ISSN 0001-4346, Mathematical Notes, 2014, Vol. 96, No. 6, pp. 878–896. © Pleiades Publishing, Ltd., 2014.
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Best Mean-Square Approximations by Entire Functions of Exponential Type and Mean ν-Widths of Classes of Functions on the Line

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Received August 9, 2013; in final form, December 10, 2013

Abstract—For the classes $L_2^r(\mathbb{R})$, $r \in \mathbb{Z}_+$, we establish the upper and lower bounds for the quantities

$$\chi_{\sigma,k,r,\mu,p}(\psi,t) := \sup\biggl\{\mathcal{A}_\sigma(f^{(r-\mu)}) \Big/ \biggl(\int_0^t \omega_k^p(f^{(r)},\tau)\psi(\tau)\,d\tau\biggr)^{1/p} : f \in L_2^r(\mathbb{R})\biggr\},$$

where $\mu, r \in \mathbb{Z}_+$, $\mu \le r$, $k \in \mathbb{N}$, $0 < p \le 2$, $0 < \sigma < \infty$, $0 < t \le \pi/\sigma$, and ψ is a nonnegative, measurable function summable on the closed interval $[0,t]$ and not equivalent to zero. In the cases $\chi_{\sigma,k,r,\mu,p}(1,t)$, where $\mu \in \mathbb{N}$, $1/\mu \le p \le 2$, and $\chi_{\sigma,k,r,\mu,2/k}(1,t)$, where $0 < t \le \pi/(2\sigma)$, we obtain the exact values of these quantities. We also obtain the exact values of the average ν-widths of classes of functions defined in terms of the modulus of continuity ω^* and the majorant Ψ.

DOI: 10.1134/S000143461411025X

Keywords: *entire function of exponential type, best mean-square approximation, average ν-width, modulus of continuity, Jackson-type inequality, Fourier transform, Plancherel's theorem, Paley–Wiener theorem, Hölder's inequality, majorant, Kolmogorov width, Bernstein width, Bernstein's inequality.*

1. INTRODUCTION

Let $L_2(\mathbb{R})$, where $\mathbb{R} := \{x : -\infty < x < -\infty\}$, be the space of all measurable real functions f on the real axis whose modulus is Lebesgue square-integrable on any finite interval and the norm is finite:

$$\|f\| := \biggl\{\int_{-\infty}^{\infty} |f(x)|^2\,dx\biggr\}^{1/2} < \infty.$$

By $L_2^r(\mathbb{R})$, where $r \in \mathbb{N}$, we mean the class of functions $f \in L_2(\mathbb{R})$ whose $(r-1)$th derivatives ($f^{(0)} \equiv f$) are locally absolutely continuous and rth derivatives $f^{(r)}$ belong to the space $L_2(\mathbb{R})$. Here $L_2^r(\mathbb{R})$ is a Banach space with norm $\|f\| + \|f^{(r)}\|$. We set $L_2^0(\mathbb{R}) \equiv L_2(\mathbb{R})$. By $B_{\sigma,2}$, where $0 < \sigma < \infty$, we denote the set of restrictions on $\mathbb{R}$ of all entire functions of exponential type σ belonging to the space $L_2(\mathbb{R})$. For an arbitrary function $f \in L_2(\mathbb{R})$, the quantity

$$\mathcal{A}_\sigma(f) := \inf\{\|f - g\| : g \in B_{\sigma,2}\}, \qquad 0 < \sigma < \infty$$

is called the *value of the best approximation* of f by elements of the subspace $B_{\sigma,2}$ in the metric of $L_2(\mathbb{R})$. For any class $\mathfrak{M} \subset L_2(\mathbb{R})$, we set

$$\mathcal{A}_\sigma(\mathfrak{M}) := \sup\{A_\sigma(f) : f \in \mathfrak{M}\}.$$

Recall that studies dealing with the approximation of functions defined on the whole real axis $\mathbb{R}$ were initiated by Bernstein (see, for example, [1]) and the subspace of entire functions of finite exponential

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type served as an approximation tool. More recently, various aspects of the theory of approximation of functions on $\mathbb{R}$ by entire functions of exponential type were studied in the papers of Akhiezer, A. F. Timan, M. F. Timan, Nikol'skii, Ibragimov, Nasibov, Popov, Ligun, Babenko, Arestov, and others (see, for example, [2]–[16]). Let us present some results dealing with the calculation of definitive (in a certain sense) relations containing the values of best approximations $\mathcal{A}_\sigma(f)$ and the moduli of continuity of the functions under consideration. Note that by the *modulus of continuity* of kth order of a function $f \in L_2(\mathbb{R})$ we mean the quantity

$$\omega_k(f,t) := \sup\{\|\Delta_h^k(f)\| : |h| \le t\}, \qquad \text{where} \quad k \in \mathbb{N}, \quad 0 \le t < \infty,$$

while

$$\Delta_h^k(f,x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh)$$

is the *finite difference* of kth order of the function f at the point x with step h.

Ibragimov and Nasibov [7] showed that, for any $0 < \sigma < \infty$ and an arbitrary function $f \in L_2(\mathbb{R})$, the following inequality holds:

$$\mathcal{A}_\sigma(f) < \frac{1}{\sqrt{2}}\,\omega_1\biggl(f, \frac{\pi}{\sigma}\biggr), \tag{1.1}$$

in which the constant $1/\sqrt{2}$ is exact.

Popov noted in [8] that, for any function $f \in L_2^r(\mathbb{R})$, where $r \in \mathbb{Z}_+$, and, for $0 < \sigma < \infty$, the following relation holds:

$$\mathcal{A}_\sigma(f) < \frac{1}{\sqrt{2}\,\sigma^r}\biggl\{\frac{\sigma}{2}\int_0^{\pi/\sigma} \omega_1^2(f^{(r)}, t)\sin\sigma t\,dt\biggr\}^{1/2}. \tag{1.2}$$

Here, for any fixed $\sigma \in (0,\infty)$, the constant $1/(\sqrt{2}\,\sigma^r)$ for the class $L_2^r(\mathbb{R})$ is exact. For $r = 0$, inequality (1.2) implies the upper bound in (1.1). In the same paper [8], for an arbitrary function $f \in L_2(\mathbb{R})$, the following inequality was also obtained:

$$\mathcal{A}_\sigma(f) < \eta_{\sigma,k}\biggl\{\int_0^{2\pi/\sigma} \omega_k^2(f,t)\varphi_\sigma(t)\,dt\biggr\}^{1/2}, \tag{1.3}$$

where $k \in \mathbb{N}$, $2 \le \sigma < \infty$,

$$\varphi_\sigma(t) := \sin\biggl(\sigma\frac{t}{2}\biggr) + \frac{1}{2}\sin\sigma t, \qquad \eta_{\sigma,k} := \frac{1}{2}\sqrt{\sigma(C_{2k}^k)^{-1}};$$

in this inequality, the constant $\eta_{\sigma,k}$ is exact on $L_2(\mathbb{R})$ for all fixed values of k and $\sigma > k$. As a consequence, from relation (1.3) we obtain the inequality

$$\mathcal{A}_\sigma(f) < (C_{2k}^k)^{-1}\omega_k\biggl(f, \frac{2\pi}{\sigma}\biggr), \tag{1.4}$$

in which the constant $(C_{2k}^k)^{-1}$ cannot also be decreased.

In connection with the result (1.4), note that an important relationship between the exact constants in the L_2-Jackson inequalities on the period and on the line was indicated in Arestov's paper [13]. In particular, it follows from the results of Vasil'ev [17] and Arestov [13] that, in inequality (1.4), the point $2\pi/\sigma$ in the argument of the highest modulus of continuity can be replaced by the smaller point $1,4\pi/\sigma$.

In the same direction of studies, Ligun and Doronin showed [11] that, for arbitrary numbers $k \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $0 < \sigma < \infty$, $0 < t \le \pi/\sigma$ and any nonnegative measurable summable (on $[0,t]$) function ψ not equivalent to zero, the following two-sided inequality holds:

$$\frac{1}{\alpha_{\sigma,r,t,k}(\psi)} \le \sup_{f \in L_2^r(\mathbb{R})} \frac{\mathcal{A}_\sigma^2(f)}{\int_0^t \omega_k^2(f^{(r)},\tau)\psi(\tau)\,d\tau} \le \frac{1}{\inf\{\alpha_{u,r,t,k}(\psi) : \sigma \le u < \infty\}}, \tag{1.5}$$

where

$$\alpha_{u,r,t,k}(\psi) := 2^k u^{2r} \int_0^t (1-\cos u\tau)^k \psi(\tau)\,d\tau.$$

A number of definitive results was also obtained as a consequence of inequality (1.5) in [11].

The extremal relation

$$\sup_{f\in L_2^r(\mathbb{R})} \frac{\sigma^{2r}\mathcal{A}_\sigma^2(f)}{\left\{\int_0^t \omega_k^{2/k}(f^{(r)},\tau)\,d\tau\right\}^k} = \left\{\frac{\sigma}{2(\sigma t-\sin\sigma t)}\right\}^k, \tag{1.6}$$

where $0<\sigma<\infty$, $0<t\le\pi/(2\sigma)$, $k\in\mathbb{N}$, $r\in\mathbb{Z}_+$, was obtained in the the author's paper [14]. For $0<t\le\pi/\sigma$, the validity of the two-sided inequality

$$\frac{1}{(\sigma t)^{2k}\sigma^{2r}} \le \sup_{f\in L_2^r(\mathbb{R})} \frac{\mathcal{A}_\sigma^2(f)}{\omega_k^2(f^{(r)},t)} \le \frac{1}{\sigma^{2r}}\left\{\frac{1}{(\sigma t)^2}+\frac{1}{2}\right\}^k \tag{1.7}$$

was also shown there. Note that, for $r=0$, the upper bounds in relations (1.5)–(1.7) are calculated for all functions $f\in L_2(\mathbb{R})$ not equivalent to zero. This also applies to inequalities (1.1)–(1.4) which, for $r=0$, also involve functions $f\in L_2(\mathbb{R})$ not equivalent to zero.

It should be noted that, in the case $k=1$, the periodic analog of relations (1.6), (1.7) was established earlier by Taikov in [18, Theorem 1]. In the case of approximation by entire functions of exponential type on the line, inequalities (1.7) for $r=0$ are contained in Arestov's paper [13, Sec. 1.3]. The present paper can be regarded as a continuation of the studies carried out earlier in [6]–[9], [11], [12], [14]–[16].

2. JACKSON-TYPE INEQUALITIES IN THE CASE OF BEST APPROXIMATION BY ENTIRE FUNCTIONS OF EXPONENTIAL TYPE IN THE SPACE $L_2(\mathbb{R})$

Let us present some preliminaries dealing with Fourier transforms (see, for example, [3, Chap. III, pp. 170–173; Chap. IV, pp. 211–212]).

It follows from Plancherel's theorem that, for any function $f\in L_2(\mathbb{R})$, the integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\frac{e^{-itx}-1}{-it}\,dt$$

almost everywhere has the finite derivative

$$\mathcal{F}(f,x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty} f(t)\frac{e^{-itx}-1}{-it}\,dt, \tag{2.1}$$

for which

$$\int_{-\infty}^{\infty} |\mathcal{F}(f,x)|^2\,dx = \int_{-\infty}^{\infty} |f(x)|^2\,dx, \tag{2.2}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty} \mathcal{F}(f,t)\frac{e^{itx}-1}{it}\,dt. \tag{2.3}$$

The function $\mathcal{F}(f)$ is called the *Fourier transform* of the function $f\in L_2(\mathbb{R})$. In addition, as $\lambda\to\infty$,

$$\int_{-\infty}^{\infty}\left|f(x)-\frac{1}{\sqrt{2\pi}}\int_{-\lambda}^{\lambda}\mathcal{F}(f,t)e^{itx}\,dt\right|^2 dx \to 0.$$

For the case in which the function f (respectively, $\mathcal{F}(f)$) in formula (2.1) (respectively, (2.3)) is absolutely integrable on $\mathbb{R}$, it can be differentiated under the sign of the integral. Then the Fourier transform (2.1) (respectively, the transform (2.3)) takes the form

$$\mathcal{F}(f,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-itx}\,dt \quad \left(\text{respectively, } f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}(f,t)e^{itx}\,dt\right). \tag{2.4}$$

Formula (2.4) defines the Fourier transform for all functions absolutely integrable on $\mathbb{R}$. Recall that, for entire functions $g\in B_{\sigma,2}$, the following statement is valid.

Theorem A (Paley, Wiener). *A function g is an entire function of finite exponential type not exceeding σ and satisfies the condition*

$$\int_{-\infty}^{\infty} |g(x)|^2\,dx < \infty$$

if and only if it can be expressed as

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \varphi(t) e^{itx}\,dt,$$

where φ is a square-integrable function on the closed interval $[-\sigma, \sigma]$.

It is well known (see, for example, [19]) that if a function f and its first derivative $f^{(1)}$ belong to the space $L_2(\mathbb{R})$ and f is locally absolutely continuous, then the Fourier transform (2.1) of $f^{(1)}$ is expressed via the Fourier transform of the function f by the formula $\mathcal{F}(f^{(1)}, x) = ix\mathcal{F}(f, x)$. In the case $f \in L_2^r(\mathbb{R})$, $r \in \mathbb{N}$, $r \geq 2$, all of its intermediate derivatives $f^{(r-\mu)}$, where $1 \leq \mu \leq r-1$, will also be locally absolutely continuous functions belonging to the space $L_2(\mathbb{R})$ (see [20, Chap. V, Sec. 4, Theorem 3]), and the following relation holds:

$$\mathcal{F}(f^{(r-\mu)}, x) = (ix)^{(r-\mu)} \mathcal{F}(f, x), \tag{2.5}$$

where $\mu = 0, \dots, r-1$. In the connection with the above remarks, the study, along with that of $\mathcal{A}_\sigma(f)$, also of the behavior of the values $\mathcal{A}_\sigma(f^{(r-\mu)})$ of best approximation of the intermediate derivatives $f^{(r-\mu)}$, where $1 \leq \mu \leq r-1$, by the subspace $B_{\sigma,2}$ for the class $L_2^r(\mathbb{R})$, $r \in \mathbb{N}$, $r \geq 2$, is of considerable interest. This question will be dealt with in Theorem 1. Before its formulation, let us introduce the following notation:

$$(1-\cos x)_* := \begin{cases} 1-\cos x & \text{if } |x| \leq \pi, \\ 2 & \text{if } |x| \geq \pi, \end{cases} \tag{2.6}$$

$$\beta_{u,k,\mu,p}(\psi, t) := 2^{k/2} u^\mu \left\{ \int_0^t (1-\cos u\tau)_*^{kp/2} \psi(\tau)\,d\tau \right\}^{1/p}, \tag{2.7}$$

$$\chi_{\sigma,k,r,\mu,p}(\psi, t) := \sup_{f \in L_2^r(\mathbb{R})} \frac{\mathcal{A}_\sigma(f^{(r-\mu)})}{\left\{ \int_0^t \omega_k^p(f^{(r)}, \tau) \psi(\tau)\,d\tau \right\}^{1/p}}, \tag{2.8}$$

where $\mu,\ r \in \mathbb{Z}_+,\ \mu \leq r,\ k \in \mathbb{N},\ 0 < p \leq 2,\ 0 < t,\ \sigma < \infty$, and ψ is a nonnegative, measurable, summable (on $[0, t]$) function not equivalent to zero. Note that, for $r = 0$, the upper bound in relation (2.8) is calculated for all functions $f \in L_2(\mathbb{R})$ not equivalent to zero.

Note that, in the 2π-periodic case, extremal characteristics similar (in a certain sense) to (2.8) were considered earlier, for example, in the papers of Chernykh [21] ($k = 1,\ n \in \mathbb{N},\ r \in \mathbb{Z}_+,\ \mu = r,\ p = 2,\ t = \pi/n,\ \psi(t) = \sin nt$), Taikov [18] ($k = 1,\ n \in \mathbb{N},\ r \in \mathbb{Z}_+,\ \mu = r,\ p = 2,\ 0 < t \leq \pi/n,\ \psi(t) \equiv 1$), Ligun [22] ($k, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $\mu = r$, $p = 2$, $0 < t \leq \pi/n$, ψ is a nonnegative, measurable, summable (on $[0, \pi/n]$) function, not identically zero), Shalaev [23] ($k, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $\mu = r$, $p = 2/k$, $t = \pi/n$, $\psi(t) = \sin nt$), Vakarchuk [24] ($k, n \in \mathbb{N},\ r \in \mathbb{Z}_+,\ \mu = r,\ p = 2/k,\ 0 < t \leq \pi/(2n),\ \psi(t) \equiv 1$), and Shabozov and Yusupov [25] ($k, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $\mu = r$, $0 < p \leq 2$, $0 < t \leq \pi/n$, the weight function ψ satisfies the conditions given in [22]).

Theorem 1. *Let $0 < t \leq \pi/\sigma$, and let $0 < p \leq 2$. Then the following two-sided inequality holds*:

$$\frac{1}{\beta_{\sigma,k,\mu,p}(\psi, t)} \leq \chi_{\sigma,k,r,\mu,p}(\psi, t) \leq \frac{1}{\inf\{\beta_{u,k,\mu,p}(\psi, t) : \sigma \leq u < \infty\}}. \tag{2.9}$$

Proof. In [7], Ibragimov and Nasibov showed that, for a function $f \in L_2(\mathbb{R})$ having the Fourier transform (2.1) in the sense of the space $L_2(\mathbb{R})$, the entire function

$$\Lambda_\sigma(f, x) := \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \mathcal{F}(f, \tau) e^{ix\tau}\,d\tau,$$

belonging to the subspace $B_{\sigma,2}$ is least deviating from the function f in the sense of the metric of $L_2(\mathbb{R})$ i.e.,

$$\mathcal{A}_\sigma(f) = \|f - \Lambda_\sigma(f)\| = \left\{\int_{|\tau|\geq\sigma} |\mathcal{F}(f,\tau)|^2\, d\tau\right\}^{1/2}.$$

Since the function f is real, it follows that the function $|\mathcal{F}(f)|$ is even. Therefore,

$$\mathcal{A}_\sigma(f) = \left\{2\int_\sigma^\infty |\mathcal{F}(f,\tau)|^2\, d\tau\right\}^{1/2}. \tag{2.10}$$

The subsequent argument is carried out in two stages, in the first stage, we consider the case $r = \mu \in \mathbb{Z}_+$ and, in the second, the case $r \in \mathbb{N}$ and $\mu \in \mathbb{Z}_+$, where $\mu < r$.

Let $r = \mu \in \mathbb{Z}_+$. Consider an arbitrary function $f \in L_2^r(\mathbb{R})$ not equivalent to zero if $r = 0$. Using the fundamental properties of the Fourier transform [19] and, in particular, formula (2.5), where $\mu = 0$, we obtain

$$\mathcal{F}(\Delta_\tau^k(f^{(r)}), u) = (iu)^r (e^{iu\tau} - 1)^k \mathcal{F}(f,u). \tag{2.11}$$

Since $\Delta_\tau^k(f^{(r)}) \in L_2(\mathbb{R})$, it follows that, by Plancherel's theorem, $\mathcal{F}(\Delta_\tau^k(f^{(r)})) \in L_2(\mathbb{R})$, and hence these functions have identical norms by virtue of formula (2.2). Therefore, using equality (2.11), we obtain

$$\|\Delta_\tau^k(f^{(r)})\|^2 = 2\int_0^\infty |\mathcal{F}(f,u)|^2 u^{2r} 2^k (1-\cos\tau u)^k\, du. \tag{2.12}$$

Taking into account the definition of the modulus of continuity of kth order from (2.12), we see that

$$\omega_k^2(f^{(r)},\tau) \geq \|\Delta_\tau^k(f^{(r)})\|^2 \geq 2\int_\sigma^\infty |\mathcal{F}(f,u)|^2 u^{2r} 2^k (1-\cos\tau u)^k\, du. \tag{2.13}$$

Setting

$$\mathcal{L}(f;u,\tau) := 2^{(1+k)p/2} |\mathcal{F}(f,u)|^p u^{rp} (1-\cos u\tau)^{kp/2} \psi(\tau),$$

as well as using inequality (2.13), the generalized Minkowski inequality (see, for example, [5, Chap. 1, Sec. 1.3]) and the notation (2.7), we obtain

$$\begin{aligned}
&\left\{\int_0^t \omega_k^p(f^{(r)},\tau)\psi(\tau)\, d\tau\right\}^{1/p} \\
&\quad\geq \left\{\int_0^t \left[2\int_\sigma^\infty |\mathcal{F}(f,u)|^2 u^{2r} 2^k (1-\cos u\tau)^k\, du\right]^{p/2} \psi(\tau)\, d\tau\right\}^{1/p} \\
&\quad= \left\{\int_0^t \left[\int_\sigma^\infty \mathcal{L}^{2/p}(f;u,\tau)\, du\right]^{p/2} d\tau\right\}^{1/p} \geq \left\{\int_\sigma^\infty \left[\int_0^t \mathcal{L}(f;u,\tau)\, d\tau\right]^{2/p} du\right\}^{(p/2)\cdot(1/p)} \\
&\quad= \left\{2\int_\sigma^\infty |\mathcal{F}(f,u)|^2 \left[2^{kp/2} u^{rp} \int_0^t (1-\cos u\tau)^{kp/2}\psi(\tau)\, d\tau\right]^{2/p} du\right\}^{1/2} \\
&\quad= \left\{2\int_\sigma^\infty |\mathcal{F}(f,u)|^2 \beta_{u,k,r,p}^2(\psi,t)\, du\right\}^{1/2}.
\end{aligned} \tag{2.14}$$

Using formula (2.10) and relation (2.14), we can write

$$\left\{\int_0^t \omega_k^p(f^{(r)},\tau)\psi(\tau)\, d\tau\right\}^{1/p} \geq \mathcal{A}_\sigma(f) \inf\{\beta_{u,k,r,p}(\psi,t) : \sigma \leq u < \infty\}.$$

From this inequality and formula (2.8), we obtain the upper bound

$$\chi_{\sigma,k,r,r,p}(\psi,t) \leq \frac{1}{\inf\{\beta_{u,k,r,p}(\psi,t) : \sigma \leq u < \infty\}}. \tag{2.15}$$

To find a lower bound for the extremal characteristic on the left-hand side of equality (2.15), we consider an entire function λ_ε of exponential type $\sigma+\varepsilon$, where $\varepsilon \in (0,\sigma_*)$ is an arbitrary number, $\sigma_* := \min(\sigma, 1)$, and

$$\lambda_\varepsilon(x) := \sqrt{\frac{2}{\pi}}\,(q_{\sigma+\varepsilon}(x) - q_\sigma(x)). \tag{2.16}$$

Here $q_a(x) := (\sin ax)/x$, where $a > 0$. For $x = 0$, we set $q_a(0) = a$. Since the Fourier transform of the function q_a has the form

$$\mathcal{F}(q_a, x) = \sqrt{\frac{\pi}{2}}\begin{cases} 1 & \text{if } |x| < a, \\ \dfrac{1}{2} & \text{if } |x| = a, \\ 0 & \text{if } |x| > a, \end{cases}$$

(see, for example, [26, Chap. 5]), it follows that, for the function λ_ε, we have

$$\mathcal{F}(\lambda_\varepsilon, x) = \begin{cases} 1, & \text{if } \sigma < |x| < \sigma+\varepsilon, \\ \dfrac{1}{2} & \text{if } |x| = \sigma \text{ or } |x| = \sigma+\varepsilon, \\ 0 & \text{if } |x| > \sigma+\varepsilon \text{ or } |x| < \sigma. \end{cases} \tag{2.17}$$

In view of formulas (2.2) and (2.5), where $f := \lambda_\varepsilon$ and $\mu := 0$, the inclusion $\lambda_\varepsilon \in L_2^r(\mathbb{R})$ is obvious. It follows from relations (2.10) and (2.17) that

$$\mathcal{A}_\sigma(\lambda_\varepsilon) = \sqrt{2\varepsilon}. \tag{2.18}$$

Taking into account the equality $\mathcal{F}(\lambda_\varepsilon^{(r)}, x) = (ix)^r \mathcal{F}(\lambda_\varepsilon, x)$ and using formulas (2.6), (2.12), and (2.17) we can write

$$\|\Delta_h^k(\lambda_\varepsilon^{(r)})\|^2 = 2^{k+1}\int_\sigma^{\sigma+\varepsilon} u^{2r}(1-\cos hu)^k\,du \le 2^{k+1}\varepsilon(\sigma+\varepsilon)^{2r}(1-\cos(\sigma+\varepsilon)h)_*^k,$$

where $|h| \le \pi/\sigma$. Combining this with the definition of the modulus of continuity of kth order, we obtain the inequality

$$\omega_k^p(\lambda_\varepsilon^{(r)}, \tau) \le 2^{(k+1)p/2}\varepsilon^{p/2}(\sigma+\varepsilon)^{rp}(1-\cos(\sigma+\varepsilon)\tau)_*^{kp/2}, \tag{2.19}$$

where $0 < \tau \le \pi/\sigma$. Multiplying both sides of inequality (2.19) by the function ψ and integrating both sides of the resulting relation over τ from 0 to t, we obtain

$$\int_0^t \omega_k^p(\lambda_\varepsilon^{(r)}, \tau)\psi(\tau)\,d\tau \le 2^{(k+1)p/2}\varepsilon^{p/2}(\sigma+\varepsilon)^{rp}\int_0^t (1-\cos(\sigma+\varepsilon)\tau)_*^{kp/2}\psi(\tau)\,d\tau. \tag{2.20}$$

Setting

$$\beta^*_{\sigma+\varepsilon,k,r,p}(\psi, t) := 2^{k/2}(\sigma+\varepsilon)^r\left\{\int_0^t (1-\cos(\sigma+\varepsilon)\tau)_*^{kp/2}\psi(\tau)\,d\tau\right\}^{1/p} \tag{2.21}$$

and using formulas (2.8), (2.18), and (2.20), (2.21), for $0 < t \le \pi/\sigma$ we can write

$$\chi_{\sigma,k,r,r,p}(\psi, t) \ge \frac{\mathcal{A}_\sigma(\lambda_\varepsilon)}{\left\{\int_0^t \omega_k^p(\lambda_\varepsilon^{(r)}, \tau)\psi(\tau)\,d\tau\right\}^{1/p}} \ge \frac{1}{\beta^*_{\sigma+\varepsilon,k,r,p}(\psi, t)}. \tag{2.22}$$

It follows from relations (2.21) and (2.6) that the quantity $\beta^*_{\sigma+\varepsilon,k,r,p}(\psi, t)$ decreases as $\varepsilon \to 0+0$ while the other parameters appearing in (2.21) are constant. Obviously,

$$\lim_{\varepsilon\to 0+0} \beta^*_{\sigma+\varepsilon,k,r,p}(\psi, t) = \beta_{\sigma,k,r,p}(\psi, t) \tag{2.23}$$

and, for any arbitrarily small number $\delta > 0$, we can chose a number $\widetilde{\varepsilon} := \varepsilon(\delta) \in (0, \delta)$ for which, in view of (2.23), we have

$$\frac{1}{\beta^*_{\sigma+\widetilde{\varepsilon},k,r,p}(\psi,t)} > \frac{1}{\beta_{\sigma,k,r,p}(\psi,t)} - \delta. \tag{2.24}$$

Using the definition of the upper bound of the set, from (2.24), we obtain

$$\sup_{\varepsilon\in(0,\sigma_*)} \frac{1}{\beta^*_{\sigma+\varepsilon,k,r,p}(\psi,t)} = \frac{1}{\beta_{\sigma,k,r,p}(\psi,t)}\,. \tag{2.25}$$

Since the left-hand side of inequality (2.22) is independent of ε, calculating the upper bound with respect to $\varepsilon \in (0, \sigma_*)$ for its right-hand side, we have

$$\chi_{\sigma,k,r,r,p}(\psi,t) \ge \frac{1}{\beta_{\sigma,k,r,p}(\psi,t)}\,. \tag{2.26}$$

For $r = \mu \in \mathbb{Z}_+$, comparing the upper bound (2.15) with the lower bound (2.26), we obtain the two-sided inequality

$$\frac{1}{\beta_{\sigma,k,r,p}(\psi,t)} \le \chi_{\sigma,k,r,r,p}(\psi,t) \le \frac{1}{\inf\{\beta_{u,k,r,p}(\psi,t) : \sigma \le u < \infty\}}\,. \tag{2.27}$$

Passing to the second stage of the proof, we set $r \in \mathbb{N}$, $\mu \in \mathbb{Z}_+$, and $\mu < r$. Since the function $f^{(r-\mu)}$, where $f \in L_2^r(\mathbb{R})$, can be regarded as an element of the class $L_2^\mu(\mathbb{R})$ and $L_2^r(\mathbb{R}) \subset L_2^\mu(\mathbb{R})$, it follows that, using formulas (2.8) and (2.27), we obtain the following upper bound in the case under consideration:

$$\begin{aligned}\chi_{\sigma,k,r,\mu,p}(\psi,t) &= \sup_{f\in L_2^r(\mathbb{R})} \frac{\mathcal{A}_\sigma(f^{(r-\mu)})}{\left\{\int_0^t \omega_k^p(f^{(r)},\tau)\psi(\tau)\,d\tau\right\}^{1/p}} \\ &\le \sup_{F\in L_2^\mu(\mathbb{R})} \frac{\mathcal{A}_\sigma(F)}{\left\{\int_0^t \omega_k^p(F^{(\mu)},\tau)\psi(\tau)\,d\tau\right\}^{1/p}} \\ &= \chi_{\sigma,k,\mu,\mu,p}(\psi,t) \le \frac{1}{\inf\{\beta_{u,k,\mu,p}(\psi,t) : \sigma \le u < \infty\}}\,. \end{aligned} \tag{2.28}$$

To obtain a lower bound for the extremal characteristic written on the left-hand side of relation (2.28), we consider the entire function $\lambda_\varepsilon \in L_2^r(\mathbb{R})$ given by formula (2.16). By formula (2.10), we have

$$\mathcal{A}_\sigma(\lambda_\varepsilon^{(r-\mu)}) = \left\{2\int_\sigma^\infty |\mathcal{F}(\lambda_\varepsilon^{(r-\mu)},\tau)|^2\,d\tau\right\}^{1/2}. \tag{2.29}$$

From equalities (2.5) and (2.17), we obtain

$$\mathcal{F}(\lambda_\varepsilon^{(r-\mu)},x) = (ix)^{(r-\mu)}\mathcal{F}(\lambda_\varepsilon,x) = (ix)^{r-\mu}\begin{cases}1 & \text{if } \sigma < |x| < \sigma+\varepsilon,\\ \dfrac{1}{2} & \text{if } |x| = \sigma \text{ or } |x| = \sigma+\varepsilon,\\ 0, & \text{if } |x| > \sigma+\varepsilon \text{ or } |x| < \sigma.\end{cases} \tag{2.30}$$

Using equalities (2.29) and (2.30), we can write

$$\mathcal{A}_\sigma(\lambda_\varepsilon^{(r-\mu)}) = \left\{2\int_\sigma^{\sigma+\varepsilon} \tau^{2(r-\mu)}\,d\tau\right\}^{1/2} \ge \sqrt{2\varepsilon}\,\sigma^{r-\mu}. \tag{2.31}$$

Applying formulas (2.7), (2.8), (2.20), (2.21), and (2.31), we obtain

$$\chi_{\sigma,k,r,\mu,p}(\psi,t) \ge \frac{\mathcal{A}_\sigma(\lambda_\varepsilon^{(r-\mu)})}{\left\{\int_0^t \omega_k^p(\lambda_\varepsilon^{(r)},\tau)\psi(\tau)\,d\tau\right\}^{1/p}}$$

$$\geq \frac{(1+\varepsilon/\sigma)^{-r}2^{-k/2}\sigma^{-\mu}}{\left\{\int_0^t (1-\cos(\sigma+\varepsilon)\tau)_*^{kp/2}\psi(\tau)\,d\tau\right\}^{1/2}} \geq \frac{1}{(1+\varepsilon/\sigma)^r\beta^*_{\sigma+\varepsilon,k,\mu,p}(\psi,t)}. \tag{2.32}$$

It follows from relations (2.6) and (2.21) that the quantity $(1+\varepsilon/\sigma)^r\beta^*_{\sigma+\varepsilon,k,\mu,p}(\psi,t)$ decreases as $\varepsilon\to 0+0$, and all the other parameters in expression (2.21) are constant. Further,

$$\lim_{\varepsilon\to 0+0}\left(1+\frac{\varepsilon}{\sigma}\right)^r\beta^*_{\sigma+\varepsilon,k,\mu,p}(\psi,t)=\beta_{\sigma,k,\mu,p}(\psi,t).$$

Using arguments similar to those used at the end of the proof of the first stage of the theorem, we can write

$$\sup_{\varepsilon\in(0,\sigma_*)}\frac{1}{(1+\varepsilon/\sigma)^r\beta^*_{\sigma+\varepsilon,k,\mu,p}(\psi,t)}=\frac{1}{\beta_{\sigma,k,\mu,p}(\psi,t)}. \tag{2.33}$$

Calculating the upper bound with respect to $\varepsilon\in(0,\sigma_*)$ for the right-hand side of inequality (2.32) and taking into account the fact that its left-hand side is independent of ε, we obtain the lower bound

$$\chi_{\sigma,k,r,\mu,p}(\psi,t)\geq\frac{1}{\beta_{\sigma,k,\mu,p}(\psi,t)}. \tag{2.34}$$

For $r\in\mathbb{N}$, $\mu<r$, and $\mu\in\mathbb{Z}_+$, comparing the upper bound (2.28) with the lower bound (2.34) for $\chi_{\sigma,k,r,\mu,p}(\psi,t)$, we obtain the two-sided inequality

$$\frac{1}{\beta_{\sigma,k,\mu,p}(\psi,t)}\leq\chi_{\sigma,k,r,\mu,p}(\psi,t)\leq\frac{1}{\inf\{\beta_{u,k,\mu,p}(\psi,t):\sigma\leq u<\infty\}}. \tag{2.35}$$

Combining the results (2.27) and (2.35), we obtain the required relations (2.9). Theorem 1 is proved. □

Note that inequalities (1.5) obtained by Ligun and Doronin are a particular case of Theorem 1 for $\mu=r$ and $p=2$.

3. COROLLARIES OF THEOREM 1

Theorem 1 implies a number of corollaries that, in our opinion, are of interest in themselves.

Corollary 1. *Suppose that $\psi\equiv 1$, $\mu,k,r\in\mathbb{N}$, $\mu\leq r$, $1/\mu\leq p\leq 2$, $0<\sigma<\infty$, $0<t\leq\pi/\sigma$. Then the following relation holds*:

$$\chi_{\sigma,k,r,\mu,p}(1,t)=\frac{1}{\beta_{\sigma,k,\mu,p}(1,t)}. \tag{3.1}$$

Proof. Consider the auxiliary function

$$\theta(u):=\{2^{-k/2}\beta_{u,k,\mu,p}(1,t)\}^p=u^{\mu p}\int_0^t(1-\cos u\tau)^{kp/2}\,d\tau$$

depending on one variable u and taking the values in the set $[\sigma,\infty)$. Obviously, if the function θ is nondecreasing for $\sigma\leq u<\infty$, then so is also the quantity $\beta_{u,k,\mu,p}(1,t)$ with respect to the variable u for fixed values of its other parameters. On the basis of these facts, we calculate the first derivative of the function θ, obtaining

$$\theta^{(1)}(u)=\mu p u^{\mu p-1}\int_0^t(1-\cos u\tau)^{kp/2}\,d\tau+u^{\mu p}\int_0^t\frac{\partial}{\partial u}(1-\cos u\tau)^{kp/2}\,d\tau. \tag{3.2}$$

It is readily verified that the following equality holds:

$$\frac{1}{\tau}\frac{\partial}{\partial u}(1-\cos u\tau)^{kp/2}=\frac{1}{u}\frac{\partial}{\partial\tau}(1-\cos u\tau)^{kp/2}, \tag{3.3}$$

where the variables u and τ are nonzero. Using formulas (3.2) and (3.3), we can write

$$\theta^{(1)}(u) = u^{\mu p-1}\Big\{\mu p\int_0^t(1-\cos u\tau)^{kp/2}\,d\tau + \int_0^t \tau\frac{\partial}{\partial\tau}(1-\cos u\tau)^{kp/2}\,d\tau\Big\}. \tag{3.4}$$

Integrating by parts the second integral on the right-hand side of equality (3.4), we obtain

$$\theta^{(1)}(u) = u^{\mu p-1}\Big\{t(1-\cos ut)^{kp/2} + (\mu p-1)\int_0^t(1-\cos u\tau)^{kp/2}\,d\tau\Big\}. \tag{3.5}$$

For any $\sigma \le u < \infty$, taking into account the inequality $p \ge 1/\mu$ and using formula (3.5), we see that $\theta^{(1)}(u) \ge 0$. Therefore,

$$\inf\{\theta(u) : \sigma \le u < \infty\} = \theta(\sigma),$$

and this, in turn, implies

$$\inf\{\beta_{u,k,\mu,p}(1,t) : \sigma \le u < \infty\} = \beta_{\sigma,k,\mu,p}(1,t). \tag{3.6}$$

Using formulas (2.9) and (3.6), we obtain relation (3.1) in the case under consideration, which concludes the proof of Corollary 1. □

Corollary 1 can be complemented by the following corollary containing a number of possible cases of variation of the parameters of the extremal characteristic (2.8).

Corollary 2. *Suppose that $\psi \equiv 1$, $k \in \mathbb{N}$, $\mu, r \in \mathbb{Z}_+$, $\mu \le r$, $p = 2/k$, $0 < \sigma < \infty$, $0 < t \le \pi/(2\sigma)$. Then the following equality holds*:

$$\sup_{f\in L_2^r(\mathbb{R})}\frac{\sigma^\mu \mathcal{A}_\sigma(f^{(r-\mu)})}{\big\{\int_0^t \omega_k^{2/k}(f^{(r)},\tau)\psi(\tau)\,d\tau\big\}^{k/2}} = \Big\{\frac{\sigma}{2(\sigma t-\sin\sigma t)}\Big\}^{k/2}. \tag{3.7}$$

Proof. Consider the quantity

$$\widetilde{\theta}(u) := \beta_{u,k,\mu,2/k}(1,t) = 2^{k/2}u^\mu\Big\{\int_0^t(1-\cos u\tau)\,d\tau\Big\}^{k/2} = (2t)^{k/2}u^\mu\Big(1-\frac{\sin ut}{ut}\Big)^{k/2} \tag{3.8}$$

as a function of the independent variable $u \in [\sigma,\infty)$ for fixed values of the other parameters defining relation (2.7). Taking into account the constraint $0 < t \le \pi/(2\sigma)$ and the behavior of the function $\sin(x)/x$ (see, for example, [27, Sec. 1.2, pp. 129, 132]), we obtain

$$\inf_{\sigma\le u<\infty}\Big(1-\frac{\sin ut}{ut}\Big)^{k/2} = \Big(1-\sup_{\sigma\le u<\infty}\frac{\sin ut}{ut}\Big)^{k/2} = \Big(1-\frac{\sin\sigma t}{\sigma t}\Big)^{k/2}. \tag{3.9}$$

Using formulas (3.8), (3.9), we can write

$$\inf_{\sigma\le u<\infty}\widetilde{\theta}(u) = \widetilde{\theta}(\sigma) = (2t)^{k/2}\sigma^\mu\Big(1-\frac{\sin\sigma t}{\sigma t}\Big)^{k/2}. \tag{3.10}$$

The required equality (3.7) follows from relations (2.9) and (3.8), (3.10). Corollary 2 is proved. □

It is necessary to note that, for $\mu = r$, the the author's result (1.6) is contained in formula (3.7).

Guseinov and Mukhtarov showed in [28, Chap. II, Sec. 1] that if ω is a modulus of continuity, then the function

$$\omega^*(t) := \frac{1}{t}\int_0^t \omega(\tau)\,d\tau, \qquad t > 0,$$

will also be a modulus of continuity. Since, as was noted in [28], $\lim_{t\to0+0}\omega^*(t) = 0$, it is assumed that $\omega^*(0) = 0$. It follows from the above that, for an arbitrary element $f \in L_2(\mathbb{R})$, the function

$$\omega^*(f,t) := \frac{1}{t}\int_0^t \omega_1(f,\tau)\,d\tau \tag{3.11}$$

is also a modulus of continuity of first order and can be used along with the classical modulus of continuity $\omega_1(f)$ for solving extremal problems of approximation theory in the space $L_2(\mathbb{R})$.

Corollary 3. *Suppose that $\mu, r \in \mathbb{N}$ and $\mu \le r$, $0 < \sigma < \infty$, $0 < t \le \pi/\sigma$. Then the following equality holds*:

$$\sup_{f \in L_2^r(\mathbb{R})} \frac{\sigma^{\mu-1}\mathcal{A}_\sigma(f^{(r-\mu)})}{\omega^*(f^{(r)}, t)} = \frac{t}{4(1-\cos(\sigma t/2))}. \tag{3.12}$$

Proof. Using equality (3.1), in which we set $k := 1$, $p := 1$, and formula (2.8), we can write

$$\chi_{\sigma,1,r,\mu,1}(1,t) = \sup_{f \in L_2^r(\mathbb{R})} \frac{\mathcal{A}_\sigma(f^{(r-\mu)})}{\int_0^t \omega_1(f^{(r)}, \tau)\,d\tau} = \frac{1}{\beta_{\sigma,1,\mu,1}(1,t)} = \frac{\sigma^{-\mu}}{\sqrt{2}\int_0^t (1-\cos\sigma\tau)^{1/2}\,d\tau}. \tag{3.13}$$

Carrying out necessary calculations on the right-hand side of (3.13) and using formula (3.11), we obtain the required equality (3.12), which concludes the proof of the corollary. □

4. INEQUALITIES CONTAINING THE BEST APPROXIMATION AND THE MODULUS OF CONTINUITY IN $L_2(\mathbb{R})$

Along with the study of the behavior of the extremal characteristic (2.8), the study of the extremal characteristic

$$\widetilde{\chi}_{\sigma,k,r,\mu}(t) := \sup_{f \in L_2^r(\mathbb{R})} \frac{\mathcal{A}_\sigma(f^{(r-\mu)})}{\omega_k(f^{(r)}, t)}, \tag{4.1}$$

where $\mu, r \in \mathbb{Z}_+$, $\mu \le r$, $k \in \mathbb{N}$, $0 < t, \sigma < \infty$, is also of interest. Recall that, for $r = 0$, the upper bound in relation (4.1) is calculated for all functions $f \in L_2(\mathbb{R})$ not equivalent to zero. The following theorem deals with this question.

Theorem 2. *Let $0 < t \le \pi/\sigma$. Then the following two-sided inequality holds*:

$$\frac{1}{\sigma^\mu(\sigma t)^k} \le \widetilde{\chi}_{\sigma,k,r,\mu}(t) \le \frac{1}{\sigma^\mu}\left\{\frac{1}{2} + \frac{1}{(\sigma t)^2}\right\}^{k/2}. \tag{4.2}$$

Proof. From equalities (2.5) and (2.10), we obtain

$$\mathcal{A}_\sigma(f^{(r-\mu)}) = \left\{2\int_\sigma^\infty |\mathcal{F}(f^{(r-\mu)}, u)|^2\,du\right\}^{1/2} = \left\{2\int_\sigma^\infty u^{2(r-\mu)}|\mathcal{F}(f,u)|^2\,du\right\}^{1/2}. \tag{4.3}$$

Using relation (4.3), we can write

$$\begin{aligned} &\mathcal{A}_\sigma^2(f^{(r-\mu)}) - 2\int_\sigma^\infty u^{2(r-\mu)}|\mathcal{F}(f,u)|^2\cos\tau u\,du \\ &\qquad = 2\int_\sigma^\infty \{u^{r-\mu}|\mathcal{F}(f,u)|\}^{2(1-1/k)}\{u^{r-\mu}|\mathcal{F}(f,u)|\}^{2/k}(1-\cos\tau u)\,du. \end{aligned} \tag{4.4}$$

Applying Hölder's inequality to the right-hand side of equality (4.4) and taking into account the definition of the modulus of continuity of kth order and formulas (2.12) and (4.3), we obtain

$$\begin{aligned} &\mathcal{A}_\sigma^2(f^{(r-\mu)}) - 2\int_\sigma^\infty u^{2(r-\mu)}|\mathcal{F}(f,u)|^2\cos\tau u\,du \\ &\quad \le \left\{2\int_\sigma^\infty u^{2(r-\mu)}|\mathcal{F}(f,u)|^2\,du\right\}^{1-1/k}\left\{2\int_\sigma^\infty u^{2(r-\mu)}|\mathcal{F}(f,u)|^2(1-\cos\tau u)^k\,du\right\}^{1/k} \\ &\quad \le \mathcal{A}_\sigma^{2(1-1/k)}(f^{(r-\mu)})\frac{1}{2\sigma^{2\mu/k}}\left\{2^{k+1}\int_\sigma^\infty u^{2r}|\mathcal{F}(f,u)|^2(1-\cos\tau u)^k\,du\right\}^{1/k} \\ &\quad \le \mathcal{A}_\sigma^{2(1-1/k)}(f^{(r-\mu)})\frac{\omega_k^{2/k}(f^{(r)},\tau)}{2\sigma^{2\mu/k}}. \end{aligned} \tag{4.5}$$

Let us first integrate both sides of inequality (4.5) over the variable τ from 0 to v and then integrate the resulting relation over v from 0 to t, where $0 < t \leq \pi/\sigma$. As a result, we have

$$\frac{t^2}{2}\,\mathcal{A}_\sigma^2(f^{(r-\mu)}) \leq 2\int_\sigma^\infty u^{2(r-\mu-1)}|\mathcal{F}(f,u)|^2(1-\cos tu)\,du + \frac{\mathcal{A}_\sigma^{2(1-1/k)}(f^{(r-\mu)})}{2\sigma^{2\mu/k}}\int_0^t\int_0^v \omega_k^{2/k}(f^{(r)},\tau)\,d\tau\,dv. \tag{4.6}$$

Using equalities (2.10) and (2.12) as well as the definition of the modulus of continuity of kth order and Hölder's inequality, we obtain the following upper bound for the first summand on the right-hand side of equality (4.6):

$$\begin{aligned}
&2\int_\sigma^\infty u^{2(r-\mu-1)}|\mathcal{F}(f,u)|^2(1-\cos tu)\,du\\
&\leq \frac{2}{\sigma^2}\int_\sigma^\infty \{u^{r-\mu}|\mathcal{F}(f,u)|\}^{2(1-1/k)}\{u^{r-\mu}|\mathcal{F}(f,u)|(1-\cos tu)^{k/2}\}^{2/k}\,du\\
&\leq \frac{1}{\sigma^2}\biggl\{2\int_\sigma^\infty u^{2(r-\mu)}|\mathcal{F}(f,u)|^2\,du\biggr\}^{1-1/k}\biggl\{2\int_\sigma^\infty u^{2(r-\mu)}|\mathcal{F}(f,u)|^2(1-\cos tu)^k\,du\biggr\}^{1/k}\\
&\leq \frac{1}{\sigma^2}\,\mathcal{A}_\sigma^{2(1-1/k)}(f^{(r-\mu)})\biggl\{\frac{2}{2^k\sigma^{2\mu}}\int_\sigma^\infty 2^k u^{2r}|\mathcal{F}(f,u)|^2(1-\cos tu)^k\,du\biggr\}^{1/k}\\
&\leq \frac{1}{2\sigma^{2(1+\mu/k)}}\,\mathcal{A}_\sigma^{2(1-1/k)}(f^{(r-\mu)})\omega_k^{2/k}(f^{(r)},t).
\end{aligned} \tag{4.7}$$

Integrating by parts in the second summand on the right-hand side of inequality (4.6) and taking into account relation (4.7), we can rewrite inequality (4.6) as

$$t^{2k}\mathcal{A}_\sigma^2(f^{(r-\mu)}) \leq \frac{1}{\sigma^{2(k+\mu)}}\biggl\{\omega_k^{2/k}(f^{(r)},t) + \sigma^2\int_0^t (t-\tau)\omega_k^{2/k}(f^{(r)},\tau)\,d\tau\biggr\}^k.$$

Hence we have

$$t^{2k}\mathcal{A}_\sigma^2(f^{(r-\mu)}) \leq \frac{1}{\sigma^{2(k+\mu)}}\,\omega_k^2(f^{(r)},t)\biggl\{1 + \frac{(\sigma t)^2}{2}\biggr\}^k.$$

Therefore, for $0 < t \leq \pi/\sigma$, in view of formulas (4.1), we can write

$$\widetilde{\chi}_{\sigma,k,r,\mu}(t) \leq \frac{1}{\sigma^\mu}\biggl\{\frac{1}{2} + \frac{1}{(\sigma t)^2}\biggr\}^{k/2}. \tag{4.8}$$

To obtain the lower bound for the extremal characteristic (4.1), consider the function $\lambda_\varepsilon \in L_2^r(\mathbb{R})$ given by formula (2.16). For it, we can write

$$\widetilde{\chi}_{\sigma,k,r,\mu}(t) \geq \frac{\mathcal{A}_\sigma(\lambda_\varepsilon^{(r-\mu)})}{\omega_k(\lambda_\varepsilon^{(r)},t)}. \tag{4.9}$$

Setting

$$\xi_{u,k,r}(t) := u^r(1-\cos ut)^{k/2}, \tag{4.10}$$

$$\widetilde{\xi}_{u,k,r}(t) := u^r(1-\cos ut)_*^{k/2} \tag{4.11}$$

and using formula (2.19), where $p = 1$, as well as formula (2.31), from inequality (4.9), we obtain

$$\widetilde{\chi}_{\sigma,k,r,\mu}(t) \geq \frac{\sigma^{r-\mu}}{2^{k/2}\widetilde{\xi}_{\sigma+\varepsilon,k,r}(t)}. \tag{4.12}$$

In the case $u := \sigma + \varepsilon$, it follows from formulas (4.11) and (2.6) that the quantity $\widetilde{\xi}_{\sigma+\varepsilon,k,r}(t)$ decreases as $\varepsilon \to 0+0$, provided that the other parameters appearing in relation (4.11) are constant. Also,

$$\lim_{\varepsilon\to 0+0} \widetilde{\xi}_{\sigma+\varepsilon,k,r}(t) = \xi_{\sigma,k,r}(t).$$

Hence, using the definition of the upper bound, we obtain

$$\sup_{\varepsilon\in(0,\sigma_*)} \frac{1}{\widetilde{\xi}_{\sigma+\varepsilon,k,r}(t)} = \frac{1}{\xi_{\sigma,k,r}(t)}. \tag{4.13}$$

Calculating the upper bound for all $\varepsilon \in (0, \sigma_*)$ from the right-hand side of inequality (4.12) and taking into account formulas (4.13), (4.10), and (2.6), we can write

$$\widetilde{\chi}_{\sigma,k,r,\mu}(t) \geq \frac{1}{\sigma^\mu (\sigma t)^k}. \tag{4.14}$$

The required relation (4.2) is a consequence of inequalities (4.8) and (4.14). Theorem 2 is proved. $\square$

Note that, in the case $\mu := r$, relation (4.2) yields the author's result (1.7) obtained earlier.

5. MEAN ν-WIDTHS OF CLASSES OF FUNCTIONS DEFINED ON THE WHOLE REAL AXIS

5.1. The definition of mean dimension was introduced by Magaril-Il'yaev (see, for example, [29], [30]); it was a modification of the corresponding notion introduced earlier by Tikhomirov [31]; this allows us to define the asymptotic characteristics of subspaces similar to widths, in which the role of dimension was played by mean dimension. As a result, it became possible to compare the approximation properties of the subspace $B_{\sigma,2}$ with similar characteristics of other subspaces from $L_2(\mathbb{R})$ of the same mean dimension and to solve extremal problems of optimal approximation theory in $L_2(\mathbb{R})$.

Before introducing the required extremal characteristics, let us present a number of notions and definitions from [29], [30]. Let $BL_2(\mathbb{R})$ be the unit ball in $L_2(\mathbb{R})$, let $\mathrm{Lin}(L_2(\mathbb{R}))$ be the set of all linear subspaces in $L_2(\mathbb{R})$,

$$\mathrm{Lin}_n(L_2(\mathbb{R})) := \{\mathcal{L} \in \mathrm{Lin}(L_2(\mathbb{R})) : \dim \mathcal{L} \leq n\}, \qquad n \in \mathbb{Z}_+,$$

and let

$$d(Q, A, L_2(\mathbb{R})) := \sup\bigl\{\inf\{\|x-y\| : y \in A\} : x \in Q\bigr\}$$

be the best approximation of a set $Q \subset L_2(\mathbb{R})$ by the set $A \subset L_2(\mathbb{R})$. By A_T, where $T > 0$, we mean the restriction of a set $A \subset L_2(\mathbb{R})$ to the closed interval $[-T, T]$, and by $\mathrm{Lin}_C\, L_2(\mathbb{R})$ we denote the set of subspaces $\mathcal{L} \in \mathrm{Lin}(L_2(\mathbb{R}))$ for which the set $(\mathcal{L} \cap BL_2(\mathbb{R}))_T$ is precompact in $L_2([-T, T])$ for any $T > 0$.

If $\mathcal{L} \in \mathrm{Lin}_C(L_2(\mathbb{R}))$ and $T, \varepsilon > 0$, then there exist $n \in \mathbb{Z}_+$ and $\mathcal{M} \in \mathrm{Lin}_n(L_2(\mathbb{R}))$ for which [29]

$$d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{M}, L_2([-T, T])) < \varepsilon.$$

Let

$$\begin{aligned} D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R})) := \min\bigl\{n \in \mathbb{Z}_+ : \text{there exists a } \mathcal{M} \in \mathrm{Lin}_n(L_2([-T,T])) \text{ a such that}\\ d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{M}, L_2([-T, T])) < \varepsilon\bigr\}. \end{aligned}$$

This function is nondecreasing in T and nonincreasing in ε. The quantity

$$\overline{\dim}(\mathcal{L}, L_2(\mathbb{R})) := \lim\biggl\{\liminf\biggl\{\frac{D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R}))}{2T} : T \to \infty\biggr\} : \varepsilon \to 0\biggr\},$$

where $\mathcal{L} \in \mathrm{Lin}_C(L_2(\mathbb{R}))$, is called the *mean dimension* of the subspace $\mathcal{L}$ in $L_2(\mathbb{R})$. It was shown in [29] that

$$\overline{\dim}(B_{\sigma,2}, L_2(\mathbb{R})) = \frac{\sigma}{\pi}. \tag{5.1}$$

Let Q be a centrally symmetric subset from $L_2(\mathbb{R})$, and let $\nu > 0$ be an arbitrary number. Then by the *mean Kolmogorov ν-width* of the set Q in $L_2(\mathbb{R})$ we mean the quantity

$$\overline{d}_\nu(Q, L_2(\mathbb{R})) := \inf\bigl\{d(Q, \mathcal{L}, L_2(\mathbb{R})) : \mathcal{L} \in \mathrm{Lin}_C(L_2(\mathbb{R})),\ \overline{\dim}(\mathcal{L}, L_2(\mathbb{R})) \le \nu\bigr\}.$$

The subspace on which the outer lower bound is attained is called *extremal*.

By the *mean linear ν-width* of the set Q in $L_2(\mathbb{R})$ we mean the quantity

$$\overline{\delta}_\nu(Q, L_2(\mathbb{R})) := \inf\bigl\{\sup\{\|f - V(f)\| : f \in Q\} : (X, V)\bigr\},$$

where the lower bound is taken over all pairs (X, V) such that X is a normed space directly embedded in $L_2(\mathbb{R})$ and $V\colon X \to L_2(\mathbb{R})$ is a continuous linear operator for which $\mathrm{Im}\, V \in \mathrm{Lin}_C(L_2(\mathbb{R}))$ and the following inequality holds:

$$\overline{\dim}(\mathrm{Im}\, V, L_2(\mathbb{R})) \le \nu, \qquad Q \subset X.$$

Here $\mathrm{Im}\, V$ is the image of the operator V. The pair on which the lower bound is attained is called *extremal*.

The quantity

$$\begin{aligned}\overline{b}_\nu(Q, L_2(\mathbb{R})) := \sup\bigl\{ &\sup\{\rho > 0 : \mathcal{L} \cap \rho BL_2(\mathbb{R}) \subset Q\} :\\ &\mathcal{L} \in \mathrm{Lin}_C(L_2(\mathbb{R})),\ \overline{\dim}(\mathcal{L}, L_2(\mathbb{R})) > \nu,\ \overline{d}_\nu(\mathcal{L} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1\bigr\}\end{aligned}$$

is called the *mean Bernstein ν-width* of the set Q in $L_2(\mathbb{R})$. The last condition imposed on $\mathcal{L}$ in calculating the outer upper bound means that we consider only subspaces for which the analog of Tikhomirov's theorem on the width of the ball is valid. This requirement is satisfied, for example, by the subspace $B_{\sigma,2}$ if $\sigma > \nu\pi$, i.e., $\overline{d}_\nu(B_{\sigma,2} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1$.

For a set $Q \subset L_2(\mathbb{R})$ between its extremal characteristics (indicated above), the following inequalities hold:

$$\overline{b}_\nu(Q, L_2(\mathbb{R})) \le \overline{d}_\nu(Q, L_2(\mathbb{R})) \le \overline{\delta}_\nu(Q, L_2(\mathbb{R})). \tag{5.2}$$

5.2. A continuous and nondecreasing function Ψ on the set $[0, \infty)$ is called a *majorant* if $\Psi(0) = 0$ (see, for example, [32, Chap. I, Sec. 2, Sec. 3]). Let Ψ be a majorant. By $W^r(\omega^*, \Psi)$, where $r \in \mathbb{N}$, we denote the class of all functions $f \in L_2^r(\mathbb{R})$ whose rth derivatives $f^{(r)}$ satisfy the condition $\omega^*(f^{(r)}, t) \le \Psi(t)$ for any $0 \le t < \infty$. The best approximation for this class by the subspace $B_{\sigma,2}$ will be denoted by $\mathcal{A}_\sigma(W^r(\omega^*, \Psi))$, i.e.,

$$\mathcal{A}_\sigma(W^r(\omega^*, \Psi)) := \sup\{\mathcal{A}_\sigma(f) : f \in W^r(\omega^*, \Psi)\}.$$

Theorem 3. *Let ν be an arbitrary finite positive number, and let Ψ be the majorant satisfying the condition*

$$\frac{\Psi(t)}{\Psi(\pi/\sigma)} \ge \frac{\pi}{2}\begin{cases} \dfrac{4}{\sigma t}\sin^2\left(\dfrac{\sigma t}{4}\right), & \text{if } 0 < t \le \dfrac{\pi}{\sigma},\\[2mm] 1 - \dfrac{\pi - 2}{\sigma t} & \text{if } \dfrac{\pi}{\sigma} \le t < \infty, \end{cases} \tag{5.3}$$

for any finite value of σ, $\sigma > \nu\pi$. Then the following equalities hold:

$$\overline{\Pi}_\nu(W^r(\omega^*, \Psi), L_2(\mathbb{R})) = \mathcal{A}_{\nu\pi}(W^r(\omega^*, \Psi)) = \frac{\pi^{1-r}}{4\nu^r}\Psi\left(\frac{1}{\nu}\right), \tag{5.4}$$

where $\overline{\Pi}_\nu(\,\cdot\,)$ is any one of the mean ν-widths examined in Sec. 5.1. *Further, the pair $(L_2^r(\mathbb{R}), \Lambda_{\nu\pi})$, where $\Lambda_{\nu\pi}$ is the linear operator defined by the formula*

$$\Lambda_{\nu\pi}(f, x) = \frac{1}{\sqrt{2\pi}}\int_{-\nu\pi}^{\nu\pi} \mathcal{F}(f, \tau)e^{ix\tau}\, d\tau,$$

will be extremal for the mean linear ν-width $\overline{\delta}_\nu(W^r(\omega^*, \Psi), L_2(\mathbb{R}))$, and the subspace $B_{\nu\pi,2}$ will be extremal for the Kolmogorov average ν-width $\overline{d}_\nu(W^r(\omega^*, \Psi), L_2(\mathbb{R}))$. The set of majorants satisfying condition (5.3) *is not empty.*

Proof. Using formula (5.1), let us calculate the mean dimension of the subspace $B_{\nu\pi,2}$ of entire functions:

$$\overline{\dim}(B_{\nu\pi,2}, L_2(\mathbb{R})) = \nu. \tag{5.5}$$

For an arbitrary function $f \in W^r(\omega^*, \Psi)$, using relation (3.12) in which $t := \pi/\sigma$, $\mu := r$, we obtain the following upper bound for the value of the best approximation:

$$\mathcal{A}_\sigma(f) \le \frac{\pi}{4\sigma^r}\,\omega^*\left(f^{(r)}, \frac{\pi}{\sigma}\right) \le \frac{\pi}{4\sigma^r}\,\Psi\left(\frac{\pi}{\sigma}\right). \tag{5.6}$$

Setting $\sigma := \nu\pi$ in formula (5.6) and using relation (5.2), the definition of the mean linear ν-width, and formulas (5.5), (5.6), we obtain the following upper bounds:

$$\begin{aligned}\overline{\Pi}_\nu(W^r(\omega^*, \Psi), L_2(\mathbb{R})) &\le \overline{\delta}_\nu(W^r(\omega^*, \Psi), L_2(\mathbb{R})) \le \sup\{\|f - \Lambda_{\nu\pi}(f)\| : f \in W^r(\omega^*, \Psi)\} \\ &= \mathcal{A}_{\nu\pi}(W^r(\omega^*, \Psi)) \le \frac{\pi^{1-r}}{4\nu^r}\,\Psi\left(\frac{1}{\nu}\right).\end{aligned} \tag{5.7}$$

Let us pass to the derivation of lower bounds for the average ν-widths. By Sec. 5.1, the subspace of entire functions $B_{\widehat{\sigma},2}$, where $\widehat{\sigma} := \nu\pi(1+\varepsilon)$, $\varepsilon \in (0,1)$, is an arbitrary number, satisfies all the requirements imposed on subspaces appearing in the definition of the Bernstein average ν-width. Further, by formula (5.1), we have $\overline{\dim}(B_{\widehat{\sigma},2}, L_2(\mathbb{R})) = \nu(1+\varepsilon)$ and, in view of [29], [30], $\overline{d}_\nu(B_{\widehat{\sigma},2} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1$.

Further, consider the set $\mathcal{B}_{\widehat{\sigma}}(\rho)$ resulting from the intersection of the ball $\rho_\varepsilon BL_2(\mathbb{R})$ of radius

$$\rho_\varepsilon := \frac{\pi}{4(\widehat{\sigma})^r}\,\Psi\left(\frac{\pi}{\widehat{\sigma}}\right) \tag{5.8}$$

with the subspace of entire functions $B_{\widehat{\sigma},2}$ i.e.,

$$\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon) := B_{\widehat{\sigma},2} \cap \rho_\varepsilon BL_2(\mathbb{R}) = \{g \in B_{\widehat{\sigma},2} : \|g\| \le \rho_\varepsilon\}.$$

Let us show that the set $\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon)$ belongs to the class $W^r(\omega^*, \Psi)$.

By the Paley–Wiener theorem, an arbitrary element $g \in B_{\widehat{\sigma},2}$ can be expressed as

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\widehat{\sigma}}^{\widehat{\sigma}} \varphi(\tau) e^{ix\tau}\, d\tau, \tag{5.9}$$

where φ is a function whose modulus is Lebesgue square-integrable on the closed interval $[-\widehat{\sigma}, \widehat{\sigma}]$. Using formula (5.9), we can write

$$\Delta_h^1(g, x) = g(x+h) - g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\widehat{\sigma}}^{\widehat{\sigma}} (e^{i\tau h} - 1)\varphi(\tau) e^{ix\tau}\, d\tau. \tag{5.10}$$

Using formulas (2.11), in which we set $f := g$, $\tau := h$, $r := 0$, $k := 1$, as well as Plancherel's theorem and formula (5.9), from equality (5.10) we obtain

$$\|\Delta_h^1(g)\|^2 = 2 \int_{-\widehat{\sigma}}^{\widehat{\sigma}} (1 - \cos \tau h) |\varphi(\tau)|^2\, d\tau. \tag{5.11}$$

Since formula (2.6) and purely geometric considerations imply that, for any $\tau \in [-\sigma, \sigma]$, the following inequality holds:

$$1 - \cos \tau h \le (1 - \cos \widehat{\sigma} h)_*,$$

it follows that, using the definition of the modulus of continuity of first order ω_1, equality (5.11), and the relation

$$\|g\|^2 = \int_{-\widehat{\sigma}}^{\widehat{\sigma}} |\varphi(\tau)|^2\, d\tau,$$

we obtain

$$\omega_1(g,t) \le 2\left(\sin\frac{\widehat{\sigma}t}{2}\right)_* \|g\|. \tag{5.12}$$

Here

$$(\sin x)_* := \begin{cases} \sin x & \text{if } 0 \le x \le \dfrac{\pi}{2}, \\ 1 & \text{if } x \ge \dfrac{\pi}{2}. \end{cases}$$

By the definition of the modulus of continuity of ω^* (given in Sec. 3) and inequality (5.12), we have

$$\omega^*(g,t) \le \frac{2}{t}\|g\| \int_0^t \left(\sin\frac{\widehat{\sigma}\tau}{2}\right)_* d\tau. \tag{5.13}$$

Using Bernstein's inequality for entire functions $g \in B_{\widehat{\sigma},2}$, we can write (see, for example, [3], [5])

$$\|g^{(r)}\| \le (\widehat{\sigma})^r \|g\|. \tag{5.14}$$

For an arbitrary function $g \in B_{\widehat{\sigma},2}$, from inequalities (5.13) and (5.14), we obtain

$$\omega^*(g^{(r)},t) \le \frac{2}{t}(\widehat{\sigma})^r\|g\| \int_0^t \left(\sin\frac{\widehat{\sigma}\tau}{2}\right)_* d\tau. \tag{5.15}$$

Further, let us show the validity of the inequality

$$\omega^*(g^{(r)},t) \le \Psi(t), \qquad t \ge 0, \tag{5.16}$$

for any element $g \in \mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon)$. To do this, we consider the following two cases: $0 \le t \le \pi/\widehat{\sigma}$ and $\pi/\widehat{\sigma} \le t < \infty$.

First, let us consider the case $0 \le t \le \pi/\widehat{\sigma}$. For an arbitrary function $g \in \mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon)$, by using the definition of the class $W^r(\omega^*, \Psi)$, inequality (5.15), formula (5.8), and the first of the constraints on the majorant Ψ in condition (5.3), in which σ is replaced by $\widehat{\sigma}$, we obtain

$$\omega^*(g^{(r)},t) \le \frac{2}{t}(\widehat{\sigma})^r\|g\| \int_0^t \sin\frac{\widehat{\sigma}\tau}{2}\, d\tau \le \frac{2\pi}{t\widehat{\sigma}}\sin^2\left(\frac{\widehat{\sigma}t}{4}\right)\Psi\left(\frac{\pi}{\widehat{\sigma}}\right) \le \Psi(t). \tag{5.17}$$

Further, we set $\pi/\widehat{\sigma} \le t < \infty$. For any element $g \in \mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon)$, using similar considerations and the second constraint on the majorant Ψ in condition (5.3) we can write

$$\omega^*(g^{(r)},t) \le \frac{2}{t}\left\{\int_0^{\pi/\widehat{\sigma}} \sin\frac{\widehat{\sigma}\tau}{2}\, d\tau + t - \frac{\pi}{\widehat{\sigma}}\right\}(\widehat{\sigma})^r\|g\| \le \frac{2}{\pi}\left\{1 - \frac{\pi-2}{\widehat{\sigma}t}\right\}\Psi\left(\frac{\pi}{\widehat{\sigma}}\right) \le \Psi(t). \tag{5.18}$$

Relations (5.17) and (5.18) imply inequality (5.16), which means that the set $\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon)$ belongs to the class $W^r(\omega^*, \Psi)$.

Using the definition of the Bernstein average ν-width and formula (5.8), we can write

$$\overline{b}_\nu(W^r(\omega^*,\Psi), L_2(\mathbb{R})) \ge \overline{b}_\nu(\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon), L_2(\mathbb{R})) \ge \frac{\pi^{1-r}}{4\nu^r}\mathcal{K}_{1+\varepsilon}(\Psi,\nu,r), \tag{5.19}$$

where

$$\mathcal{K}_x(\Psi,\nu,r) := \frac{1}{x^r}\Psi\left(\frac{1}{\nu x}\right).$$

Obviously, as $\varepsilon \to 0+0$, the quantity $\mathcal{K}_{1+\varepsilon}(\Psi,\nu,r)$ increases and its limit is the value of $\Psi(1/\nu)$. Then, from the definition of the upper bound, we obtain

$$\sup_{\varepsilon\in(0,1)} \mathcal{K}_{1+\varepsilon}(\Psi,\nu,r) = \Psi\left(\frac{1}{\nu}\right). \tag{5.20}$$

Since the left-hand side in the chain of inequalities (5.19) is independent of ε, it follows that, using the properties of the upper bound for a number set and relations (5.20) and (5.2), we can write

$$\overline{\Pi}_\nu(W^r(\omega^*,\Psi),L_2(\mathbb{R}))\ge \overline{b}_\nu(W^r(\omega^*,\Psi),L_2(\mathbb{R}))\ge \frac{\pi^{1-r}}{4\nu^r}\,\Psi\biggl(\frac{1}{\nu}\biggr). \tag{5.21}$$

For the extremal characteristics for the class $W^r(\omega^*,\Psi)$, equalities (5.4) follow from a comparison of the upper bounds (5.6) with the lower bounds (5.21).

To conclude the proof of the theorem, we shall show that the set of majorants satisfying condition (5.3), is not empty. To do this, we consider, for example, the function $\widetilde{\Psi}(t):=t^\zeta$, where $\zeta:=\pi/2-1$; for this function, condition (5.3) can be written as

$$\biggl(\frac{t\sigma}{\pi}\biggr)^\zeta\ge\frac{\pi}{2}\begin{cases}\dfrac{4}{\sigma t}\sin^2\biggl(\dfrac{\sigma t}{4}\biggr), & \text{if } 0<t\le\dfrac{\pi}{\sigma},\\[2mm] 1-\dfrac{\pi-2}{\sigma t}, & \text{if } \dfrac{\pi}{\sigma}\le t<\infty.\end{cases} \tag{5.22}$$

Let us prove that inequalities (5.22) hold. Setting $u:=t\sigma/\pi$, instead of (5.22), we obtain the equivalent relations

$$u^\zeta\ge\frac{1}{2}\begin{cases}\dfrac{4}{u}\sin^2\biggl(\dfrac{\pi u}{4}\biggr) & \text{if } 0<u\le 1,\\[2mm] \pi-\dfrac{\pi-2}{u}, & \text{if } 1\le u<\infty.\end{cases} \tag{5.23}$$

In the case $0<u\le1$, in view of (5.23), we consider the auxiliary function

$$Y(u):=u^\zeta-\frac{2}{u}\sin^2\biggl(\frac{\pi u}{4}\biggr). \tag{5.24}$$

As $u\to0+0$, using (5.24) and the equivalence of infinitesimals, we obtain the approximate formula

$$Y(u)\doteq u^\zeta\biggl(1-\frac{\pi^2}{8}u^{1-\zeta}\biggr). \tag{5.25}$$

Since ζ belongs to $(0.57, 0.58)$, equality (5.25) implies the existence of an interval $(0,\varepsilon)\subset(0,1)$ at all of whose points the function Y takes only positive values. Let us show that the function Y is of constant sign on the whole interval $(0,1)$. Using formula (5.24), we expressed the function Y as

$$Y(u)=\frac{1}{u}Y_1(u), \tag{5.26}$$

where

$$Y_1(u):=u^{\zeta+1}-2\sin^2\biggl(\frac{\pi u}{4}\biggr). \tag{5.27}$$

In view of equality (5.26), it is necessary to show that the function Y_1 is of constant sign on the interval $(0,1)$, i.e., that $Y_1(u)>0$. In view of formulas (5.27), this is equivalent to the proof of the inequality $u^{\zeta+1}>2\sin^2(\pi u/4)$ for any $u\in(0,1)$. Obviously, this inequality is equivalent to the relation

$$u^{(\zeta+1)/2}>\sqrt{2}\,\sin\frac{\pi u}{4},\qquad 0<u<1.$$

Thus, the initial problem is reduced to proving that the function

$$Y_2(u):=u^{(\zeta+1)/2}-\sqrt{2}\,\sin\frac{\pi u}{4} \tag{5.28}$$

is of constant sign for $u\in(0,1)$. We shall prove this fact by arguing by contradiction. Suppose that Y_2 is a function with alternating signs, i.e., there exists a point $z\in(0,1)$ at which Y_2 changes sign. In view of the continuity of the function Y_2, let us consider it on the closed interval $[0,1]$. In view of formula (5.28) and the assumption about the point z, we have $Y_2(0) = Y_2(z) = Y_2(1) = 0$. Then it follows from Rolle's theorem that the first derivative of the function Y_2, i.e.,

$$Y_2^{(1)}(u) = \frac{\zeta+1}{2}\,u^{(\zeta-1)/2} - \sqrt{2}\,\frac{\pi}{4}\cos\frac{\pi u}{4}, \tag{5.29}$$

must have at least two distinct zeros on the interval $(0,1)$. Taking the equality $\zeta = \pi/2 - 1$, from formula (5.29), we obtain $Y_2^{(1)}(1) = 0$. It follows from Rolle's theorem that the second derivative of the function Y_2, i.e.,

$$Y_2^{(2)}(u) = \sqrt{2}\left(\frac{\pi}{4}\right)^4 \sin\frac{\pi u}{4} - \frac{(\zeta-1)^2}{4}\,u^{(\zeta-3)/2}, \tag{5.30}$$

must also have at least two distinct zeros on the interval $(0,1)$. However, in view of the form of formula (5.30), the derivative $Y_2^{(2)}$ can have at most one zero on $(0,1)$, because it is the difference of two functions the first of which is monotone increasing and convex upward, while the second is monotone decreasing and is convex downward. The resulting contradiction proves the validity of the inequality $Y_2(u) > 0$ for any $u \in (0,1)$, and hence also the validity of the first of the inequalities in condition (5.22).

To prove the validity of the second inequality in this condition, in view of relation (5.23), we consider the auxiliary function

$$Z(u) := u^{\zeta} - \frac{\pi}{2} + \frac{\pi-2}{2u}, \tag{5.31}$$

where $1 \le u < \infty$. The first derivative of this function

$$Z^{(1)}(u) = \zeta u^{\zeta-1} - \frac{\pi-2}{2u^2} = \frac{1}{u^2}\left\{\left(\frac{\pi}{2}-1\right)u^{\zeta+1} + 1 - \frac{\pi}{2}\right\}$$

takes positive values for $1 < u < \infty$. Since, in view of formula (5.31), $Z(1) = 0$, it follows that the function Z is, obviously, positive and monotone increasing on the interval under consideration. Therefore, the second inequality in condition (5.22) also holds and the function $\widetilde{\Psi}$ is the majorant satisfying condition (5.3). Theorem 3 is proved. $\square$

In our view, the calculation of best approximations for the intermediate derivatives of functions $f \in W^r(\omega^*, \Psi)$ by subspaces of entire functions of exponential type is of considerable interest. This question is dealt with in the following statement.

Statement. *Suppose that $\mu, r \in \mathbb{N}$, $\mu \le r$, ν is an arbitrary finite positive number, and Ψ is the majorant satisfying condition* (5.3) *for any finite value of σ, $\sigma > \nu\pi$. Then the following relation holds*:

$$\sup\{\mathcal{A}_{\nu\pi}(f^{(r-\mu)}) : f \in W^r(\omega^*, \Psi)\} = \frac{\pi^{1-\mu}}{4\nu^{\mu}}\,\Psi\left(\frac{1}{\nu}\right). \tag{5.32}$$

Proof. Setting $\sigma := \nu\pi$, $t := \pi/\sigma$ in Corollary3 and using the definition of the class $W^r(\omega^*, \Psi)$, we obtain the upper bound

$$\sup\{\mathcal{A}_{\nu\pi}(f^{(r-\mu)}) : f \in W^r(\omega^*, \Psi)\} \le \frac{\pi^{1-\mu}}{4\nu^{\mu}}\,\Psi\left(\frac{1}{\nu}\right). \tag{5.33}$$

Further, let $\widetilde{\varepsilon} := \varepsilon\nu\pi$, where $\varepsilon \in (0, \sigma_*)$ is an arbitrary number, and suppose that $\sigma_* = \min(\sigma, 1)$ and $\widehat{\sigma} := \sigma + \widetilde{\varepsilon} = \nu\pi(1+\varepsilon)$. To obtain the lower bound for the extremal characteristic on the left-hand side of equality (5.33), we consider the entire function $\lambda_{\widetilde{\varepsilon}}$ defined by (2.16); it is an entire function of exponential type $\widehat{\sigma}$. By (2.2) and (2.17), we have

$$\|\lambda_{\widetilde{\varepsilon}}\|^2 = 2\int_{\sigma}^{\sigma+\widetilde{\varepsilon}} |\mathcal{F}(\lambda_{\widetilde{\varepsilon}}, \tau)|^2\,d\tau = 2\widetilde{\varepsilon}. \tag{5.34}$$

For the entire function

$$\lambda^*_{\widehat{\varepsilon}}(x) := \frac{\rho_\varepsilon}{\sqrt{2\widehat{\varepsilon}}}\lambda_{\widehat{\varepsilon}}(x), \tag{5.35}$$

where ρ_ε is given by (5.8), using equality (5.34), we obtain $\|\lambda^*_{\widehat{\varepsilon}}\| = \rho_\varepsilon$. Therefore, the entire function (5.35) belongs to the set $\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon)$ introduced in the proof of Theorem 3. Since $\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon) \subset W^r(\omega^*, \Psi)$, it follows that the function $\lambda^*_{\widehat{\varepsilon}}$ is an element of class $W^r(\omega^*, \Psi)$. Using relations (2.31), (5.8), and (5.35), we obtain

$$\begin{aligned}\sup\{\mathcal{A}_{\nu\pi}(f^{(r-\mu)}) : f \in W^r(\omega^*, \Psi)\} &\ge \mathcal{A}_{\nu\pi}((\lambda^*_{\widehat{\varepsilon}})^{(r-\mu)}) = \frac{\rho_\varepsilon}{\sqrt{2\widehat{\varepsilon}}}\mathcal{A}_{\nu\pi}(\lambda^{(r-\mu)}_{\widehat{\varepsilon}}) \ge \rho_\varepsilon(\nu\pi)^{r-\mu} \\ &= \frac{\pi(\nu\pi)^{r-\mu}}{4(\widehat{\sigma})^r}\Psi\left(\frac{\pi}{\widehat{\sigma}}\right) = \frac{\pi^{1-\mu}}{4\nu^\mu}\mathcal{K}_{1+\varepsilon}(\Psi, \nu, r), \end{aligned}\tag{5.36}$$

where the quantity $\mathcal{K}_{1+\varepsilon}(\Psi, \nu, r)$ is determined at the end of the proof of Theorem 3. Using arguments similar to those used in the derivation of the lower bound (5.21) and equality (5.20), from relation (5.36), we can write

$$\sup\{\mathcal{A}_{\nu\pi}(f^{(r-\mu)}) : f \in W^r(\omega^*, \Psi)\} \ge \frac{\pi^{1-\mu}}{4\nu^\mu}\Psi\left(\frac{1}{\nu}\right). \tag{5.37}$$

Comparing inequalities (5.33) and (5.37), we obtain the required equality (5.32), which concludes the proof of the statement. □

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