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Dynamics of the Logistic Equation with Delay

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Abstract—The logistic equation supplemented with a summand characterizing delay is considered. The local and nonlocal dynamics of this equation are studied. For equations with delay, we use the standard Andronov–Hopf bifurcation methods and the asymptotic method developed by the author and based on the construction of special evolution equations defining the local dynamics of the equations containing delay. In addition, we study the existence and methods of constructing the asymptotics of nonlocal relaxation cycles. A comparison of the results obtained with those for the Hutchinson equation and some of its generalizations is given.

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1. STATEMENT OF THE PROBLEM

The logistic equation

$$\dot{u} = r[1-u]u, \qquad r > 0, \tag{1.1}$$

possesses a simple dynamics. All the solutions with positive initial conditions tend to 1 as $t \to \infty$. The dynamics of the logistic equation with delay (the Hutchinson equation)

$$\dot{u} = r[1-u(t-T)]u, \qquad r, T > 0, \tag{1.2}$$

is significantly more varied. For $rT \le 37/24$, all the solutions (1.2) with a positive initial function also tend to 1 as $t \to \infty$ [1]. For $\lambda = r\tau \le \pi/2$, the equilibrium state $u_0 \equiv 1$ is asymptotically stable and, for $\lambda = rT > \pi/2$, this equation has a periodic solution $u_0(t,\lambda)$ [2]. For sufficiently small values of $\lambda - \pi/2$ and for a sufficiently large λ, this cycle is stable [3]. In addition, note that the number of unstable periodic solutions (1.2) unboundedly increases as $\lambda \to \infty$ [4].

Here we shall consider the following "average" (in a certain sense) equation between (1.1) and (1.2):

$$\dot{u} = r[1-au-bu(t-T)]u,$$

where $a, b > 0$ and the normalization condition $a+b=1$ holds.

It is convenient to pass to the notation $a=\alpha$ and $b = 1-\alpha$, $0<\alpha<1$, and make the change of time variable $t \to Tt$. As a result, we obtain the equation

$$\dot{u} = \lambda[1-\alpha u-(1-\alpha)u(t-1)]u, \qquad \lambda = Tr. \tag{1.3}$$

The initial functions $\varphi \in C_{[-1,0]}$ for Eq. (1.3) are assumed nonnegative. The solutions with these initial functions remain nonnegative as t increases. Just as in [3], [5], we can show that, for sufficiently large t, all (nonnegative) solutions of (1.3) satisfy the estimate

$$u \le \min(\alpha^{-1}, \exp(\lambda(1-\alpha))).$$

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The zero solution of all Eqs. (1.1), (1.2), and (1.3) is unstable. The stability of the equilibrium state $u_0 \equiv 1$ in (1.3) is defined by the distribution of the roots of the characteristic equation

$$\mu = -\lambda[\alpha + (1-\alpha)\exp(-\mu)]. \tag{1.4}$$

Let us state two simple statements on the distribution of the roots of Eq. (1.4).

Lemma 1. *Let $\alpha > 1/2$. Then all the roots of* (1.4) *have negative real parts.*

Lemma 2. *Let $\alpha < 1/2$. Then all the roots of* (1.4) *satisfy the inequality*

$$\operatorname{Re}\mu \le \ln\frac{b}{a}.$$

In the assumption of Lemma 1, all (positive) solutions of (1.3) tend to 1 as $t \to \infty$, and, in the assumption of Lemma 2, they are bounded in λ as $\lambda \to \infty$. These assertions already indicate an essential difference in the dynamical properties of (1.2) and (1.3).

The main results of the paper are given in Secs. 2–4. The first two sections deal with the behavior of the solutions of (1.3) in a sufficiently small neighborhood of the equilibrium state $u_0 \equiv 1$. In Sec. 2, we consider the simplest "regular" case based on the Andronov–Hopf bifurcation. An important role is played by the distribution of the roots of (1.4). For each $\alpha < 1/2$, there exists a value of $\lambda_0(\alpha) > 0$ such that, for $\lambda < \lambda_0(\alpha)$, all the roots of (1.4) have negative real parts and, for $\lambda > \lambda_0(\alpha)$, there are roots with positive real part. We shall study the behavior of the solutions of (1.3) under the condition

$$|\lambda - \lambda_0(\alpha)| \ll 1.$$

Note that $\lambda_0(0) = \pi/2$ and $\lambda_0(\alpha) \to \infty$ as $\alpha \to 1/2 - 0$. The difference in the dynamical properties of the solutions of Eqs. (1.2) and (1.3) is more conspicuous under the condition

$$\epsilon = \lambda^{-1} \ll 1. \tag{1.5}$$

In Sec. 3, we assume that, along with condition (1.5), the following condition holds:

$$\alpha = \frac{1}{2} - \nu, \qquad \text{where} \quad |\nu| \ll 1. \tag{1.6}$$

We study the local dynamics of the solutions of Eq. (1.3) in a sufficiently small (and independent of ϵ and ν) neighborhood of the equilibrium state u_0 under conditions (1.5) and (1.6).

For a linearized (on u_0) Eq. (1.3),

$$\epsilon\dot{v} = -\frac{1}{2}[v + v(t-1) + \mu(v - v(t-1))], \tag{1.7}$$

the characteristic quasipolynomial is expressed as

$$2\epsilon\mu = -(1 + \exp(-\mu) - \nu(1 - \exp(-\mu))). \tag{1.8}$$

It is easy to show that (1.8) has no roots with positive and real parts separated from zero as $\epsilon, \nu \to 0$. It is important that the real parts of infinitely many roots of (1.8) tend to zero as $\epsilon, \nu \to 0$. Therefore, we can say that, in the problem of the stability of the equilibrium state $u_0 \equiv 1$ the infinite-dimensional critical case, is realized. Various methods for studying the dynamical properties of solutions in similar cases were developed in [6]–[10]. In what follows, they will be used in the study of the local dynamics of Eq. (1.3). As main results, special nonlinear equations of parabolic and degenerate–parabolic types not containing small parameters will be constructed. Their nonlocal dynamics essentially determines the behavior of the solutions of the original equation for small ϵ and ν.

In the next section, we shall study the "basic" case in which $\nu = c\epsilon^2$. As it turns out, under this condition, the corresponding solutions are mostly formed of lower modes (in the parabolic equation). Therefore, it is natural to call such solutions *slowly oscillating*.

Under the condition $\nu = c\epsilon^{2\gamma}$, where $0 < \gamma < 1$, the solutions of the constructed evolution equations will contain rapidly oscillating (in the spatial variable) components.

Finally, in Sec. 4, we study the nonlocal behavior of the solutions of (1.3) under condition (1.5). Let us note that the behavior of the solutions of the degenerate (for $\epsilon = 0$) Eq. (1.3) does not provide any information about the behavior of the solutions for $\epsilon \neq 0$. We use the special asymptotic method developed in [3], [4], [11]. With its help, it can be shown that, for $0 < \alpha < 1/2$ and for sufficiently small ϵ, Eq. (1.3) has a stable relaxation periodic solution and, for it, asymptotic formulas can be obtained. A comparison is given with the asymptotics of the relaxation cycle for Eq. (1.2) and for the logistic equation with constraint and retarded argument,

$$\dot{u} = \lambda[1 - u(t-1)]u(A - u), \qquad A > 1. \tag{1.9}$$

2. THE ANDRONOV–HOPF BIFURCATION

Consider the behavior of the solutions of Eq. (1.3) with initial conditions from a sufficiently small neighborhood of the equilibrium state $u_0 \equiv 1$. In many respects, this behavior depends on the distribution of the roots of the characteristic quasipolynomial (1.4). It follows from Lemma 1 that it is necessary to consider only cases in which

$$\alpha < \frac{1}{2}. \tag{2.1}$$

The following simple statement is valid.

Lemma 3. *Suppose that inequality* (2.1) *holds. Then there exists a* $\lambda_0 = \lambda_0(\alpha)$ *such that, for* $\lambda < \lambda_0(\alpha)$, *all the roots of* (1.4) *have negative real parts and, for* $\lambda > \lambda_0(\alpha)$, *Eq.* (1.4) *has a root with positive real part.*

This implies that, for $\lambda < \lambda_0$, all the solutions of (1.3) with initial conditions sufficiently close to 1 tend to 1 as $t \to \infty$ and, for $\lambda > \lambda_0$, the equilibrium state $u_0 \equiv 1$ is unstable and the problem is no longer a local one. Here we shall study the "boundary" case. Let us fix the value of α_0 so that $0 < \alpha_0 < 1/2$, and suppose that, for some constants λ_1 and α_1,

$$\lambda = \lambda_0(\alpha_0) + \epsilon\lambda_1, \qquad \alpha = \alpha_0 + \epsilon\lambda_1, \tag{2.2}$$

where ϵ is a small parameter:

$$0 < \epsilon \ll 1. \tag{2.3}$$

For $\epsilon = 0$, Eq. (1.4) has a pair of pure imaginary roots $\mu_{1,2} = \pm i\omega$, $\omega > 0$, while all the other roots have negative real parts. Under conditions (2.2), (2.3), the well-known Andronov–Hopf bifurcation exists: this means that, in a sufficiently small (independent of ϵ) neighborhood $u_0 \equiv 1$, Eq. (1.3) has a local stable two-dimensional invariant integral manifold (see, for example, [12], [13]). On it, this equation is expressed (under some nondegeneracy conditions) as the following complex equation of first order:

$$\frac{d\xi}{dt} = \epsilon a_1 \xi + d|\xi|\xi + O(\epsilon^2 + |\xi|^2). \tag{2.4}$$

After the normalizing changes $\epsilon = \epsilon t$ and $\xi \to \epsilon^{1/2}\xi$, we see that, essentially, the behavior of the solutions of (2.4) is determined by the following equation (the normal form):

$$\frac{d\xi}{d\tau} = a_1\xi + d|\xi|^2\xi \tag{2.5}$$

(see, for example [14]). The solutions of Eq. (2.5) are related to the solutions of Eq. (1.3) by the formula

$$u = 1 + \epsilon^{1/2}\big(\xi(\tau)\exp(i\omega t) + \overline{\xi}(\tau)\exp(-i\omega t)\big) + \epsilon u_2(\tau, t) + \epsilon^{3/2}u_3(\tau, t) + \cdots, \tag{2.6}$$

in which the functions $u_j(\tau, t)$ are $2\pi/\omega$-periodic in t. Thus, in a neighborhood of u_0, the dynamical properties of (1.3) are determined essentially by the behavior of the solutions of (2.5); therefore, it only remains to determine the coefficients a_1 and d. To do this, let us use the formal series (2.6). We substitute it into (1.3) and, in the resulting formal identity, we shall successively equate the coefficients of identical powers of ϵ. So, at the second step, collecting the coefficients of ϵ^1, we determine the function $u_2(\tau, t)$:

$$u_2(\tau, t) = g\xi^2\exp(2i\omega t) + \overline{g}\,\overline{\xi}^2\exp(-2i\omega t),$$

where

$$g = \lambda_0(1-\alpha_0) i \sin\omega \cdot [2i\omega + \alpha_0\lambda_0 + (1-\alpha_0)\lambda_0 \exp(-2i\omega)]^{-1}.$$

At the third step, we collect the coefficients of $\epsilon^{3/2}$. Using the solvability conditions for the resulting equation with respect to the 2π-periodic (in t) function $u_3(\tau, t)$, we obtain Eq. (2.5) for the unknown amplitude $\xi(\tau)$ in which

$$a_1 = -[1-(1-\alpha_0)\lambda_0\exp(-i\omega)]^{-1} \cdot [\lambda_1(1+(1-\alpha_0)\exp(-i\omega)) + \lambda_0\alpha_1(1+\exp(-i\omega))],$$

$$d = -\lambda_0^2[1-\lambda_0(1-\alpha_0)\exp(-i\omega)]^{-1} \cdot [2\alpha_0 + (1-\alpha_0)(\exp(i\omega)+\exp(-2i\omega))]$$
$$\times \left[(1-\alpha_0) i \sin\omega \cdot [2i\omega + \alpha_0\lambda_0 + (1-\alpha_0)\lambda_0]\exp(-2i\omega)\right]^{-1}.$$

Note that the stability of the zero solution in (2.5) is determined by the sign of expression $\operatorname{Re} a_1$, and, under the conditions $\operatorname{Re} a_1 > 0$ and $\operatorname{Re} d < 0$, this equation has the stable periodic solutions

$$\xi_0(\tau) = \rho_0 \exp(i\varphi_0\tau),$$

where $\rho_0 = [(-\operatorname{Re} a_1)(\operatorname{Re} d)^{-1}]^{1/2}$ and $\varphi_0 = \operatorname{Im} a_1 + \rho_0^2 \operatorname{Im} d$.

Let us state, as an example, the corresponding result.

Theorem 1. *Let* $\operatorname{Re} a_1 > 0$*, and let* $\operatorname{Re} d < 0$*. Then there exists an* $\epsilon_0 > 0$ *such that, for all* $\epsilon \in (0, \epsilon_0]$*, Eq.* (1.3) *has the asymptotically orbitally stable periodic solution*

$$u_0(t,\epsilon) = 1 + \epsilon^{1/2}\rho_0\cos[(\omega + \epsilon\varphi_0 + o(\epsilon^2)t] + o(\epsilon).$$

of period

$$T(\epsilon) = 2\pi\omega^-1(1 + \epsilon\varphi\omega^{-1} + o(\epsilon^2)).$$

Note that, for $\alpha_1 = 0$ and $\lambda_1 > 0$, the condition $\operatorname{Re} a_1 > 0$ holds, while, for $\lambda_1 = 0$ and $\alpha_1 > 0$, we have inequality $\operatorname{Re} a_1 < 0$. For sufficiently small values of the parameter α_0, the parameters in (1.3) are close to those in (1.2), and hence $\omega \sim \pi/2$ and $\operatorname{Re} d < 0$ [3].

Let us show that the number of different periodic solutions of Eq. (1.3) increases unboundedly as $\lambda \to \infty$. We shall use some constructions from [4]. Assume that $\alpha_1 = 0$. First, note that, as the parameter ϵ changes from 0 to ϵ_0, the period $T(\epsilon)$ of the periodic solution $u_0(t,\epsilon)$ varies from $2\pi/\omega$ to $2\pi/\omega(1 + \epsilon\varphi_0\omega^{-1} + o(\epsilon^2))$ and its length is close to $2\pi\omega^{-2}\varphi_0\epsilon_0$. Note that the function $u_0(t,\epsilon)$ is simultaneously the solution of the equation

$$\dot{u} = \lambda[1 - \alpha u - (1-\alpha)u(t - nT(\epsilon))]u.$$

Let us make the normalizing changes $t \to (1 + nT(\epsilon))t$ and $u((1 + nT(\epsilon))t) = V(t)$. As a result, for the function V, we obtain Eq. (1.3), the only difference being that, instead of the coefficient λ, we now have the coefficient $\lambda_n = \lambda(1 + nT(\epsilon))$. This equation has the periodic solution $V_n(t,\epsilon) = u_0((1 + nT(\epsilon))t, \epsilon)$ of period $T_n(\epsilon) = T(\epsilon)(1 + nT(\epsilon))^{-1}$. It necessarily exists for all

$$\lambda \in \{(\lambda_0 + \epsilon\lambda_1)(1 + nT(\epsilon)),\ 0 < \epsilon \le \epsilon_0\}.$$

Hence it immediately follows that, as $\lambda \to \infty$ ($n \to \infty$), the number of periodic solutions of Eq. (1.3) increases unboundedly.

3. CONSTRUCTION OF QUASINORMAL FORM FOR LARGE VALUES OF THE PARAMETER λ

In this section, we continue the study of the local (in a small neighborhood of $u_0 \equiv 1$) dynamics of Eq. (1.3). The main assumption is that the following condition holds:

$$\lambda^{-1} = \epsilon \ll 1.$$

In this case, $\alpha_0 = 1/2$. As already noted, in the case under consideration, the real parts of infinitely many roots of the characteristic equation (1.4) tend to zero as $\epsilon \to 0$. The standard methods used for the study of local dynamics and based on the theory of invariant manifolds and the theory of normal forms turn out to be inapplicable. We shall use the approach developed in [6]–[10]. First, consider the simplest case in which the parameter ν is of order ϵ^2 in ϵ, and then we turn to the more complicated case in which $\nu = \epsilon^{2\gamma}$ and $0 < \gamma < 1$.

3.1. Slowly Oscillating Solutions

Let us set

$$\nu = c\epsilon^2 \tag{3.1}$$

and write out the asymptotics (as $\epsilon \to 0$) of the roots $\lambda_k(\epsilon)$ ($k = 0, \pm 1, \pm 2, \dots$) of Eq. (1.4) whose real parts tend to zero as $\epsilon \to 0$. Using standard methods, we obtain

$$\lambda_k(\epsilon) = \lambda_{k_0} + \epsilon\lambda_{k_1} + \epsilon^2\lambda_{k_2} + \cdots,$$

where

$$\lambda_{k_0} = \pi(2k+1)i, \qquad \lambda_{k_1} = -2\pi(2k+1)i, \qquad \lambda_{k_2} = -2\pi^2(2k+1)^2 + 4\pi(2k+1)i - 2c.$$

The functions $\mathrm{const}\cdot\exp(\lambda_k(\epsilon)t)$, $k = 0, \pm 1, \pm 2, \dots$, are solutions of the linear equation (1.7). They can be written as the product of a slowly oscillating (with respect to time, i.e., depending on $\tau = \epsilon^2 t$) function by an oscillating (periodic) function: $\xi(\tau)\exp[i\pi(2k+1)(1-2\epsilon)t]$.

Following the approach in [6]–[10], consider the formal series

$$v = \epsilon^{1/2}\sum_{k=-\infty}^{\infty}\xi_k(\tau)\exp[i(2k+1)\pi y] + \epsilon^2 v_1(\tau, y) + \epsilon^{5/2}v_2(\tau, y) + \cdots$$

in which $y = (1-2\epsilon)t$ and the dependence on y of the function $v_i(\tau, y)$ is periodic. We introduce another convenient notation, setting

$$\xi(\tau, y) = \sum_{k=-\infty}^{\infty}\xi_k(\tau)\exp[i\pi(2k+1)y].$$

Then

$$v(t, \epsilon) = \epsilon^{1/2}\xi(\tau, \epsilon) + \epsilon^2 v_1(\tau, y) + \epsilon^{5/2}v_2(\tau, y) + \cdots. \tag{3.2}$$

Further, we set $u = 1 + v$ in (1.3). As a result, for $v(t)$, we obtain the equation

$$\epsilon\dot{v} = -\frac{1}{2}[v + v(t-1) + c\epsilon^2(v - v(t-1))](1+v). \tag{3.3}$$

Let us substitute (3.2) into (3.3) and collect the coefficients of identical powers of ϵ. At the second step, we obtain the relation

$$v_1(\tau, y) = -\xi(\tau, y)\frac{\partial\xi(\tau, y)}{\partial y}.$$

At the third step, collecting the coefficients of $\epsilon^{5/2}$, we find the equation for $v_2(\tau, y)$. Using the solvability condition for this equation, we obtain the following equation for the unknown amplitude $\xi(\tau, y)$:

$$\frac{1}{2}\frac{\partial\xi}{\partial\tau} = \frac{\partial^2\xi}{\partial y^2} + 2\frac{\partial\xi}{\partial y} - c\xi + \xi^2\frac{\partial\xi}{\partial y} \tag{3.4}$$

with the antiperiodic boundary conditions

$$\xi(\tau, y+1) \equiv -\xi(\tau, y). \tag{3.5}$$

This immediately yields the following statement.

Theorem 2. *Suppose that, for all $\tau > \tau_0$, the boundary-value problem* (3.4), (3.5) *has a bounded solution $\xi_0(\tau, y)$ together with its derivatives with respect to τ and y. Then Eq.* (3.3) *has an asymptotic* (*with respect to the residual up to $O(\epsilon^{5/2})$*) *solution for which*

$$v_0(t, \epsilon) = \epsilon^{1/2}\xi_0(\tau, y) - \epsilon^2\xi_0^2(\tau, y)\frac{\partial\xi_0(\tau, y)}{\partial y}$$

and $\tau = \epsilon^2 t$, $y = (1-2\epsilon)t$.

Note that, in some cases, e.g., for the equilibrium states or time-periodic solutions of system (3.4), (3.5), it is possible to assert the existence of a close solution of (3.3) and study its stability.

3.2. On the Solution of the Boundary-Value Problem (3.4), (3.5)

Consider the boundary-value problem (3.4), (3.5). In it, it is convenient to make the changes $\tau_1 = 2\tau$, $x = y - 2\tau$, and $\alpha = -c$. Further, we replace τ_1 by τ, obtaining the final equation

$$\frac{\partial \xi}{\partial \tau} = \frac{\partial^2 \xi}{\partial x^2} + \alpha\xi + \xi^2 \frac{\partial \xi}{\partial x} \tag{3.6}$$

with antiperiodic boundary conditions (3.5).

First, note that the equilibrium state $\xi \equiv 0$ is stable for $\alpha \le \pi^2$ and unstable for $\alpha > \pi^2$. It is easy to justify the assertion that, for $\alpha < 0$, this equilibrium state is globally stable: all solutions of (3.6), (3.5) tend to zero as $\tau \to \infty$. Indeed, the following statement describes the behavior of the solutions of (3.6), (3.5), and hence also of Eq. (3.3) for $\alpha < 0$.

Theorem 3. *Let $\alpha < 0$, and let the solution of the boundary-value problem* (3.6), (3.5) *be determined for all $\tau \ge \tau_0$ and be continuously differentiable with respect to τ and y. Then*

$$\lim_{\tau\to\infty} \int_0^2 \xi^2(\tau, y)\, dy = 0.$$

Proof. To prove the theorem, it suffices to multiply (3.6) by $\xi(\tau, y)$ and integrate from 0 to 2 over y (in view of (3.5)). □

Numerical studies show that, for $0 \le \alpha \le \pi^2$ as well, all the solutions of the boundary-value problem (3.6), (3.5) also tend to zero as $\epsilon \to \infty$.

Consider the case in which

$$\alpha = \pi^2 + \nu, \qquad 0 < \nu \ll 1.$$

The boundary-value problem (3.6), (3.5) linearized at zero is of the form

$$\frac{\partial \eta}{\partial \tau} = \frac{\partial^2 \eta}{\partial x^2} + (\pi^2 + \nu)\eta, \qquad \eta(\tau, x+1) = -\eta(\tau, x). \tag{3.7}$$

The characteristic equation

$$\mu = -\pi^2(2k+1) + (\pi^2 + \nu), \qquad k = 0, \pm 1, \pm 2, \dots, \tag{3.8}$$

for (3.7) has two zero roots for $\nu = 0$. In (3.7), these roots correspond to the solutions $c \cos \pi x$ and $c \sin \pi x$. The other roots of (3.8) are negative. For a sufficiently small ν, the boundary-value problem (3.6), (3.5) has a stable two-dimensional local invariant integral manifold in a sufficiently small (and independent of ν) neighborhood of the equilibrium state $\xi \equiv 0$ [12], [13]. On this manifold, the boundary-value problem under consideration can be reduced to normal form, i.e., to a system of two ordinary differential equations with a special nonlinearity. The distinctive feature of such a system is that there are several "degeneracies." First, the nonlinearity in (3.6) is cubic (there are no quadratic summands). Second, it turns out that the Lyapunov quantity [15] is zero. The system of two equations on an invariant manifold can be written up to $o(\nu)$ as one scalar equation of the form

$$\frac{d\eta}{ds} = i\delta|\eta|^2\eta + \nu[a + b|\eta|^4]\eta, \tag{3.9}$$

where $s = \nu^{1/2}\tau$ and the coefficients δ, a, and b are to be determined. The relationship between the solutions of (3.6), (3.5) on this manifold and the solutions of (3.9) is given by the asymptotic formula

$$\xi(\tau, \epsilon) = \nu^{1/4}[\eta(s)\exp(i\pi x) + \overline{\eta}(s)\exp(-i\pi x)] + \nu^{1/2}\xi_2(s, x) + \nu^{3/4}\xi_3(s, x) + \nu\xi_4(s, x) + \nu^{5/4}\xi_5(s, x) + \cdots . \tag{3.10}$$

To find all the coefficients in (3.10), and hence also the solutions of (3.8), we substitute expression (3.10) into the boundary-value problem (3.6), (3.5) and collect the coefficients of identical powers of ν, taking (3.9) into account. First, we obtain $\xi_2(s,x) \equiv 0$ and, from the condition of the solvability of the resulting equation for $\xi_3(s,x)$, we derive the equality

$$\delta = \pi, \qquad \xi_3(s,x) = \frac{i\eta^3}{8\pi} e^{3\pi x} + \frac{-i\overline{\eta}^3}{8\pi} e^{-3\pi x}.$$

At the following step, we obtain the relation $\xi_4(s,x) \equiv 0$ and, from the condition of the solvability of the corresponding equation for $\xi_5(s,x)$, we determine the coefficients a and b: $a = 1$, $b = -1/8$. For these δ, a, and b, Eq. (3.9) has a unique stable cycle $\eta_0(s)$ for which

$$\eta_0(s) = \sqrt[4]{8}\, \exp(i\pi\sqrt{8}\, s).$$

This implies that, for small ν, the boundary-value problem (3.4), (3.5) has the following stable cycle:

$$\xi_0(\tau, \epsilon) = 2\sqrt[4]{8\epsilon}\, \cos(\pi x + \pi\tau\sqrt{8\epsilon} + \gamma) + O(\sqrt{\epsilon}\,).$$

Numerical simulations show that, for all $\alpha > \pi^2$, the boundary-value problem (3.6), (3.5) has a unique stable cycle. For large α, this cycle has a pronounced relaxation structure.

Let us present several graphs (Figs. 1–3) obtained on the basis of numerical analysis.

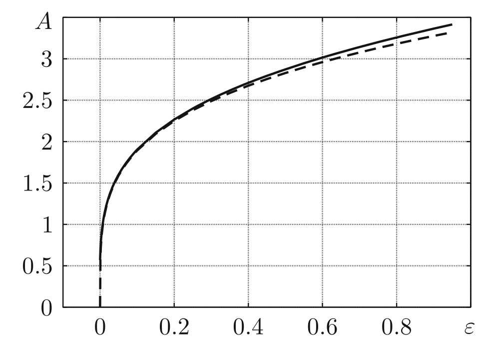

Fig. 1. The solid line depicts the value of the amplitude obtained by numerical calculations. The dotted line shows the analytical value of the amplitude.

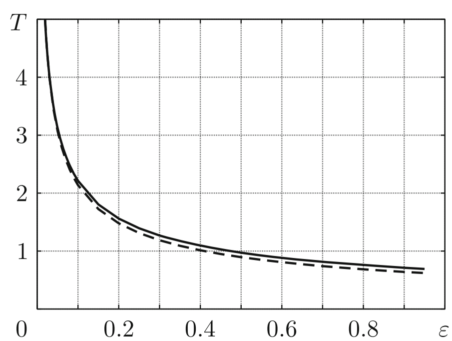

Fig. 2. The solid line depicts the value of the period obtained by numerical calculations. The dotted line shows the analytical value of the period.

The three-dimensional graph of the oscillations of the solution of the boundary-value problem under consideration for $\varepsilon = 2$ is shown in Fig. 4. It is seen from the figure that the surface formed by the solution is the so-called “traveling-wave” mode.

3.3. Rapidly Oscillating Solutions

Set

$$y = c\epsilon^{2\gamma}, \qquad 0 < \gamma < 1. \tag{3.11}$$

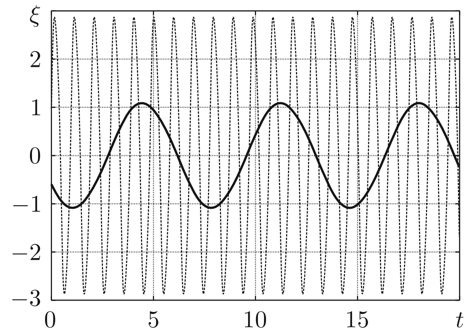

Fig. 3. The increase in the frequency and the amplitude of oscillations as the parameter ε increases. The solid line corresponds to $\varepsilon = 0.01$ and the dotted line to $\varepsilon = 0.5$. The time shift is $t = \tau - 100\,980$.

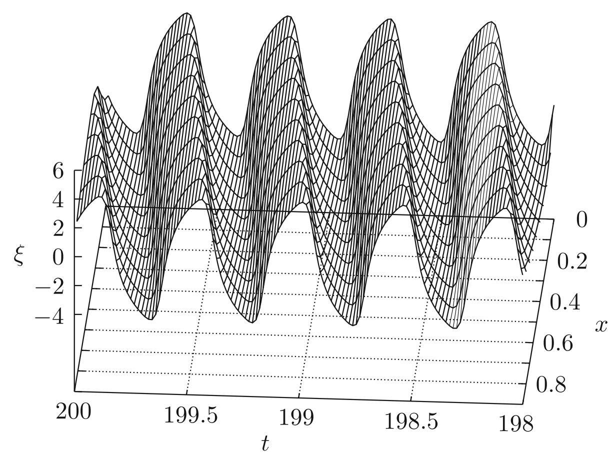

Fig. 4. Traveling wave for $\varepsilon = 2$.

It was shown above that if condition (3.1) holds (i.e., for $\alpha = 1$), the local dynamics of (3.3) is determined by the nonlocal behavior of the solutions of the boundary-value problem (3.4), (3.5); see formula (3.2). Therefore, we can say that the main contribution to the formation of the structures (3.6), (3.5) are made by modes with relatively small numbers. Therefore, they were called slowly oscillating.

But, under condition (3.11), this is not so. In that case, it will be shown that the main contribution to the formation of the structures are made by modes with asymptotically large (as $\epsilon \to 0$) numbers.

Let us introduce some notation. We arbitrarily fix the parameter $z \in (-\infty, \infty)$ and $z \neq 0$, and by $\theta_z = \theta_z(\epsilon)$ we denote a value from the half-interval $[0, 2)$ such that the quantity $z\epsilon^{\gamma-1} + \theta_z$ is an odd integer. Further, consider the set of integers

$$K_z = \{(z\epsilon^{\gamma-1} + \theta_z)(2k+1), \qquad k = 0, \pm 1, \pm 2, \dots\}.$$

For the values of k from K_z, the asymptotic formulas (2.5) take the form

$$\lambda_k(\epsilon) = (z\epsilon^{\gamma-1} + \theta_z)\pi(2k+1)(1-2\epsilon)i - z^2\epsilon^{2\gamma}\pi^2(2k+1)^2 - c\epsilon^{2\gamma} + o(\epsilon^{2\alpha}).$$

In the case under consideration, the analog of the formal series (3.2) is

$$v = \epsilon^{\gamma/2}\xi(\tau, x) + \epsilon^{2\gamma}v_1(\tau, x) + \epsilon^{5\gamma/2}v_2(\tau, x) + \cdots,$$

where $\tau = \epsilon^{2\gamma t}$, $x = (z\epsilon^{\gamma-1} + \theta)(1-2\epsilon)t$. Substituting this expression into (3.3) and performing standard actions, we obtain the following boundary-value problem for determining the function $\xi(\tau, x)$:

$$\frac{1}{2}\frac{\partial \xi}{\partial \epsilon} = z^2\frac{\partial^2 \xi}{\partial x^2} - c\xi + z\xi^2\frac{\partial \xi}{\partial x}, \tag{3.12}$$

$$\xi(\tau, x+1) \equiv -\xi(\tau, x). \tag{3.13}$$

Let us state the final result.

Theorem 4. *Suppose that, for some $z = z_0$, for all $\tau > \tau_0$, the boundary-value problem* (3.12), (3.13) *has a bounded solution $\xi_0(\tau, y)$ together with its derivatives with respect to τ and y. Then, for $\alpha = 1/2$ and $y = c\epsilon^{2\gamma}$, Eq.* (1.3) *has an asymptotic* (*with respect to the residual up to $O(\epsilon^{5\gamma/2})$*) *solution $u_0(t, \epsilon)$ for which*

$$u_0(t,\epsilon) = 1 + \epsilon^{\gamma/2}\xi_0(\tau, x) - \epsilon^{2\gamma} z\xi_0(\tau, x)\frac{\partial \xi_0(\tau, x)}{\partial x},$$

where $\tau = \epsilon^{2\gamma}t$, $x = (z_0\epsilon^{\gamma-1} + \theta)(1 - 2\epsilon)t$.

3.4. A More Complicated Construction

Let condition (3.11) hold. Let us fix an arbitrary integer $n > 1$ and consider an arbitrary set of numbers $z_1, \dots, z_n$, $z_j \neq 0$. By $\theta_j = \theta_j(z_j, \epsilon) \in [0, 2)$ we denote a value for which the expression $z_j\epsilon^{\gamma-1} + \theta$ is an odd integer. The role of the boundary-value problem (3.12), (3.13) is played by the equations

$$\frac{1}{2}\frac{\partial \xi}{\partial \tau} = \left(z_1\frac{\partial}{\partial x_1} + \dots + z_n\frac{\partial}{\partial x_n}\right)^2 \xi - c\xi + \xi^2\left(z_1\frac{\partial}{\partial x_1} + \dots + z_n\frac{\partial}{\partial x_n}\right)\xi \tag{3.14}$$

with 1-periodic or 1-antiperiodic (in $x_1, \dots, x_n$) boundary conditions, and the number of antiperiodic boundary conditions is odd.

The assertion of Theorem 3 holds and the solutions of (1.3) and (3.14) are related by

$$u = 1 + \epsilon^{\gamma/2}\xi(\tau, x_1, \dots, x_n) + \epsilon^{2\gamma}\xi(\tau, x_1, \dots, x_n)\left(z_1\frac{\partial}{\partial x_1} + \dots + z_n\frac{\partial}{\partial x_n}\right)\xi(\tau, x_1, \dots, x_n),$$

where $\tau = \epsilon^{2\gamma}t$, $x_j = (z_j\epsilon^{\gamma-1} + \theta_j)(1 - 2\epsilon)t$.

4. ASYMPTOTICS OF THE RELAXATION CYCLE FOR LARGE λ

Here we assume that the condition $\alpha < 1/2$ holds and the parameter λ is sufficiently large. To study of the dynamical properties of Eq. (1.3), we shall use a special asymptotic method developed in [3], [5], [11]. With regard to Eq. (1.3), this is as follows. First, we fix a set of initial functions S in the phase space $C_{[-1,0]}$ of Eq. (1.3). Further, we study the asymptotics as $\lambda \to \infty$ of all solutions with initial conditions from S. As it turns out, after some time has elapsed, each such solution is again contained in the original set S. Therefore, from the solutions of (1.3), we can construct a first-return operator that takes the set S into itself. Using well-known assertions about mappings of such type, we conclude that, in S, there exists a fixed point (a function $\varphi_0(s) \in C_{[-1,0]}$) of the operator under consideration. The solution (1.3) with the initial condition $\varphi_0(s)$ is, obviously, periodic. This solution is usually called *slowly oscillating*, because, as will be shown in what follows, the distance between the "splashes" (the roots of the equation $u(t) = 1$) is not less than the delay time 1.

The set $S \subset C_{[-1,0]}$ is defined as follows. Let S be a set of all such nonnegative continuous functions $\varphi(s)$ from $C_{[-1,0]}$ for which the following conditions hold:

1) $\varphi(0) = 1$;

2) $0 \le \varphi(s) \le \overline{\varphi}(s)$, where

$$\overline{\varphi}(s) = \begin{cases} \delta & \text{if } -1 \le s \le -\delta \\ (\delta^{-1} - 1)s + 1 & \text{if } -\delta \le s \le 0,\ \delta = \lambda^{-1/2} \ll 1. \end{cases}$$

By $u(t,\varphi)$ we denote the solution of (1.3) with initial (for $t=0$) condition $u(0,\varphi)=\varphi(s)\in S$. Let us now construct the asymptotics of $u(t,\varphi)$ as $\lambda\to\infty$.

First, consider two equations

$$\dot{u}=\lambda u[1-\alpha u-(1-\alpha)\delta], \tag{4.1}$$

$$\dot{u}=\lambda u[1-\alpha u]. \tag{4.2}$$

Denote by $u_0(t)$ and $u^0(t)$, respectively, the solutions of these equations with initial conditions $u_0(0)=u^0(0)=1$. Therefore,

$$\begin{aligned} u_0(t)&=[1-(1-\alpha)\delta]\exp(\lambda(1-(1-\alpha)v)t)\\ &\qquad\times[1-\alpha-(a-\alpha)\delta+\alpha\exp(\lambda(1-(1-\alpha)\delta)t)]^{-1},\\ u^0(t)&=\exp(\lambda t)[1-\alpha+\alpha\exp(\lambda t)]^{-1}. \end{aligned}$$

Using well-known results on differential inequalities, we see that

$$u_0(t)\le u(t,\varphi)\le u^0(t)\qquad\text{for}\quad t\in[0,1-\delta], \tag{4.3}$$

$$u^0(t)\ge u(t,\varphi)\qquad\text{for}\quad t\in[0,1]. \tag{4.4}$$

Hence we conclude that, as $\lambda\to\infty$, the asymptotic relations

$$u(t,\varphi)=\alpha^{-1}+o(1)\qquad\text{and}\qquad u(1,\varphi)\le\alpha^{-1} \tag{4.5}$$

hold uniformly on the closed interval $[\delta,1-\delta]$ and over all $\varphi(s)\in S$.

Now let $t\in[1,2-\delta]$. Consider the equation

$$\dot{u}=\lambda u[1-\alpha u-(1-\alpha)u_0(t-1)]$$

and by $u_{00}(t)$ denote its solution with initial (for $t=1$) condition $u_{00}(1)=\alpha^{-1}$. It follows from the inequality $u(t,\varphi)\le u_{00}(t)$ and from the properties of $u_{00}(t)$ that, uniformly in $t\in[1+\delta,2-\delta]$, we have

$$u(t,\varphi)=o(1) \tag{4.6}$$

and, for the first positive root $t_1(\varphi)$ of the equation $u(t,\varphi)=1$, the following relation holds:

$$t_1=1+o(\lambda^{-1/2}). \tag{4.7}$$

Further, for $t\in[1+\delta,2-\delta]$, Eq. (1.3) can be expressed as

$$\dot{u}=\lambda u[1-(1-\alpha)\alpha^{-1}+o(1)];$$

therefore,

$$u(t,\varphi)=\exp\biggl[\lambda\frac{2\alpha-1}{\alpha}(1+o(1))(t-t_1)\biggr] \tag{4.8}$$

and, in particular,

$$u(2-\delta,\varphi)=\exp\biggl[\lambda\frac{2\alpha-1}{\alpha}(1+o(1))\biggr].$$

Hence, from (1.3), we conclude that

$$u(2,\varphi)=\exp\biggl[\lambda\frac{2\alpha-1}{\alpha}(1+o(1))\biggr].$$

Denote by $t_2(\varphi)$ the second positive root of the equation $u(t,\varphi)=1$. For $t\in[2+\delta,3-\delta]$ and, as long as $u(t,\varphi)=o(1)$, the function $u(t,\varphi)$ is a solution of the equation $\dot{u}=\lambda u[1-\alpha u+o(1)]$, and hence

$$u(t,\varphi)=u(2,\varphi)\exp[\lambda(1+o(1))(t-2)]. \tag{4.9}$$

Thus, we conclude that the root $t_2(\varphi)$ exists and the following asymptotic equality holds:

$$t_2(\varphi)=2+(1-2\alpha)\alpha^{-1}+o(1)=\alpha^{-1}+o(1).$$

Consider the operator

$$\Pi(\varphi(s)) = u(s + t_2(\varphi), \varphi).$$

From (4.7) and (4.8), we then see that $\Pi(\varphi(s)) \in S$, i.e., $\Pi S \subset S$. It follows from well-known results (see, for example, [16]) that, in S, there exists a fixed point $\varphi_0(s)$ of the operator Π: $\Pi(\varphi_0(s)) = \varphi_0(s)$. The solution $u_0(t, \lambda)$ of Eq. (1.3) with initial condition $\varphi_0(s)$ is periodic with period $T_\alpha(\lambda) = t_2(\varphi_0(s))$ and, as $\lambda \to \infty$,

$$T_\alpha(\lambda) = \alpha^{-1} + o(1). \tag{4.10}$$

Thus, we have proved the following statement.

Theorem 5. *Under the condition $0 < \alpha < 1/2$, there exists a $\lambda_0 > 0$ such that, for all $\lambda \ge \lambda_0$, Eq.* (1.3) *has a slowly oscillating periodic solution $u_\alpha(t, \lambda)$ with period $T_\alpha(\lambda)$ for which the asymptotic equalities* (4.10) *as well as the following equalities hold*:

$$\begin{aligned} &u_\alpha(0, \lambda) = 1, \\ &u_\alpha(t, \lambda) = \alpha + o(1) \qquad \textit{for each} \quad t \in (0, 1), \\ &u_\alpha(0, \lambda) = o(1) \qquad \textit{for each} \quad t \in (1, T_0(\lambda)). \end{aligned}$$

Applying methods from [3], [4], we can show that the solution $u_0(t, \lambda)$ is asymptotically orbitally stable.

Let us make a few remarks.

Remark 1. The set S can be significantly enlarged. So, for example, the parameter δ appearing in the definition of S need not depend on λ.

Remark 2. Once the "zero" approximation $u_\lambda(t, \lambda)$ is obtained, we can find the asymptotic expansion with arbitrary accuracy with respect to λ.

It is interesting to compare the asymptotics for $\lambda \gg 1$ of the slowly oscillating solutions $u_0(t, \lambda)$ (with period $T_0(\lambda)$) of the equation

$$\dot{u} = \lambda u[1 - u(t - 1)] \tag{4.11}$$

with the solution $u_\alpha(t, \lambda)$ of Eq. (1.3).

So, for $u_0(t, \lambda)$, we have [3], [4]:

$$\begin{aligned} T_0(\lambda) &= \lambda^{-1}(1 + o(1)) \exp(\lambda), \\ \max_t u_0(t, \lambda) &= (1 + o(1)) \exp(\lambda - 1), \\ \min_t u_0(t, \lambda) &= \exp[-\exp(\lambda(1 + o(1)))]. \end{aligned}$$

Note that, as $\alpha \to 0$, the following relations hold:

$$\max_t (u_{\alpha(t,\lambda)}) \to \infty, \qquad T_\alpha(\lambda) \to \infty.$$

With a view for a discussion and propagation of the results obtained in this paper, let us consider the logistic equation (1.9) with delay and constraints:

$$\dot{u} = \lambda[1 - u(t - 1)]u(A - u), \qquad A > 1.$$

For all $t > t_0$, the solutions with initial (for some t_0) condition $0 \le \varphi(s) \le A$ remain in the range between 0 and A. Applying the asymptotic method proposed above, we fix, for the initial set, the same set $S \subset C_{[-1,0]}$ as in the proof of Theorem 5. Let us state the corresponding result for Eq. (1.9).

Theorem 6. *Under the condition $A > 1$, there exists a λ_0, $\lambda_0 > 0$, such that, for all $\lambda \geq \lambda_0$, Eq. (1.9) has a slowly oscillating periodic solution $u_A(t, \lambda)$ with period $T_A(\lambda)$ for which the asymptotic equalities*

$$T_A(\lambda) = A^2(A-1)^{-1},$$

as well as

$$u_A(0, \lambda) = 1,$$
$$u_A(t, \lambda) = A + 0(1) \qquad \text{for each} \quad t \in (0, A(A-1)^{-1}),$$
$$u_A(t, \lambda) = 0(1) \qquad \text{for each} \quad t \in (A(A-1)^{-1}, T_A(\lambda)).$$

hold.

The graphs of $u_0(t, \lambda)$, $u_\alpha(t, \lambda)$, and $u_A(t, \lambda)$ are given in Fig. 5. The main difference between $u_\alpha(t, \lambda)$ and $u_A(t, \lambda)$ is only in that the "steps" are intervals, where $u_{\alpha,A} > 1$, which have width $1 + 0(1)$ for $u_\alpha(t)$ and greater width for $u_A(t, \lambda)$.

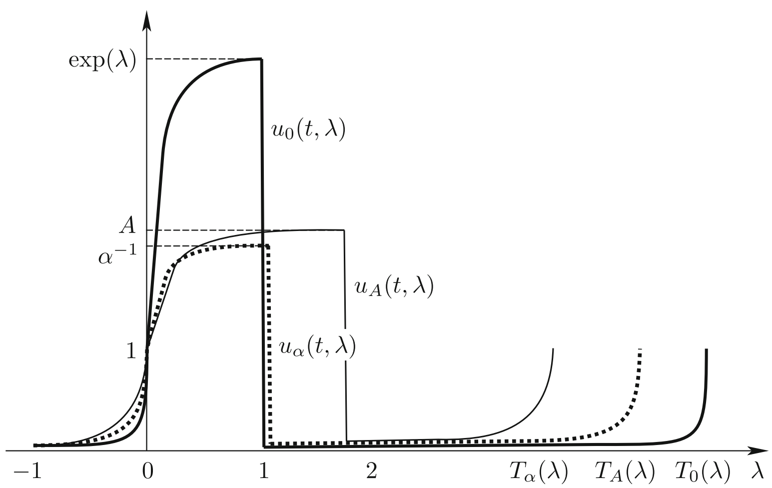

Fig. 5. The shape of the functions $u_0(t, \lambda)$, $u_\alpha(t, \lambda)$, and $u_A(t, \lambda)$.

The structure of the slowly oscillating periodic solution of the equation

$$\dot{u} = \lambda[1 - \alpha u - (1-\alpha)u(t-1)]u(A-u)$$

varies (compared to $u_{\alpha,A}(t, \lambda)$) insignificantly. Certainly, for all the equations examined here, the following statement is valid: the number of periodic (rapidly oscillating) solutions increases unboundedly as $\lambda \to \infty$.

5. CONCLUSIONS

The dynamical properties of the logistic equation containing the delay (1.3), essentially differs from those of the logistic equation (1.1) and the logistic equation with delay (1.2). For $\alpha > 1/2$, the equilibrium state $u_0 \equiv 1$ is stable for all positive λ. For $0 < \alpha < 1/2$, bifurcation phenomena may arise as soon as the parameter λ reaches some threshold value $\lambda(\alpha)$. For $\alpha \sim 1/2$ and $\lambda \gg 1$, the local dynamical properties are described by the nonlocal behavior of the solutions of special nonlinear boundary-value problems of parabolic type. It is natural to expect that, in this case, we deal with complicated and nonregular oscillations. But if $\alpha < 1/2$ and $\lambda \gg 1$, then the oscillations are of clear-cut relaxation type. The existence of a relaxation cycle is established and its asymptotics is obtained. A comparison with the asymptotic characteristics of the cycle for the Hutchinson equation. (1.2) and some of its generalizations is given.

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