

Stratified Flows Due to Spatial Inhomogeneities of Exchange Coefficients

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Abstract—Attention is drawn to the unexplored mechanism of generation of density currents in stratified media. Horizontal inhomogeneities of the exchange coefficients in a stratified medium lead to inhomogeneities of the vertical diffusion flow of buoyancy and its horizontal distribution and, consequently, to the emergence of horizontal inhomogeneities of hydrostatic pressure and the generation of currents. The appearance of ordered flows in a temperature (density) stratified turbulent medium in a gravity field near an inclined surface is considered as an example. This is due to the existence of a region of weakened turbulent exchange near the solid surface. In this case, horizontal components of the temperature, density and, consequently, pressure gradients appear near the inclined surface. This, in turn, leads to the emergence of an average (nonturbulent) slope current even in the absence of heat and momentum sources/sinks.

Keywords: stratified media, density currents, inhomogeneities of exchange coefficients, slope flows, turbulent exchange, atmospheric currents, absence of static states

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INTRODUCTION

Density-stratified geophysical media, primarily the atmosphere, are nonequilibrium systems that can be at rest only when density gradients are parallel (antiparallel) to the direction of gravity. The static state is impossible, for example, in the case of horizontally inhomogeneous heating (cooling), which is associated with many classes of flows in such media. This note draws attention to the fact that horizontal density inhomogeneities in a stratified medium can arise even without inhomogeneous sources (sinks) of buoyancy. They can be caused by horizontal inhomogeneities of the effective transfer coefficients. The strong vertical variability of the intensity of turbulent exchange in the atmosphere is well known and does not require comments; however, for the effects considered below, horizontal variations in the transfer coefficients, first and foremost, are significant.

It is possible to indicate a number of factors that cause the effective coefficients of turbulent exchange to appreciably change horizontally. The simplest example is the case of an inclined underlying surface, when the turbulent exchange essentially depends on the coordinate normal to the surface, i.e., changes not only vertically, but also horizontally. Horizontally heterogeneous turbulence can also be associated, for example, with jet streams that are manifestations of internal gravity waves, with high heterogeneous vege-

tation or buildings [1], and with heavy traffic on the highway. Another example relates to the artificial mixing of the surface air layer (dynamic methods of frost control—artificial mixing of air in protected areas using powerful fans or helicopters). Recently, the modification of the atmospheric boundary layer by large arrays of wind power plants has been actively studied [2].

The physical mechanism of the flows in question is explained in Fig. 1, where the area where the exchange coefficients differ from the background is shaded. The inhomogeneities of these coefficients in a stratified medium obviously lead to the appearance of horizontal inhomogeneities in the distributions of buoyancy and pressure and, consequently, to the occurrence of horizontal currents.

SLOPE FLOWS DUE TO THE HETEROGENEITY OF THE EXCHANGE COEFFICIENTS

As the simplest example, consider a stratified medium bounded from below by a solid inclined surface. If the surface, for example, is colder than the medium, then the denser layer of the medium cooled at the surface flows down the slope under its own weight. Such slope currents are quite common in the atmosphere and are being intensively studied (see, for

example, [3–10] and the bibliography in these sources). The geometry of the problem is shown schematically in Fig. 2, where the dashed lines represent the vertical profiles of potential temperature Θ , whose increase with height z corresponds to a stable stratification of the environment. The potential temperature, up to a constant reference value, can be represented as

$$\Theta = \gamma z + \theta(n). \quad (1)$$

Here, n is the coordinate normal to the inclined boundary, $\gamma > 0$ is the background vertical gradient of potential temperature (assumed to be constant), and θ is the temperature deviation from the background caused by the influence of the lower limit.

In the 1940s, a one-dimensional stationary Prandtl’s model was put forward [3–5, 9], which, being very transparent, is now often considered the benchmark in describing geophysical slope currents:

$$0 = K \frac{d^2 u}{dn^2} + \alpha g \theta \sin \varphi, \quad \gamma u \sin \varphi = K \frac{d^2 \theta}{dn^2}. \quad (2)$$

Here, u is the sought-after speed of the slope current, φ is the angle of the slope to the horizon; α is the thermal coefficient of expansion of the medium, and g is the acceleration of gravity. We consider turbulent exchange coefficient K to be the same for all substances for simplicity (generalization to the case of differences in the values of the coefficients of viscosity and thermal diffusivity is not difficult). The Boussinesq approximation is used.

On an inclined border $n = 0$, the deviation of the potential temperature is set θ_0 (negative if the slope is cooled) and the non-slip condition

$$\theta = \theta_0, \quad u = 0 \quad \text{at} \quad n = 0. \quad (3)$$

The damping of disturbances is assumed far from the surface ($n \rightarrow \infty$). System of equations (2) is homogeneous and, in the case of homogeneity of the boundary conditions ($\theta_0 = 0$), the problem obviously has a zero static solution.

Prandtl’s model assumes constant exchange coefficients. However, the effective coefficients of turbulent exchange in the atmosphere depend substantially on the coordinate normal to the surface n (usually decrease with $n \rightarrow 0$). Therefore, many works claim to generalize this model (see, for example, [4–7, 9]). The generalized system of equations is usually considered

$$\begin{aligned} 0 &= \frac{d}{dn} \left(K(n) \frac{du}{dn} \right) + \alpha g \theta \sin \varphi, \\ \gamma u \sin \varphi &= \frac{d}{dn} \left(K(n) \frac{d\theta}{dn} \right). \end{aligned} \quad (4)$$

This system with variable coefficients is much more difficult to analyze than (2). The literature contains extensive experience in numerical and approximate analytical solutions of system (4) (the bibliography can be found, for example, in [5, 9]).

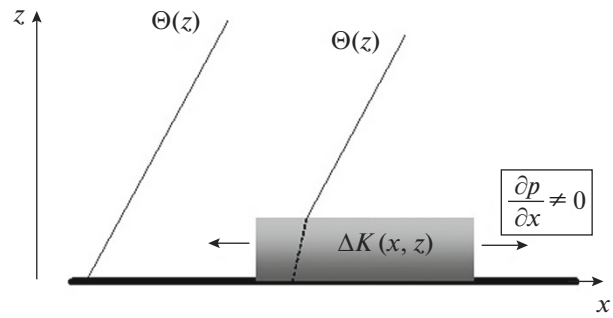


Fig. 1. Scheme of the occurrence of flows in the case of horizontal inhomogeneity of the thermal conductivity coefficient. Horizontal arrows are possible directions of pressure gradient forces arising from the appearance of thermal inhomogeneities.

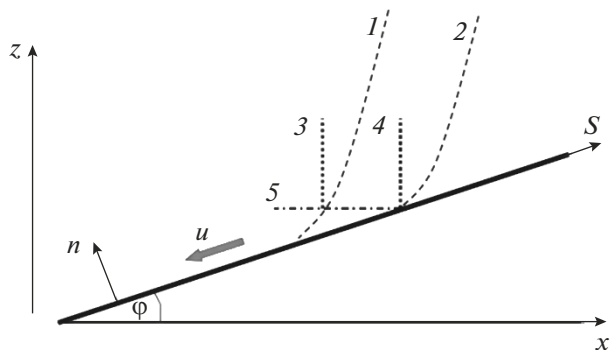


Fig. 2. Flow diagram (indicated by an arrow) near an inclined surface. Turbulent exchange is weakened near this surface; therefore, horizontal temperature and pressure inhomogeneities arise. The difference in hydrostatic pressure on verticals 3 and 4 is explained in the text. In contrast to the classical Prandtl model, the flow can arise in the absence of sources/sinks at the lower boundary due to only spatial inhomogeneity $K(n)$.

System (4) is homogeneous, like (2), and also has a zero static solution. The term *static solution* in this context means the absence of regular flows in the plane (x, z) . However, in this case, a background horizontal flow along another horizontal axis y is possible (in a direction transverse to the slope). Moreover, the presence of such a background flow is necessary for the correct formulation of the physical problem, since the exchange is assumed to be turbulent and, in the complete absence of regular flows, there is no source of turbulence.

However, the possibility of the mentioned static solution contradicts the simple physical considerations given below.

In a static state, the continuity of the diffusion heat flux implies the constancy (independence of height) of the product of the exchange coefficient and the gradient of the potential temperature. At $n \rightarrow 0$, the

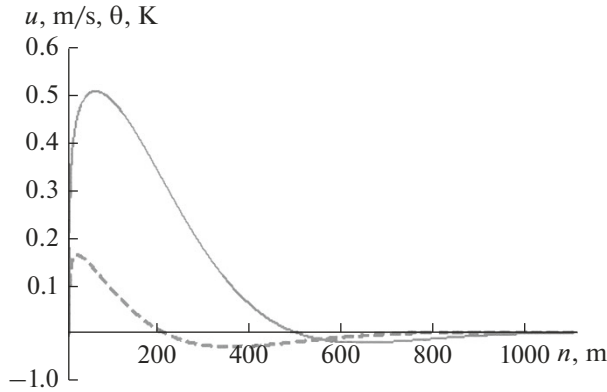


Fig. 3. Profiles of potential temperature (dashed line) and velocity (solid line) deviation for the considered numerical example.

exchange coefficient decreases, so the vertical gradient in a static state should increase. This is shown schematically in Fig. 2.

If we compare the weight of the pillars of the medium on verticals 3 and 4, then on the same horizontal surface 5 this weight will differ: vertical 4 includes a colder (denser) area near the inclined boundary. Thus, the hydrostatic pressure on the abovementioned horizontal 5 will not be uniform—in the case under consideration, it decreases to the left of vertical 4. From these qualitative considerations it is already clear that dependence K and the vertical gradient of the potential temperature from n leads to the presence in the medium of a horizontal gradient of temperature and pressure and, consequently, to the presence of a horizontal component of the force of the pressure gradient and the occurrence of a slope current. In other words, there is no static solution in a stratified environment over an inclined surface, since the weight of the pillars along verticals 3 and 4 will always be different; this fact does not depend on the set temperature of the inclined surface. Thus, in the usually considered system of equations (4), the fundamental fact of the absence of a static solution is lost.

The error in using system (4), as far as we understand, is as follows. In fact, the second equation in (4) must include the potential temperature $\Theta = \gamma z + \theta(n)$, not just its rejection $\theta(n)$. This was not paid attention to for a long time because, at $K = \text{const}$, the indicated error does not affect the result, so the Prandtl model gives the correct result. Note that, in [6], the correct equation with the right-hand side was considered initially:

$$\frac{d}{dn} \left[K(n) \left(\gamma \cos \varphi + \frac{d\theta}{dn} \right) \right]. \quad (5)$$

But then the author expressed the opinion that the term $\gamma \cos \varphi$ in parentheses is relatively small and

neglected it. In many subsequent works, this term is not considered without any comments.

The associated quantitative error in many cases is indeed relatively small, since in the surface layer usually $\gamma \ll d\theta/dz$. However, with small deviations of the surface temperature, not only a noticeable quantitative, but also a qualitative error arises: the fact of the absence of a static state is lost. At $K = K(n)$, the second equation in (4) after the correction in question becomes inhomogeneous:

$$\begin{aligned} -\gamma u \sin \varphi + K(n) \frac{d^2 \theta}{dn^2} + \frac{dK(n)}{dn} \frac{d\theta}{dn} \\ = -\gamma \frac{dK(n)}{dn} \cos \varphi. \end{aligned} \quad (6)$$

Therefore, the static solution $\theta = 0$, $u = 0$ becomes impossible and the contradiction with the above physical considerations is removed.

On the right-hand side of (6), a previously unaccounted for effective heat source/sink appeared. Let's estimate its intensity. The total effective heat dissipation per unit area of inclined surface between levels n_1 and n_2 ($n_2 > n_1$), obviously,

$$c_p \bar{\rho} \gamma [K(n_2) - K(n_1)] \cos \varphi, \quad (7)$$

where c_p is heat capacity and $\bar{\rho}$ is the average density of the medium, respectively. If $c_p = 10^3$ J/(kgK), $\bar{\rho} = 1$ kg/m³, $\gamma = 3 \times 10^{-3}$ K/m, $K(n_2) - K(n_1) = 5$ m²/s, and $\cos \varphi \approx 1$, then expression (7) gives an effective heat source of about 15 W/m². Such a source can make a significant contribution to the heat balance at relatively weak katabatic winds [10].

EXAMPLE OF A NUMERICAL SOLUTION

As a concrete example, consider the solution for the turbulent exchange model profile considered in the [4] (pp. 268–269), Dorodnitsyn's relation:

$$K(n) = K_0 + (K_1 - K_0) \left[1 - \exp \left(-\frac{n}{h_*} \right) \right]. \quad (8)$$

Let us take values of the parameters close to [4]: $\varphi = 3 \times 10^{-2}$, $\gamma = 3 \times 10^{-3}$ K/m, $K_1 = 5$ m²/s with, $K_0 = 10^{-2}$ m²/s, $\alpha = 3.4 \times 10^{-3}$ K⁻¹, and $h_* = 50$ m. Relation (8) describes a monotonic increase in the coefficient of turbulent exchange from a small value K_0 near the lower boundary to the value characteristic of the atmospheric boundary layer K_1 on vertical scales on the order of the thickness of surface layer h_* . On the inclined lower boundary, we assume homogeneous boundary conditions $\theta = 0$, $u = 0$. Figure 3 shows the results of a numerical solution of the system

consisting of the first equation in (4) and Eq. (6). Usually, the considered system (4) leads in this case to an identically zero solution (inconsistent with the above physical considerations); with this refined formulation of the problem, a flow arises, albeit a rather weak one. Note that, with a decrease in the angle of inclination φ , the flow can increase significantly and the thickness of the layer covered by it increases, but in this case the time for reaching the stationary regime also increases.

CONCLUSIONS

In this paper, attention is drawn to the fact that flows in a stratified medium can arise in the absence of sources of buoyancy and momentum due to the redistribution of heat diffusing through the medium due to the spatial inhomogeneity of the transfer coefficients. For example, the presence of a region of low turbulent exchange intensity near a solid surface (and, hence, a region of low thermal conductivity) can lead to the accumulation of heat entering due to diffusion or (depending on the boundary conditions), conversely, to a slowdown in diffusion heating. If the mentioned region is horizontally inhomogeneous, then these thermal effects lead to the appearance of horizontal pressure gradients and the appearance of density flows. This means, in particular, the appearance of an ordered flow in a turbulent stratified medium at an inclined surface.

The horizontal inhomogeneity of the transfer coefficients leads to the absence of static states in a stratified medium above an inclined surface—a slope current arises under any boundary conditions on this surface. In the literature on slope flows, this qualitative feature of the problem remained unnoticed for a long time, especially since the quantitative corrections to the calculated currents in many cases are relatively small.

Turbulent slope currents are only the simplest example of currents caused by horizontal variations in the effective coefficients of turbulent exchange. For example, let there be intensive mixing of a stably stratified surface layer in a certain area above the horizontal underlying surface (some situations related to this are mentioned in the Introduction). Temperature deviations ΔT arising under such mixing can be of the order γh , where h is the thickness of the mixed layer; the amplitude of hydrostatic pressure deviations is on

the order of $\alpha \bar{\rho} g \Delta T h$. Equating the horizontal force of the pressure gradient of the turbulent viscosity $\sim Ku/h^2$, we obtain an estimate of the speed of the horizontal flow arising from the spatial inhomogeneity of the exchange coefficients: $u \sim \alpha g \gamma h^4 / KL$, where L is the horizontal scale of the mixed area. If, for example, $K = 10 \text{ m}^2/\text{s}$, $h = 20 \text{ m}$, $L = 50 \text{ m}$, and $\gamma = 0.1 \text{ K/m}$, then $u \sim 1 \text{ m/s}$. This rough estimate is probably overestimated, since the emerging currents lead to a decrease in temperature and pressure gradients. However, it seems that it is useful to bear in mind the possibility and even the inevitability of the emergence of flows of such a previously unexplored nature.

REFERENCES

1. *The Climate of Moscow in Global Warming Conditions*, Ed. by A. V. Kislova (MGU, Moscow, 2017) [in Russian].
2. S. Emeis, *Wind Energy Meteorology. Atmospheric Physics for Wind Power Generation Wind Energy* (Springer, Berlin, 2018).
3. L. Prandtl, *Hydro- und Aeromechanik* (Julius Springer, Berlin, 1929; *Regulyarnaya i khaoticheskaya dinamika*, Moscow–Izhevsk, 2000).
4. L. N. Gutman, *Introduction to the Nonlinear Theory of Mesometeorological Processes* (Gidrometeoizdat, Leningrad, 1969) [in Russian].
5. *Mountain Weather Research and Forecasting. Recent Progress and Current Challenges*, Ed. by F. K. Chow, S. F. J. DeWekker, and B. Snyder (Springer, Berlin, 2013).
6. L. N. Gutman, “On the theory of the katabatic slope wind,” *Tellus A* **35**, 213–218 (1983).
7. B. Grisogono and J. Oerlemans, “Katabatic flow: Analytic solution for gradually varying eddy diffusivities,” *J. Atmos. Sci.* **58** (21), 3349–3354 (2001).
8. L. K. Ingel, “Towards a nonlinear theory of katabatic winds,” *Fluid Dyn.* **46** (4), 505–513 (2011).
9. M. G. Giometto, R. Grandi, J. Fang, P. A. Monkewitz, and M. B. Parlange, “Katabatic flow: A closed-form solution with spatially-varying eddy diffusivities,” *Boundary-Layer Meteorol.* **162**, 307–317 (2017).
10. J. Oerlemans and B. Grisogono, “Glacier winds and parameterisation of the related surface heat fluxes,” *Tellus A* **54**, 440–452 (2002).