functional Anal. Appl. **46** (4), 279—286 (2012)].
8. S. I. Pohozaev [Pokhozhaev], "Blow-up of smooth solutions of the Kortweg—de Vries equation," Nonlinear Anal., Theory Methods Appl. **75** (12), 4688—4698 (2012).
9. M. O. Korpusov and E. V. Yushkov, "Solution blowup for systems of shallow-water equations," Teoret. Mat. Fiz. **177** (2), 264—275 (2013) [Theoret. and Math. Phys. **177** (2), 1505—1514 (2013)].
10. M. O. Korpusov and A. A. Panin, "Local solvability and solution blowup for the Benjamin—Bona—Mahony—Burgers equation with a nonlocal boundary condition," Teoret. Mat. Fiz. **175** (2), 159—172 (2013) [Theoret. and Math. Phys. **175** (2), 580—591 (2013)].
11. M. O. Korpusov, A. G. Sveshnikov, and E. V. Yushkov, "Blow-up of solutions of the Kadomtsev—Petviashvili and Zakharov—Kuznetsov nonlinear equations," Izv. Ross. Akad. Nauk Ser. Mat. (in press).
12. J. P. Boyd, "Weakly non-local solutions for capillary-gravity waves: Fifth-degree Korteweg—de Vries equation," Phys. D **48**, 129—146 (1991).
13. E. V. Yushkov, "Blowup in Korteweg—de Vries-type systems," Teoret. Mat. Fiz. **173** (2), 197—206 (2012) [Theoret. and Math. Phys. **173** (2), 1498—1506 (2012)].