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The Energy Function of Gradient-Like Flows and the Topological Classification Problem

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Abstract—For gradient-like flows without heteroclinic intersections of the stable and unstable manifolds of saddle periodic points all of whose saddle equilibrium states have Morse index 1 or $n-1$, the notion of consistent equivalence of energy functions is introduced. It is shown that the consistent equivalence of energy functions is necessary and sufficient for topological equivalence.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let M^n be a smooth closed orientable manifold. A twice differentiable function $\varphi\colon M^n \to \mathbb{R}$ is called a *Morse function* if all of its critical points are nondegenerate, i.e., for any critical point $p \in M^n$ the determinant of the Hessian matrix $(\partial^2\varphi/(\partial x_i \partial x_j))|_p$ at this point is nonzero. According to Morse's lemma (see [1, Lemma 2.2]), in a neighborhood of a nondegenerate critical point p, there exist local coordinates $y_1, \dots, y_n$, called *Morse coordinates*, in which the function φ has the form

$$\varphi(y_1, \dots, y_n) = \varphi(p) - y_1^2 - \dots - y_k^2 + y_{k+1}^2 + \dots + y_n^2.$$

The number $k \in \{0, \dots, n\}$ does not depend on the choice of the local coordinates and is called the *index of the point* p. We denote the index of a critical point p by $\operatorname{ind}(p)$.

The smooth flow induced by the vector field $X = -\operatorname{grad}\varphi$ is called a *gradient flow*. If φ is a Morse function, then the gradient flow has no closed orbits, all of its equilibrium states are hyperbolic, the set of equilibrium states coincides with the set of critical points of φ, and the dimension of the unstable manifold W^u_p of any equilibrium state p (*Morse index*) equals $\operatorname{ind}(p)$.

According to Smale's results in [2] (Theorem A), a gradient flow can be arbitrarily closely approximated (in the C^1-topology) by a Morse–Smale flow.

Recall that a smooth flow f^t is called a *Morse–Smale flow* if its nonwandering set $\Omega(f^t)$ consists of finitely many hyperbolic equilibrium states and finitely many hyperbolic closed orbits; the stable and unstable manifolds of singular points and of periodic solutions intersect transversally. A Morse–Smale flow without closed orbits is called a *gradient-like flow*.

It follows from results of [2] (Theorem B) that, for any gradient-like flow f^t on M^n, there exists a *self-indexing energy function*, i.e., a function $\varphi\colon M^n \to [0, n]$ with the following properties:

(1) φ is a Morse function;

(2) the critical point set of φ coincides with $\Omega(f^t)$;

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(3) $\varphi(f^t(x)) < \varphi(x)$ for any $x \notin \Omega(f^t)$ and any $t > 0$;

(4) $\varphi(p) = \mathrm{ind}(p)$ for any $p \in \Omega(f^t)$.

Moreover, Smale noticed in [2] that there exists a metric on M^n with respect to which the flow f^t is the gradient flow for its energy function φ.

Meyer [3] generalized Smale's result by constructing, for any Morse–Smale flows on M^n, a Morse–Bott energy function, i.e., a Morse function whose Hessian at each critical point is nondegenerate in the normal direction to the critical level set. Moreover, it follows from results of Meyer's paper that the self-indexing energy function can be used for the topological classification of gradient-like flows. To give the precise statement of this result, we recall that flows f^t and f'^t on a manifold M^n are said to be *topologically equivalent* if there exists a homeomorphism $h\colon M^n \to M^n$ taking the orbits of f^t to orbits of f'^t and preserving the orientation of orbits. In classifying the Morse–Smale flows according to the topological equivalence relation by using a self-indexing energy function, we employ the following definition of the topological equivalence of functions due to Thom [4].

Definition 1. Two smooth functions $\varphi\colon M^n \to \mathbb{R}$ and $\varphi'\colon M^n \to \mathbb{R}$ are said to be *topologically equivalent* if there exist orientation-preserving homeomorphisms $H\colon M^n \to M^n$ and $\chi\colon \mathbb{R} \to \mathbb{R}$ for which $\varphi' H = \chi\, \varphi$.

Meyer proved that the topological equivalence of self-indexing energy functions is a necessary condition for the topological equivalence of the corresponding Morse–Smale flows, and in the case of gradient-like flows on manifolds of dimension $n = 2$, this condition is also sufficient.[1]

Employing energy functions in solving the topological classification problem turns out to be useful in mathematical modeling, when the energy function is known from physical considerations, e.g., as the energy function of a dissipative system in mechanics, the potential of an electrostatic field, etc.

The purpose of this paper is to obtain necessary and sufficient conditions in terms of energy functions for the topological equivalence of systems in the class $G(M^n)$, $n > 2$, which consists of gradient-like flows without heteroclinic intersections all of whose saddle equilibrium states have Morse index 1 or $n - 1$.

Given a flow $f^t \in G(M^n)$, we denote the set of fixed points of Morse index $i \in \{0, 1, n-1, n\}$ by Ω_i and the cardinality of Ω_i by $|\Omega_i|$. The topology of the manifold M^n and the structure of the set of equilibrium states of the flow f^t are described by the following proposition.

Proposition 1. *Let $f^t \in G(M^n)$. Then*

$$g = \frac{|\Omega_1 \cup \Omega_{n-1}| - |\Omega_0 \cup \Omega_n| + 2}{2}$$

is a nonnegative integer and the following assertions hold:

(1) *if $g = 0$, then M^n is the sphere S^n*;

(2) *if $g > 0$, then M^n is homeomorphic to a connected sum*[2] *of g copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

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[1] It was stated in [3, Proposition] that a self-indexing energy function is a complete topological invariant for Morse–Smale flows on orientable surfaces. Oshemkov and Sharko [5] gave an example of topologically nonequivalent Morse–Smale flows (with closed orbits) on the torus which have equivalent self-indexing energy functions and mentioned that Meyer's result remains valid only for gradient-like flows.

[2] A *connected sum* $M_1^n \sharp M_2^n$ of two orientable connected n-manifolds M_1^n and M_2^n is defined as the manifold $M_1^n \sharp M_2^n$ obtained by removing balls $B_1^n \subset M_1^n$ and $B_2^n \subset M_2^n$ from M_1^n and M_2^n and attaching the remaining manifolds with boundary to each other by means of a homeomorphism $\phi\colon \partial B_1^n \to \partial B_2^n$ reversing the natural orientation of ∂B_1^n.

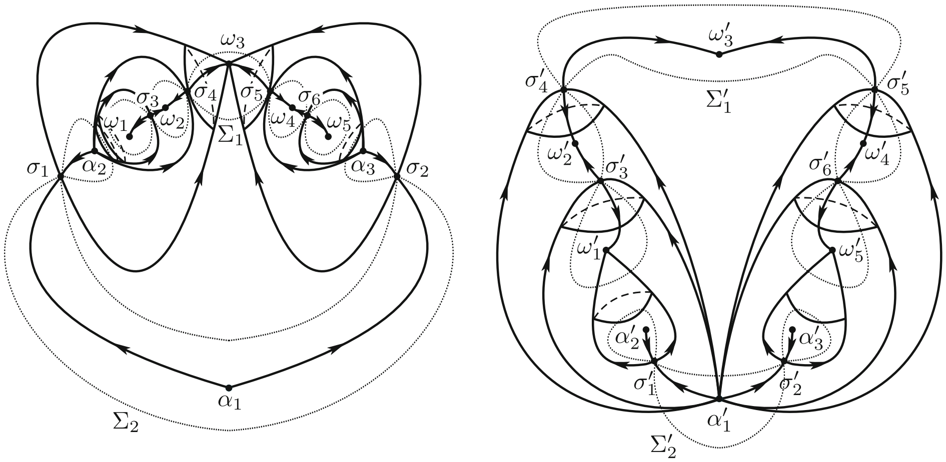

Figure: The phase portraits of topologically nonequivalent flows f^t, $f'^t \in G(S^3)$ with equivalent self-indexing energy functions.

Proposition 1 is proved by the same algorithm as similar statements in [1] (the theorem) and [7] (Theorem 1).

Unlike the two-dimensional situation, for $n \geq 3$, the class $G(M^n)$ contains topologically nonequivalent flows with equivalent self-indexing energy functions. Examples of the phase portraits of such flows f^t and f'^t belonging to the class $G(S^3)$ are given in the figure. The critical level sets of the self-indexing energy functions φ and φ' (the dotted lines in the figure represent the level sets corresponding to the values 1 and 2) of the flows f^t and f'^t are ambient homeomorphic, which implies the topological equivalence of the functions φ and φ'. The flows f^t and f'^t are not topologically equivalent, because the sink equilibrium state ω_3 of f^t contains four unstable separatrices of saddle equilibrium states in its basin, while the basin of any sink of f'^t contains at most two separatrices.

The presence of such examples leads to the necessity of introducing additional invariants distinguishing between topologically nonequivalent flows with equivalent energy functions. To describe the invariant that we propose in this paper, we represent the manifold M^n as the union of three connected sets, the attractor $A = W^u_{\Omega_1 \cup \Omega_0}$, the repeller $R = W^s_{\Omega_{n-1} \cup \Omega_n}$, and the set $V = M^n \setminus (A \cup R)$ of wandering orbits of f^t going from A to R. Following [6], we refer to V as the *characteristic manifold*.

Let $\varphi\colon M^n \to [0, n]$ be a self-indexing energy function for f^t. We refer to a hypersurface Σ of level $c \in (1, n-1)$ as a *characteristic section* (it intersects each orbit of the characteristic space in precisely one point).

Definition 2. We say that self-indexing energy functions φ and φ' for flows $f, f'^t \in G(M^n)$ are *consistently equivalent* if there exist orientation-preserving homeomorphisms $H\colon M^n \to M^n$ and $\chi\colon [0, n] \to [0, n]$ with the properties

(1) $\varphi' H = \chi\, \varphi$ and

(2) $H(W^s_{\Omega_1} \cap \Sigma) = W^s_{\Omega'_1} \cap H(\Sigma)$, $H(W^u_{\Omega_{n-1}} \cap \Sigma) = W^u_{\Omega'_{n-1}} \cap H(\Sigma)$ for some characteristic section Σ.

The main result of this paper is the following theorem.

Theorem 1. *Flows $f^t, f'^t \in G(M^n)$ are topologically equivalent if and only if their energy functions are consistently equivalent.*

In Sec. 3, we define a class $G_0(M^n) \subset G(M^n)$ of flows for which the equivalence of self-indexing functions implies consistent equivalence and, thereby, the self-indexing energy function is a complete topological invariant. The class $G_0(M^n)$ consists of flows all of whose saddle equilibrium states have Morse index 1. Theorem 1 in [7] implies the following proposition.

Proposition 2. *For any flow $f^t \in G_0(M^n)$, the nonwandering set $\Omega(f^t)$ contains precisely one source, $k \geq 0$ saddles, and $k+1$ sinks, and the ambient manifold M^n is diffeomorphic to the n-sphere.*

The main result of Sec. 3 is the following theorem.

Theorem 2. *Flows $f^t, f'^t \in G_0(M^n)$ are topologically equivalent if and only if their self-indexing energy functions are equivalent.*

The topological classification problem for Morse–Smale flows on manifolds of dimension three and higher was attacked, in particular, in [8]–[11]. Fleitas [9] obtained a topological classification of polar flows[3] on manifolds of dimensions 2 and 3 by using Heegaard diagrams. Umanskii [8] obtained a topological classification of Morse–Smale flows on 3-manifolds with finitely many heteroclinic orbits. He used a combinatorial invariant describing the mutual arrangement of singular orbits of a flow, which is similar to the dynamical system scheme introduced by Leontovich and Mayer to classify flows with finitely many equilibrium states on the sphere S^2. Pilyugin [10] observed that, in the case $M^n = S^n$, $n \geq 3$, the class $G(S^n)$ coincides with the class of all gradient-like flows with no heteroclinic intersections of stable and unstable manifolds of saddle periodic points and obtained a complete topological classification of such flows by using the Smale diagram. Prishlyak [11] obtained a complete topological classification of Morse–Smale flows on three-dimensional manifolds. The invariant introduced in [11], as well as the Fleitas invariant, contains information on the traces of two-dimensional separatrices on the characteristic section.[4] Thus, Theorem 1 is in fact a generalization of results of the papers mentioned above to the case of flows in the class $G(M^n)$.

2. PROOF OF THEOREM 1

First, we prove *necessity*. Let φ and φ' be self-indexing energy functions of topologically equivalent Morse–Smale flows f^t and f'^t from $G(M^n)$, and let $h\colon M^n \to M^n$ be a homeomorphism taking the orbits of f^t to orbits of f'^t. It follows from the definition of a self-indexing function and properties of the homeomorphism h that, for any equilibrium state p of the flow f^t, we have $\varphi(p) = \varphi'(h(p))$. Thus, we use the identity map as the homeomorphism χ and shall construct a homeomorphism H satisfying the conditions in Definition 2 as follows.

Let $x \in M^n$ be any point which is not an equilibrium state of the flow f^t. We denote the orbit of f^t (of f'^t) passing through x by l_x and the equilibrium state which is the α-limit (respectively, the ω-limit) of the set of orbits l_x by $\alpha(l_x)$ (respectively, by $\omega(l_x)$). For $x' = h(x)$, we have $l_{x'} = h(l_x)$. It follows from properties of the homeomorphism h that

$$\alpha(l_{x'}) = h(\alpha(l_x)), \qquad \omega(l_{x'}) = h(\omega(l_x)).$$

Moreover,

$$\varphi(\alpha(l_x)) = \varphi'(\alpha(l_{x'})), \qquad \varphi(\omega(l_x)) = \varphi'(\omega(l_{x'})).$$

Let

$$c \in (\varphi(\omega(l_x)), \varphi(\alpha(l_x))), \qquad \Sigma_c = \varphi^{-1}(c) \quad (\Sigma'_c = (\varphi')^{-1}(c)).$$

Note that, by the definition of an energy function, the intersection $\Sigma_c \cap l_x$ ($\Sigma'_c \cap l_{x'}$) consists of only one point. Thus, the map $\widetilde{H}$ taking each point $y = \Sigma_c \cap l_x$ to the point $y' = \Sigma'_c \cap l_{x'}$ is well defined on $M^n \setminus \Omega(f^t)$. By construction, $\widetilde{H}$ is a homeomorphism between

$$M^n \setminus \Omega(f^t) \qquad \text{and} \qquad M^n \setminus \Omega(f'^t),$$

[3] A *polar flow* is a gradient-like flow for which the set of equilibrium states contains precisely one source, one sink, and any finite number of saddles.

[4] Prishlyak and Fleitas did no use the term *characteristic section*, but the secant surface which they used to define their invariants was essentially the characteristic section in our terminology.

which transforms the orbits of f^t and the set of regular levels of the function φ into orbits of f'^t and the set of regular levels of φ'.[5] Since

$$\varphi(\alpha(l_x)) = \varphi'(\alpha(l_{x'}), \qquad \varphi(\omega(l_x)) = \varphi'(\omega(l_{x'})$$

for any point x which is not an equilibrium state, it follows that the homeomorphism $\widetilde{H}$ constructed above can be uniquely extended to a homeomorphism H with the required properties satisfying the conditions $\varphi(x) = \varphi'(H(x))$ and

$$H(W^s_{\Omega_1} \cap \Sigma) = W^s_{\Omega'_1} \cap H(\Sigma), \qquad H(W^u_{\Omega_{n-1}} \cap \Sigma) = W^u_{\Omega'_{n-1}} \cap H(\Sigma)$$

for any characteristic section Σ.

Let us prove *sufficiency*.

Suppose that the self-indexing energy functions φ and φ' are consistently equivalent, i.e., there exist orientation-preserving homeomorphisms

$$H\colon M^n \to M^n \qquad \text{and} \qquad \chi\colon [0,n] \to [0,n]$$

such that $\varphi' H = \chi\varphi$ and

$$H(W^s_{\Omega_1} \cap \Sigma) = W^s_{\Omega'_1} \cap H(\Sigma), \qquad H(W^u_{\Omega_{n-1}} \cap \Sigma) = W^u_{\Omega'_{n-1}} \cap H(\Sigma)$$

for some characteristic section Σ. For any point $x \in \Sigma$, we set $x' = H(x)$ and denote the orbits of the flows f^t and f'^t passing through x and x' by l_x and $l_{x'}$, respectively. For $c \in [0,n]$, we set $\Sigma_c = \varphi^{-1}(c)$. Consider the homeomorphism $H_V\colon V \to V'$ defined by

$$H_V(y) = l_{x'} \cap H(\Sigma_c)$$

for any $y = l_x \cap \Sigma_c$, $c \in (0,n)$.

Since $H|_\Sigma$ takes the traces of the $(n-1)$-dimensional invariant manifolds of the saddle points of f^t on Σ to the traces of similar objects of f'^t on $H(\Sigma)$ and the closure of any separatrix of dimension $n-1$ of a saddle fixed point σ of f^t consists of the point σ and a source or sink of f^t (depending on the index of σ), it follows that the homeomorphism H_V can be uniquely extended to the sets Ω_0, Ω_1, Ω_{n-1}, and Ω_n. We keep the notation H_V for the homeomorphism thus obtained and extend this homeomorphism to the one-dimensional separatrices of the saddle fixed points.

To this end, we see that, for any $c \in (0,1)$, we have

$$H_V(\Sigma_c \setminus W^u_{\Omega_1}) = H(\Sigma_c) \setminus W^u_{\Omega'_1},$$

and the sets $W^u_{\Omega_1} \cap \Sigma_c$ and $W^u_{\Omega'_1} \cap H(\Sigma_c)$ are finite unions of equally many points. It follows that the homeomorphism H_V extends by continuity to a homeomorphism $H_1\colon W^u_{\Omega_1} \to W^u_{\Omega'_1}$. The homeomorphism

$$H_{n-1}\colon W^s_{\Omega_{n-1}} \to W^s_{\Omega'_{n-1}}$$

is constructed in a similar way. The required homeomorphism $h\colon M^n \to M^n$ is defined by

$$h(x) = \begin{cases} H_V(x) & \text{if } x \in M^n \setminus (W^u_{\Omega_1} \cup W^s_{\Omega_{n-1}}), \\ H_1(x) & \text{if } x \in W^u_{\Omega_1}, \\ H_{n-1}(x) & \text{if } x \in W^s_{\Omega_{n-1}}. \end{cases}$$

This completes the proof of the theorem. □

[5]By a *regular level* of a Morse function we mean a level containing no critical points.

3. THE ENERGY FUNCTION AS A COMPLETE TOPOLOGICAL INVARIANT OF FLOWS IN THE CLASS $G_0(M^n)$

This section is devoted to the proof of Theorem 2, which is a direct corollary of Theorem 1 and the following lemma.

Lemma 1. *Let φ and φ' be self-indexing energy functions of flows $f^t, f'^t \in G_0(M^n)$, respectively. Then the functions φ and φ' are consistently equivalent if and only if they are equivalent.*

Proof. It suffices to show that, for flows $f^t, f'^t \in G_0(M^n)$, the equivalence of the corresponding self-indexing functions φ and φ' implies the consistent equivalence of these functions. We set

$$\Sigma_c = \varphi^{-1}(c), \qquad M_c = \varphi^{-1}([0,c])$$

for each $c \in [0,n]$. Since the Lyapunov function decreases along the wandering orbits of a flow, we have $M_1 \cap W^s_{\Omega_1} = \Omega_1$. It follows that, for each connected component Q of $M_1 \setminus \Omega_1$, there exists a unique sink $\omega_Q \in \Omega_0$ such that $Q \subset W^s_{\omega_Q}$. Let $x \in Q$. Then the orbit l_x of f^t passing through x has an ω-limit point ω_Q and an α-limit point α (by Proposition 2, the set Ω_n consists of a single source α). Moreover, for any $c \in (0,n)$, the intersection $\Sigma_c \cap l_x$ consists of only one point.

We set $x' = H(x)$. For the objects related to f'^t similar to those introduced above for f^t, we use the same notation with primes. For any characteristic section Σ, the homeomorphism H induces a homeomorphism

$$h\colon \Sigma \setminus W^s_{\Omega_1} \to H(\Sigma) \setminus W^s_{\Omega'_1}$$

taking $y = l_x \cap \Sigma$ to $h(y) = l_{x'} \cap h(\Sigma)$.

According to Morse theory, the hypersurfaces $\Sigma' = H(\Sigma)$ of level Σ are smooth $n-1$-spheres. Since φ (φ') is the energy function of f^t (of f'^t), it follows that the set

$$C = \Sigma \cap W^s_{\Omega_1} \qquad (C' = \Sigma' \cap W^s_{\Omega'_1})$$

consists of k $(n-2)$-spheres (one sphere on each stable manifold of $W^s_{\Omega_1}$ (of $W^s_{\Omega'_1}$)). According to the annulus theorem[6], there exists a homeomorphism $\widetilde{h}\colon \Sigma \to \Sigma$ with the following properties:

(a) $\widetilde{h}(C) = C'$;

(b) the set C has a neighborhood $V(C)$ for which $\widetilde{h}|_{\Sigma\setminus V(C)} = h|_{\Sigma\setminus V(C)}$.

The homeomorphism $\widetilde{h}$ extends to a homeomorphism $\widetilde{H}_V\colon V \to V'$ taking

$$y \in l_x \cap \Sigma_c \qquad \text{to} \qquad \widetilde{H}_V(y) = l_{x'} \cap H(\Sigma_c), \quad c \in (0,n).$$

The homeomorphism $\widetilde{H}_V$ is extended to the required homeomorphism $\widetilde{H}\colon M^n \to M^n$ in the same way as in the proof of the necessity part of Theorem 1. □

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[6]The annulus theorem can be stated as follows: *If $h_i\colon \mathbb{S}^{n-1} \to \mathbb{R}^n$, $i \in \{1,2\}$, $n \geq 2$, is a topological embedding and $K^n \subset \mathbb{R}^n$ is the open domain bounded by the spheres $h_1(\mathbb{S}^{n-1})$ and $h_1(\mathbb{S}^{n-1})$, then the closure of K^n is homeomorphic to the direct product* $\mathbb{S}^{n-1} \times [0,1]$. The validity of the annulus conjecture in the case $n = 2$ follows from Antoine's 1921 theorem and in the case $n = 3$, from Sanderson's 1960 theorem; for $n > 4$, the conjecture was proved by R. Kirby in 1969 and for $n = 4$, by F. Quinn in 1984.

REFERENCES

1. J. Milnor, *Morse Theory* (Princeton Univ. Press, Princeton, 1963; Mir, Moscow, 1965).
2. S. Smale, "On gradient dynamical systems," Ann. of Math. (2) **74** (1), 199–206 (1961).
3. K. R. Meyer, "Energy Functions for Morse–Smale Systems," Amer. J. Math. **90** (4), 1031–1040 (1968).
4. R. Thom, "La stabilité topologique des applications polynomiales," Enseign. Math. (2) **1** (2), 24–33 (1962).
5. A. A. Oshemkov and V. V. Sharko, "Classification of Morse–Smale flows on two-dimensional manifolds," Mat. Sb. **189** (8), 93–140 (1998) [Sb. Math. **189** (8), 1205–1250 (1998)].
6. V. Z. Grines and O. V. Pochinka, *Introduction to Topological Classification of Cascades on Manifolds of Dimension Two and Three* (Inst. Komp'yuternykh Issledovanii, Izhevsk, 2011) [in Russian].
7. V. Z. Grines, E. Ya. Gurevich, and V. S. Medvedev, "Peixoto graph of Morse-Smale diffeomorphisms on manifolds of dimension greater than three," in *Trudy Mat. Inst. Steklov*, Vol. 261: *Differential Equations and Dynamical Systems* (Nauka, Moscow, 2008), pp. 61–86 [Proc. Steklov Inst. Math. **261** (1), 59–83 (2008)].
8. Ya. L. Umanskii, "Necessary and sufficient conditions for topological equivalence of three-dimensional Morse–Smale dynamical systems with a finite number of singular trajectories," Mat. Sb. **181** (2), 212–239 (1990) [Math. USSR-Sb. **69** (1), 227–253 (1991)].
9. G. Fleitas, "Classification of gradient-like flows in dimension two and three," Bol. Soc. Brasil. Mat. **6** (2), 155–183 (1975).
10. S. Yu. Pilyugin, "Phase diagrams defining Morse–Smale systems without periodic trajectories on spheres," Differ. Uravn. **14** (2), 245–254 (1978).
11. A. O. Prishlyak, "Complete topological invariants of Morse–Smale flows and handle decompositions of 3-manifolds," Fundam. Prikl. Mat. **11** (4), 185–196 (2005) [J. Math. Sci. (New York) **144** (5), 4492–4499 (2007)].