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Finite Groups with Large Irreducible Character

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Abstract—In the general case, the order of a finite nonidentity group G is substantially larger than the squared degree of every irreducible character Θ of G, i.e., $\Theta(1)^2 < |G|$. In the present paper, we study finite groups with an irreducible character Θ such that

$$|G| \leq 2\Theta(1)^2.$$

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1. INTRODUCTION

Let G be a finite nonidentity group having an irreducible representation over the field of complex numbers with character Θ. According to the orthogonality relations for irreducible characters (Chap. 4 in [1]), the sum of squared degrees of these characters is equal to the order of the group. In particular,

$$\Theta(1)^2 < |G|.$$

In the general case, the order of the group is significantly greater than the squared degree of any irreducible character of the group. However, according to [2], every simple non-Abelian group G has an irreducible character of degree exceeding $|G|^{1/3}$. Snyder [3] studied groups with irreducible character of degree d for which $|G| = d(d+e)$. He proved that, in this case, the order of the group G is bounded by a function of e for $e > 1$. In the case of $e = 1$, G is a Frobenius group. The objective of the present paper is the investigation of finite groups with an irreducible character Θ such that

$$|G| \leq 2\Theta(1)^2.$$

Definition 1. We refer to a finite group G of order greater than two which has an irreducible character Θ such that $2\Theta(1)^2 \geq |G|$ as a $LC(\Theta)$-*group*.

The following theorem shows that every irreducible character of an $LC(\Theta)$-group G always enters the decomposition of the squared character Θ under the assumption that G is not a 2-group.

Theorem 1. *Let G be a $LC(\Theta)$-group. If the order of G is not a power of 2, then every irreducible character of G is a constituent of the character Θ^2.*

Note that the extraspecial 2-group of order 2^{2n+1} has an irreducible character of degree 2^n. The equation $|G| = 2\Theta(1)^2$ for an irreducible character Θ of a group G need not hold for 2-groups only. The direct product of the alternating group A_4 and the symmetric group S_3 has an irreducible character Θ of degree 6, and thus $|G| = 2 \cdot 6^2$.

In the general case, the problem of describing the structure of a group having an irreducible character Θ such that $|G| \leq c\Theta(1)^2$ for a small constant c seems to be rather complicated. The five

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Mathieu groups, the Thompson sporadic group, and the Janko group of order 604800 have a character of this kind for $c < 3.1$.

It is natural to study finite $LC(\Theta)$-groups for which the degree of the character Θ is subjected to some additional restrictions. Below we prove some results concerning $LC(\Theta)$-groups for which $\Theta(1)$ is a degree of a prime p. Recall that a group is said to be *p-nilpotent* if all elements of orders coprime to p form a subgroup.

Theorem 2. *Let G be an $LC(\Theta)$-group. If, for some prime p, the Sylow p-subgroup of G is Abelian and $\Theta(1) = p^m$ for some positive integer m, then G is a p-nilpotent group, $O_p(G) = 1$, and the Sylow p-subgroup of G is of order p^m.*

The following theorem gives a complete description of $LC(\Theta)$-groups with $\Theta(1) = p^m$ and an Abelian Sylow p-subgroup.

Theorem 3. *Let G be an $LC(\Theta)$-group with Abelian Sylow p-subgroup and an irreducible character Θ of degree p^m, where p is a prime and m is a positive integer. In this case, either p is a Mersenne prime and the group G is a direct product of m Frobenius groups of order $p(p+1)$ or $p = 2$ and G is a direct product of groups each of which is a Frobenius group of order $q_i 2^{m_i}$ (where $q_i = 2^{m_i} + 1$ are Fermat primes) or is a Frobenius group of order $3^2 2^3$ (in this case, $m_i = 3$), where $\sum_i m_i = m$.*

All groups are assumed to be finite. The symbol p is always used to denote a prime. For the necessary information concerning ordinary representations of finite groups and standard notation, see [1]. For every element a of a group G, denote by h_a the number of elements of G conjugate to a. The inner product of characters χ and ψ of a group G is denoted by $\langle \chi, \psi \rangle$. In particular, if these characters are irreducible, then $\langle \chi, \psi \rangle = \delta_{\chi,\psi}$. Denote by $\mathrm{Irr}(G)$ the set of irreducible characters of a group G.

2. PROOF OF THEOREM 1

Let G and Θ satisfy the conditions of Theorem 1. By the equation

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|,$$

the character Θ is a unique irreducible character of G of maximal degree. Therefore, all characters conjugate to Θ, under the action of the Galois group of the extension $\mathbb{Q}(\Theta)$ of the field of rational numbers $\mathbb{Q}$ ($\mathbb{Q}(\Theta)$ is obtained by adjoining the elements $\Theta(g)$ for all $g \in G$ to $\mathbb{Q}$), coincide with Θ. Hence all values $\Theta(g)$, $g \in G$, are rational integers. In particular, $\overline{\Theta(g)} = \Theta(g)$ for every $g \in G$. In any case, Θ is a faithful character.

Let $\Phi = \Theta^2$, and let the order of G do not exceed $2\Phi(1)$. Note that, since the values of Θ are real, it follows that $\Phi(g) \geq 0$ for every $g \in G$. Assume that an irreducible character χ has the property $\langle \Phi, \chi \rangle = 0$, i.e., the multiplicity of χ in Φ is equal to 0. Then

$$\sum_{t \in G^\#} \Phi(t)\chi(t) + \Phi(1)\chi(1) = 0.$$

Since Θ is an irreducible character of G, it follows that

$$|G|\langle \Theta, \Theta \rangle = \sum_{t \in G^\#} \Phi(t) + \Phi(1) = |G| \leq 2\Phi(1).$$

Therefore,

$$\left| \sum_{t \in G^\#} \Phi(t) \right| = \sum_{t \in G^\#} \Phi(t) \leq \Phi(1).$$

Further, since all values of $\Phi(t)$ are nonnegative, we obtain the inequality

$$\Phi(1)\chi(1) \geq \sum_{t\in G^\#} \Phi(t)\chi(1) = \left|\sum_{t\in G^\#} \Phi(t)\right|\chi(1) \geq \left|\sum_{t\in G^\#} \Phi(t)\chi(t)\right| = \Phi(1)\chi(1).$$

Thus, the intermediate inequalities obtained above become equalities, which is possible only if

$$|G| = 2\Phi(1) = 2\Theta(1)^2.$$

Here $\Phi(t) \neq 0$ if and only if $|\chi(t)| = \chi(1)$. In particular, it follows from Theorems 4.1.2 and 4.1.3 in [1] that, in this case, $\lambda = \chi(t)/\chi(1)$ is a root of unity. Note that the theorem is already proved for any group G of odd order, and we may assume below that the order of the group G is even. Moreover, $|G| = 2\Theta(1)^2$.

Assume first that $\chi = \Theta$ and χ is not a constituent of Φ. It follows from what was said above that there is a nonidentity element $t_0 \in G$ for which $\Theta(t_0) \neq 0$. However, $\Theta(t_0) \neq 0$ for $\Theta(t_0) = |\Theta(1)|$ only. Indeed, since Θ is an irreducible character of G, it follows from the first orthogonality relation that

$$|G|\langle\Theta, \Theta\rangle = \sum_{t\in G} |\Theta(t)|^2 = |G| = 2\Theta(1)^2.$$

In this case, there is precisely one value $t_0 \in G^\#$ for which $|\Theta(t_0)| = \Theta(1)$. In particular, $Z(G) = \{1, t_0\}$. Moreover, for every $t \in G \setminus \{1, t_0\}$ we have $\Theta(t) = 0$.

Let $\chi_1, \chi_2, \dots, \chi_k$ be all irreducible characters of G whose kernel contains $Z(G)$. Obviously, these characters exhaust all irreducible characters of the group $G/Z(G)$. Therefore,

$$\sum_{i=1}^{k} \chi(1)^2 = \frac{|G|}{2} = \Theta(1)^2.$$

Thus,

$$|G| = |G/Z(G)| + \Theta(1)^2.$$

Therefore, the number of conjugacy classes of G is greater by one than the number of conjugacy classes of the group $G/Z(G)$. In this case, $Z(G)$ is contained in the kernel of every irreducible character of G which differs from Θ.

If $g \in G \setminus Z(G)$, then $\Theta(g) = 0$. It follows now from the second orthogonality relation (Theorem 4.2.8 of [1]) that every element gt_0 for $g \in G \setminus Z(G)$ is conjugate to the element g. However, an element $x \in G$ of odd prime order m cannot be conjugate to xt_0 which has the order $2m$. Therefore, $|G|$ is a power of two.

Since the character Θ vanishes outside $Z(G)$, for every $\chi \in \mathrm{Irr}(G)$ we have

$$\langle\Phi, \chi\rangle = |G|^{-1}(\chi(1)\Theta^2(1) + \chi(t_0)\Theta^2(t_0)) = \chi(1).$$

However, for $\chi = \Theta$ we obtain $\langle\Phi, \Theta\rangle = 0$.

We can now assume that $\chi \neq \Theta$. Recall that an irreducible character χ can be absent in the decomposition of Φ only if $|G| = 2\Theta(1)^2$. Since $\chi \neq \Theta$, it follows from the orthogonality relation that $\langle\chi, \Theta\rangle = 0$. Therefore,

$$\sum_{g\in G^\#} \chi(g)\Theta(g) = -\chi(1)\Theta(1).$$

Since

$$|G|\langle\Theta, \Theta\rangle = \sum_{t\in G^\#} \Phi(t) + \Phi(1) = |G| = 2\Phi(1),$$

it follows that

$$\sum_{t\in G^\#} \Phi(t) = \Phi(1).$$

Let $t_0 = 1, t_1, t_2, \dots, t_k$ be all elements of G for which $\Phi(t) \neq 0$. In this case, we also have $\chi(t) \neq 0$. Recall that $|\chi(t_i)| = \chi(1)$ in this case and, therefore, $\chi(t_i) = \lambda_i \chi(1)$, where λ_i is a root of unity for every i. Thus,

$$\sum_{i=1}^{k} \Phi(t_i)\chi(t_i) + \Phi(1)\chi(1) = 0, \qquad \sum_{i=1}^{k} \Phi(t_i) = \Phi(1).$$

Write $\lambda_i = \chi(t_i)/\chi(1)$ for $i = 1, \dots, k$. As was proved above, $|\lambda_i| = 1$ for any i. Hence

$$\sum_{i=1}^{k} \Phi(t_i)\lambda_i + \Phi(1) = 0.$$

Denote by $a_i = \operatorname{Re}\lambda_i$ the real part of the number λ_i. It can readily be seen that $|a_i| \leq 1$. Therefore, since $\Phi(t) \geq 0$ for every $t \in G$, we obtain

$$\Phi(1) = \left|\sum_{i=1}^{k} \Phi(t_i)a_i\right| \leq \sum_{i=1}^{k} \Phi(t_i)|a_i| \leq \Phi(1).$$

Hence $\lambda_i = -1$ for any $i \leq k$. It follows from what was said above and from the orthogonality of χ and Θ that

$$0 = \sum_{g \in G^{\#}} \chi(g)\Theta(g) + \chi(1)\Theta(1) = -\chi(1) \sum_{g \in G^{\#}} \Theta(g) + \chi(1)\Theta(1).$$

Therefore,

$$-\sum_{g \in G^{\#}} \Theta(g) + \Theta(1) = 0.$$

However, $\langle \Theta, 1_G \rangle = 0$, and hence

$$\sum_{g \in G^{\#}} \Theta(g) + \Theta(1) = 0.$$

A contradiction. Thus, for every $\chi \in \operatorname{Irr}(G) \setminus \{\Theta\}$ we have $\langle \Phi, \chi \rangle \neq 0$. This completes the proof of the theorem. □

3. PROOF OF THEOREM 2

Let the Sylow p-subgroup of G be Abelian, and let $\Theta(1) = p^m$. We claim that the order of the Sylow p-subgroup P of G is equal to p^m and that $O_p(G) = 1$. If $z \in P$ and $h_z = |G : C_G(z)|$, then $h_z\Theta(z)/\Theta(1)$ is an algebraic integer, and it follows from the fact that h_z and p are coprime and from the Burnside lemma (Lemma 4.3.1 of [1]) that either $\Theta(z) = 0$ or $z \in Z(\Theta)$, i.e., $|\Theta(z)| = \Theta(1)$.

Since $\Theta \in \operatorname{Irr}(G)$, it follows from the first orthogonality relation (Theorem 4.2.2 of [1]) that

$$|Z(\Theta)|\Theta(1)^2 = \sum_{z \in Z(\Theta)} |\Theta(z)|^2 \leq \sum_{t \in G} |\Theta(t)|^2 = |G|\langle \Theta, \Theta \rangle = |G| \leq 2\Theta(1)^2.$$

Therefore, either $p = 2$ and $|G| = |P|$, which fails to hold because P is Abelian, or $\Theta(z) = 0$ for every $z \in P^{\#}$.

Suppose first that $O_p(G) \neq 1$. Since P is Abelian, it follows that the subgroup $O_p(M)$ is contained in the center of the normal subgroup $M = C_G(O_p(G))$, and M contains P, and thus the index $|G : M|$ is not divisible by p. Let $N = O_p(G)$. By Clifford theory (Theorem 6.2 of [4]),

$$\Theta|_M = e\sum_{i=1}^{s} \theta_i,$$

where θ_i stand for the conjugate irreducible characters of M (in particular, $\theta_i(1) = \theta_1(1)$ for every $i \le s$), the number e divides the index of the inertia subgroup $I_G(\theta_1)$ in G, and s divides $|I_G(\theta_1) : M|$. Therefore,

$$p^m = \Theta(1) = es\theta_1(1).$$

Since the numbers e and s divide $|G : M|$, these numbers are coprime to p. Hence $e = s = 1$ and $\Theta|_M \in \mathrm{Irr}(M)$. The latter means that $N \le Z(\Theta|_M)$. In particular, $|\Theta(g)| = \Theta(1)$ for every $g \in O_p(G)$. Therefore,

$$|O_p(G)| = 2 = p \qquad \text{and} \qquad |G| = 2p^{2m},$$

i.e., G is a 2-group. As can be seen from what was said above, this leads to a contradiction. Thus, $O_p(G) = 1$.

By the Brodkey theorem [5], there is a Sylow p-subgroup P_1 of G for which $P \cap P_1 = 1$. Let $N = N_G(P)$ and $N_1 = N_G(P_1)$. Since P and P_1 are the only Sylow p-subgroups in N and N_1, it follows that $|N \cap N_1|$ is not divisible by p. Here N and N_1 are conjugate. Therefore, $G \ne NN_1$. On the other hand,

$$|NN_1| = \frac{|N|^2}{|N \cap N_1|} \ge |N : P||P|^2.$$

If $N \ne P$, then $|G| > 2p^{2m}$, which contradicts the assumption. Thus, $P = N_G(P)$ and, by Burnside's transfer theorem (Theorem 7.4.3 of [1]), the group G is p-nilpotent. This completes the proof of the theorem. □

4. PROOF OF THEOREM 3

Let us preclude the proof of Theorem 3 by two auxiliary assertions.

Lemma 1. *Let G be an $LC(\Theta)$-group having the Abelian Sylow p-subgroup P and an irreducible character Θ of degree p^m. Then $G = F(G) \rtimes P$ is a semidirect product of the Fitting subgroup $F(G)$ of G and P. Here $F(G)$ is a direct product of elementary Abelian groups.*

Proof. By Theorem 2, the group G is p-nilpotent, i.e.,

$$G = M \rtimes P,$$

where $|G| < 2|P|^2$, $|M| < 2|P|$, and $M = O_{p'}(G)$. Let us show first that

$$M = O_{p'}(G) = F^*(G).$$

Recall that $F^*(G)$ is a central product of the subgroup $L(G)$, which is a central product of quasisimple groups normal in $L(G)$, and the Fitting subgroup of G (the case in which one of the mentioned subgroups is trivial is not excluded) (see [6, p. 50–51 (Russian transl.)]). Since G is p-nilpotent and $O_p(G) = 1$ by Theorem 2, it follows that the subgroup $F^*(G)$ is contained in M.

Suppose that $L(G)$ and $F(G)$ are nontrivial. By Proposition 1.27 in [6], we have the inequality $C_G(F^*(G)) \le F^*(G)$. By the Brodkey theorem [5], $P \cap P^g = 1$ for some $g \in F^*(G)$. Representing the group $F^*(G)P$ as the union of double cosets $F^*(G)P = \bigcup_i Pg_iP$, we obtain

$$|F^*(G)|\,|P| \ge |P| + |PgP| = |P| + |P|^2.$$

Thus, $|F^*(G))| \ge |P| + 1$. Since $|G| \le 2|P|^2$, we have

$$2|P|^2 \ge |G| \ge |P|(|P| + 1)|M : F^*(G)|.$$

Hence $M = F^*(G)$.

Let

$$|P/C_P(F(G))| = r \qquad \text{and} \qquad |P/C_P(L(G))| = s.$$

By Proposition 1.27 of [6] and by Theorem 2, the intersection of the subgroups $C_P(F(G))$ and $C_P(L(G))$ is trivial. The length of the orbit of P on $F(G)$ is $|P : C_P(F(G))| = r$, and the length of the orbit of P on

$L(G)$ is equal to $|P : C_P(L(G))| = s$. Since the intersection of the subgroups $C_P(F(G))$ and $C_P(L(G))$ is trivial, we have

$$r \geq |C_P(L(G))| \qquad \text{and} \qquad s \geq |C_P(F(G))|.$$

Let $\overline{L} = L(G)/Z(L(G))$. The group P induces a group of automorphisms on $\overline{L}$. If an element a of P acts identically on $\overline{L}$, then it acts identically on $L = L(G)$ as well. Indeed, if $[L, \langle a\rangle] \leq Z(L)$, then

$$[Z(L), L] = [L, \langle a\rangle, L] = [\langle a\rangle, L, L] = 1.$$

By the three-subgroup lemma [1, Theorem 2.2.3], we have $[L, L, \langle a\rangle] = 1$. Since L is a quasisimple group, it follows that $[L, L] = L$. Therefore, $[L, \langle a\rangle] = 1$. Applying the Brodkey theorem [5], we conclude that the group $P/C_P(L)$ has a faithful orbit on $L/Z(L)$. In particular,

$$|L(G)| > |Z(L(G))|s.$$

Therefore, $|F(G)| > r$ and $|L(G)| > s|Z(G)|$. In particular,

$$|M| = |F(G)L(G)| = \frac{|F(G)||L(G)|}{|F(G) \cap L(G)|} > rs.$$

Since $C_P(L(G)) \times C_P(F(G))$ is a subgroup of P, it follows that

$$rs = |P/C_P(L(G))||P/C_P(F(G))| = \frac{|P|^2}{|C_P(F(G))||C_P(L(G))|} > |P|.$$

Here $|M| < 2|P|$ only if $C_P(F(G)) \times C_P(L(G)) = P$. Thus,

$$G = MP = F(G)C_P(L(G))L(G)C_P(F(G)).$$

Denoting

$$C_P(F(G)) = P_1 \qquad \text{and} \qquad C_P(L(G)) = P_2,$$

we obtain two groups $M_1 = L(G) \leftthreetimes P_1$ and $M_2 = F(G) \leftthreetimes P_2$ with the lengths of the orbits of the p-subgroups P_1 on $L(G)$ and P_2 on $F(G)$ that are equal to $r = |P_1|$ and $s = |P_2|$, respectively. Thus, the problem is reduced to the cases in which $F^*(G) = L(G)$ and those in which $F^*(G) = F(G)$.

We claim that, in the case of $G = L(G) \leftthreetimes P$, where P is an Abelian p-group of order coprime to the order of $L(G) = L$, we have $|L| \geq 2|Z(L)||P|$. As was noted above, P has a faithful orbit on $L/Z(L)$. Therefore, without loss of generality, we may assume that $Z(L) = 1$ and L is a direct product of $k \geq 1$ simple non-Abelian groups. Let us prove our assertion by induction on the number k. If $k = 1$, then it follows from Lemma 7 in [7] that $|\operatorname{Out}(L)|$ is less than $|L|/2$. Our assertion is proved in this case. Let $L = L_1 \times L_2$ be the direct product of two proper P-admissible subgroups of L. By the induction assumption,

$$|L_1| \geq 2|P/C_P(L_1)|, \qquad |L_2| \geq 2|P/C_P(L_2)|.$$

Since $C_P(L_1) \cap C_P(L_2) = 1$, it follows that

$$|L| = |L_1||L_2| \geq 4|P/C_P(L_1)|\,|P/C_P(L_2)| = 4|P|\frac{|P|}{|C_P(L_1)||C_P(L_2)|}\,.$$

Since $C_P(L_1) \cap C_P(L_2) = 1$, we obtain the desired conclusion, namely, $|L| > 2|P|$.

We can now assume that L is a direct product $L_1 \times L_2 \times \cdots \times L_k$ of simple non-Abelian groups conjugate in G. The subgroup P acts transitively on the set $\{L_1, \ldots, L_k\}$ and, therefore, P contains a subgroup P_0, of index $k \geq 2$, which normalizes the subgroup L_1. Here P_0 normalizes also any other subgroup L_i, $i \leq k$. Therefore, P_0 acts faithfully on every L_i. By the induction assumption, $|L_i| > 2|P_0|$ for every $i \leq k$ and

$$|L| > 2^k|P_0| > 2k|P_0| = 2|P|.$$

Thus, we can assume that $F^*(G) = F(G)$. We claim that $F(G)$ is a direct product of elementary Abelian groups. As above, $G = F(G) \leftthreetimes P$. Suppose first that $s = 1$ and $Q_1 = Q \neq 1$ is a Sylow

q-subgroup of the group $F = F(G)$. By the Thompson theorem (Theorem 5.3.11 of [1]), there is a characteristic subgroup Q_0 of Q such that $Q_0/Z(Q_0)$ is an elementary Abelian group and an arbitrary q'-automorphism of Q acts nontrivially on $Q_0/\Phi(Q_0)$. If $F = Q$, then P acts faithfully on $Q_0/\Phi(Q_0)$ and, therefore,

$$|Q_0/\Phi(Q_0)| \geq |P| + 1.$$

However, if $Q_0 \neq Q$, then $|Q| > 2(|P| + 1)$, and thus $|G| > 2|P|^2$, a contradiction. Therefore, Q is an elementary Abelian group.

If F is not a Sylow q-subgroup of G (i.e., $s > 1$), then, using the induction, we obtain the desired conclusion. This completes the proof of the lemma. □

Lemma 2. *Let G be an $LC(\Theta)$-group satisfying the conditions of Theorem 3, let P be the Sylow p-subgroup of G, and let $F = F(G)$ be the Fitting subgroup. Then G is a direct product of Frobenius groups of orders $q_i^{n_i}p^{m_i}$ with a Frobenius complement of order p^{m_i}.*

Proof. By Lemma 1, the subgroup $F(G) = Q_1 \times Q_2 \times \cdots \times Q_s$ is a direct product of q_i-subgroups of G which are elementary Abelian P-admissible groups, where the q_i are primes. By Lemma 1, every Sylow q-subgroup Q of $F(G)$ is elementary Abelian and, therefore, it can be viewed as a linear vector space V of dimension n, where $|Q| = q^n$, and the group $P/C_P(V)$ has a faithful representation on V of degree n. By the Maschke theorem (Theorem 3.3.1 of [1]), for every minimal normal subgroup L of Q, there is a P-invariant (and therefore normal in $F(G)P$) subgroup M of Q for which $L \times M = Q$. Therefore, we may assume that our notation is chosen in such a way that the group $F(G)$ is the direct product of *minimal normal subgroups* $Q_1 \times Q_2 \cdots \times Q_s$ each of which is elementary Abelian of order $q_i^{n_i}$, where q_i is some prime coprime to p.

The group $P_i = P/C_P(Q_i)$ has an irreducible representation of degree n_i over Q_i, which is regarded as a vector space over the field $GF(q_i)$. It is clear that the group $G/C_G(Q_i)$ is isomorphic to a semidirect product $F_i = Q_i \leftthreetimes P_i$ and is a Frobenius group of order $q_i^{n_i}p^{m_i}$, where m_i is a positive integer.

Let T_i be the direct product of all Q_j, $1 \leq j \leq s$, which differ from Q_i. Then

$$C_P(T_i) \cap C_P(Q_i) \subseteq Z(G) = 1.$$

At the same time, the length of the orbit P on Q_i is equal to $|P/C_P(Q_i)|$. Similarly, $|P/C_P(T_i)|$ is the length of the orbit of P on T_i; here $C_P(Q_i) \times C_P(T_i)$ is a subgroup of P. It can readily be seen that $|P/C_P(Q_i) \times P/C_P(T_i)| = |P|$, and hence

$$Q_iC_P(L_i) \times L_iC_P(Q_i) = G.$$

Here $Q_iC_P(L_i) = F_i$ is a Frobenius group. The induction completes now the proof of Lemma 2. □

Proof of Theorem 3. By Lemma 2, the order of the Fitting subgroup of G is equal to $\prod_{i=1}^{s} q_i^{n_i}$, while the order of the Sylow p-subgroup P of this group, which is equal to the degree of the character Θ, is equal to $\prod_{i=1}^{s} p^{m_i} = p^m$.

Since p^{m_i} is the order of the Frobenius complement of the Frobenius group F_i, it follows that p^{m_i} divides $q_i^{n_i} - 1$. Since $q_i^{n_i} - 1 = l_ip^{m_i}$, it follows that

$$|G| \geq |P|^2 \prod_{i=1}^{s} \left(l_i + \frac{1}{p^{m_i}} \right).$$

However, $|G| \leq 2|P|^2$ and, therefore, $l_i = 1$ for all i, i.e., $p^{m_i} = q_i^{n_i} - 1$ for any $i \leq s$.

It follows from the Zsigmondy theorem [8] that, if a and b are positive integers, where $a \geq 2$, $b \geq 3$, and $(a, b) \neq (2, 6)$, then there is a prime r dividing $a^b - 1$ and such that r does not divide $a^i - 1$ for $1 \leq i \leq b - 1$. If q is a prime, $q > 2$, $n \geq 3$, and $q^n - 1 = p^m$, then $r = p = 2$, because $q^n - 1$ is even. Therefore, $n \leq 2$. If $n = 2$, then

$$q^2 - 1 = (q - 1)(q + 1) = 2^m$$

only for $q = 3$ and $m = 3$. For $n = 1$, we have $q - 1 = 2^m$, i.e., $q = 2^m + 1$ is a Fermat prime. If $q = 2$, then $2^n - 1 = p^m$ for a prime p only for $m = 1$ by the same Zsigmondy theorem. That is, p is a Mersenne prime.

Thus, for $p = 2$, the group G is a direct product of Frobenius groups of orders $q_i(q_i - 1)$, where q_i is a Fermat prime, and a Frobenius group of order $3^2 \cdot 2^3$, and, for $p > 2$, G is a direct product of Frobenius groups of order $p(p + 1)$, where p is a Mersenne prime. This completes the proof of the theorem. □

5. EXAMPLES

Example 1. Let G be the direct product of two Frobenius groups of orders 6 and 72. Then G is an $LC(\Theta)$-group with the greatest degree of an irreducible character equal to $16 = 2^4$ and with the Abelian Sylow 2-subgroup.

Example 2. The group $G = A_4 \times A_4$ is an $LC(\Theta)$-group with the greatest degree of an irreducible character equal to $9 = 3^2$ and with the Abelian Sylow 3-subgroup.

Example 3. The group $G = A_4 \times S_3$ is a group with an irreducible character Θ of degree 6, and thus $|G| = 2\Theta(1)^2$; the Sylow p-subgroups of G are Abelian for $p \in \{2, 3\}$; however, G is not a 2-group.

Example 4. The group $G = AGL_2(3)$ is a semidirect product of an elementary Abelian group V of order 9 and the group $GL_2(3)$ acting on V as the group of nondegenerate linear transformations. In this case,

$$|G| = 9 \cdot 48 = 432.$$

Zenkov [9] showed that the group G has an irreducible character Θ of degree 2^4, and thus $|G| < 2\Theta(1)^2$, but the Sylow 2-subgroup of G is non-Abelian.

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