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The Maupertuis–Jacobi Principle for Hamiltonians of the Form $F(x, |p|)$ in Two-Dimensional Stationary Semiclassical Problems

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Abstract—We consider two-dimensional asymptotic formulas based on the Maslov canonical operator arising in stationary problems for differential and pseudodifferential equations. In the case of Lagrangian manifolds invariant with respect to Hamiltonian flow with Hamiltonians of the form $F(x, |p|)$, we show how asymptotic formulas can be simplified by using the well-known (in classical mechanics) Maupertuis–Jacobi correspondence principle to replace the Hamiltonians $F(x, |p|)$ by Hamiltonians of the form $C(x)|p|$ arising, in particular, in geometric optics and related to the Finsler metric. As examples, we consider Hamiltonians corresponding to the Schrödinger equation, the two-dimensional Dirac equation, and the pseudodifferential equations for surface water waves.

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1. INTRODUCTION

The Maupertuis–Jacobi principle [1]–[3] allows us to pass from one Hamiltonian system to another; it can be used, for example, in constructing and analyzing the trajectories. It was shown in [4], [5] that this principle is helpful in constructing certain compact Lagrangian manifolds and the corresponding asymptotic eigenvalues and eigenfunctions (quasimodes) for self-adjoint differential and pseudodifferential operators. In this paper, we wish to show how to use this principle also for the construction of noncompact Lagrangian manifolds arising in the scattering problem and the problem of the asymptotics of the Green function. We consider the case in which the original classical Hamiltonian depends on the modulus of the momentum, i.e., is of the form $F(x, |p|)$, while the second Hamiltonian has the form $C(x)|p|$ arising, in particular, in geometric optics and related to the Finsler metric (see [6], [7]). Our main observation is that, for several reasons, it is more convenient to use the Hamiltonian $C(x)|p|$ in practical situations than the original Hamiltonian $F(x, |p|)$. In this paper, we use the recently obtained new formulas for the Maslov canonical operator in a neighborhood of focal points (see [8]). We restrict our consideration to the two-dimensional case and study examples related to the Schrödinger equation, the Dirac equation, and the pseudodifferential equation for surface water waves.

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2. INVARIANT LAGRANGIAN MANIFOLDS, EIKONAL COORDINATES AND THE MAUPERTUIS–JACOBI CORRESPONDENCE PRINCIPLE

Consider a smooth function $F(x,z)$, $x\in\mathbb{R}^2$, $z\in[0,\infty)$ and a parameter E. Suppose that, for some value of the parameter E, the equation $F(x,z)=E$ has a unique solution $z=1/C(x,E)$, where $C(x,E)$ is a bounded function bounded away from zero: $C(x,E)\ge c_0(E)>0$. Also let

$$\left|\frac{\partial F}{\partial z}\left(x,\frac{1}{C(x,E)}\right)\right|\ge c_1(E)>0$$

for positive constants $c_0(E)$, $c_1(E)$. Consider the Hamiltonians

$$\mathcal{H}(x,p,E)=F(x,|p|)-E,\quad H(x,p,E)=C(x,E)|p|-1,\qquad p\in\mathbb{R}^2,$$

and the corresponding Hamiltonian systems

$$\text{(a)}\quad \frac{dp}{dt}=-\mathcal{H}_x,\quad \frac{dx}{dt}=\mathcal{H}_p,\qquad \text{(b)}\quad \frac{dp}{d\tau}=-H_x,\quad \frac{dx}{d\tau}=H_p. \tag{1}$$

Let $\mathcal{Q}=\mathbb{R}$ or $\mathcal{Q}=\mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{Q}\to T^*\mathbb{R}^2$, and let $\varphi\mapsto(P^0(\phi,E),X^0(\phi,E))$ be a smooth embedding with image $\Lambda^1=\{p=P^0(\phi,E),x=X^0(\phi,E)\}$ in the phase space $\mathbb{R}^4_{p,x}$ lying on the zero-level surface of the Hamiltonians

$$\mathcal{H}(X^0(\phi,E),P^0(\phi,E),E)=0,\qquad \text{where}\quad |P^0(\phi,E)|C(X^0(\phi,E))=1.$$

Consider the solutions $(\mathcal{P}(t,\phi,E),\mathcal{X}(t,\phi,E))$ and $(P(\tau,\phi,E),X(\tau,\phi,E))$ of systems (1) (a) and (b), respectively, with initial data on the curve Λ^1. By the general properties of Hamiltonian systems, we have

$$\mathcal{H}(\mathcal{X}(t,\phi,E),\mathcal{P}(t,\phi,E),E)=0\qquad\text{and}\qquad |P(\tau,\phi,E)|C(X(\tau,\phi,E))=1.$$

By the Maupertuis–Jacobi principle, the trajectories

$$(\mathcal{P}(t,\phi,E),\mathcal{X}(t,\phi,E)),\qquad (P(\tau,\phi,E),X(\tau,\phi,E))$$

can be expressed in terms of each other. Namely,

$$\begin{aligned}\frac{d\mathcal{P}}{dt}&=-\mathcal{H}_x(\mathcal{P},\mathcal{X})=-R(\mathcal{X})H_x(\mathcal{P},\mathcal{X})=R(\mathcal{X})\frac{dP}{d\tau},\\ \frac{d\mathcal{X}}{dt}&=\mathcal{H}_p(\mathcal{P},\mathcal{X})=R(\mathcal{X})H_p(\mathcal{P},\mathcal{X})=R(\mathcal{X})\frac{dX}{d\tau},\end{aligned}\tag{2}$$

where

$$R(x)=\lim_{z\to 1/C(x,E)}\frac{F(x,z)-E}{zC(x,E)-1}=z\frac{\partial F}{\partial z}(x,z)_{z=1/C(x,E)}=\frac{1}{C(x,E)}\,\frac{\partial F}{\partial z}\left(x,\frac{1}{C(x,E)}\right).$$

Replacing the time t by the time $\tau=\tau(t,\phi,E)$ using the equation

$$\frac{d\tau}{dt}=R(\mathcal{X}(t,\phi,E)),\qquad \tau|_{t=0}=0,\tag{3}$$

we establish the correspondence of the solutions

$$(\mathcal{P}(t,\phi,E),\mathcal{X}(t,\phi,E))=(P(\tau,\phi,E),X(\tau,\phi,E))|_{\tau=\tau(t,\phi,E)}.$$

Inverting the equality $\tau=\tau(t,\phi,E)$, we obtain the function $t=t(\tau,\phi,E)$ and

$$(P(\tau,\phi,E),X(\tau,\phi,E))=(\mathcal{P}(t,\phi,E),\mathcal{X}(t,\phi,E))|_{t=t(\tau,\phi,E)}.$$

In the four-dimensional space $\mathbb{R}^4_{p,x}$, the families of solutions

$$(P(\tau,\phi,E),X(\tau,\phi,E))\qquad\text{and}\qquad(\mathcal{P}(t,\phi,E),\mathcal{X}(t,\phi,E))$$

define the set

$$\begin{aligned}\Lambda^2&=\{(p,x)=(\mathcal{P}(t,\phi,E),\mathcal{X}(t,\phi,E)),\ \phi\in Q,\ t\in(-\infty,\infty)\}\\ &=\{(p,x)=(P(\tau,\phi,E),X(\tau,\phi,E)),\ \phi\in Q,\ \tau\in(-\infty,\infty)\}.\end{aligned}\tag{4}$$

Suppose that Λ^2 is a smooth manifold. Then it is Lagrangian (see [9]), and (t,ϕ) and (τ,ϕ) are just different coordinate systems on Λ^2. Denote the phase flows corresponding to the Hamiltonian systems (1) (a) and (b), respectively, by $g^t_{\mathcal{H}}$ and g^t_H. Then

$$\Lambda^2 = \bigcup_{-\infty<t<\infty} g^t_{\mathcal{H}}\Lambda^1 = \bigcup_{-\infty<\tau<\infty} g^\tau_H\Lambda^1.$$

By construction, the Lagrangian manifold Λ^2 is invariant with respect to these flows. The parameters t and τ are called *proper times on* Λ^2.

The following statement shows the advantage of passing to mathematical objects related to the Hamiltonian $H(p,x)$.

Lemma. *The following relations hold*:

1) The Jacobians for passing from (t,ϕ) *to* (τ,ϕ) *and conversely are*

$$\det\frac{\partial(\tau(t,\phi,E),\phi)}{\partial(t,\phi)} = \frac{d\tau}{dt} = R(\mathcal{X}(t,\phi,E)), \tag{5}$$

$$\det\frac{\partial(t(\tau,\phi,E),\phi)}{\partial(\tau,\phi)} = \frac{dt}{d\tau} = \frac{1}{R(X(\tau,\phi,E))}. \tag{6}$$

Thus, the Jacobians in the coordinates (t,ϕ) *and* (τ,ϕ) *are related by*

$$\mathcal{J} \equiv \det\frac{\partial(\mathcal{X})}{\partial(t,\phi)} = JR(\mathcal{X}(t,\phi,E)) \equiv R(\mathcal{X}(t,\phi,E))\det\frac{\partial(X)}{\partial(\tau,\phi)}\bigg|_{\tau=\tau(t)}.$$

2) The action (eikonal) on the manifold Λ^2 *is*

$$s(t,\phi) \equiv \int_{(0,0)}^{(t,\phi)} \mathcal{P}(t,\phi,E)\,d\mathcal{X}(t,\phi,E) = s_0(\phi)+\tau, \qquad s_0(\phi) = \int_0^\phi P^0(\phi,E)\,dX^0(\phi,E). \tag{7}$$

3) The pair $(\tau' = s_0(\phi)+\tau,\phi)$ *specifies the so-called eikonal coordinates on* Λ^2 *(see [8]).*

4) For the Jacobian $J = \det(\partial(X)/\partial(\tau,\phi))$, *the following relation holds*:

$$|J| = C(X(\tau,\phi))|X_\phi|, \tag{8}$$

which implies that, for the Jacobian $\mathcal{J} = \det(\partial(\mathcal{X})/\partial(t,\phi))$,

$$|\mathcal{J}| = R(\mathcal{X}(t,\phi))C(\mathcal{X}(t,\phi))|X_\phi|.$$

Proof. The first two equalities are obtained from the equalities $\partial\phi/\partial\tau = \partial\phi/\partial t = 0$. Formula (7) follows from the relations

$$\begin{aligned} s(t,\phi) &= \int_{(0,0)}^{(t,\phi)} \mathcal{P}(t,\phi,E)\,d\mathcal{X}(t,\phi,E) = \int_{(0,0)}^{(\tau,\phi)} P(\tau,\phi,E)\,dX(\tau,\phi,E) \\ &= \int_0^\phi P^0(\phi,E)\,dX^0(\phi,E) + \int_0^\tau P(\tau,\phi,E)\,\frac{dX}{d\tau}(\tau,\phi,E)\,d\tau \\ &= \int_0^\phi P^0(\phi,E)\,dX^0(\phi,E) + \int_0^\tau |P(\tau,\phi,E)|C(X(\tau,\phi,E),E)\,d\tau = s_0(\phi)+\tau. \end{aligned}$$

Assertion 3) follows from 1) and 2). Formula (8) was proved in [8], [10]. □

Let us now consider two examples of Λ^1 that are important for applications: the straight line

$$\Lambda^1_s = \{p_1 = 0,\ p_2 = k,\ x_1 = \phi,\ x_2 = a,\ \phi\in\mathbb{R}\},$$

arising in scattering problems and the circle

$$\Lambda^1_G = \{p_1 = b\cos\phi,\ p_2 = b\sin\phi,\ x_1 = a_1,\ x_2 = a_2,\ \phi\in\mathbb{S}\},$$

related to the problem of the construction of the asymptotics of the Green function. It is easy to verify that, for these curves, $s_0(\phi) = 0$, and the action on Λ^2 is $\tau' = \tau$. For an illustration of a Lagrangian manifold with caustic, see the figure. In the figure, the Lagrangian manifold is given in the phase space (in coordinates $(x_1, x_2, p_1) \subset \mathbb{R}^4_{x,p}$) and in the projection on the configuration space $\mathbb{R}^2_x$. The Lagrangian manifold $\Lambda^2 = \bigcup g^t_H \Lambda^1$ corresponds to the scattering problem with the initial curve $\Lambda^1 = \{p = (0,2),\ x = (\phi, 0),\ \phi \in \mathbb{R}\}$, the Hamiltonian is

$$H(x,p) = \frac{|p|}{E - U(x)} - 1, \qquad \text{where} \quad E = 2, \quad U(x) = \mathbf{e}(x) e^{-(x_1-5)^2-(x_2-3)^2},$$

and $\mathbf{e}(x)$ is the cut-off function

$$\mathbf{e}(x) = 0, \quad x_2 \le 0, \qquad \mathbf{e}(x) = 1, \quad x_2 \ge 1.$$

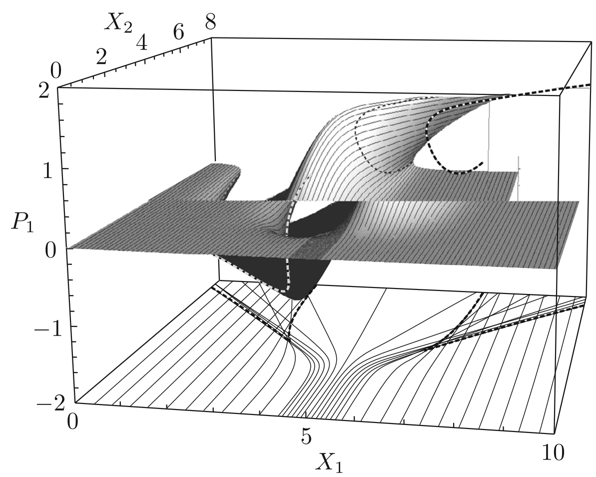

Figure: Lagrangian manifold, characteristics (thin lines) and the cycle of singularities (the dotted line).

3. APPLICATION TO THE MASLOV CANONICAL OPERATOR

Consider the Lagrangian manifold Λ^2 (constructed above) with measure $d\mu = dt \wedge d\phi$ and a smooth function $A(t,\phi)$ on it. Consider the function constructed in the form of the Maslov canonical operator

$$\psi = K^h_{\Lambda^2} A(t,\phi). \tag{9}$$

In $K^h_{\Lambda^2} A(t,\phi)$, we wish to pass from some initial coordinates (t,ϕ) to the eikonal coordinates (τ,ϕ), preserving the measure $d\mu$.

Theorem. *The following equalities hold*:

$$\begin{aligned}\psi &= K^h_{\Lambda^2}\left[A(t(\tau,\phi),\phi) \Big/ \sqrt{\det \frac{\partial(\tau,\phi)}{\partial(t,\phi)}}\right] = K^h_{\Lambda^2}\left[\frac{A(t(\tau,\phi),\phi)}{\sqrt{R(X(\tau,\phi))}}\right] \\ &= \frac{1}{\sqrt{R(x)}} K^h_{\Lambda^2}[A(t(\tau,\phi),\phi)](1 + O(h)).\end{aligned} \tag{10}$$

Proof. The proof immediately follows from equality (5) and the commutation formula for the pseudodifferential operator $\widehat{Q} = Q(x, -ih\partial/\partial x)$ with the canonical operator [9], [11]:

$$\widehat{Q} K^h_{\Lambda^2}[A(t,\phi)] = K^h_{\Lambda^2}[Q(x,p)|_{\Lambda^2} A(t,\phi)](1 + O(h)).$$

□

Recall that the canonical operator has different representations in a neighborhood of regular (nonsingular) points (where $J = \det(\partial X/\partial(\tau,\phi)) \neq 0$) and in a neighborhood of singular (focal) points (where $J = \det(\partial X/\partial(\tau,\phi)) = 0$). By equality (8), the point $(P(\tau,\phi), X(\tau,\phi)) \in \Lambda^2$ is focal if $X_\phi(\tau,\phi) = 0$. As proved in [8], under the condition that the eikonal coordinates exist in a neighborhood of the focal points, the determinant $\det(P, P_\phi) \neq 0$ does not vanish (here the 2×2 matrix (P, P_ϕ) is composed of the column vectors P and P_ϕ). This implies that the Lagrangian manifold can be covered by nonsingular charts Ω_j^{reg} in which $X_\phi(\tau,\phi) \neq 0$ and by singular charts Ω_j^{sing} in which $\det(P, P_\phi) \neq 0$. Let the functions $\{\mathbf{e}_j(\tau,\phi)\}$ constitute the partition of unity related to the charts Ω_j^{reg} and Ω_j^{sing}. By the lemma, the contribution of the regular charts to the canonical operator can be calculated by the formula

$$\psi_j = \frac{e^{-(i\pi/2)\mathbf{m}_j}}{\sqrt{R(x)C(x)|X_\phi|}}\, e^{i\tau/h} A(\tau,\phi)\mathbf{e}_j(\tau,\phi)|_{(\tau,\phi)=(\tau_j(x),\phi_j(x))}, \tag{11}$$

where $(\tau_j(x), \phi_j(x))$ is the solution of the (vector) equation $X(\tau,\phi) = x$ in the chart Ω_j^{reg} and $\mathbf{m}_j$ is the Maslov index of the nonsingular chart Ω_j^{reg} (see below). The contribution of the singular charts is given by summands of the form [8]

$$\psi_j = \frac{e^{-(i\pi/2)\mathbf{m}_j^s} e^{i\pi/4}}{\sqrt{2\pi h R(x)}} \int_{\mathbb{R}} e^{i\tau/h} \sqrt{|\det(P, P_\phi)|}\, A(\tau,\phi)\mathbf{e}_j(\tau,\phi)|_{\tau=\tau_j(x,\phi)}\, d\phi, \tag{12}$$

where $\tau_j(x,\phi)$ is the solution of the scalar equation $\langle P(\tau,\phi), x - X(\tau,\phi)\rangle = 0$ in the singular chart Ω_j^{sing} and $\mathbf{m}_j^s$ is the Maslov index of the singular chart Ω_j^{sing}.

By [8] the Maslov index $\mathbf{m}_j$ coincides with the Morse index of the trajectory issuing from the point (P, X) with coordinates $(\tau = +0, \phi)$ and ending at the point $(P(\tau,\phi), X(\tau,\phi))$ in the nonsingular chart Ω_j^{reg}: the index is equal to the number of zeros of the Jacobian $J = \det(\partial X/\partial(\tau,\phi))(\tau',\phi)$ (or the function $X_\phi(\tau',\phi)$) for $\tau' \in (+0, \tau)$. To find the Maslov index $\mathbf{m}_j^s$ of the singular chart Ω_j^{sing}, it is necessary to consider an arbitrary regular point $(P(\tau,\phi), X(\tau,\phi)) \in \Omega_j^{\mathrm{sing}}$ and compare the signs of the Jacobians $J = \det(\partial X/\partial(\tau,\phi))$ and $\det(P, P_\phi)$ at this point. If the signs coincide, then $\mathbf{m}_j^s$ is equal to the Morse index of this point $(P(\tau,\phi), X(\tau,\phi))$; otherwise, the Maslov index of the focal chart is greater than the Morse index by 1.

Finally, in order to obtain the canonical operator, we must sum all the functions ψ_j (see [9], [11]). Note that the integral (12) can be calculated in terms of the Airy or Pearcey functions (explicit formulas are given in [8]), assuming that the corresponding subset of the cycle of singularities on the Lagrangian manifold $\{(P(\tau,\phi), X(\tau,\phi))|_{X_\phi=0}\}$ is in general position ([1], [9]).

Let us consider an example. The manifold presented in the figure has two caustics (shown by dotted lines). The Lagrangian manifold over the domain of the configuration space located "inside the caustic" is folded into three sheets and there we must sum three functions of the form (11). Over the domain "outside the caustics" we have the unique sheet of the manifold, the equation $X(\tau,\phi) = x$ is uniquely solvable and the canonical operator consists of one function of the form (11). In a neighborhood of the arcs of the caustics (singularity A2-fold), the canonical operator consists of the sum of the regular (11) and nonregular (12) parts, while, near caustic cusps (singularity A3), it is sufficient to consider one singular chart.

4. EXAMPLES

Let us present a few examples of the application of the Maupertuis–Jacobi correspondence principle to the construction of the Maslov canonical operator. We shall not consider further applications to the construction of semiclassical asymptotics for equations of mathematical physics, which can be found in other papers.

Example 1 (arising from the Schrödinger equation; see [12], [13]). Consider a smooth bounded potential $U(x)$, $U(x) < E$, and the Hamiltonian

$$\mathcal{H}(x,p) = F(x,|p|) - E = \frac{p^2}{2} + U(x) - E.$$

The necessary functions are

$$C(x,E) = \frac{1}{\sqrt{2(E-U(x))}}, \qquad R(x) = z^2|_{z=1/C(x)} = 2(E-U(x)), \tag{13}$$

and the canonical operator is of the form

$$\psi(x) = \frac{1}{\sqrt{2(E-U(x))}} K^h_{\Lambda^2}[A(t(\tau,\phi),\phi)]. \tag{14}$$

Example 2 (arising in the two-dimensional Dirac equation for graphene; see [14]). Consider smooth bounded functions $U(x)$, $m(x)$ and the effective Hamiltonians

$$\mathcal{H}^{\pm}(x,p) = F(x,|p|) - E = U(x) \pm \sqrt{p^2 + m(x)^2}.$$

Here

$$C(x,E) = \frac{1}{\sqrt{(E-U)^2 - m^2}}, \qquad R = \pm\frac{z^2}{\sqrt{z^2+m(x)^2}}\bigg|_{z=1/C} = \frac{(E-U(x))^2 - m^2(x)}{E-U(x)}, \tag{15}$$

$$\psi = \frac{\sqrt{E-U(x)}}{\sqrt{(E-U(x))^2 - m(x)^2}} K^h_{\Lambda^2}[A(t(\tau,\phi),\phi)]. \tag{16}$$

Example 3 (arising in the theory of linear surface water waves; see [15], [16]). Consider a smooth positive function $D(x) > 0$ and the effective Hamiltonian

$$\mathcal{H}(x,p,E) = F(x,|p|,E) - E = \sqrt{|p|\tanh(|p|D(x))} - E.$$

It is easy to see that there exists a unique smooth positive solution $y(\mathcal{E})$ of the equation

$$\sqrt{y\tanh(y)} = \mathcal{E} = \sqrt{D(x)}\,E,$$

and thus

$$\begin{aligned} C(x,E) &= \frac{D(x)}{y(E\sqrt{D(x)})}, \\ R &= z\frac{Dz/\cosh^2(zD) + \tanh(zD)}{2\sqrt{z\tanh(zD(x))}}\bigg|_{z=1/C} \\ &= \frac{(y^2 - y^2\tanh^2(y)) + y\tanh(y))}{2\sqrt{D}\sqrt{y\tanh(y)}}\bigg|_{y=y(\sqrt{D(x)}\,E)} \\ &= \frac{y^2 - D(x)^2E^4 + D(x)E^2}{2D(x)E}\bigg|_{y=y(\sqrt{D(x)}\,E)}. \end{aligned} \tag{17}$$

The Hamiltonian system with Hamiltonian $H(x,p,E) = C(x,E)|p|$ is of the form

$$\frac{dp}{d\tau} = -|p|\frac{\partial}{\partial x}\left(\frac{D(x)}{y(E\sqrt{D(x)})}\right), \qquad \frac{dx}{d\tau} = \frac{p}{|p|}\frac{D(x)}{y(E\sqrt{D(x)})}. \tag{18}$$

This equation contains the function $y(\mathcal{E})$ and its derivative y', which is not very convenient, because its determination involves the inversion of the equation. Let us show how this system can be rewritten so as to get rid of $y(\mathcal{E})$.

Along the trajectories (P, X) of the Hamiltonian system, the following relations hold:

$$H(X,P,E)=0 \iff C(X,E)|P|=1 \implies y(E\sqrt{D(X)}\,)=\frac{D(X)}{C(X,E)}=D(X)|P|. \qquad (19)$$

Differentiating the equation

$$\sqrt{y\tanh(y)}=\mathcal{E}$$

for $y(\mathcal{E})$ with respect to $\mathcal{E}$, we obtain

$$y'(\mathcal{E})(\tanh y + y(1-\tanh^2 y)) = 2\sqrt{y\tanh y}, \qquad (20)$$

which implies that

$$y'(E\sqrt{D(X)}\,)=\frac{2y\mathcal{E}}{y^2+\mathcal{E}^2-\mathcal{E}^4}\bigg|_{(P,X)}=\frac{2|P|\sqrt{D(X)}\,E}{D(X)|P|^2+E^2-D(X)E^4} \qquad (21)$$

on the solutions (P, X) of the Hamiltonian system. Substituting expressions (19) and (21) into the Hamiltonian system, we obtain

$$\frac{dp}{d\tau}=-\frac{p^2-E^4}{D(x)p^2+E^2-D(x)E^4}\cdot\frac{\partial D(x)}{\partial x}, \qquad \frac{dx}{d\tau}=\frac{p}{p^2}. \qquad (22)$$

To write the expression for the canonical operator, we also need to substitute $y|_{(P,X)}$ into expression (17) for R:

$$R|_{(P,X)} \equiv R(X,P,E)=\frac{D(X)P^2-D(X)E^4+E^2}{2E}. \qquad (23)$$

Taking into account the resulting expression for R, as well as $C(X) = 1/|P|$, we rewrite formula (11) for the canonical operator in a neighborhood of the nonsingular point as

$$\psi_j(x)=\frac{e^{i\tau/h}e^{-(i\pi/2)\mathbf{m}_j}}{\sqrt{|X_\phi(\tau,\phi)|}}\sqrt{\frac{2E|P(\tau,\phi)|}{D(X(\tau,\phi))P^2-D(X(\tau,\phi))E^4+E^2}} \times A(\tau,\phi)\mathbf{e}_j(\tau,\phi)|_{(\tau,\phi)=(\tau_j(x),\phi_j(x))}. \qquad (24)$$

Since $R = R(x, p, E)$ depends on p, it is better not to place the (pseudodifferential) operator $1/\sqrt{R(x,\widehat{p},E)}$ before the canonical operator, but write the answer in a neighborhood of the focal point as

$$\psi_j(x)=e^{-(i\pi/2)\mathbf{m}_j^s}e^{i\pi/4} \times\frac{\sqrt{2E}}{\sqrt{2\pi h}}\int_{\mathbb{R}}\frac{\sqrt{|\det(P(\tau,\phi),P_\phi(\tau,\phi))|}\,e^{i\tau/h}A(\tau,\phi)\mathbf{e}_j(\tau,\phi)}{\sqrt{D(X(\tau,\phi))P^2(\tau,\phi)-D(X(\tau,\phi))E^4+E^2}}\bigg|_{\tau=\tau_j(x,\phi)}d\phi. \qquad (25)$$

Example 4 (arising in the theory of water waves with the consideration of surface tension; see [4], [15], [16])**.** Let us modify the Hamiltonian from Example 3 as follows:

$$\mathcal{H}(x,p)=F(x,|p|)-E=\sqrt{|p|\tanh(|p|D(x))(1+\mu(x)|p|^2)}-E, \qquad x\in\mathbb{R}^2.$$

where $\mu(x) > 0$ is the smooth function determining the surface tension of the water. Set $\nu(x) = E(\mu(x))^{1/4}$ and $\mathcal{E}(x) = E(D(x))^{1/2}$. The equality $\mathcal{H}(x, p) = 0$ can be rewritten as $f(y, \mathcal{E}, \nu) = 0$, where

$$f(y,\mathcal{E},\nu)=y\tanh(y)-\mathcal{E}^2\left(1+\frac{y^2\nu^4}{\mathcal{E}^4}\right)^{-1}$$

is a smooth function on $\mathbb{R}^3_+$. Since $\partial f/\partial y(y, \mathcal{E}, \nu) > 0$, it follows that, by the implicit-function theorem, the equality $f(y, \mathcal{E}, \nu) = 0$ can be inverted: $y = Y(\mathcal{E}, \nu)$, where Y is a smooth function in $(\mathcal{E}, \nu) \in \mathbf{R}^2_+$. Just as above, the Hamiltonian system with Hamiltonian $C(x, E)|p|$ is of the form

$$\frac{dp}{d\tau}=-|p|\frac{\partial}{\partial x}\left(\frac{D(x)}{Y(E\sqrt{D(x)}\,,E\mu(x)^{1/4})}\right), \qquad \frac{dx}{d\tau}=\frac{p}{|p|}\frac{D(x)}{Y(E\sqrt{D(x)}\,,E\mu(x)^{1/4})} \qquad (26)$$

and can be simplified to the form

$$\frac{dp}{d\tau} = -|p|\left(\frac{1}{y}\frac{\partial D}{\partial x} - \frac{D(x)}{y^2}\frac{dY}{dx}\right)\Big|_{y=D(x)|p|}, \qquad \frac{dx}{d\tau} = \frac{p}{p^2}. \tag{27}$$

Here dY/dx is determined by differentiating the equation $y = Y(\mathcal{E}(x), \nu(x))$:

$$\frac{dY}{dx} = -\left(\frac{\partial f}{\partial y}\right)^{-1}\left(\frac{\partial f}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial x} + \frac{\partial f}{\partial \nu}\frac{\partial \nu}{\partial x}\right),$$

$$\frac{\partial f}{\partial y} = \tanh(y) + y(1 - \tanh^2(y)) + 2y\mathcal{E}^{-2}\nu^4\left(1 + y^2\frac{\nu^4}{\mathcal{E}^4}\right)^{-2} > 0.$$

Substituting these derivatives into (27) and taking the equality $y(E\sqrt{D(X)}) = |P|D(X)$ into account, we obtain a system, similar to (22). Also we can write an expression for R, which is similar to (23), and thus obtain a representation for the Maslov canonical operator just as in (24) and (25).

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