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The Kantorovich and Variation Distances between Invariant Measures of Diffusions and Nonlinear Stationary Fokker–Planck–Kolmogorov Equations*

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Abstract—We obtain upper bounds for the total variation distance and the quadratic Kantorovich distance between stationary distributions of two diffusion processes with different drifts. More generally, our estimate holds for solutions to stationary Kolmogorov equations in the class of probability measures. This estimate is applied to nonlinear stationary Fokker–Planck–Kolmogorov equations for probability measures.

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The investigation of Kantorovich-type distances between probability measures related to diffusion processes has become a popular subject over the past decade, see [1]–[7] and references therein. The goal of our paper is to obtain an upper bound for the total variation and the quadratic Kantorovich distance W_2 between stationary distributions of two diffusion processes. Actually, a more general situation is considered, where in place of invariant measures, we deal with solutions to the stationary Kolmogorov equation in the class of probability measures. These estimates are applied to nonlinear stationary Fokker–Planck–Kolmogorov equations for probability measures.

Let b be a locally bounded Borel vector field on $\mathbb{R}^d$ or on a Riemannian manifold. Suppose that there exists a diffusion process ξ_t with generator

$$L_b f = \Delta f + \langle b, \nabla f\rangle,$$

for example, a solution of the stochastic differential equation

$$d\xi_t = b(\xi_t)\,dt + \sqrt{2}\,dW_t.$$

In this case b is called the drift coefficient. Suppose also that the diffusion process ξ_t possesses an invariant probability measure μ. Then this measure μ satisfies the elliptic equation

$$L_b^*\mu = 0, \tag{1}$$

which is understood in the sense of the integral identity

$$\int_{\mathbb{R}^d} L_b u(x)\,\mu(dx) = 0 \qquad \forall u \in C_0^\infty(\mathbb{R}^d).$$

This equation can be also written as

$$\Delta\mu - \operatorname{div}(b\cdot\mu) = 0$$

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in the sense of distributions. Moreover, under our basic assumptions any solution μ has a density ϱ_μ with respect to Lebesgue measure and this density satisfies the elliptic equation

$$\Delta \varrho_\mu - \operatorname{div}(\varrho_\mu b) = 0$$

in the classical weak sense.

We recall (see [8]–[12]) that, for the existence and uniqueness of an invariant probability measure it suffices to have a function $V \in C^2(\mathbb{R}^d)$, called a Lyapunov function, such that

$$\lim_{|x|\to\infty} V(x) = +\infty \qquad \text{and} \qquad \lim_{|x|\to\infty} L_b V(x) = -\infty.$$

In the manifold case, these two conditions are modified as follows: in place of the former we require that the sets $\{V \le R\}$ be compact (in the case of $\mathbb{R}^d$ this is an equivalent condition for continuous functions) and the latter condition is replaced by

$$\lim_{R\to\infty} \sup_{x:\ V(x)>R} LV(x) = -\infty.$$

Under this condition, the associated diffusion process with unique probability invariant measure μ also exists. In what follows our main object will be Eq. (1) with respect to probability measures and the existence of associated diffusions will not be assumed, but it is useful to have in mind that the established estimates of distances between measures corresponding to different drift coefficients give some information on the dependence of stationary distributions of diffusions on their drifts.

Suppose now that we are given two Borel probability measures μ and ν satisfying the equations $L_{b_\mu}^*\mu = 0$ and $L_{b_\nu}^*\nu = 0$ with certain locally bounded Borel vector fields b_μ and b_ν. This is the case if we have two diffusion processes with invariant measures. The main problem studied in this paper concerns estimating the quadratic Kantorovich distance between these measures via a suitable distance between b_μ and b_ν. Since the coefficients b_μ and b_ν are locally bounded, the measures μ and ν have continuous positive densities ϱ_μ and ϱ_ν with respect to Lebesgue measure (see [12], [13]), therefore, we can assume that

$$\nu = v \cdot \mu,$$

where $v = \varrho_\nu/\varrho_\mu$ is a continuous positive function.

The main result of this paper gives the following estimate under broad conditions on b_μ and b_ν:

$$\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{v}\, d\mu \le \int_{\mathbb{R}^d} |b_\mu - b_\nu|^2\, d\nu. \tag{2}$$

A similar estimate holds in the manifold case.

Estimate (2) resembles the inequality

$$\int_{\mathbb{R}^d} \frac{|\nabla \varrho_\nu|^2}{\varrho_\nu^2} d\nu \le \int_{\mathbb{R}^d} |b_\nu|^2 d\nu,$$

established in [14] and meaning that ϱ_ν has a finite Fisher information number. The latter inequality can be formally obtained from the former one if for μ we take Lebesgue measure with $b_\mu = 0$. This substitution is possible on Riemannian manifolds of finite volume. For manifolds, the latter inequality is also true (see [15]), but requires additional assumptions; for example, it fails if the given manifold possesses a nonconstant positive integrable harmonic function ϱ_ν, since, in that case, we can take for μ the Riemannian volume, and then $\nu = \varrho_\nu \cdot \mu$ satisfies the equation $\Delta\nu = 0$, i.e., $b_\nu = 0$, but $\nabla\varrho_\nu \ne 0$. Clearly, (2) also fails in this case, since μ is another solution to the Laplace equation with zero drift.

Suppose now that the measures μ and ν have second moments, i.e., $|x| \in L^2(\mu+\nu)$. In the manifold case, the corresponding condition is that all Lipschitzian functions belong to $L^2(\mu+\nu)$. If it is given in addition that the measure μ satisfies the logarithmic Sobolev inequality

$$\operatorname{Ent}_\mu(f^2) := \int_{\mathbb{R}^d} f^2 \ln f^2\, d\mu - \int_{\mathbb{R}^d} f^2\, d\mu \ln \int_{\mathbb{R}^d} f^2\, d\mu \le C \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu$$

for all $f \in W^{2,1}(\mu)$ (equivalently, for all $f \in C_b^1(\mathbb{R}^d)$), then, as is well-known (see [16], [7, Section 9.3] or [2, Theorem 3.3.2]), the probability measure $\nu = v \cdot \mu$ satisfies the so-called transport inequality

$$W_2^2(\mu, v\mu) \le C\mathrm{Ent}_\mu v = C \int_{\mathbb{R}^d} v \ln v \, d\mu, \tag{3}$$

hence, applying the logarithmic Sobolev inequality with $f = \sqrt{v}$, we arrive at the estimate

$$W_2^2(\mu, \nu) \le 4^{-1} C^2 \int_{\mathbb{R}^d} |b_\mu - b_\nu|^2 \, d\nu. \tag{4}$$

Here W_2 is the quadratic Kantorovich metric (see [2], [7]):

$$W_2^2(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^2 \, \pi(dx, dy),$$

where $\Pi(\mu, \nu)$ is the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with projections μ and ν on the first and second factors. There is a vast literature on the logarithmic Sobolev inequality, see [17], [18], and [19].

If in place of (3) we apply the Pinsker–Kullback–Csiszár inequality (see [20, Theorem 2.12.24]) for the total variation distance

$$\|\mu - \nu\| \le \sqrt{2\mathrm{Ent}_\mu v}$$

or its weighted generalizations obtained in [21], then we arrive at the estimate

$$\|\mu - \nu\|^2 \le \frac{C}{2} \int_{\mathbb{R}^d} |b_\mu - b_\nu|^2 \, d\nu. \tag{5}$$

We apply estimate (4) to the study of the problem of the existence and uniqueness of a probability solution to a nonlinear stationary Fokker–Planck–Kolmogorov equation. Note that the problem of existence of solutions to nonlinear elliptic equations for probability measures was studied in [22].

Let $W^{2,1}(\mu)$ denote the weighted Sobolev class with respect to the measure μ obtained by completing the class $C_0^\infty(\mathbb{R}^d)$ of smooth functions with compact support with respect to the weighted Sobolev norm given by the equality

$$\|\varphi\|_{2,1,\mu}^2 = \int_{\mathbb{R}^d} |\varphi|^2 \, d\mu + \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, d\mu.$$

The main result of our work is the following.

Theorem 1. *Let μ and ν be two probability solutions to Eq.* (1) *with locally bounded Borel coefficients b_μ and b_ν, respectively. Suppose that*

$$|b_\mu - b_\nu| \in L^2(\nu)$$

and that at least one of the following two conditions is fulfilled:

(i) $(1 + |x|)^{-1} |b_\mu(x)| \in L^1(\nu)$,

(ii) *there exists a function $V \in C^2(\mathbb{R}^d)$ such that $L_{b_\mu} V(x) \le M V(x)$ for all x and some $M > 0$ and*

$$\lim_{|x| \to \infty} V(x) = +\infty, \qquad \frac{\langle b_\mu - b_\nu, \nabla V \rangle}{1 + V} \in L^1(\nu).$$

Then the estimate

$$\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{v} \, d\mu \le \int_{\mathbb{R}^d} |b_\mu - b_\nu|^2 \, d\nu$$

holds, which yields, in particular, the inclusion $\sqrt{v} \in W^{2,1}(\mu)$. In case (ii)*, a similar assertion is true for a smooth Riemannian manifold in place of $\mathbb{R}^d$, provided that the condition*

$$\lim_{|x| \to \infty} V(x) = +\infty$$

is replaced by the requirement that the sets $\{V \le R\}$ be compact.

Proof. As shown in [13], in the case of a locally bounded coefficient b, any solution of Eq. (1) is given by a positive continuous density with respect to Lebesgue measure and this density belongs to the Sobolev class $W^{p,1}_{loc}(\mathbb{R}^d)$ for every $p \geq 1$. Therefore, we can write the solution ν as a measure given by density v with respect to the measure μ, i.e., $\mu = \varrho\, dx$, $\nu = v \cdot \mu = v\varrho\, dx$. The function v satisfies the equation

$$\operatorname{div}(\varrho\nabla v - va - vh\varrho) = 0, \tag{6}$$

where

$$a = \varrho b_\mu - \nabla\varrho, \qquad h = b_\nu - b_\mu.$$

We observe that, by virtue of the equation $L^*_{b_\mu}\mu = 0$, we have the equality

$$\operatorname{div} a = 0.$$

Let $f \in C^1(0, +\infty)$ and let f' satisfy the Lipschitz condition. Let also $\psi \in C_0^\infty(\mathbb{R}^d)$. Multiplying the equation by the function $f'(v)\psi$ and integrating by parts (exactly as in [11, Lemma 1] or [12, Lemma 4.1.4.]), we obtain

$$\int_{\mathbb{R}^d} |\nabla v|^2 f''(v)\psi\varrho\, dx = \int_{\mathbb{R}^d} \varrho f(v) L_{b_\mu}\psi\, dx + \int_{\mathbb{R}^d} \big[\langle h, \nabla v\rangle f''(v)\psi + \langle h, \nabla\psi\rangle f'(v)\big] v\varrho\, dx.$$

Suppose that $f'' \geq 0$ and $\psi \geq 0$. Then by the Cauchy inequality

$$\int_{\mathbb{R}^d} \langle h, \nabla v\rangle f''(v)\psi v\varrho\, dx \leq \frac{1}{2}\int_{\mathbb{R}^d} |\nabla v|^2 f''(v)\psi\varrho\, dx + \frac{1}{2}\int_{\mathbb{R}^d} |h|^2 f''(v) v^2\psi\varrho\, dx.$$

Therefore, we arrive at the inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2 f''(v)\psi\varrho\, dx \leq 2\int_{\mathbb{R}^d} \varrho f(v) L_{b_\mu}\psi\, dx + \int_{\mathbb{R}^d} |h|^2 f''(v) v^2\psi\varrho\, dx + 2\int_{\mathbb{R}^d} \langle h, \nabla\psi\rangle f'(v) v\varrho\, dx. \tag{7}$$

Since $L^*_{b_\mu}\mu = 0$ and $L^*_{b_\nu}\nu = 0$, for any numbers α, β, we have the following equality:

$$\int_{\mathbb{R}^d} \varrho(f(v) - \alpha v - \beta) L_{b_\mu}\psi\, dx = \int_{\mathbb{R}^d} \varrho f(v) L_{b_\mu}\psi\, dx - \alpha\int_{\mathbb{R}^d} \langle h, \nabla\psi\rangle\, d\nu. \tag{8}$$

Let $m, k \geq 1$ and

$$f_{m,k}(t) = \begin{cases} -t\ln k & \text{if } t \leq k^{-1} \\ t\ln t - t + k^{-1} & \text{if } k^{-1} < t < m \\ t\ln m - m + k^{-1} & \text{if } t \geq m. \end{cases}$$

We observe that $f'_{m,k}(t) = \ln\big((k^{-1} \vee t) \wedge m\big)$ and $f''_{m,k}(t) = t^{-1} I_{(k^{-1},m}(t)$, where $I_{(k^{-1},m)}$ is the indicator of the interval (k^{-1}, m). In addition, $|f_{m,k}(t)| \leq C(m,k)t$ for any fixed m, k.

We now consider separately cases (i) and (ii). Let (i) be fulfilled. Set

$$\psi_N(x) = \psi(x/N),$$

where $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi \geq 0$, $\psi(x) = 1$ if $|x| < 1$ and $\psi(x) = 0$ if $|x| > 2$. Substituting in (7) the functions f_m and ψ_N, we arrive at the inequality

$$\int_{k^{-1}<v<m} \frac{|\nabla v|^2}{v}\psi_N\varrho\, dx \leq 2\int_{\mathbb{R}^d} \varrho f_{m,k}(v) L_{b_\mu}\psi_N\, dx + \int_{k^{-1}<v<m} |h|^2\psi_N\, d\nu + 2\int_{\mathbb{R}^d} \langle h, \nabla\psi_N\rangle \ln\big((k^{-1} \vee v) \wedge m\big)\, d\nu.$$

We observe that $|f_m(v)| \leq C(m,k)v$ and

$$\left|\int_{\mathbb{R}^d} \varrho f_{m,k}(v) L_{b_\mu}\psi_N\, dx\right| \leq C(m,k)\int_{\mathbb{R}^d} |L_{b_\mu}\psi_N|\, d\nu.$$

We have

$$|L_{b_\mu}\psi_N| \le N^{-2}|\Delta\psi| + N^{-1}|b_\mu||\nabla\psi|$$

and $L_{b_\mu}\psi_N = 0$ outside of the set $\{N < |x| < 2N\}$. Therefore, it follows from our assumption $(1+|x|)^{-1}|b_\mu(x)| \in L^1(\nu)$ that

$$\lim_{N\to\infty}\int_{\mathbb{R}^d} \varrho f_{m,k}(v) L_{b_\mu}\psi_N\,dx = 0.$$

In addition,

$$\int_{\mathbb{R}^d}\langle h, \nabla\psi_N\rangle \ln\bigl((k^{-1}\vee v)\wedge m\bigr)\,d\nu \le (\ln k + \ln m)N^{-1}\max|\nabla\psi|\int_{\mathbb{R}^d}|h|\,d\nu.$$

Letting first $N\to\infty$ and then $m,k\to\infty$, we arrive at the desired estimate.

Let us now consider case (ii). Set

$$\psi_N(x) = \zeta(V(x)/N),$$

where $\zeta \in C^\infty(\mathbb{R}^1)$, $\zeta' \le 0$, $\zeta'' \ge 0$, $\zeta(0) = 1$ and $\zeta(t) = 0$ if $t > 1$. We observe that

$$L_{b_\mu}\psi_N = N^{-1}\zeta'(V/N)L_{b_\mu}V + N^{-2}\zeta''(V/N)|\nabla V|^2.$$

Let us substitute the functions f_m and ψ_N into (7) and use equality (8). We have

$$\begin{aligned}\int_{k^{-1}<v<m}\frac{|\nabla v|^2}{v}\psi_N\varrho\,dx \le 2\int_{\mathbb{R}^d}&\varrho(f_{m,k}(v) - v\ln m - k^{-1})L_{b_\mu}\psi_N\,dx\\ &- 2\ln m\int_{\mathbb{R}^d}\langle h,\nabla\psi_N\rangle\,d\nu + \int_{k^{-1}<v<m}|h|^2\psi_N\,d\nu\\ &+ 2\int_{\mathbb{R}^d}\langle h,\nabla\psi_N\rangle\ln\bigl((k^{-1}\vee v)\wedge m\bigr)\,d\nu.\end{aligned}$$

We observe that

$$(f_{m,k}(v) - v\ln m - k^{-1}) \le 0, \qquad |f_{m,k}(v) - v\ln m - k^{-1}| \le C(m,k)(1+v)$$

and

$$\varrho(f_{m,k}(v) - v\ln m - k^{-1})L_{b_\mu}\psi_N \le C(\zeta,m,k)MN^{-1}(\varrho + v\varrho)V.$$

For every $\delta \in (0,1)$ the following inequality holds:

$$N^{-1}\int_{V<N}V\,d(\mu+\nu) \le \delta\int_{V<\delta N}1\,d(\mu+\nu) + \int_{\delta N<V<N}1\,d(\mu+\nu);$$

therefore,

$$\limsup_{N\to\infty}N^{-1}\int_{V<N}V\,d(\mu+\nu) \le 2\delta.$$

Since δ was arbitrary, we have

$$\lim_{N\to\infty}N^{-1}\int_{V<N}V\,d(\mu+\nu) = 0,$$

whence we find that

$$\limsup_{N\to\infty}\int_{\mathbb{R}^d}\varrho(f_{m,k}(v) - v\ln m - k^{-1})L_{b_\mu}\psi_N\,dx \le 0.$$

Finally, we observe that $\langle h,\nabla\psi_N\rangle = N^{-1}\zeta'(V/N)\langle h,\nabla V\rangle$ and, for every $\delta\in(0,1)$,

$$N^{-1}\int_{V<N}\bigl|\langle h,\nabla V\rangle\bigr|\,d\nu \le (N^{-1}+\delta)\int_{V<\delta N}\bigl|\langle h,\nabla V\rangle\bigr|(1+V)^{-1}\,d\nu$$

$$+ (N^{-1}+1)\int_{\delta N<V<N} \big|\langle h,\nabla V\rangle\big|(1+V)^{-1}\,d\nu.$$

As above, we conclude that

$$\limsup_{N\to\infty}\int_{\mathbb{R}^d}\langle h,\nabla\psi_N\rangle \ln\big((k^{-1}\vee v)\wedge m\big)\,d\nu \le 0.$$

Letting first $N\to\infty$ and then $m,k\to\infty$, we arrive at the desired estimate. The same reasoning applies in the manifold case. □

Note that condition (i) is satisfied if $|b_\mu(x)|\le C_1+C_2|x|$ or if $|b_\mu(x)|\le C_1+C_2|x|^k$ and the function $|x|^{k-1}$ is ν-integrable. If ν has all moments, then any polynomial bound on $|b_\mu|$ is sufficient.

Corollary 1. *Suppose that, in addition to the hypotheses of Theorem* 1*, it is assumed that the solutions μ and ν have second moments and the measure μ satisfies the logarithmic Sobolev inequality with constant C. Then*

$$W_2(\mu,\nu)^2 \le \frac{C^2}{4}\int_{\mathbb{R}^d}|b_\mu-b_\nu|^2\,d\nu,$$

$$\|\mu-\nu\|^2 \le \frac{C}{2}\int_{\mathbb{R}^d}|b_\mu-b_\nu|^2\,d\nu.$$

In case (ii) *of the theorem, these estimates hold on a Riemannian manifold.*

Proof. This assertion follows immediately from the theorem and the transport and Pinsker–Kullback–Csiszár inequalities. □

Let us consider an important partial case illustrating the last corollary.

Let b be a Borel locally bounded vector field on $\mathbb{R}^d$ satisfying the following condition:

(H) $\langle b(x)-b(y),x-y\rangle \le -\kappa|x-y|^2$ for some $\kappa>0$ and all $x,y\in\mathbb{R}^d$.

It is known (see [23] or [12, Theorem 5.6.36]) that condition (H) ensures that every probability solution μ to the equation $L_b^*\mu=0$ satisfies the logarithmic Sobolev inequality with constant $2/\kappa$.

Moreover, in this case $\langle b(x),x\rangle\to-\infty$ as $|x|\to\infty$ and condition (ii) of Theorem 1 is fulfilled with $V(x)=|x|^2$. In addition, the equation $L_b^*\mu=0$ has the unique probability solution μ and $|x|^2\in L^1(\mu)$.

Thus, we arrive at the following assertion.

Corollary 2. *Let b_μ and b_ν be locally bounded Borel vector fields satisfying condition* (H) *and let μ and ν be the corresponding probability solutions to equation* (1) *such that $|b_\mu-b_\nu|\in L^2(\mu+\nu)$. Then, for every number $\gamma\in[0,1]$, the following estimate*

$$W_2(\mu,\nu)\le\kappa^{-1}\|b_\mu-b_\nu\|_{L^2(\mu_\gamma)},$$

where $\mu_\gamma=\gamma\mu+(1-\gamma)\nu$, holds.

Remark 1. If condition (H) is fulfilled just for b_μ and $|b_\mu-b_\nu|\in L^2(\nu)$, then

$$W_2(\mu,\nu)\le\kappa^{-1}\|b_\mu-b_\nu\|_{L^2(\nu)}.$$

We now apply Corollary 2 to a nonlinear stationary Fokker–Planck–Kolmogorov equation.

Let $\mathcal{P}_2(\mathbb{R}^d)$ denote the space of all Borel probability measures on $\mathbb{R}^d$ with finite second moment.

Suppose that, for every measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we are given a locally bounded Borel vector field $b(\cdot,\mu)$ on $\mathbb{R}^d$. Set

$$L_\mu u(x) = \Delta u(x) + \langle b(x,\mu), \nabla u(x)\rangle, \qquad u \in C^2(\mathbb{R}^d).$$

As in the linear case, the measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ is called a solution to the nonlinear stationary Fokker–Planck–Kolmogorov equation

$$L_\mu^*\mu = 0 \tag{9}$$

if we have

$$\int_{\mathbb{R}^d} L_\mu u(x)\,\mu(dx) = 0 \qquad \forall u \in C_0^\infty(\mathbb{R}^d).$$

Corollary 3. *Suppose that, for every measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the vector field $b(\cdot,\mu)$ satisfies condition* (H) *with a common* (*for all measures μ*) *constant $\kappa > 0$. Assume also that there exists a number $C > 0$ such that*

$$|b(x,\mu) - b(x,\nu)| \le CW_2(\mu,\nu) \qquad \textit{for all } x \in \mathbb{R}^d,\ \mu,\nu \in \mathcal{P}_2(\mathbb{R}^d).$$

If $C < \kappa$, then there exists a unique solution to equation (9) *in the class $\mathcal{P}_2(\mathbb{R}^d)$.*

Proof. We define a mapping $F\colon\ \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathcal{P}_2(\mathbb{R}^d)$ as follows:

$$\mu = F(\sigma) \iff L_\sigma^*\mu = 0.$$

By Corollary 2, we have the estimate

$$W_2(\mu_1,\mu_2) \le \frac{C}{\kappa} W_2(\sigma_1,\sigma_2), \qquad \mu_1 = F(\sigma_1),\ \mu_2 = F(\sigma_2).$$

Then the mapping F is contractive if $C < \kappa$. By the Banach contracting mapping theorem, there exists a unique solution of Eq. (9). ☐

Corollary 4. *Suppose that, for every measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the vector field $b(x,\mu)$ satisfies condition* (H) *with a common* (*for all μ*) *constant $\kappa > 0$. Suppose that there exist a number $C > 0$ and a function $\Psi > 0$ with*

$$\lim_{|x|\to\infty} |x|^{-2}\Psi(x) = +\infty$$

such that, for every measure μ, there is a Lyapunov function V_μ for which

$$L_\mu V_\mu \le C - \Psi.$$

Suppose also that if a sequence $\{\mu_n\}$ of measures in the set

$$K = \Bigl\{\mu \in \mathcal{P}_2(\mathbb{R}^d)\colon\ \int_{\mathbb{R}^d} \Psi\,d\mu \le C\Bigr\}$$

converges to a measure μ with respect to the metric W_2, then the fields $b(\cdot,\mu_n)$ converge to $b(\cdot,\mu)$ in $L^2(\mu)$. Then equation (9) *has a solution in $\mathcal{P}_2(\mathbb{R}^d)$.*

Proof. We observe that K is compact with respect to the metric W_2. Indeed, given a sequence of measures $\mu_n \in K$, we can pick a weakly convergent subsequence with a limit $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, since the measures μ_n have uniformly bounded second moments due to our condition on Ψ. Moreover, the same condition yields that the integrals of the function $|x|^2$ with respect to μ_n converge to the integral of this function with respect to μ, which shows that $W_2(\mu_n,\mu) \to 0$ (see [2, Theorem 1.1.9]). Obviously, K is convex.

Let F be the same as in the proof of the previous corollary. By Corollary 2, we have the estimate

$$W_2(F(\mu), F(\sigma)) \le \kappa^{-1}\|b(\cdot,\mu) - b(\cdot,\sigma)\|_{L^2(\sigma)},$$

which, by our assumptions, yields the continuity of F on the convex compact set K. By Schauder's theorem, there exists a fixed point and this point is a solution of Eq. (9). □

Corollary 5. *Suppose that, for every measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the vector field $b(\cdot,\mu)$ satisfies condition* (H) *with a constant $\kappa(\mu) > 0$. Assume also that there exists a positive Borel function Q on $\mathbb{R}^d$ such that*

$$|b(x,\mu) - b(x,\sigma)| \le Q(x)W_2(\mu,\sigma)$$

for all $x \in \mathbb{R}^d$, $\mu,\sigma \in \mathcal{P}_2(\mathbb{R}^d)$. If $\mu,\sigma \in \mathcal{P}_2(\mathbb{R}^d)$ satisfy equation (9) *and $\|Q\|_{L^2(\sigma)} < \kappa(\mu)$, then $\mu = \sigma$.*

Proof. According to Remark 1 one has the estimate

$$W_2^2(\mu,\sigma) \le \kappa^{-2}\|Q\|^2_{L^2(\sigma)} W_2^2(\mu,\sigma).$$

If $\|Q\|_{L^2(\sigma)} < \kappa(\mu)$, then $W_2^2(\mu,\sigma) = 0$ and $\mu = \sigma$. □

In conclusion, we give examples illustrating these corollaries.

Example 1. Let

$$b(x,\mu) = -kx - \nabla U(x) - \int \nabla W(x-y)\,\mu(dy),$$

where U and W are continuously differentiable convex functions such that

$$|\nabla W(z_1) - \nabla W(z_2)| \le C|z_1 - z_2|.$$

It is clear that b satisfies the hypotheses of Corollary 3. If $k > C$, then the corresponding equation possesses a unique probability solution.

Let us now give an example of non-uniqueness that exhibits a substantial difference between the nonlinear and linear cases.

Example 2. Let $d = 1$ and $b(x,\mu) = -x + B(\mu)$, where the vector $B(\mu)$ is the mean of the measure μ, i.e.,

$$B(\mu) = \int x\,\mu(dx).$$

Any measure μ_a with density $\varrho_a(x) = (2\pi)^{-1/2}\exp\bigl(-|x-a|^2/2\bigr)$ is a solution to our nonlinear equation $L^*_\mu\mu = 0$. Indeed, $B(\mu_a) = a$ and

$$b(x,\mu_a) = \frac{\nabla\varrho_a(x)}{\varrho_a(x)} = -(x-a).$$

We observe that condition (H) for such b is fulfilled with $\kappa = 1$. Let us verify the second condition in Corollary 3:

$$|b(x,\mu) - b(x,\sigma)| \le \int\int |x-y|\,\pi(dx,dy) \le \sqrt{\int\int |x-y|^2\,\pi(dx,dy)},$$

where π is an arbitrary probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ having projections μ and σ on the factors. Since π is arbitrary, we arrive at the required estimate with constant $C = 1$. On the other hand, if $\kappa > 1$, then, by Corollary 3, our equation has a unique solution given by density

$$k^{1/2}(2\pi)^{-1/2}\exp\bigl(-k|x|^2/2\bigr).$$

Thus, the condition $C < \kappa$ in Corollary 3 is sharp.

Analogous results are true for more general elliptic operators with variable second order coefficients and also in the infinite-dimensional case. These questions will be considered in a separate paper.

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