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Bounded Subharmonic Functions Possess the Lebesgue Property at Each Point

A. S. Sadullaev[1*], S. A. Imomkulov[2], and K. Kh. Rakhimov[1***]**

[1]*National University of Uzbekistan, Tashkent, Uzbekistan*

[2]*Navoi State Pedagogical Institute, Navoi, Uzbekistan*

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Abstract—It is proved that the restriction of a bounded subharmonic function in a domain $D \subset \mathbb{C}$ to any real line $l \subset \mathbb{C}$ possesses the Lebesgue property at each point of $l \cap D$.

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1. INTRODUCTION

Let $\varphi(x)$ be a summable function defined on $(a, b) \subset \mathbb{R}$. A point $x_0 \in (a, b)$ is called a *Lebesgue point* of the function φ if

$$\lim_{h\to 0} \frac{1}{h} \int_{x_0}^{x_0+h} |\varphi(x) - \varphi(x_0)|\, dx = 0. \tag{1.1}$$

Obviously, any point of continuity of the function $\varphi(x)$ is a Lebesgue point. In the general case, almost all (with respect to Lebesgue measure) points of a summable function are Lebesgue points. If $x_0 \in (a, b)$ is a Lebesgue point of the function φ, then the function

$$f(x) = \int_a^x \varphi(t)\, dt$$

has, at this point, the derivative $f'(x_0) = \varphi(x_0)$. Note that functions of one real variable $y = f(x)$ having derivatives at each point $x \in (a, b)$ possess many important properties. In particular, if the derivative $f'(x)$ of such a function is summable, then this derivative completely defines the function itself:

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

2. MAIN RESULT

Theorem 1. *Suppose that $u(z)$ is a bounded subharmonic function in a domain $D \subset \mathbb{C}$, $u(z) \in \mathrm{sh}(D) \cap L^\infty(D)$, and l is an arbitrary fixed real line whose intersection with D is nonempty. Then, for the restriction $\varphi = u|_l$, each point $x \in l \cap D$ is its Lebesgue point.*

*E-mail: sadullaev@mail.ru
**E-mail: sevdiyor_i@mail.ru
***E-mail: karimjon1705@gmail.com

In order to prove the theorem, we use the well-known notion of thinness and Wiener's criterion (see, for example, [1, Chap. VII] and [2, Chap. V]). A set E is said to be *thin* at a point $z_0 \in \overline{E}$ if, in a neighborhood U of the point z_0, there exists a subharmonic function $u(z) \in \operatorname{sh}(U)$ such that

$$\varlimsup_{z \to z_0,\, z \in E \setminus \{z_0\}} u(z) < u(z_0).$$

For convenience, the points $z_0 \notin \overline{E}$ are also assumed thin points of the set E.

Wiener's criterion. *A Borel set $E \subset \mathbb{C}$ is thin at a point $z_0 \in \overline{E}$ if and only if*

$$\sum_{n=1}^{+\infty} \frac{n}{\ln(1/C(E_n))} < \infty, \tag{2.1}$$

where

$$E_n = E \cap \{z : q^{n+1} \le |z - z_0| \le q^n\}, \qquad 0 < q < 1,$$

and $C(E_n)$ is the logarithmic capacity of the set E_n.

Since, for Borel sets $E \subset l \subset \mathbb{C}$, the following inequality holds:

$$C(E) \ge \frac{1}{4} \operatorname{mes}(E),$$

it follows from (2.1) that, for a thin point $x_0 \in \overline{E}$, the following inequality holds:

$$\sum_{n=1}^{+\infty} \frac{n}{\ln(1/\operatorname{mes}(E_n))} < \infty. \tag{2.2}$$

For example, for a set $E \subset l \subset \mathbb{C}$ to be thin at a point $x_0 \in \overline{E}$, it is necessary that the following relation be valid:

$$\varliminf_{n \to \infty} \frac{\operatorname{mes}(E_n)}{q^n} = 0, \tag{2.3}$$

because, otherwise, the common term of the series (2.2) does not tend to zero.

Proof of the theorem. Suppose that $u(z)$ is a bounded subharmonic function in the domain D. Without loss of generality, we can assume that $l = \{z = x + iy : y = 0\}$ and $l \cap D = (a, b) \subset \mathbb{R}$. It follows from (2.3) that a thin point $x_0 \in \mathbb{R}$ of the set E is a point of density 0, i.e., the following equality holds:

$$\lim_{h \to 0} \frac{\operatorname{mes} E \cap \{|x - x_0| < h\}}{2h} = 0. \tag{2.4}$$

Let $\varphi(t)$ be the restriction of u to the interval (a, b). Choose an arbitrary point $x \in (a, b)$ and consider the integral

$$J(x, h) = \frac{1}{h} \int_x^{x+h} |\varphi(t) - \varphi(x)|\, dt \qquad \text{for all} \quad h\colon \quad |h| < 1, \quad x + h \in (a, b).$$

Let us fix an arbitrary number $\varepsilon > 0$ and set

$$I(\varepsilon) = \{t \in (a, b) : |\varphi(t) - \varphi(x)| > \varepsilon\}.$$

Obviously, the set $I(\varepsilon)$ is thin at the point x. By (2.4), the density of the set $I(\varepsilon)$ at the point x is zero, i.e.,

$$\lim_{h \to 0} \frac{\operatorname{mes} I(\varepsilon) \cap [x - h, x + h]}{2h} = \lim_{h \to 0} \frac{1}{2h} \int_{I(\varepsilon) \cap [x-h, x+h]} dt = 0.$$

The last equality and the fact that the function φ is bounded imply that

$$\lim_{h\to 0}\frac{1}{2h}\int_{I(\varepsilon)\cap[x-h,x+h]}|\varphi(t)-\varphi(x)|\,dt=0. \tag{2.5}$$

Since $|\varphi(t)-\varphi(x)|\le\varepsilon$ for any $t\in(a,b)\setminus I(\varepsilon)$, combining this inequality with (2.5), we obtain

$$\varlimsup_{h\to 0}\frac{1}{h}\int_x^{x+h}|\varphi(t)-\varphi(x)|\,dt\le\varepsilon.$$

Since the number ε is arbitrary, we obtain (1.1), i.e., the point $x\in(a,b)$ is a Lebesgue point. □

The theorem (proved above) for subharmonic functions in the space $\mathbb{R}^m$, $m\ge 2$, holds in the following statement.

Theorem 2. *Let $D\subset\mathbb{R}^m$, and let $u(z)\in\operatorname{sh}(D)\cap L^\infty(D)$. Then, for any hyperplane*

$$P\subset\mathbb{R}^m,\qquad \dim P=m-1,\quad P\cap D\ne\varnothing,$$

each point $x^0\in P\cap D$ is a Lebesgue point of the restriction $\varphi=u|_P$, i.e.,

$$\lim_{h\to 0}\frac{1}{h^{m-1}}\int_{P\cap B(x,h)}|\varphi(x)-\varphi(x^0)|\,dV_P=0,$$

where dV_P is an area element in the hyperplane P.

Note that, for a plane of smaller dimension, Theorem 2, in general, is not valid.

3. APPLICATIONS

Koepke (see [3]) constructed a function $f(x)$ defined on the interval (a,b) for which there exists a derivative $f'(x)$ for all $x\in(a,b)$ and the set of zeros $Z_f=\{x\in(a,b):f'(x)=0\}$ and the complement $(a,b)\setminus Z_f$ are everywhere dense in the interval (a,b). Subsequent modifications of Koepke's example were given in the papers [4] of Pompeiu and [5] of Denjoy. Recently, Kalyabin [6] proved that, for any countable set $X=\{x_k\}_{k=1}^{\infty}\subset[0,1]$, the function

$$\varphi(x)=\inf_{k\in\mathrm{N}}|x-x_k|^{1/k}$$

possesses the following properties:

1) $\varphi(x)$ is *upper semicontinuous on the closed interval* $[0,1]$;

2) $\varphi(x_k)=0$ *for all* $k\in\mathbb{N}$ *and* $\varphi(x)>0$ *at almost all points* $x\in[0,1]$;

3) *each point* $x\in[0,1]$ *is a Lebesgue point for* φ.

Theorem 1 proved in Sec. 2 allows us to assert that, for each complete polar set $X\subset\mathbb{R}$, there exists a function $\varphi(x)$ possessing the following properties:

1) $\varphi(x)$ is *upper semicontinuous*;

2) $\varphi(x)=0$ *for any* $x\in X$ *and* $\varphi(x)>0$ *outside* X;

3) *each point* $x\in\mathbb{R}$ *is a Lebesgue point for* φ.

Thus, for all $x \in \mathbb{R}$, the function $\varphi(x)$ is the derivative of its antiderivative:

$$f(x) = \int_0^x \varphi(t)\,dt, \qquad f'(x) = \varphi(x) \qquad \text{for all} \quad x \in \mathbb{R}.$$

In addition, by 2), the function $f(x)$ is monotone increasing on $\mathbb{R}$.

In order to prove 1)–3), it suffices to note that, in view of the fact that the set X is complete and polar, there exists a function $\vartheta(z) \in \operatorname{sh}(\mathbb{C})$ such that $X = \{z \in \mathbb{C} : \vartheta(z) = -\infty\}$. Now it suffices to apply Theorem 1 to the locally bounded subharmonic function $u(z) = e^{\vartheta(z)}$ and to the restriction of u to the line $l = \mathbb{R} \subset \mathbb{C}$.

Example. Consider a countable set of rational numbers

$$X = \{p_k/q_k\}_{k=1}^{\infty} \subset \left[-\frac{1}{2}, \frac{1}{2}\right], \qquad q_k \geq 1,$$

everywhere dense on the closed interval $[-1/2, 1/2] \subset \mathbb{R}$ and construct the following function in the plane:

$$u(z) = \sum_{k=1}^{\infty} \frac{\ln|z - p_k/q_k|}{q_k^{2+\sigma}}, \qquad \sigma > 0. \tag{3.1}$$

Then

$$u(z) \in \operatorname{sh}(\mathbb{C}), \qquad u|_X \equiv -\infty, \qquad u(0) = \sum_{k=1}^{\infty} \frac{\ln|p_k/q_k|}{q_k^{2+\sigma}} \neq -\infty.$$

Therefore, the set $\widehat{X} = \{z \in \mathbb{C} : u(z) = -\infty\}$ is a polar Borel set of type G_δ and $X \subset \widehat{X} \subset [-1/2, 1/2]$. Since X is not a set of type G_δ, it follows that $\widehat{X} \neq X$.

Note that the subharmonic function constructed in (3.1) is related to Diophantine numbers: if the point x^0 is a Diophantine number, then $u(x^0) \neq -\infty$ (recall that a real number $\alpha \in \mathbb{R}$ is said to be *Diophantine* if there exist numbers $c > 0$, $\mu > 0$ such that $|\alpha - p/q| \geq c/q^\mu$ for all $p, q \in \mathbb{N}$). The set of all Diophantine numbers has full Lebesgue measure; moreover, its complement has zero Hausdorff dimension.

Applying Theorem 1 to the function $\exp u$, we find that the restriction

$$\varphi(x) = \exp u|_{y=0} = \exp \sum_{k=1}^{\infty} \frac{\ln|x - p_k/q_k|}{q_k^{2+\sigma}},$$

possesses the following properties:

a) $\varphi(x)$ is upper semicontinuous on the closed interval $[-1/2, 1/2]$, it is continuous at all points of the set $\widehat{X}$, which is everywhere dense in $[-1/2, 1/2]$;

b) $\varphi(x) = 0$ for all $x \in \widehat{X}$;

c) $\varphi(x) > 0$ for any Diophantine number $x \in [-1/2, 1/2]$.

REFERENCES

1. M. Brelot, *Eléments de la théorie classique du potentiel* (Paris, 1961; Mir, Moscow, 1964).
2. N. S. Landkof, *Foundations of Modern Potential Theory* (Nauka, Moscow, 1966; Springer-Verlag, New York-Heidelberg, 1972).
3. A. M. Bruckner, *Differentiation of Real Functions*, in *Lecture Notes in Math.* (Springer-Verlag, Berlin, 1978), Vol. 659.
4. D. Pompeiu, "Sur les fonctions dérivées," Math. Ann. **63** (3), 326–332 (1907).
5. A. Denjoy, "Sur les fonctions dérivées sommables," Bull. Soc. Math. France **43**, 161–248 (1915).
6. G. A. Kalyabin, "New examples of Pompeiu functions," in *International Conference "The Theory of Functions and Functional Analysis"*, Abstracts (Astana, 2012), pp. 47–48 [in Russian].