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Coupled Fixed-Point Results for T-Contractions on Cone Metric Spaces with Applications*

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Abstract—The notion of coupled fixed point was introduced in 2006 by Bhaskar and Lakshmikantham. On the other hand, Filipović et al. [M. Filipović et al., "Remarks on "Cone metric spaces and fixed-point theorems of T-Kannan and T-Chatterjea contractive mappings"," Math. Comput. Modelling **54**, 1467–1472 (2011)] proved several fixed and periodic point theorems for solid cones on cone metric spaces. In this paper we prove some coupled fixed-point theorems for certain T-contractions and study the existence of solutions of a system of nonlinear integral equations using the results of our work. The results of this paper extend and generalize well-known comparable results in the literature.

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1. INTRODUCTION AND PRELIMINARIES

The following famous fixed-point theorem was proved by Banach in 1922 [1]: "Suppose that (X, d) is a complete metric space and a self-map f of X satisfies $d(fx, fy) \leq \lambda d(x, y)$ for all $x, y \in X$ where $\lambda \in [0, 1)$; that is, f is a contraction. Then f has a unique fixed point. Later, other people considered various definitions of contractive mappings and proved several fixed-point theorems [2]–[7]. On the other hand, the notion of cone metric space was introduced in 2007 by Huang and Zhang [8]. Then several fixed and common fixed-point results on cone metric spaces were obtained in [9]–[18].

In 2009, Beiranvand et al. [19] defined T-contractions in a metric space. Afterward, some fixed-point results dealing with Kannan contraction and the Zamfirescu operator were proved for T-contractions in [20], [21]. Soon afterwards, Morales and Rajes [22] introduced T-Kannan and T-Chatterjea contractive mappings in cone metric spaces and proved some fixed-point theorems. Then other authors [23], [24] obtained some fixed-point results under T-contractions on cone metric spaces. Later, Filipović et al. [25] defined T-Hardy-Rogers contraction in a cone metric space and proved some fixed and periodic point theorems. Also, recently, Rahimi et al. proved some new fixed and periodic point theorems for T-contractions of two maps on cone metric spaces in [26], [27].

In 2006, Bhaskar and Lakshmikantham [28] introduced the concept of coupled fixed point in partially ordered metric spaces. Then, other authors generalized this concept and proved several common coupled fixed and coupled fixed-point theorems in ordered metric and ordered cone metric spaces (see [29]–[39] and the references contained therein).

In this paper, we define the concept of T-contraction in coupled fixed-point theory and obtain some coupled fixed-point results on cone metric spaces without normality condition. Our theorems extend, unify and generalize the results of Sabetghadam et al. [37] and Bhaskar and Lakshmikantham [28].

We begin with some important definitions.

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Definition 1 (see [40], [8]). Let E be a real Banach space, θ the null element of E and P a subset of E. Then P is called a *cone* if and only if

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;

(c) if $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone $P \subset E$, a partial ordering $\preceq$ with respect to P is defined by $x \preceq y \Longleftrightarrow y - x \in P$. We shall write $x \prec y$ to mean $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of P). If $intP \neq \varnothing$, the cone P is said to be *solid*. A cone P is said to be *normal* if there exists a number $K > 0$ such that, for all $x, y \in E$,

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the *normal constant* of P.

Definition 2 (see [8]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then, d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Definition 3 (see [8]). Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ converges to x if, for every $c \in E$ with $\theta \ll c$, there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim_{n\to+\infty} x_n = x$.

(ii) $\{x_n\}$ is called a *Cauchy sequence* if, for every $c \in E$ with $\theta \ll c$, there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.

The notation $\theta \ll c$ for $c \in intP$ of a positive cone is used by Krein and Rutman [41]. Also, a cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. In the sequel we shall always suppose that E is a real Banach space, P is a solid cone in E, and $\preceq$ is a partial ordering with respect to P.

Lemma 1 (see [25]). *Let (X, d) be a cone metric space over an ordered real Banach space E. Then the following properties are often used, particularly when dealing with cone metric spaces in which the cone need not be normal.*
(P_1) If $x \preceq y$ and $y \ll z$, then $x \ll z$.
(P_2) If $\theta \preceq x \ll c$ for each $c \in intP$, then $x = \theta$.
(P_3) If $x \preceq \lambda x$ where $x \in P$ and $0 \leq \lambda < 1$, then $x = \theta$.
(P_4) Let $x_n \to \theta$ in E and $\theta \ll c$. Then there exists a positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.

Definition 4 (see [25]). Let (X, d) be a cone metric space, P a solid cone and $S : X \to X$. Then

(i) S is said to be *sequentially convergent* if, for every sequence $\{x_n\}$, such that $\{Sx_n\}$ is convergent, then $\{x_n\}$ also is convergent.

(ii) S is said to be *subsequentially convergent* if, for every sequence $\{x_n\}$, such that $\{Sx_n\}$ is convergent, then $\{x_n\}$ has a convergent subsequence.

(iii) S is said to be *continuous* if $\lim_{n\to+\infty} x_n = x$ implies that $\lim_{n\to+\infty} Sx_n = Sx$, for all $\{x_n\}$ in X.

Definition 5 (see [25]). Let (X, d) be a cone metric space and $T, f : X \to X$ be two mappings. A mapping f is called a *T-Hardy-Rogers contraction* if there exist $\alpha_i \geq 0$, $i = 1, \dots, 5$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ such that for all $x, y \in X$,

$$d(Tfx, Tfy) \preceq \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Tfx) + \alpha_3 d(Ty, Tfy) + \alpha_4 d(Tx, Tfy) + \alpha_5 d(Ty, Tfx).$$

In Definition 5, if one assumes that

$$\alpha_1 = \alpha_4 = \alpha_5 = 0 \text{ and } \alpha_2 = \alpha_3 \neq 0 \quad (\text{respectively, } \alpha_1 = \alpha_2 = \alpha_3 = 0 \text{ and } \alpha_4 = \alpha_5 \neq 0),$$

then one obtains a T-Kannan (respectively, T-Chatterjea) contraction.

Theorem 1 (see [26], [42]). *Let (X, d) be a complete cone metric space, P a solid cone and $T : X \to X$ a continuous and one-to-one mapping. Moreover, let f be a mapping on X satisfying*

$$d(Tfx, Tfy) \preceq \alpha d(Tx, Ty) + \beta[d(Tx, Tfx) + d(Ty, Tfy)] + \gamma[d(Tx, Tfy) + d(Ty, Tfx)],$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$; that is, f be a T-contraction. Then

(1) *for each $x_0 \in X$, $\{Tf^n x_0\}$ is a Cauchy sequence*;

(2) *there exists a $z_{x_0} \in X$ such that $\lim_{n\to\infty} Tf^n x_0 = z_{x_0}$*;

(3) *if T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence*;

(4) *there exists a unique $w_{x_0} \in X$ such that $fw_{x_0} = w_{x_0}$, that is, f has a unique fixed point*;

(5) *if T is sequentially convergent, then, for each $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to w_{x_0}.*

Example 1 (see [22], [26]). Let $X = [0, 1]$, $E = C^2_{\mathbb{R}}[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $P = \{f \in E | f \geq 0\}$ and $d(x, y) = |x - y| 2^t$ where $2^t \in P$. Moreover, suppose that $Tx = x^2$ and $fx = x/2$, for all $x \in X$. (X, d) is a cone metric space with non-normal solid cone [8], [16]. Also, T is a one-to-one, continuous mapping, and f is not a Kannan contraction [22]. All of the conditions of Theorem 1 are satisfied with $\alpha = \gamma = 0$ and $\beta = 1/3$. Therefore, $x = 0$ is the unique fixed point of f.

In the sequel, we review some definitions for coupled fixed-point fields.

Definition 6 (see [37]). Let (X, d) be a cone metric space. An element $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Note that if (x, y) is a coupled fixed point of F, then also (y, x) is a coupled fixed point of F.

Theorem 2 (see [37]). *Let (X, d) be a complete cone metric space and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$d(F(x, y), F(x^*, y^*)) \preceq \frac{k}{2}[d(x, x^*) + d(y, y^*)] \tag{1}$$

where $k \in [0, 1)$ is a constant. Then F has a unique coupled fixed point.

Example 2 (see [37]). Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$, $X = [0, 1]$ and $d : X \times X \to E$ be defined by $d(x, y) = (|x - y|, |x - y|)$. Then (X, d) is a complete cone metric space. Define the mapping $F : X \times X \to X$ by $F(x, y) = (x + y)/6$. Then F satisfies the contractive condition (1) with $k = 1/3 \in [0, 1)$; that is,

$$d(F(x, y), F(x^*, y^*)) \preceq \frac{1}{6}[d(x, x^*) + d(y, y^*)].$$

According to Theorem 2, F has a unique coupled fixed point which, in this case, is $(0, 0)$.

2. MAIN RESULTS

The main results of this work are divided into two parts. In the first part, we prove some coupled fixed-point theorems for T-contractions. Next, we explain a general approach to our theorems in the first part.

2.1. Coupled Fixed-Point Theorems

Definition 7. Let (X,d) be a cone metric space and $T : X \to X$ be a mapping. A mapping $F : X \times X \to X$ is called a *T-Sabetghadam-contraction* if there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that for all $x, y \in X$,

$$d(TF(x,y), TF(x^*,y^*)) \preceq \alpha d(Tx, Tx^*) + \beta d(Ty, Ty^*)$$

Theorem 3. *Suppose that (X,d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one-to-one mapping. Moreover, let $F : X \times X \to X$ be a mapping satisfying*

$$d(TF(x,y), TF(x^*,y^*)) \preceq \alpha d(Tx, Tx^*) + \beta d(Ty, Ty^*) \tag{2}$$

for all $x, y, x^*, y^* \in X$, where $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Then

(i) for each $x_0, y_0 \in X$, $\{TF^n(x_0,y_0)\}$ and $\{TF^n(y_0,x_0)\}$ are Cauchy sequences;

(ii) there exist $z_{x_0}, z_{y_0} \in X$ such that

$$\lim_{n\to+\infty} TF^n(x_0,y_0) = z_{x_0} \qquad \text{and} \qquad \lim_{n\to+\infty} TF^n(y_0,x_0) = z_{y_0};$$

(iii) if T is subsequentially convergent, then $\{TF^n(x_0,y_0)\}$ and $\{TF^n(y_0,x_0)\}$ have a convergent subsequence;

(iv) there exist unique $w_{x_0}, w_{y_0} \in X$ such that $F(w_{x_0}, w_{y_0}) = w_{x_0}$ and $F(w_{y_0}, w_{x_0}) = w_{y_0}$, that is, F has a unique coupled fixed point;

(v) if T is sequentially convergent, then, for each $x_0, y_0 \in X$, the sequence $\{TF^n(x_0,y_0)\}$ converges to $w_{x_0} \in X$ and the sequence $\{TF^n(y_0,x_0)\}$ converges to $w_{y_0} \in X$.

Proof. Let $x_0, y_0 \in X$ and set

$$x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0,y_0),\, y_{n+1} = F(y_n, x_n) = F^{n+1}(y_0,x_0)$$

for all $n \in \mathbb{N} \cup \{0\}$. Now, according to (2), we have

$$d(Tx_n, Tx_{n+1}) = d(TF(x_{n-1}, y_{n-1}), TF(x_n, y_n)) \preceq \alpha d(Tx_{n-1}, Tx_n) + \beta d(Ty_{n-1}, Ty_n), \tag{3}$$

$$d(Ty_n, Ty_{n+1}) = d(TF(y_{n-1}, x_{n-1}), TF(y_n, x_n)) \preceq \alpha d(Ty_{n-1}, Ty_n) + \beta d(Tx_{n-1}, Tx_n). \tag{4}$$

Let $d_n = d(Tx_n, Tx_{n+1}) + d(Ty_n, Ty_{n+1})$. From (3) and (4), we obtain

$$d_n \preceq (\alpha + \beta)(d(Tx_{n-1}, Tx_n) + d(Ty_{n-1}, Ty_n)) = \lambda d_{n-1},$$

where $\lambda = \alpha + \beta < 1$. Thus, for all n,

$$\theta \preceq d_n \preceq \lambda d_{n-1} \preceq \lambda^2 d_{n-2} \preceq \cdots \preceq \lambda^n d_0. \tag{5}$$

If $d_0 = \theta$ then (x_0, y_0) is a coupled fixed point of F. Now, let $d_0 > \theta$. If $m > n$, we have

$$d(Tx_n, Tx_m) \preceq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m) \tag{6}$$

and similarly,

$$d(Ty_n, Ty_m) \preceq d(Ty_n, Ty_{n+1}) + d(Ty_{n+1}, Ty_{n+2}) + \cdots + d(Ty_{m-1}, Ty_m). \tag{7}$$

Adding up (6) and (7) and using (5), since $\lambda < 1$, we have

$$d(Tx_n, Tx_m) + d(Ty_n, Ty_m) \;\preceq\; d_n + d_{n+1} + \cdots + d_{m-1}$$

$$\preceq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})d_0$$
$$\preceq \frac{\lambda^n}{1-\lambda} d_0 \to \theta \qquad \text{as} \qquad n \to +\infty.$$

Now, by (P_1) and (P_4), it follows that for every $c \in intP$ there exists a positive integer N such that $d(Tx_n, Tx_m) + d(Ty_n, Ty_m) \ll c$ for every $m > n > N$, so $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in X. Since X is a complete cone metric space, there exist $z_{x_0}, z_{y_0} \in X$ such that

$$\lim_{n\to+\infty} TF^n(x_0, y_0) = z_{x_0}, \qquad \lim_{n\to+\infty} TF^n(y_0, x_0) = z_{y_0}. \tag{8}$$

Now if T is subsequentially convergent, $F^n(x_0, y_0)$ and $F^n(y_0, x_0)$ have convergent subsequences. Thus, there exist $w_{x_0}, w_{y_0} \in X$ and two sequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$ such that

$$\lim_{i\to+\infty} F^{n_i}(x_0, y_0) = w_{x_0}, \qquad \lim_{i\to+\infty} F^{n_i}(y_0, x_0) = w_{y_0}.$$

Because of the continuity of T, we have

$$\lim_{i\to+\infty} TF^{n_i}(x_0, y_0) = Tw_{x_0}, \qquad \lim_{i\to+\infty} TF^{n_i}(y_0, x_0) = Tw_{y_0}. \tag{9}$$

Now, by (8) and (9), we conclude that

$$Tw_{x_0} = z_{x_0}, \qquad Tw_{y_0} = z_{y_0}.$$

On the other hand, from $(d3)$ and (2), we have

$$\begin{aligned} d(TF(w_{x_0}, w_{y_0}), Tw_{x_0}) &\preceq d(TF(w_{x_0}, w_{y_0}), TF(x_{n_i}, y_{n_i})) + d(Tx_{n_i+1}, Tw_{x_0}) \\ &\preceq \alpha d(Tw_{x_0}, Tx_{n_i}) + \beta d(Tw_{y_0}, Ty_{n_i}) + d(Tx_{n_i+1}, Tw_{x_0}). \end{aligned}$$

Using Lemma 1, it follows that $d(TF(w_{x_0}, w_{y_0}), Tw_{x_0}) = \theta$, which implies the equality

$$TF(w_{x_0}, w_{y_0}) = Tw_{x_0}.$$

Since T is one-to-one, then $F(w_{x_0}, w_{y_0}) = w_{x_0}$. Analogously, one finds that $F(w_{y_0}, w_{x_0}) = w_{y_0}$. Therefore, (w_{x_0}, w_{y_0}) is a coupled fixed point of F. Now if (u_{x_0}, u_{y_0}) is another coupled fixed point of F, then

$$d(Tw_{x_0}, Tu_{x_0}) = d(TF(w_{x_0}, w_{y_0}), TF(u_{x_0}, u_{y_0})) \preceq \alpha d(Tw_{x_0}, Tu_{x_0}) + \beta d(Tw_{y_0}, Tu_{y_0}) \tag{10}$$

and

$$d(Tw_{y_0}, Tu_{y_0}) = d(TF(w_{y_0}, w_{x_0}), TF(u_{y_0}, u_{x_0})) \preceq \alpha d(Tw_{y_0}, Tu_{y_0}) + \beta d(Tw_{x_0}, Tu_{x_0}). \tag{11}$$

Adding up (10) and (11), we obtain

$$d(Tw_{x_0}, Tu_{y_0}) + d(Tw_{x_0}, Tu_{y_0}) \preceq \lambda[d(Tw_{x_0}, Tu_{x_0}) + d(Tw_{y_0}, Tu_{y_0})]. \tag{12}$$

Since $\lambda = \alpha + \beta < 1$, from (12) follows that $d(Tw_{x_0}, Tu_{x_0}) + d(Tw_{y_0}, Tu_{y_0}) = \theta$. Hence,

$$d(Tw_{x_0}, Tu_{x_0}) = d(Tw_{y_0}, Tu_{y_0}) = \theta.$$

So, $Tw_{x_0} = Tu_{x_0}$ and $Tw_{y_0} = Tu_{y_0}$. Since T is one to one, we have $(w_{x_0}, w_{y_0}) = (u_{x_0}, u_{y_0})$. Finally if T is sequentially convergent, then we can replace n by n_i. Thus, we have

$$\lim_{n\to+\infty} TF^n(x_0, y_0) = w_{x_0}, \qquad \lim_{n\to+\infty} TF^n(y_0, x_0) = w_{y_0}.$$

This completes the proof of Theorem 3. □

Proceeding as in the proof of Theorem 3, we can obtain the following theorems.

Theorem 4. *Suppose that (X, d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one-to-one mapping. Moreover, let $F : X \times X \to X$ be a mapping satisfying*

$$d(TF(x, y), TF(x^*, y^*)) \preceq \alpha d(TF(x, y), Tx) + \beta d(TF(x^*, y^*), Tx^*)$$

for all $x, y, x^*, y^* \in X$, where $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Then, the results of Theorem 3 hold.

Theorem 5. *Suppose that (X, d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one-to-one mapping. Moreover, let $F : X \times X \to X$ be a mapping satisfying*

$$d(TF(x, y), TF(x^*, y^*)) \preceq \alpha d(TF(x, y), Tx^*) + \beta d(TF(x^*, y^*), Tx)$$

for all $x, y, x^*, y^* \in X$, where $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Then, the results of Theorem 3 hold.

The following corollaries are obtained from Theorems 3, 4 and 5.

Corollary 1. *Suppose that (X, d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one-to-one mapping. Moreover, let $F : X \times X \to X$ be a mapping satisfying*

$$d(TF(x, y), TF(x^*, y^*)) \preceq \frac{k}{2}[d(Tx, Tx^*) + d(Ty, Ty^*)] \tag{13}$$

for all $x, y, x^*, y^* \in X$, where $k \in [0, 1)$. Then, the results of Theorem 3 hold.

Corollary 2. *Suppose that (X, d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one-to-one mapping. Moreover, let $F : X \times X \to X$ be a mapping satisfying*

$$d(TF(x, y), TF(x^*, y^*)) \preceq \frac{k}{2}[d(TF(x, y), Tx) + d(TF(x^*, y^*), Tx^*)]$$

for all $x, y, x^*, y^* \in X$, where $k \in [0, 1)$. Then, the results of Theorem 3 hold.

Corollary 3. *Suppose that (X, d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one-to-one mapping. Moreover, let $F : X \times X \to X$ be a mapping satisfying*

$$d(TF(x, y), TF(x^*, y^*)) \preceq \frac{k}{2}[d(TF(x, y), Tx^*) + d(TF(x^*, y^*), Tx)]$$

for all $x, y, x^*, y^* \in X$, where $k \in [0, 1)$. Then, the results of Theorem 3 hold.

Remark 1. If in each of Theorems 3, 4 and 5, we take $T = I_X$ where I_X is the identity mapping on X, then we obtain the main results of Sabetghadam et al. [37, Theorems 2.2, 2.5, 2.6]. Also if in each of Corollaries 1, 2 and 3, we take $T = I_X$, then we obtain the other results of Sabetghadam et al. [37, Corollaries 2.3, 2.7, 2.8]. Therefore, our theorems and corollaries extend and generalize well-known comparable results in the literature.

Example 3. Let $X = [0, 1]$. Take $E = C^1_{\mathbb{R}}[0, 1]$ endowed with order induced by

$$P = \{\phi \in E : \phi(t) \geq 0 \text{ for } t \in [0, 1]\}.$$

The mapping $d : X \times X \to E$ is defined by $d(x, y)(t) = |x - y|3^t$. In this case (X, d) is a complete cone metric space with a cone having nonempty interior. Define the mappings $F : X \times X \to X$ and $T : X \to X$ by

$$Tx = \frac{x}{2}, \qquad F(x, y) = \frac{x + y}{4}.$$

Then F satisfies the contractive condition (13) with $k = 1/2$; that is,

$$d(TF(x, y), TF(u, v)) \preceq \frac{1}{4}[d(Tx, Tu) + d(Ty, Tv)].$$

Therefore, by Corollary 1, F has a unique coupled fixed point, which, in this case, is $(0, 0)$. Also, let $F : X \times X \to X$ be defined by $F(x, y) = (x + y)/2$. Then F satisfies the contractive condition (13) with $k = 1$. In this case, $(0, 0)$ and $(1, 1)$ are both coupled fixed points of F. Thus, the coupled fixed point of F is not unique.

For more examples, one can see [43], [44], [37] and translate into this framework our theorems and corollaries.

2.2. General Approach

We start with the following Lemma.

Lemma 2. *The following assertions hold*:

(1) *Suppose that* (X, d) *is a cone metric space. Then,* $(X \times X, d_1)$ *is a cone metric space with*

$$d_1((x,y),(u,v)) = d(x,u) + d(y,v). \tag{14}$$

Further, (X, d) *is complete if and only if* $(X \times X, d_1)$ *is complete.*

(2) *The mapping* $F : X \times X \to X$ *has a coupled fixed point if and only if the mapping*

$$S_F : X \times X \to X \times X$$

defined by $S_F(x,y) = (F(x,y), F(y,x))$ *has a fixed point in* $X \times X$.

Proof. (1) Clearly, d_1 satisfies conditions $(d1)$ and $(d2)$ of Definition 2. Thus, we only need to prove condition $(d3)$ for d_1. Since (X, d) is a cone metric space, we have

$$d(x,u) \preceq d(x,z) + d(z,u) \qquad \text{for all} \quad x, z, u \in X \tag{15}$$

and

$$d(y,v) \preceq d(y,w) + d(w,v) \qquad \text{for all} \quad y, v, w \in X. \tag{16}$$

Adding up (15) and (16), we obtain

$$\begin{aligned} d_1((x,y),(u,v)) &= d(x,u) + d(y,v) \\ &\preceq (d(x,z) + d(z,u)) + (d(y,w) + d(w,v)) \\ &= (d(x,z) + d(y,w)) + (d(z,u) + d(w,v)) \\ &= d_1((x,y),(z,w)) + d_1((z,w),(u,v)). \end{aligned}$$

Thus, $(X \times X, d_1)$ is a cone metric space. The proof of completeness is easy and is left to the reader.

(2) Let (x,y) be a coupled fixed point of F. In this case, $F(x,y) = x$ and $F(y,x) = y$. Thus,

$$S_F(x,y) = (F(x,y), F(y,x)) = (x,y).$$

Therefore, $(x,y) \in X \times X$ is a fixed point of S_F. Conversely, suppose that $(x,y) \in X \times X$ is a fixed point of S_F, then $S_F(x,y) = (x,y)$. Consequently, $F(x,y) = x$ and $F(y,x) = y$. □

Theorem 6. *Suppose that* (X, d) *is a complete cone metric space,* P *is a solid cone, and* $T : X \to X$ *is a continuous and one-to-one mapping. Moreover, let* $F : X \times X \to X$ *be a mapping satisfying*

$$d(TF(x,y), TF(x^*,y^*)) + d(TF(y,x), TF(y^*,x^*)) \preceq \lambda[d(Tx,Tx^*) + d(Ty,Ty^*)] \tag{17}$$

for all $x, y, x^*, y^* \in X$*, where* $\lambda \in [0,1)$. *Then, the results of Theorem 3 hold.*

Proof. According to T-Hardy-Rogers contraction (Definition 5) and Lemma 2(2), the contractive condition (17) for all $Y = (x,y), V = (u,v) \in X \times X$ becomes

$$d_1(S_F(Y), S_F(V)) \preceq \lambda d_1(TY, TV).$$

Since $\lambda < 1$, the conclusion follows by setting $a_1 = \lambda$ and $a_2 = a_3 = a_4 = a_5 = 0$ in Theorem 2.1 of [25]. □

3. AN APPLICATION

In this section, we study the existence of solutions of a system of nonlinear integral equations using the results proved in Section 2.

Consider the following system of integral equations:

$$F(x,y)(t) = \int_0^T K(t,s) f(s, x(s), y(s))\, ds + g(t), \tag{18}$$

$$F(y,x)(t) = \int_0^T K(t,s) f(s, y(s), x(s))\, ds + g(t), \tag{19}$$

where $t \in [0,T]$, $T > 0$.

Let $X = C([0,T], \mathbb{R})$ be the set of continuous functions defined on $[0,T]$ endowed with the metric given by

$$d(u,v) = \sup_{t\in[0,T]} |u(t) - v(t)| \text{ for all } u, v \in X.$$

We make the following assumptions:
(a) $K : [0,T] \times [0,T] \to \mathbb{R}$ is a continuous function.
(b) $g \in C([0,T], \mathbb{R})$.
(c) $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
(d) For all $t \in [0,T]$, for all $x, y, u, v \in X$, we have

$$|f(t, x(t), y(t)) - f(t, u(t), v(t))| \le \alpha |x(t) - u(t)| + \beta |y(t) - v(t)|,$$

where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$.
(e) $\displaystyle\int_0^T |K(t,s)| \le 1$.

Now, we formulate our result.

Theorem 7. *Under hypotheses (a)–(e), the system* (18)–(19) *has at least one solution in* $C([0,T], \mathbb{R})$.

Proof. We consider the operator $F : X \times X \to X$ defined by

$$F(x,y)(t) = \int_0^T K(t,s) f(s, x(s), y(s))\, ds + g(t), \quad t \in [0,T].$$

It is easy to show that (x,y) is a solution to (18)–(19) if and only if (x,y) is a coupled fixed point of F. To establish the existence of such a point, we will use our Theorem 3 with T the identity mapping. Then we must check that all the hypotheses of Theorem 3 are satisfied.
Let $x, y, u, v \in X$. For all $t \in [0,T]$, we have

$$|F(x,y)(t) - F(u,v)(t)| \le \int_0^T |K(t,s)|\, |f(t, x(s), y(s)) - f(t, u(s), v(s))|\, ds.$$

Using condition (d), we obtain

$$\begin{aligned} |F(x,y)(t) - F(u,v)(t)| &\le \int_0^T |K(t,s)|\, [\alpha |x(s) - u(s)| + \beta |y(s) - v(s)|]\, ds \\ &\le \left(\int_0^T |K(t,s)|\, ds \right) [\alpha d(x,u) + \beta d(y,v)]. \end{aligned}$$

Using condition (e), we obtain

$$|F(x,y)(t) - F(u,v)(t)| \le \alpha d(x,u) + \beta d(y,v).$$

This implies that

$$d(F(x,y), F(u,v)) \le \alpha d(x,u) + \beta d(y,v)$$

for all $x, y, u, v \in X$. Then the contractive condition of Theorem 3 is satisfied. Therefore, Theorem 3 applies to F, which has a unique coupled fixed point (x^*, y^*) in $C([0,T], \mathbb{R})$, that is, (x^*, y^*) is the unique solution to system (18)–(19). □

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