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# Asymptotics of Solutions of Volterra Integral Equations with Difference Kernel

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**Abstract**—The paper deals with the asymptotics of solutions of the Volterra integral equation with difference kernel in the case where the free term has exponential growth.

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Consider the integral equation

$$x(t) = \int_0^t K(t-s)x(s)\,ds + f(t). \tag{0.1}$$

We study the asymptotics of solutions of Eq. (0.1) in the case where the free term is of the form

$$f = \sum_{l=1}^{s} e^{\mu_l t}\phi_l(t),$$

where $\mu_l = \alpha_l + i\beta_l$, $\alpha_l, \beta_l \in \mathbb{R}$, $\phi_l \in A_m$, $\mathbb{R}$ is the set of real numbers, and $A_m[0,\infty)$ is the set of functions continuous on $[0,\infty)$ and admitting the expansion

$$z(t) = \sum_{k=0}^{m} \frac{z_k}{(t+1)^k} + \frac{o(1)}{(t+1)^m}, \qquad t \to \infty.$$

It suffices to consider the case in which $\operatorname{Re}\mu_l = \alpha$ for all $l$, because if $\operatorname{Re}\gamma < \operatorname{Re}\mu_l$, then $e^{\gamma t} = o(e^{\mu_l t})$.

Let $\widehat{K}(z)$ denote the Laplace transform of the kernel $K$, and let $K * x$ denote the convolution

$$\int_0^t K(t-s)x(s)\,ds.$$

Let $P_\gamma(t)$ be a polynomial of degree $\le \gamma$.

Further, we assume that $\operatorname{Re}\mu > 0$, because, for $\operatorname{Re}\mu = 0$, the asymptotics of Eq. (0.1) was well studied (see, for example, [1]).

We shall need the following statements.

**Proposition** (see [1, p. 24]). *Suppose that $\beta > 0$, $k, r \ge 0$ are integers, and $\operatorname{Re}\lambda \ge 0$. Then*

$$\frac{e^{i\gamma t}}{(t+1)^\beta} * t^r e^{\lambda t} = \begin{cases} e^{\lambda t}P_r(t) + \displaystyle\sum_{l=\beta}^{\beta+k} c_l \frac{e^{i\gamma t}}{(t+1)^l} + \frac{O(1)}{(t+1)^{k+\beta+1}}, & \lambda \ne i\gamma, \\ c e^{i\gamma t}(t+1)^{r+1-\beta} + e^{i\gamma t}P_r(t), & \lambda = i\gamma. \end{cases}$$

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**Lemma.** *Suppose that $m > 0$, $\gamma \geq 0$ is an integer, $\lambda$, $\mu$ are complex numbers, and $\operatorname{Re}\lambda > 0$. Then, for $\phi \in A_m$, the convolution $t^\gamma e^{\lambda t} * e^{\mu t}\phi(t)$ is of the form*

$$t^\gamma e^{\lambda t} * e^{\mu t}\phi(t) = \begin{cases} P_\gamma(t)e^{\lambda t} + e^{\mu t}\biggl(\displaystyle\sum_{j=0}^{m} \frac{c_j}{(t+1)^j} + \frac{o(1)}{(t+1)^m}\biggr), & \operatorname{Re}\mu \leq \operatorname{Re}\lambda, \\ e^{\mu t}\biggl(\displaystyle\sum_{j=0}^{m} \frac{c_j}{(t+1)^j} + \frac{o(1)}{(t+1)^m}\biggr), & \operatorname{Re}\mu > \operatorname{Re}\lambda. \end{cases}$$

**Proof.** Let $\operatorname{Re}\mu < \operatorname{Re}\lambda$. Then

$$t^\gamma e^{(\lambda-\mu)t} * c_0 = P_\gamma(t)e^{(\lambda-\mu)t}.$$

Further, for $k \geq 1$, by Lemma 3 [1, p. 24], we have

$$t^\gamma e^{(\lambda-\mu)t} * \frac{1}{(t+1)^k} = e^{(\lambda-\mu)t}P_\gamma(t) + \sum_{l=k}^{m} \frac{c_l}{(t+1)^l} + \frac{o(1)}{(t+1)^m}\,.$$

Finally,

$$\begin{aligned} t^\gamma e^{(\lambda-\mu)t} * \frac{o(1)}{(t+1)^m} &= e^{(\lambda-\mu)t}\int_0^\infty (t-s)^\gamma e^{-(\lambda+\mu)s} o\biggl(\frac{1}{(s+1)^m}\biggr)\,ds \\ &\quad + \int_t^\infty (t-s)^\gamma e^{(\lambda-\mu)(t-s)} o\biggl(\frac{1}{(s+1)^m}\biggr)\,ds \\ &\leq P_\gamma(t)e^{(\lambda-\mu)t} + o\biggl(\frac{1}{(t+1)^m}\biggr)\int_0^\infty e^{-(\lambda-\mu)\theta}\theta^\gamma\,d\theta. \end{aligned}$$

Thus,

$$\begin{aligned} t^\gamma e^{\lambda t} * e^{\mu t}\phi(t) &= e^{\mu t}\int_0^t (t-s)^\gamma e^{(\lambda-\mu)(t-s)}\phi(s)\,ds \\ &= e^{\mu t}\biggl(P_\gamma(t)e^{(\lambda-\mu)t} + \sum_{j=0}^{m} \frac{c_j}{(t+1)^j} + \frac{o(1)}{(t+1)^m}\biggr) \\ &= P_\gamma(t)e^{\lambda t} + e^{\mu t}\biggl(\sum_{j=0}^{m} \frac{c_j}{(t+1)^j} + \frac{o(1)}{(t+1)^m}\biggr). \end{aligned}$$

Let $\operatorname{Re}\mu = \operatorname{Re}\lambda$. Then

$$\begin{aligned} t^\gamma e^{\lambda t} * e^{\mu t}\phi(t) &= \int_0^t (t-s)^\gamma e^{\lambda(t-s)} e^{\mu s}\phi(s)\,ds = \int_0^t (t-s)^\gamma e^{\lambda(t-s)} e^{\mu(s-t)} e^{\mu t}\phi(s)\,ds \\ &= e^{\mu t}\int_0^t (t-s)^\gamma e^{\operatorname{Im}(\lambda-\mu)(t-s)}\phi(s)\,ds = e^{\lambda t}(t^\gamma e^{\operatorname{Im}(\lambda-\mu)t} * \phi(t)). \end{aligned}$$

In view of the proof of the theorem given in [1, p. 82], we can write

$$t^\gamma e^{\operatorname{Im}(\lambda-\mu)t} * \phi(t) = P_\gamma(t)e^{\operatorname{Im}(\lambda-\mu)t} + \sum_{j=0}^{m} \frac{c_j}{(t+1)^j} + \frac{o(1)}{(t+1)^m}\,.$$

Therefore, the convolution $t^\gamma e^{\lambda t} * e^{\mu t}\phi(t)$ is of the form

$$t^\gamma e^{\lambda t} * e^{\mu t}\phi(t) = e^{\lambda t}\biggl(P_\gamma(t) + \sum_{j=0}^{m} \frac{c_j}{(t+1)^j} + \frac{o(1)}{(t+1)^m}\biggr).$$

Let $0 < \operatorname{Re}\lambda < \operatorname{Re}\mu$. Denote $H(t) = t^\gamma e^{(\lambda-\mu)t}$. Then

$$t^\gamma e^{\lambda t} * e^{\mu t}\phi(t) = e^{\mu t}\int_0^t (t-s)^\gamma e^{(\lambda-\mu)(t-s)}\phi(s)\,ds = e^{\mu t}(t^\gamma e^{(\lambda-\mu)t} * \phi(t)) = e^{\mu t}(H * \phi)(t).$$

Since

$$\int_0^\infty t^m H(t)\,dt = \int_0^\infty t^{m+\gamma} e^{(\lambda-\mu)t}\,dt < \infty,$$

we have $t^m H \in L_1[0,\infty)$. Therefore, for $\phi \in A_m$, the assumptions of Lemma 4 from [1, p. 65] hold, and

$$t^\gamma e^{\lambda t} * e^{\mu t}\phi(t) = e^{\mu t}(H * \phi(t)) = e^{\mu t}\biggl(\sum_{j=0}^m \frac{c_j}{(t+1)^j} + \frac{o(1)}{(t+1)^m}\biggr).$$

Finally, we obtain

$$t^\gamma e^{\lambda t} * e^{\mu t}\phi(t) = \begin{cases} P_\gamma(t)e^{\lambda t} + e^{\mu t}\biggl(\displaystyle\sum_{j=0}^m \frac{c_j}{(t+1)^j} + \frac{o(1)}{(t+1)^m}\biggr), & \operatorname{Re}\mu \le \operatorname{Re}\lambda, \\ e^{\mu t}\biggl(\displaystyle\sum_{j=0}^m \frac{c_j}{(t+1)^j} + \frac{o(1)}{(t+1)^m}\biggr), & \operatorname{Re}\mu > \operatorname{Re}\lambda. \end{cases}$$

□

Let $K$ be the kernel of Eq. (0.1), let $R(t)$ be its resolvent, and let $t^m K \in L_1$ for some $m$. Consider two cases.

*Case* 1. The kernel $K$ is stable. This means that $R(t) \in L_1$. Since $t^m K \in L_1$, we also have $t^m R(t) \in L_1$ [1]. In addition, stability means that the equation $\widehat{K}(z) = 1$ has no roots for $\operatorname{Re} z \ge 0$ [1].

**Theorem 1.** *Let*

$$f = \sum_{l=1}^s e^{\mu_l t}\phi_l(t),$$

*where* $\mu_l = \alpha + i\beta_l$, $\beta_l \in R$, *and* $\phi_l \in A_m$. *Then the solution of Eq.* (0.1) *is of the form*

$$x = \sum_{l=1}^s e^{\mu_l t}\xi_l(t),$$

*where* $\xi_l(t) \in A_m$.

**Proof.** The solution of Eq. (0.1) can be expressed as $x(t) = f(t) + R * f(t)$, where

$$f = \sum_{l=1}^s e^{\mu_l t}\phi_l(t), \qquad \mu_l = \alpha + i\beta_l, \qquad \phi_l \in A_m.$$

Consider the convolution $R * e^{\mu_l t}\phi_l(t)$ for $l = 1, \dots, s$:

$$R * e^{\mu_l t}\phi_l(t) = e^{\mu_l t}\int_0^t R(t-s)e^{-\mu_l(t-s)}\phi_l(s)\,ds.$$

Denote $H(t) = R(t)e^{-\mu_l t}$. Then

$$R * e^{\mu_l t}\phi_l(t) = e^{\mu_l t}(H * \phi_l(t)).$$

Since $t^m R(t) \in L_1[0,\infty)$, it follows that $t^m H \in L_1[0,\infty)$ and, by the corollary of Lemma 4 from [1, p. 70], $H * \phi_l(t) \in A_m$.

Therefore,

$$x(t) = \sum_{l=1}^s e^{\mu_l t}\biggl(\sum_{i=0}^m \frac{c_l i}{(t+1)^i} + \frac{o(1)}{(t+1)^m}\biggr) = \sum_{l=1}^s e^{\mu_l t}\xi_l(t), \qquad \text{where} \quad \xi_l(t) \in A_m. \quad \square$$

*Case* 2. Now consider the case of an unstable kernel.

**Theorem 2.** *Let the equation* $1-\widehat{K}(z)=0$ *have roots* $\lambda_1,\lambda_2,\dots,\lambda_k,\dots$ *of multiplicity* $m_k$ *in the half-plane* $\operatorname{Re} z\ge 0$ *such that* $\operatorname{Re}\lambda_1\ge\operatorname{Re}\lambda_2\ge\dots\ge\operatorname{Re}\lambda_k\ge\dots$ *. Then if*

$$f=\sum_{l=1}^{s} e^{\mu_l t}\phi_l(t),$$

*where* $\mu_l=\alpha+i\beta_l$, $\alpha,\beta_l\in R$, *and* $\phi_l\in A_m$, *then, as* $t\to\infty$, *Eq.* (0.1) *has a solution of the form*

$$x(t)=\begin{cases}\displaystyle\sum_{l=1}^{s} e^{\mu_l t}\biggl(\sum_{j=0}^{m}\frac{c_l j}{(t+1)^j}+\frac{o(1)}{(t+1)^m}\biggr), & \alpha>\operatorname{Re}\lambda_k\ge\operatorname{Re}\lambda_{k+1}\ge\cdots,\\ \displaystyle\sum_{l=1}^{k} P_{m_j-1}(t)e^{\lambda_j t} & \\ \displaystyle\qquad+\sum_{l=1}^{s} e^{\mu_l t}\biggl(\sum_{j=0}^{m}\frac{c_l j}{(t+1)^j}+\frac{o(1)}{(t+1)^m}\biggr), & \alpha\le\operatorname{Re}\lambda_k\le\operatorname{Re}\lambda_{k-1}\le\cdots.\end{cases}$$

**Proof.** Let $\alpha>\operatorname{Re}\lambda_k\ge\operatorname{Re}\lambda_{k+1}\ge\cdots$, and let $\gamma$ satisfy the chain of inequalities

$$\alpha>\gamma>\operatorname{Re}\lambda_k\ge\operatorname{Re}\lambda_{k+1}\ge\cdots.$$

Since

$$R(t)=K(t)+\int_0^t R(t-s)K(s)\,ds,$$

we have

$$R(t)e^{-\gamma t}=e^{-\gamma t}K(t)+\int_0^t e^{-\gamma(t-s)}R(t-s)K(s)e^{-\gamma s}\,ds.$$

Denote $K_1(t)=e^{-\gamma t}K(t)$ and $R_1(t)=e^{-\gamma t}R(t)$. Then

$$R_1(t)=K_1(t)+\int_0^t R_1(t-s)K_1(s)\,ds,$$

$$\widehat{K}_1(z)=\int_0^\infty e^{-zt}e^{-\gamma t}K(t)\,dt=\int_0^\infty e^{-(z+\gamma)t}K(t)\,dt=\widehat{K}(z+\gamma).$$

Therefore, the equation $1-\widehat{K}_1(z)=0$ has no roots for $\operatorname{Re} z\ge 0$, i.e., the kernel $K_1$ is stable. Then, by Theorem 1, the solution of the equation

$$x_1(t)=\int_0^t K_1(t-s)x_1(s)\,ds+f_1(1)$$

is of the form

$$x_1(t)=\sum_{l=1}^{s} e^{(\mu_l-\gamma)t}\xi_l(t),\qquad\text{where}\quad \xi_l\in A_m,$$

and hence the solution of Eq. (0.1) can be written as

$$x(t)=\sum_{l=1}^{s} e^{\mu_l t}\biggl(\sum_{j=0}^{m}\frac{c_l j}{(t+1)^j}+\frac{o(1)}{(t+1)^m}\biggr),\qquad t\to\infty.$$

Now let $\alpha\le\operatorname{Re}\lambda_k\le\operatorname{Re}\lambda_{k-1}\le\cdots$, and let $\gamma<\alpha$. Since $\gamma<\operatorname{Re}\mu\le\operatorname{Re}\lambda_k\le\operatorname{Re}\lambda_{k-1}\le\cdots$ and

$$R(t)=K(t)+\int_0^t R(t-s)K(s)\,ds,$$

it follows that

$$R(t)e^{-\gamma t} = e^{-\gamma t}K(t) + \int_0^t e^{-\gamma(t-s)}R(t-s)K(s)e^{-\gamma s}\,ds.$$

Denoting $K_1(t) = e^{-\gamma t}K(t)$ and $R_1(t) = e^{-\gamma t}R(t)$, we obtain

$$R_1(t) = K_1(t) + \int_0^t R_1(t-s)K_1(s)\,ds, \qquad \widehat{K}_1(z) = \int_0^\infty e^{-zt}e^{-\gamma t}K(t)\,dt = \widehat{K}(z+\gamma)$$

just as above. Hence the equation $1 - \widehat{K}_1(z) = 0$ has roots of the form $\lambda_j - \gamma$ such that $\operatorname{Re}(\lambda_j - \gamma) > 0$, $j < k$. For $j \ge k$, this equation has no roots.

In view of the equation

$$x(t) = f(t) + \int_0^t R(t-s)f(s)\,ds,$$

denoting $x_1(t) = e^{-\gamma t}x(t)$ and $f_1(t) = e^{-\gamma t}f(t)$, we obtain

$$x_1(t) = f_1(t) + \int_0^t R_1(t-s)f_1(s)\,ds.$$

Since

$$f = \sum_{l=1}^{s} e^{\mu_l t}\phi_l(t),$$

we have

$$f_1(t) = \sum_{l=1}^{s} e^{(\mu_l-\gamma)t}\phi_l(t)$$

and the solution is of the form

$$x_1(t) = \sum_{l=1}^{s} e^{-(\gamma-\mu_l)t}\phi_l(t) + \sum_{l=1}^{s}\int_0^t e^{-\gamma(t-s)}R(t-s)e^{(\mu_l-\gamma)s}\phi_l(s)\,ds.$$

In view of the remark to Theorem 2 from [1, p. 46], we can write

$$R_1(t) = R_2(t) + Q * R_2(t),$$

where

$$Q(t) = \sum_{j=1}^{k} P_{m_j-1}(t)e^{(\lambda_j-\gamma)t}$$

and $t^m R_2(t) \in L_1[0,\infty)$. Since

$$\int_0^t (t-s)^l e^{\lambda(t-s)}R_2(s)\,ds = e^{\lambda t}\int_0^\infty (t-s)^l e^{-\lambda s}R_2(s)\,ds + \int_t^\infty (t-s)^l e^{\lambda(t-s)}R_2(s)\,ds,$$

and

$$\int_0^\infty t^m \int_t^\infty e^{-\operatorname{Re}\lambda(s-t)}(s-t)^l|R_2(s)|\,ds\,dt \le \int_0^\infty |R_2(s)|s^m \int_0^s \tau^l e^{-\operatorname{Re}\lambda\tau}\,d\tau\,ds < \infty,$$

it follows that

$$R_1(t) = \sum_{j=1}^{k} P_{m_j-1}(t)e^{(\lambda_j-\gamma)t} + R_3(t),$$

where $t^m R_3(t) \in L_1[0,\infty)$.

By the lemma, the convolution $t^q e^{(\lambda_j-\gamma)t} * e^{(\mu_l-\gamma)t}\phi_l$, where $q=1,\dots,m_j-1$, $l=1,\dots,s$, and $j=1,\dots,k$, is of the form

$$t^q e^{(\lambda_j-\gamma)t} * e^{(\mu_l-\gamma)t}\phi_l = \begin{cases} P_q(t)e^{(\lambda_j-\gamma)t} \\ \qquad + e^{(\mu_l-\gamma)t}\biggl(\displaystyle\sum_{j=0}^{m}\frac{c_j}{(t+1)^j}+\frac{o(1)}{(t+1)^m}\biggr), & \operatorname{Re}\mu_l < \operatorname{Re}\lambda_k; \\ e^{(\mu_l-\gamma)t}\biggl(\displaystyle\sum_{j=0}^{m}\frac{c_j}{(t+1)^j}+\frac{o(1)}{(t+1)^m}\biggr), & \operatorname{Re}\mu_l > \operatorname{Re}\lambda_k. \end{cases}$$

Hence

$$Q * e^{(\mu_l-\gamma)t}\phi_l(t) = \sum_{j=1}^{k} P_{m_j-1}(t)e^{(\lambda_j-\gamma)t} + \sum_{l=1}^{s} e^{(\mu_l-\gamma)t}\biggl(\sum_{j=0}^{m}\frac{c_lj}{(t+1)^j}+\frac{o(1)}{(t+1)^m}\biggr).$$

Further,

$$\begin{aligned} R_3 * e^{(\mu_l-\gamma)t}\phi_l &= \int_0^t e^{(\mu_l-\gamma)(t-s)}\phi_l(t-s)R_3(s)\,ds = e^{(\mu_l-\gamma)t}\int_0^t e^{-(\mu_l-\gamma)s}R_3(s)\phi_l(t-s)\,ds \\ &= e^{(\mu_l-\gamma)t}[e^{-(\mu_l-\gamma)t}R_3(t)*\phi_l(t)] = e^{(\mu_l-\gamma)t}\widetilde{\phi}_l(t), \qquad \widetilde{\phi}_l \in A_m \end{aligned}$$

(see [1, p. 69]). This yields

$$R_1 * e^{(\mu_l-\gamma)t}\phi_l(t) = \sum_{j=1}^{k} P_{m_j-1}(t)e^{(\lambda_j-\mu_l)t} + e^{(\mu_l-\gamma)t}\phi_l(t).$$

Therefore,

$$x(t) = \sum_{l=1}^{k} P_{m_j-1}(t)e^{\lambda_j t} + \sum_{l=1}^{s} e^{\mu_l t}\biggl(\sum_{j=0}^{m}\frac{c_lj}{(t+1)^j}+\frac{o(1)}{(t+1)^m}\biggr), \qquad t\to\infty.$$

Combining these results, we obtain the solution of Eq. (0.1) in the form

$$x(t) = \begin{cases} \displaystyle\sum_{l=1}^{s} e^{\mu_l t}\biggl(\sum_{j=0}^{m}\frac{c_lj}{(t+1)^j}+\frac{o(1)}{(t+1)^m}\biggr), & \alpha > \operatorname{Re}\lambda_k \ge \operatorname{Re}\lambda_{k+1} \ge \dots, \\ \displaystyle\sum_{l=1}^{k} P_{m_j-1}(t)e^{\lambda_j t} \\ \qquad + \displaystyle\sum_{l=1}^{s} e^{\mu_l t}\biggl(\sum_{j=0}^{m}\frac{c_lj}{(t+1)^j}+\frac{o(1)}{(t+1)^m}\biggr), & \alpha \le \operatorname{Re}\lambda_k \le \operatorname{Re}\lambda_{k-1} \le \cdots. \end{cases} \quad \square$$

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