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Boundary-Value Problem with Nonlocal Integral Condition for Mixed-Type Equations with Degeneracy on the Transition Line

Yu. K. Sabitova*

Sterlitamak Division, Bashkir State University, Sterlitamak, Russia
Institute for Applied Studies, Sterlitamak, Russia

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Abstract—For an elliptic-hyperbolic type equation, the boundary-value problem with nonlocal Samarskii–Ionkin condition in a rectangular domain is solved. Using the spectral analysis method, a uniqueness criterion is established and the existence theorem for the solution of the problem is proved. The solution of the problem is constructed as the sum of a biorthogonal series.

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1. INTRODUCTION

In the present paper, we consider the following equation of mixed, elliptic-hyperbolic, type:

$$Lu = K(y)u_{xx} + u_{yy} = 0, \tag{1.1}$$

where $K(y) = \operatorname{sgn} y \cdot |y|^m$, $m > 0$, in the rectangular domain

$$D = \{(x, y) \mid 0 < x < 1,\ -\alpha < y < \beta\},$$

which degenerates on the change-of-type line. For Eq. (1.1) in the domain D, we pose the following problem.

Boundary-value problem. *In the domain D, find the function $u(x, y)$ satisfying the following conditions:*

$$u(x, y) \in C^1(\overline{D}) \cap C^2(D_+ \cup D_-), \tag{1.2}$$

$$Lu(x, y) \equiv 0, \qquad (x, y) \in D_+ \cup D_-, \tag{1.3}$$

$$u(x, \beta) = \varphi(x), \quad u(x, -\alpha) = \psi(x), \qquad 0 \le x \le 1, \tag{1.4}$$

$$u(1, y) = 0, \qquad -\alpha \le y \le \beta, \tag{1.5}$$

$$\int_0^1 u(x, y)dx = A = \text{const}, \qquad -\alpha \le y \le \beta, \tag{1.6}$$

where $D_- = D \cap \{y < 0\}$, $D_+ = D \cap \{y > 0\}$, $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions satisfying the conditions $\varphi(1) = \psi(1) = 0$ and

$$\int_0^1 \varphi(x)\, dx = \int_0^1 \psi(x)\, dx = A. \tag{1.7}$$

*E-mail: sabitovauk@rambler.ru

For various equations, including those of mixed type, but with constant coefficients, problems of similar type were studied, in particular, in [1]–[3].

For a fixed $y \in (-\alpha, 0) \cup (0, \beta)$, integrating Eq. (1.1) over the variable x from ε to $1-\varepsilon$, where ε is a sufficiently small number, we obtain

$$K(y) \int_{\varepsilon}^{1-\varepsilon} u_{xx}\, dx + \int_{\varepsilon}^{1-\varepsilon} u_{yy}\, dx = 0.$$

Hence, as $\varepsilon \to 0$, we have

$$K(y)[u_x(1,y) - u_x(0,y)] + \frac{d^2}{dy^2} \int_0^1 u(x,y)\, dx = 0.$$

In view of condition (1.6), the last equality becomes the other nonlocal condition

$$u_x(0,y) = u_x(1,y), \qquad -\alpha \le y \le \beta, \tag{1.8}$$

expressing the equality of flows across the lateral sides $x = 0$ and $x = 1$ of the rectangle D.

In what follows, instead of problem (1.2)–(1.6), we shall study the problem described by (1.2)–(1.5) and (1.8). In this paper, using ideas from [2]–[5],, we establish a criterion for the uniqueness of the solution of problem (1.2)–(1.5), (1.8). In justifying the existence of a solution of the problem, we must deal with the small denominators with respect to the parameter α. Under certain conditions on the number α and the functions $\varphi(x)$, $\psi(x)$, the solution can be expressed as the sum of a biorthogonal series. We prove the convergence of the constructed series for the class (1.2).

2. UNIQUENESS OF THE SOLUTION

We search for the particular solutions of Eq. (1.1) satisfying conditions (1.2), (1.5), (1.8), in the form $u(x,y) = X(x)Y(y)$ by using the method of separation of variables. Substituting this product into Eq. (1.1), we obtain the relations

$$X''(x) + \mu X(x) = 0, \qquad 0 < x < 1, \tag{2.1}$$

$$X(1) = 0, \qquad X'(0) = X'(1), \tag{2.2}$$

$$Y''(y) - \mu \operatorname{sgn} y \cdot |y|^m Y(y) = 0, \qquad y \in (-\alpha, 0) \cup (0, \beta), \tag{2.3}$$

where μ is a constant. Problem (2.1), (2.2) is non-self-adjoint; The adjoint problem is of the form

$$Y''(x) + \mu Y(x) = 0, \quad 0 < x < 1, \qquad Y'(0) = 0, \qquad Y(0) = Y(1).$$

The eigenvalues of the first problem are the numbers $\mu_n = \lambda_n^2$, $\lambda_n = 2\pi n$, $n = 1, 2, \dots,$ to which correspond to the following eigen and associated functions

$$\begin{gathered} X_0(x) = 2(1-x), \\ X_{2n-1}(x) = 4 \sin 2\pi n x, \quad X_{2n}(x) = 4(1-x) \cos 2\pi n x, \qquad n = 1, 2, \dots . \end{gathered} \tag{2.4}$$

For $n \ge 1$, the eigenvalues are double, the $X_{2n-1}(x)$ are eigenfunctions, and the $X_{2n}(x)$ are associated functions. The system of eigen and associated functions of the adjoint problem is of the form

$$\begin{gathered} Y_0(x) = 2, \\ Y_{2n}(x) = 4 \cos 2\pi n x, \quad Y_{2n-1}(x) = 4x \sin 2\pi n x, \qquad n = 1, 2, \dots . \end{gathered} \tag{2.5}$$

It is easy to verify that systems (2.4) and (2.5) are mutually biorthogonal, i.e.,

$$(X_k(x), Y_l(x)) = \delta_{kl},$$

where δ_{kl} is the Kronecker delta. Both problems under consideration are regular and, therefore, the systems $\{X_k\}$ and $\{Y_k\}$ are complete in the space $L_2(0,1)$ [6]. Since both these systems are Bessel, i.e., for all $f \in L_2(0,1)$, the following inequalities hold:

$$\Sigma |(f, X_k)|^2 < \infty, \qquad \Sigma |(f, Y_k)|^2 < \infty,$$

it follows from the Bari theorem that these systems constitute Riesz bases in $L_2(0,1)$.

Suppose that $u(x,y)$ is a solution of problem (1.2)–(1.5), (1.8). Consider the functions

$$w_n(y) = \int_0^1 u(x,y)\cos 2\pi nx\,dx, \qquad n = 1,2,\dots, \tag{2.6}$$

$$w_0(y) = \int_0^1 u(x,y)\,dx, \tag{2.7}$$

$$z_n(y) = \int_0^1 u(x,y)x\sin 2\pi nx\,dx, \qquad n = 1,2,\dots. \tag{2.8}$$

Using (2.6), we introduce the function

$$w_{\varepsilon,n}(y) = \int_\varepsilon^{1-\varepsilon} u(x,y)\cos 2\pi nx\,dx, \qquad n = 1,2,\dots, \tag{2.9}$$

where $\varepsilon > 0$ is a sufficiently small number. Twice differentiating equality (2.9) for $y \in (-\alpha,0)\cup(0,\beta)$ and taking into account Eq. (1.1), we can write

$$\begin{aligned} w''_{\varepsilon,n}(y) &= \int_\varepsilon^{1-\varepsilon} u_{yy}(x,y)\cos 2\pi nx\,dx = -\operatorname{sgn} y\cdot|y|^m\int_\varepsilon^{1-\varepsilon} u_{xx}(x,y)\cos 2\pi nx\,dx \\ &= -\operatorname{sgn} y\cdot|y|^m\int_\varepsilon^{1-\varepsilon} u_{xx}(x,y)\cos 2\pi nx\,dx. \end{aligned} \tag{2.10}$$

Integrating (2.10) by parts twice, taking into account conditions (1.5), (1.8), and passing to the limit as $\varepsilon \to 0$, we obtain the differential equation

$$w''_n(y) - \operatorname{sgn} y\cdot|y|^m(2\pi n)^2 w_n(y) = 0 \tag{2.11}$$

with the boundary conditions

$$w_n(\beta) = \int_0^1 \varphi(x)\cos 2\pi nx\,dx = \varphi_n, \qquad w_n(-\alpha) = \int_0^1 \psi(x)\cos 2\pi nx\,dx = \psi_n. \tag{2.12}$$

The general solution of Eq. (2.11) is of the form

$$w_n(y) = \begin{cases} a_n\sqrt{y}\,I_{1/(2q)}(p_n y^q) + b_n\sqrt{y}\,K_{1/(2q)}(p_n y^q), & y > 0, \\ c_n\sqrt{-y}\,J_{1/(2q)}(p_n(-y)^q) + d_n\sqrt{-y}\,Y_{1/(2q)}(p_n(-y)^q), & y < 0, \end{cases} \tag{2.13}$$

where $J_{1/(2q)}(p_n(-y)^q)$ and $Y_{1/(2q)}(p_n(-y)^q)$ are Bessel functions of the first and the second kind, respectively, $I_{1/(2q)}(p_n y^q)$ and $K_{1/(2q)}(p_n y^q)$ are modified Bessel functions, a_n, b_n, c_n and d_n is an arbitrary constants, and $q = (m+2)/2$, $p_n = (2\pi n)/q$.

In view of (1.2), we choose the constants a_n, b_n, c_n, and d_n so that the following equalities hold:

$$w_n(0+) = w_n(0-), \qquad w'_n(0+) = w'_n(0-). \tag{2.14}$$

The first of the equalities (2.14) holds for $d_n = -\pi b_n/2$ and any a_n and c_n, while the second equality holds for $c_n = \pi\cot[\pi/(4q)]b_n/2 - a_n$ and $d_n = -\pi b_n/2$.

Let us substitute the resulting expressions for the constants c_n and d_n into (2.13); then the functions $w_n(y)$ take the form

$$w_n(y) = \begin{cases} a_n\sqrt{y}\,I_{1/(2q)}(p_n y^q) + b_n\sqrt{y}\,K_{1/(2q)}(p_n y^q), & y > 0, \\ -a_n\sqrt{-y}\,J_{1/(2q)}(p_n(-y)^q) + b_n\sqrt{-y}\,\overline{Y}_{1/(2q)}(p_n(-y)^q), & y < 0, \end{cases} \tag{2.15}$$

where

$$\overline{Y}_{1/(2q)}(p_n y^q) = \frac{\pi}{2\sin[\pi/(2q)]}(J_{1/(2q)}(p_n y^q) + J_{-1/(2q)}(p_n y^q)).$$

Note that, for the functions (2.15), the following equality holds:

$$w_n''(0+) = w_n''(0-) = 0.$$

Now, using (2.12) and (2.15), we obtain the following system for finding a_n and b_n:

$$\begin{cases} a_n I_{1/(2q)}(p_n\beta^q) + b_n K_{1/(2q)}(p_n\beta^q) = \varphi_n \beta^{-1/2}, \\ -a_n J_{1/(2q)}(p_n\alpha^q) + b_n \overline{Y}_{1/(2q)}(p_n\alpha^q) = \psi_n \alpha^{-1/2}, \qquad n = 1, 2, \dots . \end{cases} \tag{2.16}$$

If the determinant of system (2.16) is nonzero,

$$\Delta_n(\alpha,\beta) = J_{1/(2q)}(p_n\alpha^q)K_{1/(2q)}(p_n\beta^q) + I_{1/(2q)}(p_n\beta^q)\overline{Y}_{1/(2q)}(p_n\alpha^q) \neq 0, \qquad n \in \mathbb{N}, \tag{2.17}$$

then this system has the unique solution

$$a_n = \frac{\varphi_n\sqrt{\alpha}\,\overline{Y}_{1/(2q)}(p_n\alpha^q) - \psi_n\sqrt{\beta}\,K_{1/(2q)}(p_n\beta^q)}{\Delta_n(\alpha,\beta)\sqrt{\alpha\beta}}, \tag{2.18}$$

$$b_n = \frac{\varphi_n\sqrt{\alpha}\,J_{1/(2q)}(p_n\alpha^q) + \psi_n\sqrt{\beta}\,I_{1/(2q)}(p_n\beta^q)}{\Delta_n(\alpha,\beta)\sqrt{\alpha\beta}}. \tag{2.19}$$

Using (2.18) and (2.19), from (2.15) we determine the final form of the functions

$$w_n(y) = \begin{cases} \dfrac{\varphi_n\sqrt{\alpha y}\,\Delta_n(\alpha,y) + \psi_n\sqrt{\beta y}\,A_n(y,\beta)}{\Delta_n(\alpha,\beta)\sqrt{\alpha\beta}}, & y > 0, \\ \dfrac{\varphi_n\sqrt{-\alpha y}\,B_n(\alpha,-y) + \psi_n\sqrt{-\beta y}\,\Delta_n(-y,\beta)}{\Delta_n(\alpha,\beta)\sqrt{\alpha\beta}}, & y < 0, \end{cases} \tag{2.20}$$

where

$$\Delta_n(\alpha,y) = J_{1/(2q)}(p_n\alpha^q)K_{1/(2q)}(p_n y^q) + \overline{Y}_{1/(2q)}(p_n\alpha^q)I_{1/(2q)}(p_n y^q), \tag{2.21}$$

$$A_n(y,\beta) = I_{1/(2q)}(p_n\beta^q)K_{1/(2q)}(p_n y^q) - I_{1/(2q)}(p_n y^q)K_{1/(2q)}(p_n\beta^q),$$

$$B_n(\alpha,-y) = \overline{Y}_{1/(2q)}(p_n(-y)^q)J_{1/(2q)}(p_n\alpha^q) - \overline{Y}_{1/(2q)}(p_n\alpha^q)J_{1/(2q)}(p_n(-y)^q), \tag{2.22}$$

$$\Delta_n(-y,\beta) = J_{1/(2q)}(p_n(-y)^q)K_{1/(2q)}(p_n\beta^q) + \overline{Y}_{1/(2q)}(p_n(-y)^q)I_{1/(2q)}(p_n\beta^q).$$

Just as for $w_n(y)$, we see that the function $w_0(y)$ defined by formula (2.7), satisfies the conditions

$$w_0''(y) = 0, \qquad y \in (-\alpha,0)\cup(0,\beta), \tag{2.23}$$

$$w_0(0+) = w_0(0-), \qquad w_0'(0+) = w_0'(0-), \tag{2.24}$$

$$w_0(\beta) = \int_0^1 \varphi(x)\,dx = \varphi_0, \qquad w_0(-\alpha) = \int_0^1 \psi(x)\,dx = \psi_0. \tag{2.25}$$

The unique solution of problem (2.23)–(2.25) is given by

$$w_0(y) = \frac{\varphi_0 - \psi_0}{\alpha+\beta}(y+\alpha) + \psi_0. \tag{2.26}$$

Repeating the same actions for the function $z_n(y)$ as for $w_n(y)$ given by formula (2.8), we obtain the inhomogeneous differential equation

$$z_n''(y) - \operatorname{sgn} y\cdot|y|^m(2\pi n)^2 z_n(y) = -4\pi n\operatorname{sgn} y\cdot|y|^m w_n(y), \tag{2.27}$$

where $y \in (-\alpha,0)\cup(0,\beta)$, with the corresponding boundary conditions

$$z_n(\beta) = \int_0^1 \varphi(x)x\sin 2\pi nx\,dx = \varphi_{1n}, \tag{2.28}$$

$$z_n(-\alpha) = \int_0^1 \psi(x)x\sin 2\pi nx\,dx = \psi_{1n} \tag{2.29}$$

and the conjugation conditions

$$z_n(0+) = z_n(0-), \qquad z_n'(0+) = z_n'(0-). \tag{2.30}$$

Using the method of variation of arbitrary constants, we find a solution of problem (2.27)–(2.30) in the form

$$z_n(y) = \begin{cases} \dfrac{\sqrt{\alpha y}\,[\varphi_{1n} - z_n^+(\beta)]\Delta_n(\alpha, y)}{\Delta_n(\alpha,\beta)\sqrt{\alpha\beta}} + \dfrac{\sqrt{\beta y}\,[\psi_{1n} - z_n^-(\alpha)]A_n(y,\beta)}{\Delta_n(\alpha,\beta)\sqrt{\alpha\beta}} + z_n^+(y), & y > 0, \\ \dfrac{\sqrt{-\beta y}\,[\psi_{1n} - z_n^-(\alpha)]\Delta_n(-y,\beta)}{\Delta_n(\alpha,\beta)\sqrt{\alpha\beta}} & \\ \qquad + \dfrac{\sqrt{-\alpha y}\,[\varphi_{1n} - z_n^+(\beta)]B_n(\alpha,-y)}{\Delta_n(\alpha,\beta)\sqrt{\alpha\beta}} + z_n^-(y), & y < 0, \end{cases} \tag{2.31}$$

where the functions $z_n^+(y)$ and $z_n^-(y)$ are defined, respectively, by the equalities

$$\begin{aligned} z_n^+(y) = {} & \frac{2\pi n}{q^2}\sqrt{y}\,[b_n K_{1/(2q)}(p_n y^q) - a_n I_{1/(2q)}(p_n y^q)] \\ & \times \biggl\{ y^{2q}\biggl[\biggl(1 + \frac{1}{4(p_n q y^q)^2}\biggr) I_{1/(2q)}(p_n y^q) K_{1/(2q)}(p_n y^q) \\ & \qquad + \frac{1}{4}(p_n q)^2 y^{2q-2}(I_{1/(2q)-1}(p_n y^q) \\ & \qquad + I_{1/(2q)+1}(p_n y^q))[K_{1/(2q)+1}(p_n y^q) + K_{1/(2q)-1}(p_n y^q)]\biggr] - (2p_n^2)^{-1}\biggr\} \\ & - \frac{\pi^2 n}{q^2 \sin(\pi/(2q))}\, b_n I_{1/(2q)}(p_n y^q) y^{2q+1/2} \\ & \times \biggl[\frac{\pi}{2\sin(\pi/(2q))}(I^2_{-1/(2q)}(p_n y^q) - I_{-1/(2q)-1}(p_n y^q) I_{1-1/(2q)}(p_n y^q)) \\ & \qquad - 2K_{1/(2q)}(p_n y^q) I_{1/(2q)}(p_n y^q) - 2I_{1+1/(2q)}(p_n y^q) K_{1/(2q)-1}(p_n y^q)\biggr] \\ & + \frac{2\pi n}{q^2}\, a_n K_{1/(2q)}(p_n y^q) y^{2q+1/2} \\ & \times [I^2_{1/(2q)}(p_n y^q) - I_{1/(2q)-1}(p_n y^q) I_{1/(2q)+1}(p_n y^q)], \qquad y > 0, \end{aligned} \tag{2.32}$$

$$\begin{aligned} z_n^-(y) = {} & -\frac{\pi^2 n}{2q^2} y^{2q+1/2}[a_n J_{1/(2q)}(p_n y^q) - b_n Y_{1/(2q)}(p_n y^q)] \\ & \times [2J_{1/(2q)}(p_n y^q) Y_{1/(2q)}(p_n y^q) - J_{1/(2q)+1}(p_n y^q) Y_{1/(2q)-1}(p_n y^q) \\ & \qquad - J_{1/(2q)-1}(p_n y^q) Y_{1/(2q)+1}(p_n y^q)] \\ & - \frac{\pi^2 n}{q^2} b_n y^{2q+1/2}[Y^2_{1/(2q)}(p_n y^q) - Y_{1/(2q)+1}(p_n y^q) Y_{1/(2q)-1}(p_n y^q)] J_{1/(2q)}(p_n y^q) \\ & + \frac{\pi^2 n}{q^2} a_n y^{2q+1/2}[J^2_{1/(2q)}(p_n y^q) - J_{1/(2q)+1}(p_n y^q) J_{1/(2q)-1}(p_n y^q)] Y_{1/(2q)}(p_n y^q) \\ & + \frac{\pi n}{p_n^2 q^3}\biggl[a_n + 2\cot\frac{\pi}{2q}\, b_n\biggr] J_{1/(2q)}(p_n y^q)\sqrt{y} - \frac{\pi n}{p_n^2 q^3} b_n Y_{1/(2q)}(p_n y^q)\sqrt{y}, \qquad y < 0. \end{aligned} \tag{2.33}$$

Note that more detailed calculations in the derivation of formula (2.31) were given in [7, Secs. 1.1, 2.1].

Under condition (2.17), formulas (2.20), (2.26), (2.31) imply the uniqueness of the solution of problem (1.2)–(1.5), (1.8), because if $\varphi(x) \equiv 0$, $\psi(x) \equiv 0$ on $[0,1]$, then $w_n(y) \equiv 0$, $w_0(y) \equiv 0$, $z_n(y) \equiv 0$ for $n = 0, 1, 2, \dots$ on $[-\alpha, \beta]$. Then, from (2.6)–(2.8), we obtain

$$4\int_0^1 u(x,y)(1-x)\cos 2\pi n x\,dx = 0, \qquad 2\int_0^1 (1-x)u(x,y)\,dx = 0,$$

$$4\int_0^1 u(x,y)\sin 2\pi nx\,dx = 0, \qquad n = 1,2,\dots .$$

Hence, in view of the completeness of the system of root functions (2.4) in the space $L_2[0,1]$, it follows that the function $u(x,y) = 0$ almost everywhere on $[0,1]$ for any $y \in [-\alpha,\beta]$. By virtue of (1.2), the function $u(x,y)$ is continuous on $\overline{D}$; therefore, $u(x,y) \equiv 0$ on $\overline{D}$.

If, for some α, β, and $n = l \in \mathbb{N}$, condition (2.17) fails, i.e., $\Delta_l(\alpha,\beta) = 0$, then the homogeneous problem (1.2)–(1.5), (1.8) (where $\varphi(x) = \psi(x) \equiv 0$) has the nontrivial solution

$$u_l(x,y) = \begin{cases} \dfrac{\sqrt{y}\,\Delta_l(\alpha,y)}{J_{1/(2q)}(p_l\alpha^q)}\,X_l(x), & y > 0, \\[2ex] \dfrac{\sqrt{-y}\,\Delta_l(-y,\beta)}{I_{1/(2q)}(p_l\beta^q)}\,X_l(x), & y < 0, \end{cases} \tag{2.34}$$

where

$$X_l(x) = A_1(1-x)\cos 2\pi lx + A_2(1-x) + A_3\sin 2\pi lx$$

and A_i, $i = 1,\dots,3$ are arbitrary constants.

It is easy to verify that the function (2.34) constructed above belongs to the class $C^2(\overline{D})$ and is a solution of Eq. (1.1) everywhere in D.

Naturally, we ask whether there exist zeros of the expression $\Delta_n(\alpha,\beta)$ for a fixed n with respect to α. Let us express $\Delta_n(\alpha,\beta)$ as

$$\Delta_n(\alpha,\beta) = I_{1/(2q)}(p_n\beta^q)\,\delta_n(\alpha,\beta), \tag{2.35}$$

where

$$\delta_n(\alpha,\beta) = J_{1/(2q)}(p_n\alpha^q)\frac{K_{1/(2q)}(p_n\beta^q)}{I_{1/(2q)}(p_n\beta^q)} + \overline{Y}_{1/(2q)}(p_n\alpha^q). \tag{2.36}$$

The existence of zeros of $\delta_n(\alpha,\beta)$ with respect to α follows from the fact that $J_{1/(2q)}(p_k z)$ and $\overline{Y}_{1/(2q)}(p_k z)$, $z = \alpha^q$, are linearly independent solutions of Bessel's equation

$$y''(z) + \frac{1}{z}y'(z) + \left[p_k^2 - \left(\frac{1}{2qz}\right)^2\right]y(z) = 0. \tag{2.37}$$

Since the functions $J_{1/(2q)}(p_k z)$ and $\delta_n(z,\beta)$ are linearly independent solutions of Eq. (2.37), it follows from the general theory of linear differential equations [8, p. 135 (Russian transl.)] that the zeros of two linearly independent solutions of Bessel's equation strictly alternate, i.e., the interval between any two successive zeros of any of these solutions contains exactly one zero of the other solution. The function $J_{1/(2q)}(p_k z)$ has a countable set of positive zeros. Then the function $\delta_n(z,\beta)$ also has a countable set of positive zeros with respect to $z = \alpha^q$.

Thus, we have proved the following statement.

Theorem 1. *If there exists a solution of problem* (1.2)–(1.5), (1.8)*, then it is unique if and only if* $\Delta_n(\alpha,\beta) \neq 0$ *for all* $n \in \mathbb{N}$.

3. EXISTENCE OF A SOLUTION

Since α and β are arbitrary positive numbers, for sufficiently large n, the expression $\Delta_n(\alpha,\beta)$ appearing in the denominators of formulas (2.20), (2.31) can become sufficiently small due to the existence of a countable set of zeros of $\delta_n(\alpha,\beta)$ with respect to α^q. Therefore, the problem of small denominators arises just as in the study of the Dirichlet problem for Eq. (1.1), [9].

As is well known [10, p. 99 (Russian transl.)], $K_\nu(z) = O(z^{-1/2}e^{-z})$ and $I_\nu(z) = O(z^{-1/2}e^{z})$ as $z \to +\infty$; hence the quantity

$$J_{1/(2q)}(p_n\alpha^q)\frac{K_{1/(2q)}(p_n\beta^q)}{I_{1/(2q)}(p_n\beta^q)}$$

is an infinitesimal of higher order than $\overline{Y}_{1/(2q)}(p_n\alpha^q)$ for large n. Therefore, it suffices to consider the expression

$$\widetilde{\delta}_n(\alpha)=\frac{2\sin(\pi/(2q))}{\pi}\overline{Y}_{1/(2q)}(2\pi n\alpha_q)=J_{1/(2q)}(2\pi n\alpha_q)+J_{-1/(2q)}(2\pi n\alpha_q),\qquad \alpha_q=\frac{\alpha^q}{q},$$

which also has a countable set of positive zeros with respect to α.

Using the asymptotic formula for the function

$$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\cos\biggl(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\biggr)+O(z^{-3/2}),\qquad z\to\infty,$$

for $n>n_0$, where n_0 is a sufficiently large natural number, we obtain

$$\begin{aligned}\sqrt{n}\,\widetilde{\delta}_n(\alpha)&=\frac{1}{\pi\sqrt{\alpha_q}}\biggl[\cos\biggl(2\pi n\alpha_q-\frac{\pi}{4q}-\frac{\pi}{4}\biggr)+\cos\biggl(2\pi n\alpha_q+\frac{\pi}{4q}-\frac{\pi}{4}\biggr)\biggr]\\&=\frac{2}{\pi}\alpha_q^{-1/2}\cos\biggl(\frac{\pi}{4q}\biggr)\cos\biggl(2\pi n\alpha_q-\frac{\pi}{4}\biggr)=A\cos\biggl(2\pi n\alpha_q-\frac{\pi}{4}\biggr).\end{aligned}$$

It is easy to see that if, for example, $\alpha_q=\mu\in\mathbb{N}$, i.e., $\alpha=(\mu q)^{1/q}$, then, for $n>n_0$,

$$|\sqrt{n}\widetilde{\delta}_n(\alpha)|=A\biggl|\cos\biggl(2\pi n\mu-\frac{\pi}{4}\biggr)\biggr|=A\biggl|\cos\frac{\pi}{4}\biggr|\ge\widetilde{C}_1>0;$$

here and further, the $\widetilde{C}_i$ are positive constants.

Now let $\alpha_q=k/m$ be a rational number, where k and m, and m and 4 are coprime numbers. Then

$$|\sqrt{n}\,\widetilde{\delta}_{k/m}(\alpha)|=A\biggl|\cos\biggl(2\pi\frac{kn}{m}-\frac{\pi}{4}\biggr)\biggr|. \tag{3.1}$$

Let us divide $2kn$ by m with remainder $2kn=sm+r$, $s,r\in\mathbb{N}\cup\{0\}$, $0\le r<m$. If $r=0$, then this case can be reduced to the one examined above. Let $r>0$, and it is obvious that $1\le r\le m-1$ and $m\ge 2$. Then expression (3.1) takes the form

$$\Bigl|\sqrt{n}\,\widetilde{\delta}_{k/m}(\alpha)\Bigr|=A\biggl|\cos\pi\biggl(\frac{r}{m}-\frac{1}{4}\biggr)\biggr|\ge\widetilde{C}_2>0.$$

Therefore, the following statement is valid.

Lemma 1. *If one of the following conditions holds*:

1) *α_q is any natural number*;

2) *α_q is any rational number, i.e., $\alpha_q=k/m\notin\mathbb{N}$, where $k,m\in\mathbb{N}$, k and m, and m and 4 are coprime numbers,*

then there exist positive constants n_0 ($n_0\in\mathbb{N}$) and C_0 depending on α and q such that, for any fixed $\beta>0$ and all $n>n_0$, the following estimate holds:

$$|\sqrt{n}\delta_n(\alpha)|\ge C_0>0. \tag{3.2}$$

If $\delta_n(\alpha,\beta)\ne 0$ for $n=1,\dots,n_0$ and estimate (3.2) holds, then the solution of problem (1.2)–(1.5), (1.8) can be expressed as the sum of the series

$$u(x,y)=2(1-x)w_0(y)+4\sum_{n=1}^{\infty}w_n(y)(1-x)\cos 2\pi nx+4\sum_{n=1}^{\infty}z_n(y)\sin 2\pi nx, \tag{3.3}$$

where the functions $w_0(y)$, $w_n(y)$, $z_n(y)$ are given, respectively, by formulas (2.26), (2.20), (2.31).

Let us now show that, under certain conditions with respect to the functions $\varphi(x)$ and $\psi(x)$, the series (3.3) and its first derivatives in the closed domain $\overline{D}$, and the second derivatives with respect to x and y, respectively, in the closed domains $\overline{D}_+$ and $\overline{D}_-$ converge uniformly.

Consider the following ratios:

$$P_n(y) = \frac{\sqrt{y}\Delta_n(\alpha, y)}{\Delta_n(\alpha,\beta)}, \qquad Q_n(y) = \frac{\sqrt{y}A_n(y,\beta)}{\Delta_n(\alpha,\beta)}, \qquad y \in [0,\beta], \tag{3.4}$$

$$M_n(y) = \frac{\sqrt{-y}B_n(\alpha, -y)}{\Delta_n(\alpha,\beta)}, \quad N_n(y) = \frac{\sqrt{-y}\Delta_n(-y,\beta)}{\Delta_n(\alpha,\beta)}, \qquad y \in [-\alpha, 0]. \tag{3.5}$$

Lemma 2. *Let condition* (3.2) *hold for all $n > n_0$. Then, for such n, the following estimates hold*:

$$\begin{aligned}
&|P_n(y)| \le C_1, &&|P_n'(y)| \le C_2 n, &&|P_n''(y)| \le C_3 n^2, \\
&|Q_n(y)| \le C_4 n^{1-\lambda}, &&|Q_n'(y)| \le C_5 n^{\lambda}, &&|Q_n''(y)| \le C_6 n^{3-\lambda}, \qquad y \in [0,\beta], \\
&|M_n(y)| \le C_7 n^{\lambda} e^{-nd}, &&|M_n'(y)| \le C_8 n e^{-nd}, &&|M_n''(y)| \le C_9 n^{2+\lambda} e^{-nd}, \\
&|N_n(y)| \le C_{10} n^{\lambda}, &&|N_n'(y)| \le C_{11} n^{1/2+\lambda}, &&|N_n''(y)| \le C_{12} n^{2+\lambda}, \qquad y \in [-\alpha, 0],
\end{aligned}$$

where $\lambda = 1/2 + 1/2q$, $d = 2\pi\beta_q$, $\beta_q = \beta^q/q$, and the C_i are positive constants here and elsewhere.

Proof. The validity of these estimates of the functions (3.4), (3.5) and of their derivatives are established using the asymptotic formulas describing the behavior of Bessel functions [10, pp. 98–99 (Russian transl.)] as $z \to 0$ and $z \to +\infty$. Here, as an example, we consider the functions $P_n(y)$ and $M_n(y)$. In view of (2.35) and (3.2), using (3.4) and (2.21) for $0 \le y \le \beta$ and large n, we obtain

$$\begin{aligned}
|P_n(y)| &\le \left|\frac{\sqrt{n}J_{1/(2q)}(p_n\alpha^q)\sqrt{y}K_{1/(2q)}(p_n y^q)}{I_{1/(2q)}(p_n\beta^q)\widetilde{C}_0}\right| + \left|\frac{\sqrt{ny}I_{1/(2q)}(p_n y^q)\overline{Y}_{1/(2q)}(p_n\alpha^q)}{I_{1/(2q)}(p_n\beta^q)\widetilde{C}_0}\right| \\
&\le \widetilde{C}_3|\sqrt{n}\overline{Y}_{1/(2q)}(p_n\alpha^q)| \le C_1,
\end{aligned} \tag{3.6}$$

because for $y \in [0,\beta]$,

$$\left|\frac{y^{1/2}I_{1/(2q)}(p_n y^q)}{I_{1/(2q)}(p_n\beta^q)}\right| \le \widetilde{C}_4, \qquad |y^{1/2}K_{1/(2q)}(p_n y^q)| \le \widetilde{C}_5 n^{-1/(2q)}.$$

Using the formulas [10, p. 90 (Russian transl.)]

$$\frac{d}{dz}[z^\nu I_\nu(z)] = z^\nu I_{\nu-1}(z), \qquad \frac{d}{dz}[z^\nu K_\nu(z)] = -z^\nu K_{\nu-1}(z),$$

we calculate the derivative

$$P_n'(y) = \frac{p_n q y^{q-1/2}}{\Delta_n(\alpha,\beta)}\big[I_{1/(2q)-1}(p_n y^q)\overline{Y}_{1/(2q)}(p_n\alpha^q) - J_{1/(2q)}(p_n\alpha^q)K_{1/(2q)-1}(p_n y^q)\big]. \tag{3.7}$$

Since, for $y \in [0,\beta]$,

$$|y^{q-1/2}I_{1/(2q)-1}(p_n y^q)| \le \widetilde{C}_6 n^{-(1-1/(2q))}, \qquad |y^{q-1/2}K_{1/(2q)-1}(p_n y^q)| \le \widetilde{C}_7 n^{-(1-1/(2q))},$$

using formula (3.7) and taking into account (2.35) and (3.2), we obtain

$$\begin{aligned}
|P_n'(y)| \le &\frac{|p_n q\sqrt{n}y^{q-1/2}I_{1/(2q)-1}(p_n y^q)\overline{Y}_{1/(2q)}(p_n\alpha^q)|}{I_{1/(2q)}(p_n\beta^q)\widetilde{C}_0} \\
&+ \frac{|p_n q\sqrt{n}y^{q-1/2}J_{1/(2q)}(p_n\alpha^q)K_{1/(2q)-1}(p_n y^q)|}{I_{1/(2q)}(p_n\beta^q)\widetilde{C}_0} \\
&\le \widetilde{C}_6 p_n\sqrt{n}|\overline{Y}_{1/(2q)}(p_n\alpha^q)| \le C_2 n.
\end{aligned}$$

It is easy to verify that the second derivative $P_n''(y)$ is $P_n''(y) = (p_n q)^2 y^{2q-2} P_n(y)$. In view of estimate (3.6), it follows from this equality that $|P_n''(y)| \le C_3 n^2$.

Similarly, using (3.5), (2.22), and (2.35), (3.2) we estimate the function $M_n(y)$:

$$|M_n(y)| \le \frac{\sqrt{n}|J_{1/(2q)}(p_n\alpha^q)\sqrt{-y}\overline{Y}_{1/(2q)}(p_n(-y)^q)|}{I_{1/(2q)}(p_n\beta^q)\widetilde{C}_0} + \frac{\sqrt{n}|\sqrt{-y}J_{1/(2q)}(p_n(-y)^q)\overline{Y}_{1/(2q)}(p_n\alpha^q)|}{I_{1/(2q)}(p_n\beta^q)\widetilde{C}_0}. \tag{3.8}$$

For any $y \in [-\alpha, 0]$, using the estimate

$$|\sqrt{-y}\overline{Y}_{1/(2q)}(p_n(-y)^q)| \le \widetilde{C}_8 n^{1/(2q)}, \qquad |\sqrt{-y}J_{1/(2q)}(p_n(-y)^q)| \le \widetilde{C}_9 n^{1/(2q)},$$

from inequality (3.8), we obtain

$$|M_n(y)| \le C_7 n^{\lambda} e^{-nd}. \tag{3.9}$$

Using the formulas [10, p. 20 (Russian transl.)]

$$\frac{d}{dz}(z^{\nu} J_{\nu}(z)) = z^{\nu} J_{\nu-1}(z), \qquad \frac{d}{dz}(z^{\nu} J_{-\nu}(z)) = -z^{\nu} J_{1-\nu}(z),$$

we calculate the derivative

$$M_n'(y) = \frac{\pi(-y)^{q-1/2} p_n q}{2\sin(\pi/(2q))\Delta_n(\alpha,\beta)} \times [J_{1/(2q)}(p_n\alpha^q)J_{1-1/(2q)}(p_n(-y)^q) + J_{-1/(2q)}(p_n\alpha^q)J_{1/(2q)-1}(p_n(-y)^q)].$$

Hence, in view of (2.35) and (3.2), we obtain $|M_n'(y)| \le C_8 n^{\lambda} e^{-nd}$.

The second derivative of the function $M_n(y)$ is of the form $M_n''(y) = (p_n q)^2 y^{2q-2} M_n(y)$. Then estimate (3.9) implies the inequality $|M_n''(y)| \le C_9 n^{2+\lambda} e^{-nd}$. □

Lemma 3. *The following assertions hold*:

1) *the functions $z_n^+(y)$ and $z_n^-(y)$ are solutions of the inhomogeneous equation* (2.27) *for $y > 0$ and $y < 0$, respectively, and satisfy the zero initial conditions*

$$z_n^+(0) = 0, \quad {z_n^+}'(0) = 0, \quad z_n^-(0) = 0, \quad {z_n^-}'(0) = 0;$$

2) *for $0 < \varepsilon \le y \le \beta$ and large n, the following estimates hold*:

$$|z_n^+(y)| \le \widetilde{C}_3 n^{-3/2}(|a_n|e^{-2\pi n\beta_q} + |b_n|e^{-2\pi n\varepsilon_q}),$$
$$|{z_n^+}''(y)| \le \widetilde{C}_4 n^{1/2}(|a_n|e^{-2\pi n\beta_q} + |b_n|e^{-2\pi n\varepsilon_q});$$

3) *for $-\alpha \le y \le -\varepsilon < 0$ and large n, the following estimates hold*:

$$|z_n^-(y)| \le \widetilde{C}_5(|c_n| + |d_n|)n^{-1/2}, \qquad |{z_n^-}''(y)| \le \widetilde{C}_6 n^{3/2}(|c_n| + |d_n|),$$

where $\varepsilon > 0$ is a sufficiently small number, $\beta_q = \beta^q/q$, $\varepsilon_q = \varepsilon^q/q$.

The proof is based on (2.32) and (2.33), and the asymptotic formulas describing the behavior of Bessel functions as $z \to 0$ and $z \to +\infty$ are used in the proof. A detailed proof is given in ([7, Sec. 1.1, Sec. 2.1]). □

Lemma 4. *Let condition* (3.2) *hold for all* $n > n_0$. *Then, for such* n, *the following estimates hold:*

$$|w_n(y)| \le C_{13}(|\varphi_n| + n^\lambda |\psi_n|), \qquad |w_n'(y)| \le C_{14}(|\varphi_n| n + n^{1/2+\lambda}|\psi_n|),$$
$$|w_n''(y)| \le C_{15}(n^2|\varphi_n| + n^{2+\lambda}|\psi_n|),$$
$$|z_n(y)| \le C_{16}(|\varphi_{1n}| + n^\lambda |\psi_{1n}|), \qquad |z_n'(y)| \le C_{17}(n|\varphi_{1n}| + n^{1/2+\lambda}|\psi_{1n}|),$$
$$|z_n''(y)| \le C_{18}(n^2|\varphi_{1n}| + n^{2+\lambda}|\psi_{1n}|).$$

The validity of these estimates follows from Lemmas 2 and 3.

Lemma 5. *If* $\varphi(x) \in C^3[0,1]$, $\psi(x) \in C^{3+\gamma}[0,1]$, $\gamma > \lambda$,

$$\varphi'(0) = \varphi'(1), \quad \psi'(0) = \psi'(1), \quad \varphi(1) = 0, \quad \psi(1) = 0, \quad \varphi''(1) = 0, \quad \psi''(1) = 0,$$

then the following estimates hold:

$$|\varphi_n| \le \frac{C_{19}|g_n|}{n^3}, \quad |\varphi_{1n}| \le C_{20}\left(\frac{|g_{1n}|}{n^3} + \frac{|g_n|}{n^4}\right), \quad |\psi_n| \le \frac{C_{21}}{n^{3+\gamma}}, \quad |\psi_{1n}| \le \frac{C_{22}}{n^{3+\gamma}}, \tag{3.10}$$

where

$$g_n = \int_0^1 \varphi'''(x)\sin(2\pi n x)\,dx, \qquad g_{1n} = \int_0^1 \varphi'''(x) x \cos(2\pi n x)\,dx,$$
$$\sum_{n=1}^{+\infty} g_n^2 < +\infty, \qquad \sum_{n=1}^{+\infty} g_{1n}^2 < +\infty. \tag{3.11}$$

Proof. Using the assumptions of the lemma, we integrate by parts three times in the integrals (2.12), (2.28), and (2.29). Further, applying the theorem on the rate of decrease of the coefficients of the Fourier series of a function satisfying, on [0,1], Hölder's condition with exponent $\gamma \in (0,1]$, we obtain estimates (3.10). The convergence of the series (3.11) can be justified in a similar way to [2]. □

Thus, in view of Lemmas 4 and 5, the series (3.3) for any (x, y) from $\overline{D}$ is majorized by the convergent series

$$C_{23}\sum_{n=1}^{+\infty}\frac{1}{n^3}\left(|g_n| + |g_{1n}| + \frac{1}{n^{\gamma-\lambda}}\right);$$

therefore, in view of the Weierstrass criterion, the series (3.3) converges absolutely and uniformly in the closed domain $\overline{D}$. The series containing the first derivatives on $\overline{D}$ and the second derivatives are majorized, respectively, on the closed domains $\overline{D}_+$ and $\overline{D}_-$ by the convergent numerical series

$$C_{24}\sum_{n=1}^{+\infty}\frac{1}{n}\left(|g_n| + |g_{1n}| + \frac{1}{n^{\gamma-\lambda}}\right).$$

Therefore, the sum $u(x, y)$ of the series (3.3) belongs to the class (1.2) and satisfies Eq. (1.1) on the set $D_+ \cup D_-$.

If, for the values of α_q indicated in Lemma 1 and $l = k_1, k_2, \dots, k_m$, where

$$1 \le k_1 < k_2 < \cdots < k_m \le n_0,$$

k_i, $i = 1, \dots, m$, and m are given natural numbers, the equality $\Delta_l(\alpha, \beta) = 0$ holds, then, to solve problem (1.2)–(1.5), (1.8), it is necessary and sufficient that the following conditions hold:

$$\varphi_l\sqrt{\alpha}J_{1/(2q)}(p_l\alpha^q) + \psi_l\sqrt{\beta}I_{1/(2q)}(p_l\beta^q) = 0, \tag{3.12}$$

$$\varphi_{1l}\sqrt{\alpha}J_{1/(2q)}(p_l\alpha^q) + \psi_{1l}\sqrt{\beta}I_{1/(2q)}(p_l\beta^q) = 0, \qquad l = k_1, k_2, \dots, k_m. \tag{3.13}$$

In that case, the solution of the problem is obtained as the sum of the series

$$u(x, y) = 2(1 - x)w_0(y)$$

$$+4\Big(\sum_{n=1}^{k_1-1}+\sum_{n=k_1+1}^{k_2-1}+\dots+\sum_{n=k_m+1}^{+\infty}\Big)[z_n(y)\sin 2\pi nx+(1-x)w_n(y)\cos 2\pi nx]$$
$$+\sum_l u_l(x,y), \tag{3.14}$$

where, in the last sum, l assumes the values $k_1, k_2, \dots, k_m$ and the functions $u_l(x,y)$ are given by the formula

$$u_l(x,y)=\begin{cases}\Big(\dfrac{\varphi_l\sqrt{y}\,I_{1/(2q)}(p_l y^q)}{\sqrt{\beta}\,I_{1/(2q)}(p_l\beta^q)}+\dfrac{\varphi_{1l}\sqrt{y}\,I_{1/(2q)}(p_l y^q)}{\sqrt{\beta}\,I_{1/(2q)}(p_l\beta^q)} \\ \qquad +C_l\dfrac{\sqrt{y}\,\Delta_l(\alpha,y)}{J_{1/(2q)}(p_l\alpha^q)}\Big)X_l(x), & y>0,\\ \Big(\dfrac{\psi_l\sqrt{-y}\,J_{1/(2q)}(p_l(-y)^q)}{\sqrt{\alpha}\,J_{1/(2q)}(p_l\alpha^q)}+\dfrac{\psi_{1l}\sqrt{-y}\,J_{1/(2q)}(p_l(-y)^q)}{\sqrt{\alpha}\,J_{1/(2q)}(p_l\alpha^q)} \\ \qquad +\dfrac{\sqrt{-y}\,\Delta_l(-y,\beta)}{I_{1/(2q)}(p_l\beta^q)}\Big)X_l(x), & y<0,\end{cases}$$

where C_l is an arbitrary constant; the finite sums in (3.14) must be assumed zeros, provided that the upper limit is smaller than lower one.

Thus, we have proved the following statement.

Theorem 2. *Let the functions $\varphi(x)$ and $\psi(x)$ satisfy the assumptions of Lemma* 5*, and let estimate* (3.2) *hold for $n>n_0$. In that case, if $\delta_n(\alpha,\beta)\ne 0$ for all $n=1,\dots,n_0$, then there exists a unique solution of problem* (1.2)–(1.5)*,* (1.8) *and this solution is given by the series* (3.3)*; if $\delta_n(\alpha,\beta)=0$ for some $n=k_1,k_2,\dots,k_m\le n_0$, then problem* (1.2)–(1.5)*,* (1.8) *is solvable only if conditions* (3.12)*,* (3.13) *hold and then the solution is given by the series* (3.14)*.*

In view of Theorem 2, the function (3.3) constructed above satisfies all the conditions of problem (1.2)–(1.5), (1.8) whenever the functions $\varphi(x)$, $\psi(x)$ satisfy the assumptions of Lemma 5, but, at the same time, the function (3.3) does not satisfy equality (1.6), i.e., the function (3.3) is not a solution of problem (1.2)–(1.6). Using Theorem 2, it is easy to prove the following statement.

Theorem 3. *Let all the assumptions of Theorem* 2 *and conditions* (1.7) *hold. In that case, if $\delta_n(\alpha,\beta)\ne 0$ for all $n=1,\dots,n_0$, then there exists a unique solution of problem* (1.2)–(1.6) *and it is given by the series* (3.3)*, where $w_0(y)=A$; if $\delta_n(\alpha,\beta)=0$ for some $n=k_1,k_2,\dots,k_m\le n_0$, then problem* (1.2)–(1.6) *is solvable only if conditions* (3.12)*,* (3.13) *hold and then the solution is given by the series* (3.14) *for $\omega_0(y)=A$.*

In view of formulas (2.20), (2.26), (2.31), it is easy to verify the validity of the equalities

$$w_n''(0+)=w_n''(0-),\qquad n=0,1,2,\dots,$$
$$z_n''(0+)=z_n''(0-),\qquad n=1,2,\dots.$$

Hence we obtain the following qualitative result concerning the deletion of the singularity on the change-of-type line of Eq. (1.1).

Corollary. *The solution $u(x,y)$* (*constructed above*) *of the problem belongs to the class $C^2(\overline{D})$, and the function $u(x,y)$ is a solution of Eq.* (1.1) *everywhere in D.*

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