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WHAT SHOULD SOCIETY MAXIMISE UNDER UNCERTAINTY?

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1. Introduction

This paper overviews the research on social decision criteria under uncertainty and attempts to provide insights for future directions.

There already exist helpful survey papers on this topic by the leading professionals, such as Mongin (2016), Mongin and Pivato (2016) and Fleurbaey (2018). Nevertheless, I decided to take the risk of being redundant by writing another survey paper on this matter, because I am keen to take the viewpoint that social decision under risk and uncertainty should: (i) be able to provide an objective function, or more formally a ranking over all potential social alternatives, rather than to give a rule which maps a particular situation to a particular choice; and (ii) possess certain decision-theoretic content, namely the rationality property which is significant especially in risky and uncertainty environments and in associated dynamic environments; and (iii) be able to handle heterogeneity and fairness.

I will try to illustrate the basic logic which governs compatibility or conflict between appealing normative requirements, as directly as possible at an axiomatic level. Hence, I will spend space on illustrating the axiomatic systems for decision making and outlining some of the proofs, perhaps at the cost of sacrificing an exhaustive coverage of the literature.

1.1 Should society try to maximise something, first of all?

It is worth questioning, in the beginning, whether a society should be maximising something, formally a complete and transitive ordering. Why not think of a rule, which maps a particular situation to a particular choice, instead of maximising a ranking or a function?

One answer is that if we want make our choice consistently we have to maximise some ranking. There are two kinds of consistency being considered here, one is static and the other is dynamic.

The static consistency conditions are well-studied (see e.g. Moulin 1991 for a comprehensive illustration of this matter), which basically say what we choose from a larger set must be chosen from a smaller set containing it. The world champion must be the state champion in his/her country. For example, if we choose x from $\{x, y, z\}$ but choose y from $\{x, y\}$, this leads to an inconsitency. It is known that if we wish to avoid this type of inconsistency we have to maximise some ranking.

One may argue that the notion of static consistency is rather hypothetical, as choice opportunity is typically given as fixed. Violation of static consistency leads to violation of dynamic consistency, which is seen to be more substantive. For example, consider that we have to choose between $\{x, y\}$ and $\{z\}$ in Period 1 and choose the final alternative in Period 2 from the set we chose in Period 1. From the ex-ante viewpoint, the decision-maker desires to have x in the end if he or she can commit to, because it is the "best" alternative out of $\{x, y, z\} = \{x, y\} \cup \{z\}$. However, to obtain x, the decision-maker has to choose $\{x, y\}$ in Period 1 and he or she will pick y from $\{x, y\}$ in Period 2, which is against the ex-ante desire of the decision-maker and results in dynamic inconsitency. If we wish to avoid this type of problem, our choice rule must be dynamically consistent.

In the deterministic setting it is shown by Hammond (1976) that dynamic consistency implies static consistency. Thus, if we want to be dynamically consistent, we must be statically consistent and, hence, must be maximising some ranking. This result is extended to the setting with uncertainty by Hayashi (2011).

1.2 Relevance of uncertainty

We are not content with the social ranking just being complete and transitive, while it is typically the case in abstract social choice/social ordering. We want the social ranking to possess certain *decision-theoretic rationality*.

There are two kinds of decision-theoretic rationality. One is consistency to resolution of uncertainty. Potentially we have a dynamic choice environment in mind, in which uncertainty gradually resolves over time. In such situations, we require that social decision must be dynamically consistent.

The other is a normative attitude toward uncertainty. We like to be able to say something about a normative degree of risk/uncertainty aversion and a well-defined concept of belief at a social level.

These two requirements put a restriction on social ranking over alternatives in environments with uncertainty. Typically, such resitriction takes the form of (either objective or subjective) expected utility theory.

1.3 Relevance of heterogeneity and fairness

The critical difference between social decision and a mere application of single-person decision theory is, of course, that we need to be able to handle heterogeneity and accommodate with fairness concerns. In particular, under uncertainty and under associated dynamic environments, the concept of fairness varies depending on our time perspective, namely from ex-ante to ex-post, which very often conflict with each other. To maintain dynamic consitency, how should we handle the tension between them? This question should be kept in mind even when we consider social decision under apparently static environments.

1.4 Outline

Just to clarify the terminologies in decision theory, there are two kinds of uncertainty in the literature. One is *risk*, which refers to situations in which probability distribution over outcomes is given as an object. The other is *subjective uncertainty* or simply *uncertainty*, which refers to situations in which no such thing as objective distribution is given and the decison-maker has to have certain belief about states of the world.

The survey proceeds as follows. In Section 2 we review social decision under risk. There we briefly review the axiomatic system for the expetced utility theory due to von-Neumann and Morgenstern, and illustrate the problem of aggregating expected utility preferences. Section 3 reviews social decision under uncertainty, in which individuals disagree not only in tastes (including risk preferences) but also in beliefs. Section 4 covers fairness in social decision under uncertainty, from ex-ante and ex-post viewpoints. Section 5 discusses the validity of the use of expected consumer surplus and its aggregate as a measure of welfare in partial equilibrium analysis. Section 6 provides concluding remarks and a discussion on future directions.

2. Social decision under risk

2.1 The von-Neumann/Morgenstern expected utility theory

Here we briefly review the von-Neumann/Morgenstern expected utility theory (EUT) of choice under risk, as an understanding of its axiomatic properties is necessary for understanding the problem.

Let X denote the set of outcomes, which is assumed to be finite for simplicity. Let $\Delta_S(X)$ denote the set of simple lotteries (probability distributions having only finitely many outcomes with positive probabilities) over X, which are choice objects. The set of lotteries $\Delta_S(X)$ is a mixture-space in the following sense: given $p, q \in \Delta_S(X)$ and $\lambda \in [0, 1]$, the mixture of lotteries $\lambda p + (1 - \lambda)q$ is defined by

$$(\lambda p + (1 - \lambda)q)(x) = \lambda p(x) + (1 - \lambda)q(x)$$

for each $x \in X$.

We consider ranking $\succsim$ over $\Delta_S(X)$.

Expected utility theory imposes the following axioms:

1. Completeness: either $p \succsim q$ or $q \succsim p$ holds for all $p, q \in \Delta_S(X)$.
2. Transitivity: $p \succsim q$ and $q \succsim r$ implies $p \succsim r$, for all $p, q, r \in \Delta_S(X)$.
3. Mixture Continuity: $\{\lambda \in [0, 1] : \lambda p + (1 - \lambda)q \succsim r\}$ and $\{\lambda \in [0, 1] : r \succsim \lambda p + (1 - \lambda)q\}$ are closed subsets of [0, 1], for all $p, q, r \in \Delta_S(X)$.
4. Independence: for all $p, q, r \in \Delta_S(X)$ and $\lambda \in (0, 1)$ it holds that

$$p \succsim q \quad \Longrightarrow \quad \lambda p + (1 - \lambda)r \succsim \lambda q + (1 - \lambda)r$$

and

$$p \succ q \quad \Longrightarrow \quad \lambda p + (1 - \lambda)r \succ \lambda q + (1 - \lambda)r.$$

Completeness and transitivity will not need an explanation, and we take it as natural under the current scope, although it may be problematic in abstract social choice environments.

Continuity is sought of a technical requirement, while it has some substantive implication that any risk is "compensatiable", which may be potentially problematic in social decision under risk. For example, consider that there are two outcomes, safety and unsafety (S and U), and that the decision-maker is intolerant of any positive probability of unsafety and takes it as equivalent to sure unsafety. Consider a sequence of positive numbers $\{p^v\}$ which converges to zero. Then it holds that

$$(S; 1-p^v, U; p^v) \sim (S; 0, U; 1)$$

for all v. However, in the limit we have

$$(S; 1, U; 0) \succ (S; 0, U; 1),$$

which is a violation of continuity.

Thus, continuity is violated when the decision-maker can never tolerate with some outcome to have positive probability, however small the probability is and however large compensation he or she is offered as the price for accepting it.

We proceed with accepting continuity as a natural axiom, however, given the scope of this review. See Fishburn (1971, 2015) for characterisation of risk preferences violating continuity and its application to social decision-making.

The most problematic one will be Independence. Mathematically, it is just linearity. Why should we care for it, beyond mathematical convenience?

Here we offer the following interpretation. A mixture lottery $\lambda p + (1-\lambda)r$ is interpreted as a "lottery" which gives lottery p with probability λ and lottery r with probability $1-\lambda$, which is generated by some possibly biased coin.

Now the ranking $\lambda p + (1-\lambda)r \succsim \lambda q + (1-\lambda)r$ is interpreted as preferring a "lottery" which gives lottery p with probability λ and lottery r with probability $1-\lambda$, over a "lottery" which gives lottery q with probability λ and lottery r with probability $1-\lambda$, and it should be the case when $p \succsim q$ because the two "lotteries" yield the common outcome r with the same probability and the only difference between them is p and q (see Figure 1).

Suppose the condition is violated, say, $\lambda p + (1-\lambda)r \prec \lambda q + (1-\lambda)r$ despite $p \succsim q$. Then, after choosing $\lambda q + (1-\lambda)r$ over $\lambda p + (1-\lambda)r$, and after the "lottery" turns to give q, the decision-maker will change his or her mind and switch to prefer to get p, or in other words will regret choosing $\lambda q + (1-\lambda)r$. Thus, violation of Independence leads to dynamic inconsistency (see Figure 2).

However, is a "cheat" in this explanation. We should note that the "lottery" which gives lottery p with probability λ and lottery r with probability $1-\lambda$ is a *two-stage* object and it is physically different from the *one-stage* object $\lambda p + (1-\lambda)r$. What is presumed here is that only the probability distribution over final outcomes should matter, and the decision-maker is indifferent in the timing of resolution of risk.

We will come back to the problem about the Independence axiom later, and state the expected utility representation theorem (see a standard textbook such as Mas-Colell *et al.*, 1995 or Herstein and Milnor, 1953).

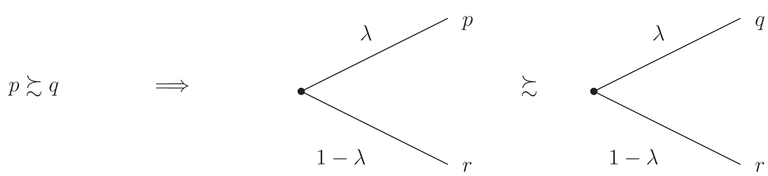

FIGURE 1. Independence

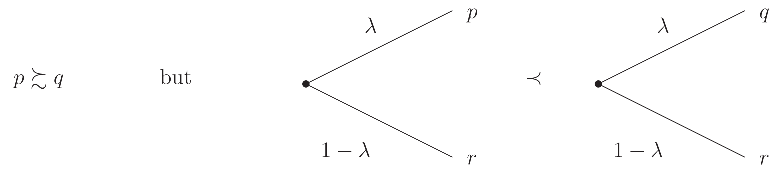

FIGURE 2. Violation of independence

Theorem 1 (vNM Expected Utility Theorem): *Let $\succsim$ be a binary relation on $\Delta_S(X)$. Then the following statements are equivalent:*

(a) $\succsim$ satisfies Completeness, Transitivity, Mixture Continuity and Independence.
(b) There exists a vNM index $u : X \to \mathbb{R}$, such that $\succsim$ is represented by the $U : \Delta_S(X) \to \mathbb{R}$ in the expected utility form

$$U(p) = \sum_{x \in S(p)} u(x)p(x),$$

where $S(p)=\{x \in X : p(x)>0\}$.

Moreover, if v is any other function that forms an expected utility representation for $\succsim$, then there exist $a > 0$ and b such that

$$v = au + b.$$

2.2 Is vNM expected utility cardinal?

A vNM index u is cardinal in the sense that it is unique up to positive affine transformations, *within the class of expected utility representation of the preference*. Thus, it has quantitative meaning as far as its curvature explains risk aversion.

Notice that the entire representation is still *ordinal*, nevertheless, in the class of *all* representations of an EUT preference. For any monotone transformation f,

$$f\left(\sum_{x \in S(p)} u(x)p(x)\right)$$

represents the same ranking as $\sum_{x \in S(p)} u(x)p(x)$ does. Thus, it cannot have a quantitative meaning like measure of happiness unless we declare some faith.

2.3 Aggregation of expected utility preferences

Let I denote the set of individuals. For each individual $i \in I$, let $\succsim_i$ denote his/her prefernce ranking over $\Delta_S(X)$, which is assumed to satisfy EUT. Note it is a *descriptive* assumption that each individual's prefernce satisfies EUT.

The social ranking over $\Delta_S(X)$ is denoted by $\succsim_0$ and it is supposed to satisfy EUT as well, but note that this is a *normative* requirement. In particular, the Independence axiom, which is equivalent to dynamic consistency, is taken to be the rationality postulate imposed on social decision. Otherwise, maximisation of such dynamically inconsistent social objective will not be implemented or credible. See also Hammond (1981, 1983) for arguments on the dynamic consistency requirement for social welfare objectives. Note, however, that the equivalence between the Independence axiom and dynamic consistency presumes consequentialism, in the sense that the decision-maker should care only about final distributions over outcomes.

Harsanyi's theorem considers versions of the Pareto condition applied to preferences over lotteries. Because the application presumes that everybody is responsible for his/her risk attiutde we call it *ex-ante Pareto condition* in particular.

Assume for simplicity of presentation that there exist $x, y \in X$ such that $x \succ_i y$ for all $i \in I$, which says that there is some minimal agreement about ranking over outcomes. One can imagine some disastrous outcome which everybody wants to avoid.

We can think of several versions of the ex-ante Pareto condition:

1. Pareto Indifference: For all $p, q \in \Delta_S(X)$, if $p \sim_i q$ for all $i \in I$ then $p \sim_0 q$.
2. Weak Preference Pareto: For all $p, q \in \Delta_S(X)$, if $p \succsim_i q$ for all $i \in I$ then $p \succsim_0 q$.
3. Weak Pareto: For all $p, q \in \Delta_S(X)$, if $p \succ_i q$ for all $i \in I$ then $p \succ_0 q$.
4. Strong Pareto: For all $p, q \in \Delta_S(X)$, if $p \succsim_i q$ for all $i \in I$ and $p \succ_i q$ for at least one $i \in I$ then $p \succ_0 q$.

Clearly, Strong Pareto implies Weak Pareto, Weak Preference Pareto implies Pareto Indifference, and Strong Pareto and Pareto Indifference imply Weak Preference Pareto. In addition, under mixture continuity and the above minimal agreement condition, Weak Pareto implies Weak Preference Pareto, and the conjunction of Pareto indifference and Strong Pareto are equivalent to Stong Pareto. See Mongin (1995) for technical details.

Now we state Harsanyi's aggregation theorem (see Harsanyi, 1955 for the seminal contribution, and see Border, 1985; Weymark, 1991; De Meyer and Mongin, 1995 for its refinements among many).

Theorem 2: *Suppose both* $\{\succsim_i\}_{i\in I}$ *and* $\succsim_0$ *follow EUT and fix the vNM indices* u_i *for each* $i \in I$ *and* u_0. *Then, they satisfy Pareto indifference if and only if there exist a vector* $\lambda \in \mathbb{R}^I$ *and a number* μ *such that*

$$u_0(\cdot) = \sum_{i\in I} \lambda_i u_i(\cdot) + \mu.$$

Likewise, for Weak Preference Pareto ($\lambda \in \mathbb{R}^I_+$), Weak Pareto ($\lambda \in \mathbb{R}^I_+ \setminus \{\mathbf{0}\}$) and Strong Pareto ($\lambda \in \mathbb{R}^I_{++}$).

Note that the above aggregation is for *fixed* profile of representations, and the obtained welfare weights are dependent on the choice of representations. For example, suppose we double somebody's vNM index, then the same social vNM index is obtained by making their welfare weight half.

More formally, suppose we adopt another profile of vNM indices for individuals

$$\widehat{u}_i = \alpha_i u_i + \beta_i$$

and we still maintain the same social vNM index u_0. Then we obtain

$$u_0(\cdot) = \sum_{i \in I} \widehat{\lambda}_i \widehat{u}_i(\cdot) + \widehat{\mu},$$

where $\{\widehat{\lambda}_i\}_{i \in I}$ and $\widehat{\mu}$ are such that

$$\widehat{\lambda}_i = \lambda_i / \alpha_i$$

for each $i \in I$ and

$$\widehat{\mu} = \mu - \sum_{i \in I} \lambda_i \beta_i / \alpha_i.$$

Therefore, unless we declare a faith that a particular choice of representing vNM indices is the right one, the welfare weight vector obtained above has no quantitative meaning like "weights on individuals' happiness".

2.4 Aggregation with individual consumptions

Although the aggregation problem as above is formulated in the domain of lotteries over social outcomes, it is not difficult to extend the argument to the domain of lotteries over profiles of individual consumptions, where each individual cares only about marginal distributions over his/her consumption.

For each $i \in I$, let X_i be a set of individual-specific outcomes for i. Let $\succsim_i$ denote i's preference over $\Delta(X_i)$, for each $i \in I$, while the social ranking $\succsim_0$ is defined over $\Delta_S(\prod_i X_i)$.

As above, assume that they follow EUT and are represented in the form

$$U_i(p_i) = \sum_{x_i \in S(p_i)} u_i(x_i) p_i(x_i), \quad U_0(p) = \sum_{x \in S(p)} u_0(x) p(x),$$

where p_i denotes the marginal of p over X_i and $S(p_i) = \{x_i \in X_i : p_i(x_i) > 0\}$.

We impose the following version of Ex-ante Pareto,

$$p_i \sim_i q_i \ \ \forall i \in I \quad \Longrightarrow \quad p \sim_0 q.$$

Then it is not difficult to show that u_0 and U_0 are given in the aggregation form:

$$u_0(x) = \sum_i \lambda_i u_i(x_i) + \mu_0, \quad U_0(p) = \sum_i \lambda_i U_i(p_i) + \mu_0.$$

2.5 Does it give any meaning to *additive* aggregation of cardinal utilities? (Harsanyi-Sen debate)

It is already stated that the welfare weight vector in the Harsanyi-type additive aggregation formula is dependent on the choice of individuals' vNM indices, and has no quantitative meaning such as weight on individuals' happiness.

Here is another question: Can Harsanyi's theorem be understood as providing a foundation of additive aggregation of cardinal utilities?

The answer is YES to some extent, if it is talking about aggregation of vNM indices explaning normative risk attitude at the social level, in which curvature of vNM indices explains the degree of risk aversion. The answer is NO if it is talking about aggregation of entire utility functions, which are *ordinal*. The negative views are expressed by Sen (1986) and later by Weymark (1991).

To illustrate, start with an additive aggregation formula:

$$U_0(p) = \sum_{x \in S(p)} u_0(x)p(x), \quad U_i(p) = \sum_{x \in S(p)} u_i(x)p(x), \quad i \in I$$

Then we obtain

$$U_0(p) = \sum_{i \in I} \lambda_i U_i(p) + \mu.$$

However, after let's say taking the exponential transformation we obtain

$$V_0(p) \equiv e^{U_0(p)} = e^{\sum_{i \in I} \lambda_i U_i(p) + \mu} \equiv e^{\mu} \prod\nolimits_{i \in I} (V_i(p))^{\lambda},$$

which represents the same social ranking.

Thus, if we are talking about aggregation of entire utility functions, additive aggregation is not a necessity, and it is just one of the arbitrarily many ways of representing the social objective.

2.6 Are risk attitudes a matter of taste?

The ex-ante Pareto condition presumes that each individual is responsible for his/her risk attiutde and their risk attitudes, however absurd they look, should be taken into account in detremining the social ranking, because they are taken to be a matter of taste.

There can be a different view, however, that individuals are not responsible for their risk attitudes and only preferences over outcomes should be taken into account.

For example, why not Ex-post Pareto instead of Ex-ante Pareto?:

$$x \succsim_i y \ \ \forall i \in I \quad \Longrightarrow \quad x \succsim_0 y.$$

Maintain the assumption that social ranking follows EUT; then it is not difficult to establish the aggregation form

$$U_0(p) = \sum_{x \in S(p)} u_0(x)p(x),$$

where

$$u_0(x) = \phi(u_1(x), \cdots, u_{|I|}(x))$$

and $u_1, \cdots, u_{|I|}$ are fixed ordinal representations of $\succsim_1, \cdots, \cdots, \succsim_{|I|}$ over X, respectively.

2.7 The Diamond critique: Should the social objective satisfy the expected utility theory?

Let's leave aside the problem about the quantitative meaning of the aggregation formula, and turn to the assumption that the social objective should follow the expected utility theory.

Diamond (1967) raises a question against the normative requirement that the social ranking should satisfy the expected utility theory, in particular the Independence axiom.

He points out that the Independence axiom excludes any desire for giving a "fair chance". To understand, consider an individual item to be given either to A or B. Assume that giving to A and giving to B are equally valuable; that is, $A \sim B$. Then the Independence axiom implies

$$\frac{1}{2}A + \frac{1}{2}B \sim \frac{1}{2}A + \frac{1}{2}A = A$$

and

$$\frac{1}{2}A + \frac{1}{2}B \sim \frac{1}{2}B + \frac{1}{2}B = B.$$

However, if we have a normative requirement that we should give a fair chance, we should have

$$\frac{1}{2}A + \frac{1}{2}B \succ A \sim B,$$

which is a violation of the Independence axiom.

This motivates us to characterise a class of social rankings over lotteries which allow for the normative violation of the Independence axiom. One prominent case is to weaken Independence to Mixture Symmetry (Chew *et al.*, 1991), which is stated as

$$p \sim q \quad \Longrightarrow \quad \lambda p + (1-\lambda)q \sim (1-\lambda)p + \lambda q.$$

Together with the other axioms maintained, the mixture symmetry axioms characterise the quadratic expected utility representation

$$U(p) = \sum_{x \in S(p)} u(x)p(x) + \sum_{x \in S(p)} \sum_{y \in S(p)} \phi(x,y)p(x)p(y).$$

As before, let $\succsim_i$ denote individual i's preference ranking over lotteries, for each $i \in I$, and assume that it satisfies EUT.

Now denote the social ranking over lotteries by $\succsim_0$, and assume that it satisfies the quadratic expected utility theory. Epstein and Segal (1992) show that under the ex-ante Pareto condition the social ranking is represented in the form of quadratic social welfare function

$$U_0(p) = \sum_i \lambda_i U_i(p) + \sum_i \sum_j a_{ij} U_i(p) U_j(p).$$

2.8 Non-expected utility and dynamic (in)consistency

As suggested in the explanation of the independence axiom as above, the departure from it is not free of issues, because the violation of it leads to dynamic inconsistency or necessitates a departure from consequentialism. Here is an example given by Machina (1989).

Example 1: *We quote from Machina*

Mom has a single indivisible item—a 'treat'—which she can give to either daughter Abigail or son Benjamin. Assume that she is indifferent between Abigail getting the treat and Benjamin getting the treat, and strongly prefers either of these outcomes to the case where neither child gets it. However, in a violation of the precepts of expected utility theory, Mom strictly prefers a coin flip over either of these sure outcomes, and in particular, strictly prefers 1/2: 1/2 to any other pair of probabilities. This random allocation procedure would be straightforward, except that Benjie, who cut his teeth on Raiffa's classic Decision Analysis, behaves as follows:

Before the coin is flipped, he requests a confirmation from Mom that, yes, she does strictly prefer a 50:50 lottery over giving the treat to Abigail. He gets her to put this in writing. Had he won the flip, he would have claimed the treat. As it turns out, he loses the flip. But as Mom is about to give the treat to Abigail, he reminds Mom of her preference for flipping a coin over giving it to Abigail (producing her signed statement), and demands that she flip again.

What would your Mom do if you tried to pull a stunt like this? She would undoubtedly say "You had your chance!" and refuse to flip the coin again. This is precisely what Mom does.

Machina continues, "By replying 'You had your chance', Mom is reminding Benjamin of the existence of the snipped-off branch (the original 1/2 probability of B) and that her preferences are not separable, so the fact that nature could have gone down that branch still matters. Mom is rejecting the property of consequentialism—and, in my opinion, rightly so". Here the term "consequentialism" is the presumption that the decision should be independent of events or outcomes which turned out not to have occurred.

Each of Mom's and Benjamin's claims leads to a problem. If we accept Benjamin's claim, and if we want to be dynamically consistent, the ex-ante probability of Abigail's winning the item should be $1/2\times1/2=1/4$ and Benjamin's winning probability ex-ante should be $1/2+1/2\times1/2=3/4$, which is unfair in any sense from the ex-ante viewpoint, or we have to flip the coin forever. Thus, there is a conflict between ex-ante fairness and ex-post fairness.

If we accept Mom's claim, rejecting consequentialism, it opens up a problem of what should be an alternative "rationality" requirement on social decisions. Can ex-post social decisions be reason-based, in the sense that we can say something more than just "it is already decided"?

Hayashi (2016) proposes a meta axiom that an axiom used for ex-ante welfare judgment must be met by ex-post judgment as well. For example, the standpoint such as "everybody should get the same expected utility" cannot be recurrent over time or stable in other words, because somebody may be lucky and somebody else may be unlucky, and we cannot give them the same conditional expected utility after this. He presents a non-consequentialist process of social rankings which satisfies this meta axiom in addition to dynamic consistency, which is, in other words, closed under consistent updating.

To illustrate, consider that there are three periods, 0, 1 and 2. Let S denote a finite set of states of the world, and let $\mathcal{E}$ denote a partition of S. Consider that nothing is revealed at Period 0, an event (element of $\mathcal{E}$) is known at Period 1, and the final state realizes at Period 2. Consider that the society has a common prior p over S, while the argument can be extended to the cases of belief disagreements.

At Period 0, the society is to rank random utility profiles. An $I \times S$-vector $\mathbf{u} \in \mathbb{R}^{I \times S}$ denotes a random utility profile, whose (i,s) entry denoted u_{is} refers to i's utility at State s. Given an event $E \in \mathcal{E}$, an $I \times E$-vector $\mathbf{u}_E \in \mathbb{R}^{I \times E}$ denotes a random utility profile conditional on E. In addition, an $I \times (S \setminus E)$-vector $\mathbf{u}_{-E} \in \mathbb{R}^{I \times (S \setminus E)}$ denotes a random utility profile conditional on $S \setminus E$.

Note, here we dare to assume that utilities are interpersonally comparable, as well as comparable across states. Therefore, the concept of welfare weight and the concept of degree of inequality aversion are seen to make sense.

In general, when the society knows Event $E \in \mathcal{E}$, its conditional decision can depend on *what would have been obtained if E had not occured*, which is namely the profile of unrealised utilities, $\mathbf{u}_{-E}$. Thus, the social ranking conditional on E at Period 1, defined over conditional random utility profiles in $\mathbb{R}^{I \times E}$, must depend on $\mathbf{u}_{-E}$ in general, hence is denoted by $\succsim_{\mathbf{u}_{-E}}$, while the non-conditional one at Period 0 is denoted by $\succsim$. Thus, the relation $\mathbf{u} \succsim \mathbf{v}$ says that the random utility profile $\mathbf{u} \in \mathbb{R}^{I \times S}$ is socially at least as good as $\mathbf{v} \in \mathbb{R}^{I \times S}$, and the relation $\mathbf{u}_E \succsim_{\mathbf{u}_{-E}} \mathbf{v}_E$ says that given the unrealised utility profile $\mathbf{u}_{-E}$ the conditional random utility profile $\mathbf{u}_E \in \mathbb{R}^{I \times E}$ is socially at least as good as $\mathbf{v}_E \in \mathbb{R}^{I \times E}$.

Here the dynamic consistency condition is formulated by

$$(\mathbf{u}_E, \mathbf{u}_{-E}) \succsim (\mathbf{v}_E, \mathbf{u}_{-E}) \quad \Longleftrightarrow \quad \mathbf{u}_E \succsim_{\mathbf{u}_{-E}} \mathbf{v}_E$$

for all E, $\mathbf{u}_E, \mathbf{v}_E$ and $\mathbf{u}_{-E}$, and the meta axiom states that when a normative postulate is met by $\succsim$ it must be met by $\succsim_{\mathbf{u}_{-E}}$ as well for all E and $\mathbf{u}_{-E}$.

Hayashi provides a set of axioms which are closed under consistent updating and characterise the process of orderings represented in the form

$$\Phi(\mathbf{u}) = -\sum_{i \in I} a_i \exp\left(-\lambda \sum_{s \in S} u_{is} p(s)\right)$$

for $\succsim$, where a denotes the vector of ex-ante welfare weights adding up to one and λ refers to the degree of inequality aversion ex-ante, and

$$\Phi(\mathbf{u}_E | \mathbf{u}_{-E}) = -\sum_{i \in I} a_i(\mathbf{u}_{-E}) \exp\left(-\lambda(\mathbf{u}_{-E}) \sum_{s \in E} u_{is} p(s|E)\right)$$

for $\succsim_{\mathbf{u}_{-E}}$, where $a_i(\mathbf{u}_{-E})$ denotes the vector of ex-post welfare weights adding up to one and $\lambda(\mathbf{u}_{-E})$ refers to the degree of inequality aversion ex-post, and the vector of welfare weights and the degree of inequality aversion follow the updating rule

$$a_i(\mathbf{u}_{-E}) = \frac{a_i \exp\left(-\lambda \sum_{s\in S\setminus E} u_{is}p(s)\right)}{\sum_{j\in I} a_j \exp\left(-\lambda \sum_{s\in S\setminus E} u_{js}p(s)\right)}$$
$$\lambda(\mathbf{u}_{-E}) = \lambda p(E).$$

Note that the ex-post welfare weight for a given individual is discounted as his or her expected utility conditional on the unrealised event is larger, and the society tends to be less inequality averse ex-post when the prior probability of the realised event is lower. This is consistent with Mom's claim in the above example.

3. Social decision under subjective uncertainty

3.1 Double disagreements in beliefs and tastes

Now we come to the problem of aggregation when individuals may disagree not only in their tastes (including risk attiudes) but also in beliefs about likelihood of states of the world.

These double disagreements lead to so-called spurious unanimity (Mongin, 2016). Following Gilboa *et al.* (2004), consider that two individuals, A and B, are to decide whether to duel or not. A believes he/she wins and he/she is happy if he/she wins. B believes he/she wins and he/she is happy if he/she wins. Thus, there is a unanimous agreement that they should duel. If we follow the ex-ante Pareto condition, the social decision is that they should duel. However, this sounds absurd.

The example shows that the ex-ante Pareto condition is not obvious. It may depend on how we phrase the example, however. For example, imagine that A believes it will rain tomorrow and B believes it will be sunny, and they agree to bet 100 dollars. In this case, more people would be happy to let them bet.

Apart from the question of whether the ex-ante Pareto condition is appealing, below I illustrate the results that this principle is incompatible with the requirement that the social objective should satisfy the subjective expected utility theory.

We might have a "crude hope" that we can aggregate individuals' vNM indices into a social one and individuals' subjective beliefs into a social one, but this cannot be compatible with the ex-ante Pareto condition.

3.2 Reviewing the Savage axioms

Here we briefly review the subjective expected utility theory due to Savage (1972). Let S denote the set of states of the world, which is assumed to be *objective*, so that any problem of unawareness is ruled out. Let Σ denote the family of events (subsets of S) which is endowed with suitable measurability properties. Let X denote the set of outcomes.

Here any decision is formalised as an *act*, which is a mapping $f: S \to X$. This requires that how a physical action relates between each possible state of the world and an outcome is objectively understood by everybody.

We restrict attention to simple acts, which take finitely many outcomes. Let $\mathcal{F}$ denote the set of simple acts.

Preference relation $\succsim$ is defined over $\mathcal{F}$.

Savge proposed six axioms, which are named as P1 to P6.

P1, Completeness and Transitivity

P2, Sure-thing Principle (or Eventwise Separability): for all $f, g, h, h' \in \mathcal{F}$ and $E \in \Sigma$, it holds that

$$fEh \succsim gEh \iff fEh' \succsim gEh',$$

where fEh denotes the act which gives the outcome of f under E and that of h under E^c, and similarly for the others.

P3, Eventwise Monotonicity: For any non-null E and $x,y \in X$ and $f \in \mathcal{F}$, it holds that

$$x \succsim y \iff xEf \succsim yEf,$$

where E is null if $xEf \sim yEf$ for any $x,y \in X$ and $f \in \mathcal{F}$.

P4, (Weak) Comparative Probability: for all $A,B \in \Sigma$, it holds that

$$xAy \succsim xBy \iff zAw \succsim zBw$$

for all $x,y,z,w \in X$ with $x \succ y$ and $z \succ w$.

P5, Nontriviality: There exist $x,y \in X$ with $x \succ y$.

P6, Small Event Continuity: For all $f \succ g$ and $x \in X$, there exists a finite partition $\{E_k\}_{k=1}^n$ of S such that

$$f \succ xE_kg, \quad \text{and} \quad xE_kf \succ g$$

holds for all $k=1, \cdots ,n$.

P1 and P5 will need no explanation. The Savage theory proceeds by defining a ranking between events, called qualitative probability, by

$$A \succsim^l B \iff \begin{pmatrix} x & \text{if} & A \\ y & \text{if} & A^c \end{pmatrix} \succsim \begin{pmatrix} x & \text{if} & B \\ y & \text{if} & B^c \end{pmatrix},$$

where $x, y \in X$ are such that $x \succ y$. That is, the decision-maker is said to believe that A is more likely than B when he or she prefers to bet on A rather than on B; that is, when he or she prefers a bet which gives a better outcome if A occurs and a worse outcome otherwise over a bet which gives the better outcome if B occurs and the worse outcome otherwise.

To make this relation well-defined, we have to be able to define "better" and "worse" outcomes independently of states and events. This is guaranteed by P3. Otherwise, for example, preferring a bet "beer if Tigers win, no beer if Tigers lose" over a bet "beer if Giants win, no beer if Giants lose" may be simply due to the fact that for this decision-maker beer tastes good when Tigers win and Giants lose, and tastes bad if Tigers lose and Giants win. P4 guarantees that the definition of qualitative probability does not depend on choice of the "better" and "worse" outcomes. For example, we should have the same ranking even when we replace "beer" and "no beer" by "vodka" and "no vodka", respectively.

P2 guarantees that such qualitative probability is additive in the sense that it holds that

$$A \succsim^l B, \quad C \cap (A \cup B) = \emptyset \quad \Longrightarrow \quad A \cup C \succsim^l B \cup C$$

for all A,B,C.

P6 says that the state space is sufficienly rich so that we can partition it arbitrarily finely. In particular, it implies that we can partition S into E_1 and E_2 with $E_1 \sim^l E_2$. By repeating this, for every n we obtain a partition $E_1^n, \ldots, E_{2^n}^n$ of equally subjectively likely events, each of which is now seen to have subjective probability $\frac{1}{2^n}$. For a general event E, its subjective probability $p(E)$ is defined by the limit of approximation by taking the union of these subjectively equally likely events, where the partition tends to be arbitrarily fine. Thus, it delivers the represetation by a finitely additive set function p: $\Sigma \rightarrow [0, 1]$, so that

$$A \succsim^l B \quad \Longrightarrow \quad p(A) \geq p(B)$$

and p is shown to be convex-ranged: for all A and $\lambda \in [0, 1]$ there is $B \subset A$ with $p(B) = \lambda p(A)$.

Given a simple act f, let $\Phi_{p,f} \in \Delta_S(X)$ denote the simple lottery generated by p and f, in the form

$$\Phi_{p,f}(x) = p(f^{-1}(x))$$

for any $x \in f(S)$.

It is shown that only preference over induced lotteries should matter:

$$\Phi_{p,f} = \Phi_{p,g} \quad \Longrightarrow \quad f \sim g.$$

This allows us to define a ranking over simple lotteries, denoted $\succsim^*$, defined by

$$l \succsim^* m \quad \Longleftrightarrow \quad \exists f, g: \quad l = \Phi_{p,f}, \ m = \Phi_{p,g}, \ f \succsim g.$$

P2 gurantees that $\succsim^*$ satisfies the Independence axiom, and P5 guarantees that it satisfies mixture continuity. Thus, $\succsim^*$ satisfies the vNM expected utility theory.

Summing up, the following theorem holds (see Savage, 1972 and Fishburn, 1970 for complete arguments).

Theorem 3: $\succsim$ *satisfies P1–P6 if and only if there is a convex-ranged finitely-additive measure* p *and a function* $u: X \rightarrow \mathbb{R}$ *such that* $\succsim$ *is represented in the form*

$$U(f) = \sum_{x \in f(S)} u(x) p(f^{-1}(x)).$$

One may add an additonal axiom, called P7, to guarantee that the subjective probability measure p is countably additive. In the study I illustrate below, Mongin (1995) adopted the version that p is countably additive and non-atomic, but μ being finitely additive and convex-ranged is enough for the current argument.

Note that state independence is vital for the concept of subjective belief to make sense, as illustrated above. To illustrate further, suppose that preference over outcomes depends on the nature of an event E. Then we cannot rule out any way for the vNM index to depend on E. Say, let u_E denote one choice of the vNM index conditional on E and u_{E^c} also be one choice of the vNM index conditional on E^c. Then the "desired" subjective expected utility representation for binary bets with regard to E and E^c would take the form

$$u_E(x)p(E) + u_{E^c}(y)p(E^c).$$

However, because there is nothing to discipline the choice of state-dependent vNM indices, one can also take, for example,

$$\overline{u}_E(x) = 2u_E(x).$$

Then by redefining the other entries by

$$\overline{p}(E) = \frac{1}{2}p(E), \quad \overline{u}_{E^c}(y) = \frac{p(E^c)}{\frac{1}{2} + \frac{1}{2}p(E^c)}u_{E^c}(y), \quad \overline{p}(E^c) = \frac{1}{2} + \frac{1}{2}p(E^c)$$

we can represent the same ranking over binary bets with regard to E and E^c in the form

$$\overline{u}_E(x)\overline{p}(E) + \overline{u}_{E^c}(y)\overline{p}(E^c).$$

In the literature of individual decision theory, P2 (Sure-Thing Principle) has been criticised as it is violated by the Ellsberg paradox. P2 is not the issue here actually, as is demonstrated below.

3.3 Aggregation of subjective expected utility preferences

Let I denote the set of individuals. For each $i \in I$, $\succsim_i$ denotes his of her preference ranking over $\mathcal{F}$, which is assumed to satisfy SEUT. Note that this is a *descriptive* assumption.

Let $\succsim_0$ denote the social ranking over $\mathcal{F}$, which is also assumed to satisfy SEUT. Note, however, that this is a *normative* requirement. The SEUT axioms as illustrated above are taken to be the rationality postulates for the social decision here. In particular, P2 is parallel to the Independence axiom as in the objective expected utility, which conveys the idea of dynamic consistency. Moreover, P3 and P4 convey the idea that social decision should be able to admit a well-defined notion of "social belief", which is separated from "social taste".

Here are several versions of the ex-ante Pareto condition:

1. Pareto Indifference: For all $f, g \in \mathcal{F}$, if $f \sim_i g$ for all $i \in I$ then $f \sim_0 g$.
2. Weak Preference Pareto: For all $f, g \in \mathcal{F}$, if $f \succsim_i f$ for all $i \in I$ then $f \succsim_0 g$.
3. Weak Pareto: For all $f, g \in \mathcal{F}$, if $f \succ_i g$ for all $i \in I$ then $f \succ_0 g$.

4. Strong Pareto: For all $f, f \in \mathcal{F}$, if $f \succsim_i g$ for all $i \in I$ and $f \succ_i g$ for at least one $i \in I$ then $f \succ_0 g$.

Assume for simplicity of presentation that there exist $x, y \in X$ such that $x \succ_i y$ for all $i \in I$.

Clearly, Strong Pareto implies Weak Pareto, Weak Preference Pareto implies Pareto Indifference, and Strong Pareto and Pareto Indifference imply Weak Preference Pareto. In addition, under mixture continuity and the above minimal agreement condition, Weak Pareto implies Weak Preference Pareto, and the conjunction of Pareto indifference and Strong Pareto are equivalent to Stong Pareto. See Mongin (1995) for technical details.

In a linear space Y, say that n vectors $y_1, \ldots, y_n \in Y$ are linearly independent if $\sum_{i=1}^{n} a_i y_i = 0$ implies $a_1 = \cdots, a_n = 0$, and they are affine independent if $\sum_{i=1}^{n} a_i y_i + b = 0$ implies $a_1 = \cdots, a_n = b = 0$.

Theorem 4 (*Mongin,* 1995*; see also Hylland and Zeckhauser, 1979*) *Suppose that* $\succsim_i$ *satisfies SEUT for all* $i \in I$ *and* $\succsim_0$ *does as well. For each* $i \in I$ *fix* (u_i, p_i) *which gives an SEU representation of* $\succsim_i$*, and fix* (u_0, p_0) *which gives an SEU representation of* $\succsim_0$*. Then:*

(i) If $(\succsim_i)_{i \in I \cup \{0\}}$ satisfies Pareto Indifference then there exist $\lambda \in \mathbb{R}^I$ and $\alpha \in \mathbb{R}^I$ with $\sum_{i \in I} \alpha_i = 1$ such that

$$u_0 = \sum_{i \in I} \lambda_i u_i, \quad p_0 = \sum_{i \in I} \alpha_i p_i.$$

(ii) However, if $\{p_1, \ldots, p_n\}$ are linearly independent there is $i^* \in I$ such that $\lambda_i = 0$ for all $i \in I \setminus \{i^*\}$.
(iii) Also, if $\{u_1, \ldots, u_n\}$ are linearly independent there is $j^* \in I$ such that $\alpha_i = 0$ for all $i \in I \setminus \{j^*\}$.
(iv) If $\{p_1, \ldots, p_n\}$ are linearly independent, unless $\{u_1, \ldots, u_n\}$ is pairwise affine dependent there is no $\succsim_0$ satisfying Strong Pareto.
(v) If $\{u_1, \ldots, u_n\}$ are affine independent, unless $\{p_1, \ldots, p_n\}$ are identical there is no $\succsim_0$ satisfying strong Pareto.

3.4 Dropping the sure-thing principle does not help

One immediate response to the above impossibility result will be to drop or weaken P2, the sure-thing principle, as the decision theorists' instinct will associate the impossiblity with the Ellesberg paradox. Gajdos *et al.* (2008) extend the impossibility result to the class of preferences including multiple-priors preferences due to Gilboa and Schmeidler (1989). As is shown in the next section, the source of impossibility is P2.

In fact, we encounter the impossibility result even when we drop the sure-thing principle, as demonstrated by Aczel and Maksa (1996), Zuber (2016) and Mongin and Pivato (2015), as far as we stick to the presumption that the two natural approaches are commutative, the *ex-ante approach* and the *ex-post approach*. The ex-ante approach first calculates each individual's ex-ante welfare, which may or may not be his/her own calculation of expected utility, based on certain decision-theoretic criteria, and then aggregates them across individuals into the social objective, based on certain social welfare criteria. The ex-post approach first calculates ex-post social welfare at each state, based on certain social welfare criteria, and then aggregates them across states into the social objective, based on certain decision-theoretic criteria.

To explain, consider that the society is to rank state-contingent utility profiles, generically denoted by $f \in \mathbb{R}^{I\times S}$, where f_{is} denotes individual i's utility at State s. Hence, the social ranking $\succsim$ is defined over $\mathbb{R}^{I\times S}$. Assume that S is finite. Here taking utilities as the primitive is rather more for mathematical convenience and does not involve an argument on interpersonal comparison of utilities.

The ex-ante approach assumes that we can define each individual's ex-ante ranking over his or her random utils, which may or may not be the individual's own expected utility preference. Denote such ranking by $f_i \succsim_i g_i$, for each $i \in I$. In order that such ex-ante ranking is well-defined, independently of random utils for the others, we need to impose the following axiom.

Axiom 1: *For all* i $\in$ I*, for all* f_i, g_i *and* h_{-i}, $\widetilde{h}_{-i}$ *it holds that*

$$(f_i, h_{-i}) \succsim (g_i, h_{-i}) \iff (f_i, \widetilde{h}_{-i}) \succsim (g_i, \widetilde{h}_{-i})$$

The ex-post approach assumes that we can define ex-post social ranking at each state Denote such ranking by $f_s \succsim_s g_s$, for each $s \in S$. In order that such ex-post ranking is well-defined for each state, independently of what will be given in the other states, we need the following axiom:

Axiom 2: *For all* s $\in$ S*, for all* f_s, g_s *and* h_{-s}, $\widetilde{h}_{-s}$ *it holds that*

$$(f_s, h_{-s}) \succsim (g_s, h_{-s}) \iff (f_s, \widetilde{h}_{-s}) \succsim (g_s, \widetilde{h}_{-s}).$$

Theorem 5: $\succsim$ *satisfies both axioms for the ex-ante approach and the ex-post approach if and only if there is a collection of functions* $(\phi_{is})_{i\in I, s\in S}$ *such that it holds*

$$f \succsim g \iff \sum_{i\in I}\sum_{s\in S}\phi_{is}(f_{is}) \geq \sum_{i\in I}\sum_{s\in S}\phi_{is}(g_{is})$$

for all f, g.
In addition, if there is another collection of functions $(\psi_{is})_{i\in I, s\in S}$ *which represents* $\succsim$ *in the above form then there exists* $\alpha > 0$*, a collection of constants* $(\beta_{is})_{i\in I, s\in S}$ *such that*

$$\psi_{is} = \alpha\phi_{is} + \beta_{is}.$$

We follow the proof as it is of interest by itself. Because the social ranking $\succsim$ is weakly separable across states there exist conditional rankings $\succsim_1, \ldots, \succsim_S$, which are represented by $u_1, \cdots, u_S$ respectively, and we obtain an aggregation formula:

$$U(f) = \Phi\left(u_1\begin{pmatrix} f_{11} \\ \vdots \\ f_{I1} \end{pmatrix}, \cdots, u_S\begin{pmatrix} f_{1S} \\ \vdots \\ f_{IS} \end{pmatrix}\right).$$

In contrast, because the social ranking $\succsim$ is weakly separable across individuals we have rankings for individuals $\succsim_1, \ldots, \succsim_I$, which may or may not be their own subjective preferences, which are represented by $U_1, \ldots, U_I$ respectively. Then we obtain an aggregation formula:

$$U(f) = W\begin{pmatrix} U_1(f_{11}, \cdots, f_{1S}) \\ \vdots \\ U_I(f_{I1}, \cdots, f_{IS}) \end{pmatrix}.$$

By combining the above two arguments we obtain a functional equation:

$$W\begin{pmatrix} U_1(f_{11}, \cdots, f_{1S}) \\ \vdots \\ U_I(f_{I1}, \cdots, f_{IS}) \end{pmatrix} = \Phi\left(u_1\begin{pmatrix} f_{11} \\ \vdots \\ f_{I1} \end{pmatrix}, \cdots, u_S\begin{pmatrix} f_{1S} \\ \vdots \\ f_{IS} \end{pmatrix}\right).$$

From Aczel and Maksa (1996), it is known that the solution to the above functional equation must have the form

$$U(f) = \sum_{i \in I} \sum_{s \in S} \phi_{is}(f_{is})$$

$$U_i(f) = \sum_{s \in S} \phi_{is}(f_{is}) \quad u_s(f) = \sum_{i \in I} \phi_{is}(f_{is}).$$

Thus, ex-ante welfare evaluation for each individual must be additively separable across states. In this sense, the sure-thing-principle is rather a *consequence* of the assumption of commutativity between the ex-ante approach and the ex-post approach.

Moreover, when we impose the ex-ante Pareto principle we have to have some common $p \in \Delta(S)$ and $\{u_i\}_{i \in I}$ such that

$$\phi_{is} = p_s u_i$$

for all $i \in I$. Hence, the common-prior condition is obtained as a *consequence* of the commutativity assumption and the ex-ante Pareto condition, and we again come back to the above impossibility when the individuals disagree on beliefs.

3.5 What is the direct cause of impossibility?

Then what is the cause of impossibility? Chambers and Hayashi (2006) show that there is a direct conflict between the ex-ante Pareto condition and each of P3 and P4. The intuition is actually pretty clear. Again consider the example that A believes it rains and B believes it shines, then Ex-ante Pareto says A should get everything if it rains and B should get everything if it shines. Such ex-post welfare ranking must be *state-dependent*.

To illustrate, let (100, 0) be the outcome in which A receives 100 and B receives 0, and (0, 100) be the opposite. Then natural symmetry will conclude that $(100, 0) \sim_0 (0, 100)$. Suppose A believes that an event D is more likely than an event E, and B believes the opposite at least weakly (meaning to believe that D^c is at least as likely as E^c). Then it holds that:

$$\begin{pmatrix} (100,0) & D \\ (0,100) & D^c \end{pmatrix} \succ_A \begin{pmatrix} (100,0) & E \\ (0,100) & E^c \end{pmatrix}, \quad \begin{pmatrix} (100,0) & D \\ (0,100) & D^c \end{pmatrix} \succsim_B \begin{pmatrix} (100,0) & E \\ (0,100) & E^c \end{pmatrix}.$$

By Strong Pareto, we obtain

$$\begin{pmatrix} (100,0) & D \\ (0,100) & D^c \end{pmatrix} \succ_0 \begin{pmatrix} (100,0) & E \\ (0,100) & E^c \end{pmatrix}.$$

However, this contradicts P3.

This can be stated more formally as follows.

Theorem 6: *Assume $\succsim_A$ and $\succsim_B$ satisfy P4, and for each i = >A, B and for all D, E it holds that*

$$D \succsim_i^l E \iff D^c \precsim_i^l E^c.$$

In addition, assume there exist $\overline{x}$, $\underline{x} \in X$ *such that* $\overline{x} \succ_A \underline{x}$, $\overline{x} \prec_B \underline{x}$, $\overline{x} \sim_0 \underline{x}$. *Then $\succsim_0$ satisfies P3 and Strong Ex-ante Pareto only when $\succsim_A^l = \succsim_B^l$.*

In addition, when we compare (100, 0) and (0, 0) only A is relevant; it holds that

$$\begin{pmatrix} (100,0) & D \\ (0,0) & D^c \end{pmatrix} \succ_A \begin{pmatrix} (100,0) & E \\ (0,0) & E^c \end{pmatrix}, \quad \begin{pmatrix} (100,0) & D \\ (0,0) & D^c \end{pmatrix} \sim_B \begin{pmatrix} (100,0) & E \\ (0,0) & E^c \end{pmatrix},$$

which by Strong Pareto implies

$$\begin{pmatrix} (100,0) & D \\ (0,0) & D^c \end{pmatrix} \succ_0 \begin{pmatrix} (100,0) & E \\ (0,0) & E^c \end{pmatrix}.$$

However, because when we compare between (0, 100) and (0, 0) only B is relevant, it holds that

$$\begin{pmatrix} (0,100) & D \\ (0,0) & D^c \end{pmatrix} \sim_A \begin{pmatrix} (0,100) & E \\ (0,0) & E^c \end{pmatrix}, \quad \begin{pmatrix} (0,100) & D \\ (0,0) & D^c \end{pmatrix} \precsim_B \begin{pmatrix} (0,100) & E \\ (0,0) & E^c \end{pmatrix},$$

which by Strong Pareto implies

$$\begin{pmatrix} (0,100) & D \\ (0,0) & D^c \end{pmatrix} \precsim_0 \begin{pmatrix} (0,100) & E \\ (0,0) & E^c \end{pmatrix},$$

which is a violation of P4.

This can be stated more formally as follows. It says that P4 and Strong Ex-ante Pareto are compatible only under common qualitative probability, under a fairly mild condition which is weaker than P3.

Theorem 7: *Assume* $\succsim_A$ *and* $\succsim_B$ *satisfy P4, and for each* i = A, B *and all* D, E, x, x' *it holds that:*

$$x \sim_i x' \implies xDx' \sim_i xEx'.$$

In addition, assume there exist $\bar{x}$, $\underline{x}$, $\bar{y}$, $\underline{y} \in X$ *such that* $\bar{x} \succ_A \underline{x}$, $\bar{x} \sim_B \underline{x}$ *and* $\bar{y} \sim_A \underline{y}$, $\bar{y} \succ_B \underline{y}$.
Then, $\succsim_0$ *satisfies P4 and is Strong Ex-ante Pareto only when* $\succsim_A^l = \succsim_B^l$.

The above two theorems state that we have to give up either of the ex-ante Pareto condition or state independence of social ranking. More specifically, the latter theorem states that the ex-ante Pareto condition and the concept of qualitative probability are incompatible.

Giving up state independence allows us to aggregate subjective expected utilities, just like in the manner of Harsanyi's theorem applied to ex-ante expected utilities. Mongin (1998) and Chambers and Hayashi (2006) obtain the form

$$U_0(f) = \sum_i \alpha_i U_i(f) = \sum_i \alpha_i \int_S u_i(f(s)) p_i(ds).$$

3.6 Is violation of state independence a problem?

Note that the above aggregation formula, when the welfare weight vectors are normalised as $\sum_i \alpha_i = 1$, may be read as

$$\sum_i \alpha_i \int_S u_i(f(s)) p_i(ds) = \int_S \left\{ \sum_i \alpha_i(s) u_i(f(s)) \right\} p_0(ds),$$

where

$$\alpha_i(s) = \frac{\alpha_i p_i(ds)}{\sum_j \alpha_j p_j(ds)}, \quad p_0 = \sum_i \alpha_i p_i.$$

It is tempting to interpret that p_0 is the "aggregate belief" and $\alpha_i(s)$ is the "state-dependent welfare weight". However, violation of P3 and P4 is understood as a failure of separation of belief and taste (which is welfare judgment when applied to a social ranking), and the above interpretation should be no more than an artifact of manipulating functional form. Generally, no meaningful notion of “social belief” can be thought under state dependence.

However, we see from the above arguments that the method of violating state independence is not an arbitrary one. In fact, if we can rely on interpersonally comparable cardinal utilities, the above form satisfies state independence over the domain of “egalitarian outcomes”. To see this, consider a class of outcomes, say, denoted X^* such that $u_i(x) = u_j(x) \equiv u(x)$ for all $i, j \in I$, and let $\mathcal{F}^* = \{f \in \mathcal{F} : f(S) \subset X^*\}$. Then, because $\sum_i \alpha_i(s) = 1$ for all $s \in S$, it holds that

$$U(f) = \int_S u(f(s))p_0(ds)$$

for all $f \in \mathcal{F}^*$, and the social ranking satisfies state independence in the restricted domain of $\mathcal{F}^*$, regardless of state-dependent ex-post welfare weights. Indeed, P3 and P4 are met over $\mathcal{F}^*$.

Isn’t it a cheat? Yes it is. However, we should be aware that this kind of cheat is used somehow in the standard theory as well. Recall that the subjective expected utility theory following Savage (1972) and Anscombe and Aumann (1963) derives first a state-independent utility function from state independence of preference over outcomes, then its aggregation is established across states. However, state independence of preference over outcomes does not neccesitate choosing a state-independent utility function u. Ti neccesitate uniqueness without any such “cheat”, one needs a richer domain such as the one in Karni *et al.* (1983). As we will see in the next section, the problem disappears once when we accept interpersonally comparable utility values straightly.

3.7 Weakening ex-ante Pareto

Given the impossiblity result, another natural direction is to weaken the ex-ante Pareto principle.

Gilboa *et al.* (2004) propose applying the ex-ante Pareto principle only to bets over events about which individuals beliefs agree.

Let

$$\Sigma^* = \{E \in \Sigma : p_i(E) = p_j(E),\ \forall i,j \in I\}$$

and let $\mathcal{F}^*$ bet the set of acts which are measurable with respect to Σ^*.

Then impose the following condition: For all $f,\ g \in \mathcal{F}^*$,

$$f \sim_i g \quad \forall \in I \quad \Longrightarrow f \sim_0 g.$$

A natural question is whether such class of events exists. The answer is yes, when the state space is rich like in the Savage theory. The Lyapunov theorem (see Rao and Rao, 1983) states that when finitely additive measures p_i, $i \in I$ are convex-ranged over Σ, the range of vector measure

$$p(\Sigma) = \{(p_1(E), \cdots, p_I(E)) \in [0,1]^I : E \in \Sigma\}$$

is convex. The rest follows from $(0, \ldots, 0),(1, \ldots, 1) \in p(\Sigma)$. However, we should again note that richness of the state space is really necessary, and this argument does not work naturally, for example, when the state space is finite.

A drastic but straightforward weakening of the Pareto principle is to apply it only to ex-post outcomes, but that is obviously too weak.

Is there any non-obvious weakening? Nehring (2004) and Chambers and Hayashi (2014) propose that in an incomplete information setting, the Pareto principle should be met when common knowledge of Pareto improvement holds, when common knowledge is formulated via suitably defined epstemic states. They show, however, that this weaker version of Pareto principle still implies the additivity of aggregation across individuals' subjective expected utilities. Again, such aggregation is possible only under common prior. In this sense, they showed that the impossibility is stronger than we initially thought.

Gilboa *et al.* (2014) propose a weakening of the ex-ante Pareto condition, saying that if f ex-ante Pareto dominates g *and* there is *some* common probaility measure such that everybody ranks f over g based on an expected utility calculation using such common probability measures.

Gayer *et al.* (2014) propose that f should be socially better than g if f generates higher expected utility than g for everyone *based on everybody's belief*. That is, when calculating individual i's expected utility, society does not use only i's belief but uses everyone's, and test the Pareto comparison. In other words, each individual is not solely responsible for his/her belief calculating own expected utility, and the whole society is responsible for calculating its members' expected utilities. Note that the condition rules out imposing some belief beyond their opinions; otherwise such imposition of particular belief will be "undemocratic". In this review let us call the condition *Consensus Ex-ante Pareto*.

Formally, given *a profile of subjective expected utility representations* $(u_i, p_i)_{i\in I}$, Consensus Ex-ante Pareto says that

$$E_{p_j}[u_i \circ f] \geq E_{p_j}[u_i \circ g] \quad \forall i,j \in I \quad \Longrightarrow \quad f \succsim_0 g.$$

This condition is read as if we treat individuals' vNM indices and beliefs as the observables. Note that this argument still does not rely on interpersonal comparability of utilities. Individuals' vNM indices and beliefs are uniquely identified from their preference rankings over acts, while of course it relies on the "cheat" mentioned above).

Consensus Ex-ante Pareto allows us to establish the condition that the social belief is a convex combination of the individuals' ones. That is, the representation takes the form

$$U(f) = \sum_{i\in I} \sum_{s\in S} u_0(f(s))p_0(s),$$

where

$$p_0 = \sum_{i\in I} \lambda_i p_i$$

for some $\lambda \in \{a \in \mathbb{R}^I_+ : \sum_{i \in I} a_i = 1\}$, and

$$u_0 = \sum_{i \in I} \alpha_i u_i + \beta$$

for some $\alpha \in \mathbb{R}^I_+$ and $\beta \in \mathbb{R}$. See Alon and Gayer (2016), which establishes this type result in a more gereral class of multiple priors. See Section 4 as well.

Another interesting proposal of compromise is made in a recent paper by Ceron and Vergopoulos (2017). They propose a weaker axiom which *nests* both ex-ante Pareto and monotonicity (state independence) of the social ranking. Formally, it says that

$$f \succsim_i g, \quad \forall i \in I, \quad f(s) \succsim_i g(s) \quad \forall s \in S \quad \Longrightarrow \quad f \succsim_0 g.$$

It characterises

$$U_0(f) = \sum_{i \in I} \lambda_i \sum_{s \in S} u_i(f(s)) p_i(s) + \sum_{s \in S} u_0(f(s)) p_0(s), \quad u_0 = \sum_{i \in I} \lambda_i u_i,$$

where p_0 may be seen as a consensus belief which can be *partially* influencing the social decision.

3.8 Allowing incompleteness

Another way of looking at the impossibility result will be consdering incomplete preferences. Assume a finite state space for simplicity here. Bewley (2002) proposed and gave a characterisation of incomplete preferenece represented in the form

$$f \succsim g \quad \Longleftrightarrow \quad \sum_{s \in S} u(f(s)) p(s) \geq \sum_{s \in S} u(g(s)) p(s) \quad \textit{forall} \quad p \in P,$$

where P is a set of probability distributions over S. This is in general incomplete ranking because different probability distributions in P may give different expected values.

Assume that individual prefernces $\{\succsim_i\}$ and the social ranking $\succsim_0$ satisfy the Bewley theory. Let (u_i, P_i) form a representation of $\succsim_i$ for each $i \in I$, respectively, and (u_0, P_0) form a fixed representation of $\succsim_0$.

Danan *et al.* (2016) show that $\{\succsim_i\}$ and $\succsim_0$ satisfy the ex-ante Pareto condition in the form

$$f \succsim_i g \quad \forall i \in I \quad \Longrightarrow \quad f \succsim_0 g$$

if and only if it holds that

$$u_0 = \sum_{i \in I} \lambda_i u_i, \quad P_0 \subset \bigcap_{i \in I} P_i$$

with $\lambda \in \mathbb{R}^I_+$.

Note that this result is empty when $\bigcap_{i \in I} P_i = \emptyset$, particularly when $P_i = \{p_i\}$ for each $i \in I$ and $p_i \neq p_j$ for some $i, j \in I$, which confirms the above-explained impossibility result.

4. Uncertainty and inequality, ex-ante and ex-post

Again, there are two natural starting points we can think of in social decision-making under uncertainty, the *ex-ante approach* and the *ex-post approach*. The ex-ante approach first calculates each individual's ex-ante welfare, which may or may not be his/her own calculation of expected utility, based on certain decision-theoretic criteria, and then aggregate across individuals into the social objective, based on certain social welfare criteria. The ex-post approach first calculates ex-post social welfare at each state, based on certain social welfare criteria, and then aggregates them across states into the social objective, based on certain decision-theoretic criteria.

We saw above that commutativity between ex-ante and ex-post approaches implies additive separability across states/individuals.

Not only being incompatible with the ex-ante Pareto principle under disagreements in beliefs, additive aggregation across states/individuals leads to a problem about fairness.

Consider that there are two individuals, two states equally likely, then the ranking

$$\begin{array}{c|cc} & s_1 & s_2 \\ \hline A & 1 & 0 \\ B & 0 & 0 \end{array} \quad \sim \quad \begin{array}{c|cc} & s_1 & s_2 \\ \hline A & 0 & 0 \\ B & 0 & 1 \end{array}$$

will be natural.

Then additivity across states/individuals implies

$$\begin{array}{c|cc} & s_1 & s_2 \\ \hline A & 1 & 1 \\ B & 0 & 0 \end{array} \quad \sim \quad \begin{array}{c|cc} & s_1 & s_2 \\ \hline A & 0 & 1 \\ B & 0 & 1. \end{array}$$

The left one is obviously unfair both ex-ante and ex-post, but at least one individual "survives" for sure. The right one is fair at every state and ex-ante as well; also there everybody gains and loses together, but we have to accept the possibility that at s_1 the entire sotiety runs into a "disaster".

It should be a non-obvious question which standpoint we should value, but the above argument leaves no flexibility for the society on how to prioritise between them.

Ben-Porath *et al.* (1997) claim that therefore uncertainty and inequality should be treated *together*. However, requiring that the two approaches yield the same ranking leads to additivity, which means that we have to make a choice between ex-ante and ex-post approaches. A number of studies attept to reconcile between the two approaches, such as Fleurbaey (2010), Fleurbaey *et al.* (2015) and Fleurbaey and Zuber (2017).

Below we follow Hayashi and Lombardi (2018), who provide two classes of social rankings that accommodate such concerns as flexibly as possible: one is based on the ex-ante approach and the other is based on the ex-post approach.

Let $\succsim$ be the social ranking defined over state-contingent utility profiles, $\mathbb{R}^{I\times S}$, where $u = (u_{is})_{i\in I, s\in S}$ denotes a generic profile.

The key axiom is that the ranking is invariant to translations of state-contingent utility profiles which shift a utility profile equally at all states and for all individuals. Formally, it is stated as

$$u \succsim v \quad \Longrightarrow \quad u + c\mathbf{1}_{I\times S} \succsim v + c\mathbf{1}_{I\times S}$$

for all $u, v \in \mathbb{R}^{I\times S}$ and $c \in \mathbb{R}$, where $\mathbf{1}_{I\times S}$ is the $|I\times S|$-vector of ones. It is a natural extension of the C-independence condition as considered by Gilboa and Schmeidler (1989) to the product set of individuals and states.

Together with other natural axioms such as order, continuity and convexity across states/individuals, we obtain the following general class:

$$W(u) = \min_{\lambda\in\Lambda} \sum_{i\in I} \sum_{s\in S} \lambda_{is} u_{is},$$

with $\Lambda \subset \Delta(I \times S)$ being the set of joint weights over states/individuals; where $\Delta(I \times S)$ denotes the set of non-negative vectors with their entries adding up to one.

As an extreme/limit case, it includes the case that the society should save the unhappiest individual at the worst possible state; namely, the form

$$W(u) = \min_i \min_s u_{is}.$$

When we impose both the ex-ante approach and the ex-post approach we go back to the additive aggregation across states/individuals, hence we have to make a choice between the two.

If we take the ex-ante approach, that is, assuming weak separability across I (existence of $\succsim_i$ for each $i \in I$), then the general class reduces to

$$W(u) = \min_{a\in A} \sum_{i\in I} a_i \left\{ \min_{p_i\in P_i} \sum_{s\in S} u_{is} p_{is} \right\}$$

with $A \subset \Delta(I)$ and $P_i \subset \Delta(S)$ for each $i \in I$. That is, first each individual's ex-ante welfare is calculated based on the multiple-priors model à la Gilboa and Schmeidler (1989), where the sets of priors may or may not be related to his/her own subjective belief; then they are aggregated into social one objective in a generalised egalitarian manner.

If we take the ex-post approach, that is, assuming weak separability across S (existence of $\succsim_s$ for each $s \in S$) then the general class reduces to

$$W(u) = \min_{p \in P} \sum_{s \in S} \left\{ \min_{a_s \in A_s} \sum_{i \in I} a_{is} u_{is} \right\} p_s,$$

with $P \subset \Delta(S)$ and $A_s \subset \Delta(S)$ for each $s \in S$. That is, first for each state, ex-post social welfare is calculated in a generalised egalitarian manner, where the set of welfare weights can be state-dependent, and they are aggregated across states based on the multiple-priors model due to Gilboa and Schmeidler (1989).

The above two classes allow comparative statics so that the concepts of inequality aversion and uncertainty are well-defined, where the social attiutde toward uncertainty is defined over the domain of "egalitarian acts".

Say that $\succsim$ is more ex-ante inequality averse than $\succsim'$ if

$$u \succsim (w_s \mathbf{1}_I)_{s \in S} \quad \Longrightarrow \quad u \succsim' (w_s \mathbf{1}_I)_{s \in S},$$

where $\mathbf{1}_I$ denotes the $|I|$-vector of ones. It is shown that $\succsim$ being more ex-ante inequality averse than $\succsim'$ is equivalent to $A \supset A'$ in the ex-ante approach.

Say that $\succsim_s$ is more ex-post inequality averse than $\succsim_s'$ if

$$u_s \succsim_s w_s \mathbf{1}_I \quad \Longrightarrow \quad u_s \succsim_s' w_s \mathbf{1}_I.$$

It is shown that $\succsim_s$ being more ex-post inequality averse than $\succsim_s'$ is equivalent to $A_s \supset A_s'$ in the ex-post approach.

Say that $\succsim$ is more socially uncertainty averse than $\succsim'$ if

$$u \succsim (w_i \mathbf{1}_S)_{i \in I} \quad \Longrightarrow \quad u \succsim' (w_i \mathbf{1}_S)_{i \in I},$$

where $\mathbf{1}_S$ denotes the $|S|$-vector of ones. Note that comparison of uncertainty attitudes at the social level makes sense over egalitarian acts, as discussed above. Then it is shown that $\succsim$ being more socially uncertainty averse than $\succsim'$ is equivalent to $P \supset P'$ in the ex-post approach.

Say that $\succsim_i$ is more individually uncertainty averse than $\succsim_i'$ for i if

$$u_i \succsim_i w_i \mathbf{1}_S \quad \Longrightarrow \quad u_i \succsim_i' w_i \mathbf{1}_S.$$

It is shown that $\succsim_i$ being more individually uncertainty averse than $\succsim_i'$ for i is equivalent to $P_i \supset P_i'$ in the ex-ante approach.

Note that Ex-ante Pareto implies separability across individuals, because $(u_i, u_{-i}) \succsim (u_i', u_{-i})$ holds if and only if $u_i \succsim_i u_i'$ accodring to i's own preference $\succsim_i$, which holds if and only if $(u_i, u_{-i}') \succsim (u_i', u_{-i}')$ for any other u_{-i}'. Therefore, when it is imposed on the ex-post approach the additive aggregation, in which each individual's ex-ante welfare is just his or her own subjective expected utility, is the only option, and the aggregation form is limited to

$$W(u) = \sum_{i \in I} a_i \sum_{s \in S} u_{is} p_{is},$$

where p_i is i's own subjective belief.

Consensus Ex-ante Pareto allows that in the ex-post approach the social set of priors is contained in the convex hull spanned by all people's beliefs. More formally, given a profile of individuals' beliefs $\{p_i\}_{i \in I}$, the social set of priors P must satisfy

$$P \subset conv\{p_i : i \in I\},$$

which is also obtained by Alon and Gayer (2016), while they do not allow ex-post inequality aversion. In addition, in the ax-ante approach each individual's set of priors is contained in the convex hull spanned by all the people's beliefs. Given a profile of individuals' beliefs $\{p_i\}_{i \in I}$, individual j's set of priors P_j, which is assigned by the society, must satisfy

$$P_j \subset conv\{p_i : i \in I\}.$$

5. Partial equilibrium welfare measure under uncertainty

The arguments up to the previous sections implicitly assume that the social decision covers all the relevant issues and can intervene allocation of resources from a blank sheet of paper. The policy-maker cannot overhaul the entire economy, however, and can intervene only through a limited channel. In such partial equilibrium settings, the policy maker is typically interested in maximising expected (aggregate) consumer surplus.

Using expected consumer surplus has an uneasy implication, however. To illustrate, consider linear inverse demand $p = a - bx$. Let p be random, then expected consumer surplus is given by

$$E\left[\frac{(a-p)^2}{2b}\right].$$

Notice that $\frac{(a-p)^2}{2b}$ is a *convex* function, which implies that the price being riskier is better. The same argument holds for local linearisation with small risk. See Waugh (1944) for classic discussions on this problem.

To understand the problem more precisely, go back to the definition of expected consumer surplus:

$$U(p) = \int_{X \times \mathbb{R}} \{v(x) + t\} dp(x, t).$$

Notice that this functional form presumes no income effect, when applied to deterministic allocations, *and* risk neutrality, as the vNM index $v(x) + t$ is linear in income and also in consumer surplus.

To accommodate with risk-aversion one may take a seemingly more general form:

$$U(p) = \int_{X \times \mathbb{R}} \phi(v(x) + t) dp(x, t),$$

which is used, for example, in analysing auction problems. However, risk attitude itself should be a structural parameter at a general equilibrium level. How much one is willing to take risk at a partial equilibrium level should be an endogenous feature.

Schlee (2003) finds that Willig's method (Willig, 1976) of approximating equivalent/compensating variation by change in consumer surplus may result in large errors.

Rogerson (1980) considers preference over probability distribution of price–income pairs in ex-post spot markets, represented in the expected utility form $U(F) = \int V(p, m)\, dF(p, m)$, in which the von-Neumann/Morgenstern index V defined over price vector p and income m is supposed to play the role of indirect utility function in the ex-post spot markets as well as to describe the consumer's risk attitude toward price–income uncertainty. Rogerson shows that expected consumer surplus from a good represents the consumer's preference over distributions of its price given the same income if and *only if* V is additively separable between the price and income. This suggests that the form of risk attitude to be allowed in the partial equilibrium setting will be a stringent one when it is required to be consistent with expected utility maximisation in general equilibrium environments. This motivates us to characterise the partial equilibrium welfare measurem which is consistent with the underlying general equilibrium model and expected utility maximisation.

Rogerson assumed that there is no asset market and income arriving at each state cannot be controlled by the consumer. Hayashi (2014) considers a more general model in which consumers can transfer wealth across states at least in a partial manner, through incomplete asset markets.

Following Radner (1968), consider a two-period model, in which at Period 0 there is no consumption or earning, and at Period 1 consumers receive state-contingent earnings. Let S be the set of states of the world. Let H denote the set of assets. Let R be an $S \times H$ return matrix. Let $q \in \mathbb{R}^H$ denote a price vector taken as given by the consumer. Let $\zeta \in \mathbb{R}^H$ denote a portfolio choice vector, which is supposed to satisfy the budget equation $\sum_{h \in H} q_h \zeta_h = 0$, and at each state s pays return $\sum_{h \in H} R_{sh} \zeta_h$.

At each state, there is a continuum [0, 1] of goods in spot markets, which is given suitable mathematical properties. Let μ denote the Lebesgue measure on it. The set of commodity characteristics is partitioned arbitrarily finely, where the order of partition is denoted by n and the corresponding partition is denoted by $\mathcal{J}^n$. Given n, the argument reduces to a model of finitely many goods where the number of goods is $|\mathcal{J}^n|$.

Pick one element $J \in \mathcal{J}^n$, which is taken to be the object of partial equilibrium analysis in the approximate sense. Under uncertainty, let $x \in \mathbb{R}^S_+$ denote a vector of state-contingent delivery of such commodity piece J. Let $a \in \mathbb{R}^S$ denote a vector of state-contingent delivery of income transfer, which may be either positive or negative.

Let $(x, a) \in (\mathbb{R}_+ \times \mathbb{R})^S$ denote a vector of state-contingent delivery of the commodity piece J and associated income transfer. Preference over such random pairs is defined according to Hicksian aggeration with incomplete asset markets. Let $z_{-J} \in \mathbb{R}^{(\mathcal{J}^n \setminus \{J\}) \times S}_{++}$ denote a vector of state-contingent delivery of the other commodity pieces in the partion $\mathcal{J}^n$.

The price system for ex-post spot markets is a vector-valued function $p : [0, 1] \to \mathbb{R}^S_{++}$, which is given suitable integrability property. Then the price of commodity piece $K \in \mathcal{J}^n$ to be delivered at State $s \in S$ is defined by $p_{sK} = \int_K p_s(t) dt$.

Here is the definition of preference induced over random deliveries of pairs of the commodity piece J and associated income transfer:

Definition 1: *Given* n, $J \in \mathcal{J}^n$ *and* $(x, a), (y, b) \in \mathbb{R}^S_{++} \times \left(-\frac{w}{\mu(J)}, \infty\right)^S$, *the relation*

$$(x, a) \succsim^{n,J} (y, b)$$

holds if

$$V^{n,J}(x, a) \geq V^{n,J}(y, b),$$

where

$$V^{n,J}(x, a) = \max_{\zeta \in \mathbb{R}^H, z_{-J} \in \mathbb{R}^{(\mathcal{J}^n \setminus \{J\}) \times S}_{++}} \sum_{s \in S} u^n\left(x_s, z_{s,-J}\right)\pi_s$$

subject to

$$\sum_{h \in H} q_h \zeta_h = 0 \quad (*)$$

$$\sum_{K \in \mathcal{J}^n \setminus \{J\}} p_{sK} z_{sK} = w_s + \sum_{h \in H} R_{sh}\zeta_h + a_s \mu(J) \quad \text{for each } s \in S \quad (**).$$

Note that associated income transfer to be spent on the other commodity pieces is adjusted to the mass of commodity piece J; that is, $\mu(J)$. This is because when J tends to be a small commodity piece the magnitude of associated income transfer tends to be small as well.

Hayashi (2014) shows that as n goes to infinity, that is, as the partition tends to be arbitrarily fine, so that any given sequence of intervals converges to a single point of "negligible commodity" (after taking a subsequence if necessary), say $\tau \in [0, 1]$, and the vector of consumption of "the other" commodity pieces converges to some functiom f_s for each state $s \in S$, which is the optimal consumption bundle in the limit, the above form converges to

$$U^\tau(x, a) = \sum_{s \in S} \left\{ \frac{\pi_s}{\lambda_0} \int_0^{x_s} \Delta u(z_s, \tau, f_s) dz_s + \frac{\lambda_s}{\lambda_0} a_s \right\},$$

where $\Delta u(z, \tau, f)$ is the so-called Voltera derivative defined by

$$\Delta u(z, \tau, f) = \lim_{J \to \{\tau\}} \frac{\frac{\partial}{\partial z} u(z\mathbf{1}_J, f\mathbf{1}_{[0,1] \setminus J})}{\mu(J)}$$

and λ_0 is the Lagrangean multiplier on budget constraint (*) and λ_s is the Lagrangean multiplier on budget constraint (**) for each $s \in S$. Because the value of income at state s is adjusted by $\frac{\lambda_s}{\lambda_0}$, let us call this the *expected adjusted consumer surplus*.

When $u(c) = \int_0^1 v_t(c_t)dt$, the expected adjusted consumer surplus reduces to

$$U^{\tau}(x,a) = \sum_{s\in S}\left\{\frac{\pi_s}{\lambda_0}v_{\tau}(x_s) + \frac{\lambda_s}{\lambda_0}a_s\right\}.$$

We should note the following points from this result. First, no income effect and risk neutrality are *tied together*. As far as we try to justify the assumption of no income effect because of smallness of the commodity piece (see Vives, 1987; Hayashi, 2013), the same reason implies that the consumer must be approximately risk-neutral. This intuition is quite simple. When the commmodity piece tends to be arbitrarily small, the risk about income transfer associated to it tends to be arbitrarily small. From Arrow (1971, 1973), we know that for small risks an expected utility maximiser behaves in an approximately risk-neutral manner.

Second when the asset markets are complete $\frac{\lambda_s}{\lambda_0}$ is equalised across individuals, regardless of disagreements in beliefs, because belief disagreements are taken into account in the determination of the Lagrange multipliers. However, this is not the case when markets are incomplete. This means that the standard expected consumer surplus as used in practice cannot come from expected utility maximisation at the general equilibrium level when markets are incomplete, and a solution for maximising the sum of expected adjusted consumer surplus is in general ex-ante Pareto-inefficient.

6. Future directions

I conclude by suggesting future directions for the research.

First, we have here assumed throughout that individuals' preferences are "rational" in the standard sense as assumed in exptected/subjective expected utility theory. It is natural to wonder what we should do when individuals' preferences violate the standard theory, in particular, violating either static or dynamic consistency. The problem arising then is that it is not clear what is better or worse even for a single individual. See Bernheim and Rangel (2009), Chambers and Hayashi (2012) and Rubinstein and Salant (2011) for proposals about what should be the criterion for individual welfare when individuals' choices are inconsistent.

Second, as we have assumed that individuals' preferences satisfy exptected/subjective expected utility theory; this means we have assumed that they are consequentialist or, in other words, path-independent. Hayashi (2016) argues that even when individuals' preferences are path-indepenent the social one must be path-dependent, and obey a non-consequentialist updating, in order to be dynamically consistent and fair over time. When individuals' preferences are, indeed, path-dependent while being dynamically consistent, a further layer of path dependence of the social ranking should be naturally expected.

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