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EFFICIENCY AND LACK OF COMMITMENT IN AN OVERLAPPING GENERATIONS MODEL WITH ENDOWMENT SHOCKS*

By KOICHI MIYAZAKI
National Taiwan University

This paper considers a pure exchange stochastic overlapping generations model in which, on each date, an economy faces an aggregate endowment shock. On each date, a young agent and an old agent simultaneously decide how much of their respective endowments to transfer to the other agent; however, a young agent cannot make promises about how much he or she will give when old. In this sense, an economy faces a limited commitment constraint. This paper characterizes an efficient intergenerational risk sharing allocation that satisfies a limited commitment constraint, and also studies the role of money and history in a stochastic overlapping generations economy.
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1. Introduction

The overlapping generations model, which was formulated by Samuelson (1958) and Diamond (1965), is one of the basic models in macroeconomics and public finance. Some more recent literature concerned with insurance is based on models of contemporaneous, infinitely-lived agents, constrained by the inability to make contractual commitments (see e.g. Thomas and Worrall, 1988; Kocherlakota, 1996). The present paper considers insurance under a limited commitment constraint in a stochastic overlapping generations economy. The situation in this environment is substantially different from that in contemporaneous, infinitely-lived agent economies. One of the present paper's goals is to characterize efficient allocations under a limited commitment constraint in the overlapping generations model.

The model in this paper considers a pure exchange overlapping generations economy that faces an endowment shock on every date. Each generation consists of one single agent. After the shock is realized, a new young agent is born. Both a young agent and an old agent decide how much of their respective endowments to transfer to the other agent, doing so both simultaneously and independently. After the transfer is made, each agent consumes and derives utility, and an old agent dies.

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Among several welfare concepts, this paper uses "interim" Pareto efficiency as a welfare concept.[1] Interim Pareto efficiency considers an agent's expected lifetime utility calculated on each agent's birthdate given the histories at that moment. Because a young agent is born after a current shock is realized, it is natural for the agent to be distinguished after different histories, even though he or she has the same name. Because of the limited commitment constraint, an equilibrium concept requires subgame perfection.[2] Hence, an equilibrium concept examined in this paper is a subgame perfect equilibrium.

In the analysis, to make the comparison of a contemporaneous, infinitely-lived agent model and an overlapping generations model clear, the paper imposes one assumption and focuses on some featured allocations. One important result of the contemporaneous, infinitely-lived agent model is that history-dependent behavior improves agents' welfare. That is, the autarkic allocation is not efficient. Thus, the paper imposes the assumption that the autarkic allocation is not interim Pareto efficient. Another important result of the contemporaneous, infinitely-lived agent model is that the first-best allocation is subgame perfect when the discount factor is large enough. In an overlapping generations environment, the corresponding first-best allocation is not well-defined. Hence, among all interim Pareto efficient allocations, this paper focuses on *golden-rule type* allocations. An allocation is a golden-rule type allocation if the allocation maximizes the weighted sum of the young agent's conditional expected lifetime utility. Notice that a young agent who has different histories is distinguished. Although this golden-rule type allocation does not exactly correspond to the first-best allocation in the contemporaneous, infinitely-lived agent model, the study of a golden-rule type allocation can provide some insight into the comparison of the two models. Because the golden-rule allocation is defined by a stationary allocation, the present paper also focuses on a stationary allocation.[3]

One of the contributions of this paper is to characterize the golden-rule type subgame perfect equilibrium allocations. The first result is that all subgame perfect equilibrium allocations are supported by an autarky-reversion trigger strategy profile. Notice that the autarkic allocation is always supported by a subgame perfect equilibrium, and gives the worst expected utility to a young agent among all subgame perfect equilibrium allocations. In a subgame perfect equilibrium, an old agent never transfers something to a young agent, because an old agent will depart from the economy after the current period. Thanks to this result, instead of considering the strategy profile, the present paper just focuses on the allocation that satisfies the limited commitment constraint. The second result is to provide an almost necessary and sufficient condition for the existence of a non-autarkic, subgame perfect equilibrium allocation. The condition is that the autarkic allocation is not interim Pareto efficient. This is intuitive if we consider a deterministic, simple overlapping generations model. In a deterministic environment where a young agent is endowed with a

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[1] For instance, another possible welfare concept is "*ex-ante*" Pareto efficiency. *Ex-ante* Pareto efficiency considers an agent's expected lifetime utility calculated on the initial date of the economy, that is, before a shock is realized. Under *ex-ante* Pareto efficiency, "the" agent who lives on a particular date is regarded as one individual.

[2] There are several papers that investigate subgame perfect efficient allocations in a deterministic, two-period-lived-agents overlapping generations model. See, for example, Hammond (1975) and Bhaskar (1998). In a stochastic environment, although the setting is different from that in this paper, Messner and Polborn (2003) extend Cremer (1986) by adding a stochastic cooperation cost.

[3] One difficulty in an overlapping generations model is that it is not easy to study non-stationary allocations. Because of this difficulty, quite a few papers focus on a stationary allocation. For instance, see Demange and Laroque (1999) and Gottardi and Kubler (2011).

larger endowment than an old agent, a transfer from a young agent to an old agent is supported by a subgame perfect equilibrium, because the consumption is smoothed after the transfer. The third result is that when the autarkic allocation is not interim Pareto efficient, there always exist golden-rule type allocations that satisfy the limited commitment constraint if agents condition their behavior on past history. Notice that if agents cannot condition their behavior on past history, only an autarkic allocation is a subgame perfect equilibrium allocation, because there is no punishment/reward for a young agent to make the transfer to an old agent. The implication of this result is that history-dependent behavior improves welfare.[4] The fourth result is that if an allocation satisfies the limited commitment constraint, then there should be a transfer from a young agent to an old agent in the shock where a young agent is relatively rich compared to an old agent.[5] This implies that the transfer that makes the consumption smoothed over time is necessary to satisfy the limited commitment constraint.

In addition to the characterization of the golden-rule type, subgame perfect equilibrium allocations, the present paper compares the role of money and history in this economic environment. The overlapping generations model was used to rationalize the existence of money (a durable and intrinsically useless good) in the economy. Wallace (1980) is the first paper to investigate this problem, and considers, for example, when money has a value in the economy and how it helps two different generations trade in a pure-exchange deterministic overlapping generations model. The second goal of this paper is to reconsider this role of money and compare it with the role of history in a pure-exchange stochastic overlapping generations model. As a benchmark, first, I consider a deterministic environment. Notice that in both deterministic and stochastic environments, if there is no money, the autarkic allocation is the only equilibrium. When the autarkic allocation is not Pareto efficient, Wallace (1980) shows the existence of a unique monetary equilibrium and the allocation is the golden-rule allocation. If the agent can exploit the history, then there are infinitely many subgame perfect equilibrium allocations, some of which are Pareto efficient allocations and one of which is the golden-rule allocation. Thus, it can be said that money and history are similar in terms of the implementation of the golden-rule allocation, while history implements more Pareto efficient allocations than money. In a stochastic environment, as Magill and Quinzii (2003) show, under the assumption that the autarkic allocation is not interim Pareto efficient, there is a unique monetary equilibrium, and the present paper shows that its allocation is the golden-rule type allocation. The history also implements this unique monetary equilibrium, but as we saw above, the history implements more golden-rule type allocations than the money, although one might think that this difference is not as crucial as in a deterministic environment. Hence, this paper considers a two-shock example economy in which there is a critical difference between history and money. The example shows that a stationary allocation that maximizes a young agent's *ex-ante* expected utility is a subgame perfect equilibrium allocation, but not a monetary equilibrium allocation.[6] This is a crucial difference, because in terms of *ex-ante* welfare, history dominates money. In a deterministic environment, the condition for a stationary allocation to be a monetary equilibrium allocation and the condition for a stationary allocation to be a golden-rule

[4] This implication is also derived in the contemporaneous, infinitely-lived agent model.

[5] The assumption that the autarkic allocation is not interim Pareto efficient implies that there is at least one shock in which a young agent's endowment is larger than an old agent's endowment.

[6] In this allocation, an agent consumes the same amount of consumption at any age in any state.

allocation are equivalent, while in a stochastic environment they are not equivalent. This implies the possibility that even if a stationary allocation is not a monetary equilibrium allocation, it is a golden-rule type allocation that is a subgame perfect equilibrium allocation. The paper shows one example in which this possibility creates a critical difference between history and money.

The remainder of this paper is organized as follows. Section 2 sets up the model and provides several definitions. Section 3 characterizes the golden-rule type allocation that satisfies the limited commitment constraint. Section 4 discusses the role of money and history in a stochastic pure-exchange overlapping generations model, and also presents a numerical example. Section 5 concludes.

2. The model

In this section, I set up the model and provide definitions of efficiency and the equilibrium.

2.1 Environment

Time is discrete and infinite, $t = 1, 2, \ldots$. On each date t, a single agent is born. An agent born on date t is called a *generation-t* agent. An agent born on date $t = 0$ is referred to as *initial old*. Agents live for two periods, *young* and *old*.

On each date t, the economy faces an endowment shock. Let $S := \{1, 2, \ldots, \mathbf{S}\}$ denote a set of shocks with generic element s, where $\mathbf{S} \geq 2$ is finite. The probability that a shock is $s \in S$ is denoted by $\pi(s)$, where $\pi : S \to [0, 1]$ is a probability function. Endowments for a young agent and an old agent are determined by a shock. $e^y(s)$ denotes a young agent's endowment when a shock is $s \in S$ and $e^o(s)$ is an old agent's endowment given the current shock $s \in S$. Let $e(s) := e^y(s) + e^o(s)$ be the total endowment for shock $s \in S$. Let $s^t := (s_1, s_2, \ldots, s_t) \in S^t$ be a history of shocks up to date t. I assume that the stochastic process, $\{s_t\}$, is i.i.d. The endowment is perishable and there is no storage technology.

An *allocation for generation-t*(≥1) is denoted by a pair of mappings, $c_t := (c_t^y, c_{t+1}^o)$, where $c_t^y : S^t \to \mathrm{R}_+$ and $c_{t+1}^o : S^{t+1} \to \mathrm{R}_+$. An *allocation for the initial old* is denoted by a mapping, $c_1^o : S \to \mathrm{R}_+$. Let $c := (c_1^o, (c_t)_{t=1}^\infty)$ be an *allocation*. An allocation, c, is *feasible* if for all $t \geq 1$,

$$c_t^y(s^t) + c_t^o(s^t) = e(s_t)$$

for all $s^t \in S^t$.

Agents derive utility from consuming in each period of life. Let $u: \mathrm{R}_+ \to \mathrm{R}$ be an agent's periodic utility function, which is strictly concave, strictly increasing and twice-continuously differentiable. I assume that the agent's lifetime utility is additively separable with no discounting the future.[7] For later convenience, I define the agent's expected lifetime utility as being conditional on a history of shocks, $s^t \in S^t$. For any date $t \geq 1$, if the generation-t agent's allocation is c_t, his or her expected lifetime utility conditional on $s^t \in S^t$ is denoted by

[7] This assumption is harmless for all of the following results.

$$U(c_t, s^t) := u(c_t^y(s^t)) + \sum_{\hat{s}\in S} \pi(\hat{s}) u(c_{t+1}^o(s^t, \hat{s})). \tag{1}$$

2.2 Welfare concept

In this paper, because a current shock is realized before a young agent is born, it is natural to use *interim Pareto efficiency* as a social welfare concept. To put it differently, the agent is regarded as a different individual if the histories before she is born are different under interim Pareto efficiency. The formal definition is as follows:

Definition 1: *An allocation, c, is* ***interim Pareto efficient*** *if c is feasible and there does not exist another feasible allocation,* $\tilde{c}$*, such that*

$$U(\tilde{c}_t, s^t) \geq U(c_t, s^t) \tag{2}$$

for all $t \geq 1$ *and all* $s^t \in S^t$*, and*

$$u(\tilde{c}_1^o(s^1)) \geq u(c_1^o(s^1)) \tag{3}$$

for all $s^1 \in S$*, and either Equation (2) holds with strict inequality for some* $t \geq 1$ *and some* $s^t \in S^t$ *or Equation (3) holds with strict inequality for some* $s^1 \in S$*.*

2.3 Specification of transfer game without commitment

In this economy, both a young agent and an old agent can voluntarily transfer part of their own endowments to the other agent who is currently alive. I assume that no young agent can commit his or her amount of transfer in old age and, hence, each agent decides how much he or she will transfer at each age. Moreover, there is no externally enforced commitment device. An agent's strategy determines how much of his or her endowment to transfer to the other agent conditional on the histories of transfers and shocks up to this time. Let $H^t := ([0, \bar{e}]^2)^t$, where $\bar{e} := \max_{s\in S} e(s)$, $H^0 = \emptyset$ and $S^0 := \emptyset$. Let

$$\sigma_t^y : H^{t-1} \times S^t \to [0, \bar{e}]$$

be a mapping from a history of transfers before date t, $h^{t-1} \in H^{t-1}$, and a history of shocks up to and including date t's shock, $s^t \in S^t$, to a real number that satisfies $\sigma_t^y(h^{t-1}, s^t) \in [0, e^y(s_t)]$. Let

$$\sigma_{t+1}^o : H^t \times S^{t+1} \to [0, \bar{e}]$$

be a mapping from a history of transfers before date $t+1$, $h^t \in H^t$, and a history of shocks up to and including date $t+1$'s shock, $s^{t+1} \in S^{t+1}$, to a real number that satisfies $\sigma_{t+1}^o(h^t, s^{t+1}) \in [0, e^o(s_{t+1})]$. Then, a strategy of the generation-t (≥ 1) agent is defined by

$$\sigma_t := (\sigma_t^y, \sigma_{t+1}^o).$$

A strategy of the initial old agent is

$$\sigma_0 := \sigma_1^o : S \rightarrow [0, \bar{e}],$$

which satisfies $\sigma_1^o(s) \in [0, e^o(s)]$ for all $s \in$ S. Let Σ_t^y and Σ_t^o be a set of all mappings, σ_t^y and σ_t^o, respectively, as defined above. Let $\Sigma_t := \Sigma_t^y \times \Sigma_{t+1}^o$ be a set of all strategies of generation-t ($t \geq 0$). When a strategy profile is σ, the after-transfer allocations for the young and the old agents at date t, given $s^t \in S^t$ and $h^{t-1} \in H^{t-1}$, are

$$c_t^y(s^t) = e^y(s_t) - \sigma_t^y(h^{t-1}, s^t) + \sigma_t^o(h^{t-1}, s^t)$$

$$c_t^o(s^t) = e^o(s_t) - \sigma_t^o(h^{t-1}, s^t) + \sigma_t^y(h^{t-1}, s^t).$$

2.4 Equilibrum concept in a transfer game

I use a subgame perfect equilibrium of this commitment structure as an equilibrium concept. In a subgame perfect equilibrium, at any point in time and history, each agent optimally chooses the amount of transfers at that time.

Definition 2: *A strategy profile σ* is a **subgame perfect equilibrium (SPE)** if for all $t \geq 1$, all $s^t \in S^t$, and all $h^{t-1} \in H^{t-1}$,*

$$\sigma_t^* \in \arg\max_{\sigma_t \in \Sigma_t} \left\{ u\begin{pmatrix} e^y(s_t) - \sigma_t^y(h^{t-1}, s^t) \\ +\sigma_t^{o*}(h^{t-1}, s^t) \end{pmatrix} + \sum_{\hat{s} \in S} \pi(\hat{s}) u \begin{pmatrix} e^o(\hat{s}) - \sigma_{t+1}^o(h^t, s^t, \hat{s}) \\ +\sigma_{t+1}^{y*}(h^t, s^t, \hat{s}) \end{pmatrix} \right\}, \tag{4}$$

where $h^t = (h^{t-1}, (\sigma_t^y(h^{t-1}, s^t), \sigma_t^o(h^{t-1}, s^t)))$, *and for all $t \geq 1$, all $s^t \in S^t$, and all $h^{t-1} \in H^{t-1}$,*

$$\begin{aligned} & u(e^o(s_t) - \sigma_t^{o*}(h^{t-1}, s^t) + \sigma_t^{y*}(h^{t-1}, s^t)) \\ & \geq u(e^o(s_t) - \sigma_t^o(h^{t-1}, s^t) + \sigma_t^{y*}(h^{t-1}, s^t)), \end{aligned} \tag{5}$$

*for all $\sigma_t^o \in \Sigma_t^o$. An allocation, c, is **subgame perfect** or an **SPE allocation** if c is feasible and is induced by some SPE σ.*

In the definition, Equation (4) is the incentive condition for the young agent, and Equation (5) is the incentive condition for the old agent.

Now, I show that the autarky allocation is always supported by an SPE and it gives the lowest utility to the agents. This property is important and is used in later sections.

Lemma 1: *The autarky allocation is always an SPE allocation and provides lower consumption to every old agent and lower conditional expected lifetime utility (according to Equation (1)) to every young agent in every history than any other SPE allocation provides.*

Proof: See the Appendix.

Lemma 1 guarantees that the autarky-reversion trigger strategy is the best strategy when we consider an SPE, which concurs with the findings of Abreu (1988), Kocherlakota (1996) and Thomas and Worrall (1988).

By using Lemma 1, the following result regarding the SPE allocation holds:

Proposition 1: *An allocation c is an SPE allocation if and only if c is feasible and for all t and all* $s^t \in S^t$,

$$c_t^o(s^t) \geq e^o(s_t) \tag{6}$$

and

$$U(c_t, s^t) \geq u(e^y(s_t)) + \sum_{\hat{s} \in S} \pi(\hat{s}) u(e^o(\hat{s})). \tag{7}$$

Proof: See the Appendix.

Equation (6) implies that the old agent prefers an allocation $c_t^o(s^t)$ to their own endowment. This implies that any old agent transfers nothing to a young agent. This is because an old agent will exit the game right after the current period, and, hence, an old agent will not be punished even if he or she transfers nothing to a young agent. Thus, an old agent does not have an incentive to transfer something to a young agent. Equation (7) implies that a young agent at date t prefers consuming his or her allocation c_t to consuming his or her endowment. If one of Equations (6) and (7) is violated, an allocation c cannot be an SPE allocation. Hence, two equations are a necessary condition for c to bve an SPE allocation. For sufficiency, if both Equations (6) and (7) hold, we can construct the autarky-reversion trigger strategy whose outcome is c. By Lemma 1, the autarky allocation itself is an SPE allocation; hence, the autarky-reversion trigger strategy is an SPE.

3. Golden-rule type, subgame perfect equilibrium allocations

First, to make the analysis interesting, I impose one assumption on the model hereafter.

Assumption 1: *The autarkic allocation is not interim Pareto efficient.*

If the autarkic allocation is interim Pareto efficient, a transfer between generations is meaningless. Based on Aiyagari and Peled (1991) and Chattopadhyay and Gottardi (1999), this assumption is translated into the following equation:

$$\sum_{s \in S} \frac{\pi(s) u'(e^o(s))}{u'(e^y(s))} > 1. \tag{8}$$

One interest in a two-sided limited commitment model such as Thomas and Worrall (1988) and Kocherlakota (1996) concerns how to solve the conflict between risk sharing and incentives. To achieve efficient risk sharing, the long-run relationship is useful, while in each period each agent has an incentive to break the relationship. As shown in Thomas and Worrall (1988) and Kocherlakota (1996), when the discount factor is high enough, depending on the initial shock and the division of the surplus, an efficient risk-sharing allocation, or first-best risk-sharing allocation, is subgame perfect.

In an overlapping generations model, one difficulty with the analysis is that it is not easy to judge whether an allocation is interim Pareto efficient or not. Chattopadhyay and Gottardi (1999) provide a complete characterization of interim Pareto efficient allocations, although it is not convenient to use their condition in practice. Following studies such as Gottardi and Kubler (2011), I focus on a *stationary* allocation hereafter.

Definition 3: *An allocation c is* **stationary** *if c is feasible and there is a mapping* $\tilde{c} := (\tilde{c}^y, \tilde{c}^o)$, *where* $\tilde{c}^y(s) \in [0, e(s)]$ *and* $\tilde{c}^o(s, s') \in [0, e(s')]$, *such that for all t and all* $s^t \in S^t$, $c_t^y(s^t) = \tilde{c}^y(s_t)$ *and* $c_t^o(s^t) = \tilde{c}_t^o(s_{t-1}, s_t)$.

In words, if an allocation is stationary, the young agent's allocation just depends on the current shock and the old agent's allocation depends on shocks while he or she is alive. Specifically, in this model, $c_t^y(s^t) = \tilde{c}^y(s_t)$ implies that a stationary allocation for an old agent also depends only on today's shock because $c_t^o(s^t) = e(s_t) - c_t^y(s^t) = e(s_t) - c^y(s_t)$. Hence, hereafter, a stationary allocation c is written as $c = (c^y, c^o)$, where $c^y: S \to R_+$ and $c^o: S \to R_+$, with $c^y(s) + c^o(s) = e(s)$. An (almost) necessary and sufficient condition for the existence of a stationary, non-autarkic SPE allocation is presented as follows:

Proposition 2: *There exists a stationary, non-autarkic SPE allocation c if Equation (8) holds. A stationary, non-autarkic SPE allocation exists only if*

$$\sum_{s\in S} \frac{\pi(s)u'(e^o(s))}{u'(e^y(s))} \geq 1. \tag{9}$$

Proof: See the Appendix.

Assumption 1 is (almost) a necessary and sufficient condition for the existence of a stationary, non-autarkic SPE allocation. For sufficiency, the proof in the Appendix uses the result derived by Magill and Quinzii (2003). Here, I simply provide an intuition for the result. If Equation (8) holds, the autarkic allocation is not interim Pareto efficient, which implies that there is another feasible allocation that makes some agent strictly better off without hurting any other agents. Equation (8) guarantees that there is at least one $s \in S$ such that $e^y(s) > e^o(s)$.[8] Then, consider a positive transfer from a young agent whose endowment is larger than an old agent's endowment. It is clear that such a transfer makes an old agent strictly better off in terms of welfare as well as making a young agent better off because the consumption becomes smoothed in the current state if the future shock is the same as the current shock and the consumption when old in other states does not decrease.[9] This implies that the allocation after the transfer satisfies incentive conditions for both the young agent and the old agent, and the resulting allocation is an SPE allocation. For necessity, if Equation (9) does not hold, then the autarkic allocation is interim Pareto efficient. For some non-autarkic allocation to be an SPE allocation, a positive transfer from a young agent to an old agent should be made. Because, however, the autarkic allocation is now interim Pareto efficient, such a transfer makes at least one agent worse off. Because an old agent's welfare is improved, a young agent's welfare should decline. This implies that

[8] If there is no $s \in S$ such that $e^y(s) > e^o(s)$, then for all $s \in S$, $u'(e^y(s)) > u'(e^o(s))$. This implies that $u'(e^o(s))/u'(e^y(s)) < 1$ for all s. Then, $\Sigma_{s\in S}\pi(s)u'(e^o(s))/u'(e^y(s)) < \Sigma_{s\in S}\pi(s) = 1$.

[9] An allocation after this transfer is not always an SPE. In such a case, a transfer from a young agent whose endowment is smaller than an old agent's endowment also has to involve a transfer of something to an old agent. Then, an allocation after the transfer is an SPE allocation. For more details, see the Appendix.

the young agent's expected lifetime utility conditional on some $s \in S$ is lower than the expected lifetime utility from the autarkic allocation. This means that such an allocation is not subgame perfect.

By Chattopadhyay and Gottardi (1999), a stationary allocation c is interim Pareto efficient if and only if

$$\sum_{s \in S} \frac{\pi(s)u'(c^o(s))}{u'(c^y(s))} \leq 1. \tag{10}$$

Because our focus is on a feasible allocation, $c^y(s) + c^o(s) = e(s)$ for all $s \in S$, once we know either a young agent's consumption, $c^y(s)$, or an old agent's consumption, $c^o(s)$, we also know another agent's consumption. Therefore, hereafter, I focus on an old agent's state-contingent consumption, $(c^o(s))_{s \in S}$. Let

$$SPE := \{(c^o(s))_{s \in S} | (c^o(s))_{s \in S} \text{ is a stationary SPE allocation}\}$$

be a set of all stationary SPE allocations.

Also worth mentioning is that the set up in this paper does not have a notion corresponding to the first-best risk sharing commonly used in an infinitely-lived agent model (see e.g. Kocherlakota, 1996).[10] However, this paper considers the following *golden-rule type* allocation:

Definition 4: *A stationary allocation c is a* ***golden-rule type*** *allocation if c is feasible; i.e.* $c^y(s) + c^o(s) = e(s)$ *for all* $s \in S$, *and a stationary allocation c maximizes*

$$\sum_{s \in S} \lambda(s)\left\{u(c^y(s)) + \sum_{s' \in S} \pi(s')u(c^o(s'))\right\} \tag{11}$$

for some $\lambda(s) \in (0, 1)$ *and* $\sum_{s \in S}\lambda(s) = 1$.

In words, if a stationary allocation is a golden-rule type allocation, no young agents are made better off by some stationary allocation. Notice that "young agents" here means young agents who have a different history of shocks. Because u is strictly concave and the feasibility constraint is linear, the necessary and sufficient conditions for c being a solution to Equation (11) are

$$\frac{\pi(s)u'(c^o(s))}{u'(c^y(s))} = \lambda(s) \tag{12}$$

for all $s \in S$. This condition is equivalent to

$$\sum_{s \in S} \frac{\pi(s)u'(c^o(s))}{u'(e(s) - c^o(s))} = 1. \tag{13}$$

Clearly, a golden-rule type allocation is interim Pareto efficient. Let

[10] Because a young agent is born after the shock is realized, the first-best risk-sharing allocation where consumption is stabilized at $e(s)/2$ for every $s \in S$ is not possible.

$$GR := \{(c^o(s))_{s\in S} | (c^o(s)) \text{ satisfies Equation (13)}\}$$

be a set of all golden-rule type allocations. In the following, the set, *GR* ⋂ *SPE*, is characterized.

Proposition 3: *Under Assumption 1,*

(i) GR ⋂ SPE is non-empty and compact in $\mathrm{R}^{\mathbf{S}}_{+}$.
(ii) GR ⋂ SPE is connected in $\mathrm{R}^{\mathbf{S}}_{+}$.

Proof: See the Appendix.

Under Assumption 1, Proposition 2 shows the existence of a stationary, non-autarkic SPE allocation. Thus, if this allocation is a golden-rule type allocation, then the first part is shown. As Magill and Quinzii (2003) show, under Assumption 1, there is a mapping $\gamma : S \to (0, e^y(s))$ such that

$$u'(e^y(s) - \gamma(s))\gamma(s) = \sum_{s'\in S} \pi(s')u'(e^o(s') + \gamma(s'))\gamma(s') \tag{14}$$

for all $s \in S$. Proposition 2 in this paper shows that this allocation is an SPE allocation. This implies that the expected lifetime utility for a young agent for shock $s \in S$, $u(e^y(s) - \gamma(s)) + \sum_{s'\in S} \pi(s')u(e^o(s') + \gamma(s'))$, is maximized. Therefore, an allocation satisfying Equation (14) must be a golden-rule type allocation. Because both GR and SPE are closed and bounded in $\mathrm{R}^{\mathbf{S}}_{+}$, GR ⋂ SPE is compact. For connectedness, it is not difficult to show that SPE and GR are connected in $\mathrm{R}^{\mathbf{S}}_{+}$. Because $SPE \cap GR \neq \emptyset$, SPE ⋂ GR is also connected.[11] The numerical example in the following section will be useful.

From Proposition 3, when the autarkic allocation is not interim Pareto efficient, there always exists a stationary, golden-rule type allocation that is supported by SPE. Furthermore, a stationary, golden-rule type SPE allocation is not unique, and there are infinitely many stationary, golden-rule type SPE allocations.

Let

$$\mathbf{C} := \{(c^o(s))_{s\in S} | c^o(s) \geq e^o(s), \forall s \in S\}$$

be a set of all feasible allocations where the consumption for an old agent is not less than his/her endowment for any $s \in S$. Because of Equation (6), if a feasible stationary allocation $(c^o(s))_{s\in S}$ is an SPE allocation, $(c^o(s))_{s\in S}$ should be in **C**. Let

$$S^y := \{s \in S | e^y(s) > e^o(s)\}$$

and

$$S^o := \{s \in S | e^y(s) < e^o(s)\}.$$

[11] As for the connectedness, see, for instance, Munkres (2000).

The set S^y is a set of all shocks in which a young agent's endowment is larger than an old agent's endowment, and S^o is its complement. Assume that $S = S^y \cup S^o$.[12] Let

$$\hat{\mathbf{C}} := \{(c^o(s))_{s\in S} \in \mathbf{C} | c^o(s) > e^o(s) \text{ only if } s \in S^o\}$$

be a set of all allocations in **C** in which a positive transfer is made from a young agent to an old agent only if an old agent's endowment is larger than a young agent's endowment.

Proposition 4: $SPE \cap \hat{\mathbf{C}} = \{(e^o(s))_{s\in S}\}$.

Proof: See the Appendix.

This proposition states that if a non-autarkic stationary allocation is an SPE allocation, then a young agent whose endowment is larger than an old agent's endowment has to transfer something to the old agent. If a young agent transfers something to an old agent when the young agent's endowment is smaller than the old agent's endowment, the young agent's welfare will be lowered. Note that, in a stationary allocation, the marginal disutility from a transfer when the agent is young, $u'(e^y(s))$, is greater than the marginal benefit from receiving the same amount when he or she is old, $u'(e^o(s))$, because $e^y(s) < e^o(s)$ and u is strictly concave. Hence, such a transfer lowers the young agent's welfare, and the young agent prefers an autarkic allocation and deviates from the transfer scheme. This proposition leads to the following result:

Corollary 1: *Under Assumption 1,* $[SPE \cap GR] \cap \hat{\mathbf{C}} = \varnothing$.

The next section focuses on a specific situation and further characterizes the set $SPE \cap GR$. In addition, the section investigates the relationship between the role of money and the role of history in this environment.

4. Money and history

The necessity for money in the economy was first studied in an overlapping generations model (see e.g. Wallace, 1980). In a typical pure-exchange overlapping generations model, the autarkic allocation is a unique competitive equilibrium allocation and it is not efficient. In such an environment, however, if money is introduced, then the efficient allocation is supported in a decentralized economy.[13] In this sense, money implements the efficient allocation. In contrast, as we have seen so far, if an agent can observe previous histories (under a certain belief), the golden-rule type allocation is supported as an SPE allocation. In this section, I compare the role of money and the role of history in this paper's setting.

4.1 Definition of a monetary equilibrium

In this section, I describe the model with money. The model is the same as in Magill and Quinzii (2003). There is an infinitely-lived asset, called money, available in positive supply,

[12] This assumption excludes the state in which $e^y(s) = e^o(s)$. Even if such a state exists, the implications of the following results will not change.

[13] For more detail, see, for example, Wallace (1980).

normalized to 1. The money is originally given to the initial old agent, and is then exchanged in each period between a young agent and an old agent. Let $q(s)$ denote the price of the money when the shock is s.[14] After the trade is made, a young agent's consumption is

$$c^y(s) = e^y(s) - q(s)m(s)$$

and an old agent's consumption is

$$c^o(s) = e^o(s) + q(s)m(s),$$

where $m(s)$ is the amount of money traded between a young agent and an old agent. A (stationary) monetary equilibrium is defined in a standard way.[15] In monetary equilibrium, $m(s) = 1$ for all $s \in S$ and $q(s) > 0$ for all s.

4.2 Benchmark: Deterministic environment

As for the benchmark, let us consider a deterministic environment. Suppose $S = \{1\}$ and $e^y(1) > e^o(1) > 0$.[16] Because there is only one shock, I do not express the shock as "1" hereafter. Notice that the autarkic allocation is not Pareto efficient.

Lemma 2: *There exists a unique (stationary) monetary equilibrium, in which*

$$\frac{u'(c^o)}{u'(c^y)} = 1 \quad \text{and} \quad q = \frac{e^y + e^o}{2}.$$

Moreover, this equilibrium allocation is a golden-rule allocation.

Proof: See, for example, Wallace (1980).

If the economy possesses money, a golden-rule allocation is supported by a unique monetary equilibrium. The next lemma shows that history can also implement the golden-rule type allocation.

Lemma 3: *There is a continuum of SPE allocations, and an SPE allocation c^o satisfies*

$$c^o \in [e^o, e^y].$$

Moreover, among these SPE allocations, the allocation satisfying $c^o \geq (e^y + e^o)/2$ is Pareto efficient.

[14] Rigorously speaking, the price should depend on the history of shocks. However, because the focus is on a stationary allocation, the price only depends on the shock in the current period.

[15] For the precise definition, see, for example, Magill and Quinzii (2003).

[16] Note that when $e^o > e^y > 0$, the autarkic allocation is Pareto efficient.

Proof: See the Appendix.

These two lemmas imply that: (i) a monetary equilibrium allocation is also an SPE allocation; (ii) some SPE allocations are not monetary equilibrium allocations; and (iii) in terms of the implementation of the golden-rule type allocation, money and history are equivalent.

4.3 Stochastic environment

Let us now consider a stochastic environment. As shown in Magill and Quinzii (2003), under Assumption 1, if $\lim_{c\to 0} u'(c) = +\infty$ and $-cu''(c)/u'(c) \leq 1$ for all $c > 0$, there exists a unique monetary equilibrium, and this monetary equilibrium allocation is a golden-rule type allocation. Therefore, one more assumption on u is imposed throughout this section.

Assumption 2: *The periodic utility function, u, satisfies* $\lim_{c\to 0} u'(c) = +\infty$ *and*

$$-c\frac{u''(c)}{u'(c)} \leq 1$$

for all $c > 0$.

Proposition 5: *Under Assumptions 1 and 2, there exists a unique monetary equilibrium, and its allocation is in* $SPE \cap GR$.

Proof: By proposition 1 in Magill and Quinzii (2003) and the proof of Proposition 3 in this paper.

As in the deterministic case, a monetary equilibrium allocation is an SPE allocation in the stochastic case, and, because $GR \cap SPE$ is not a singleton, some SPE allocations are not monetary equilibrium allocations in the stochastic case. The third observation in the deterministic case is not true in the stochastic case. More precisely, history can implement more golden-rule type allocations than money. This difference comes from Equations (13) and (14). In a deterministic case, i.e. $|S| = 1$, Equations (13) and Equation (14) are equivalent. This implies that the monetary equilibrium allocation and the golden-rule type allocation are the same in a deterministic case. However, in a stochastic case, this is not true. Even if Equation (14) does not hold, Equation (13) can hold. This suggests that there are some golden-rule type allocations that are not supported by monetary equilibrium. Because history can implement more allocations than money, it is possible that history can support more golden-rule type allocations than money. This difference between history and money is critical under some situations.

Let us consider a two-shock example, i.e. $S = \{1, 2\}$.[17] Suppose that two shocks are equally likely to occur. The endowments are assumed to satisfy

$$e^y(1) = 2 - x_1, e^o(1) = x_1$$

and

[17] This example is suggested by one referee. The author would like to thank the referee for suggesting this example.

$$e^y(2) = 2 - x_2, e^o(2) = x_2,$$

where $x_s \in (0,1)$ for all $s \in S$. Notice that the total endowments are the same for two shocks. Consider a stationary allocation c such that $c^o(s) = 1$ for all $s \in S$. That is, an agent consumes 1 unit of consumption goods in any state at any age. This allocation is a golden-rule type allocation, and it maximizes an agent's ex-ante expected utility; that is,

$$\sum_{s \in S} \frac{1}{2}\left[u(c^y(s)) + \sum_{s' \in S} \frac{1}{2} u(c^o(s, s'))\right]$$

subject to $c^o(s, s') = 2 - c^y(s')$ for all $s \in S$ and all $s' \in S$.

Proposition 6: *For sufficiently small $x_1 > 0$ and $x_2 > 0$ with $x_1 \neq x_2$, a stationary allocation c such that $c^o(s) = 1$ for all $s \in S$ is an SPE allocation, but not a monetary equilibrium allocation.*

Proof: See the Appendix.

For the stationary allocation in Proposition 6 to be a monetary equilibrium, Equation (14) must hold; that is, for $s = 1$,

$$u'(1) \cdot (1 - x_1) = \frac{1}{2} u'(1) \cdot (1 - x_1) + \frac{1}{2} u'(1) \cdot (1 - x_2)$$

and for $s = 2$,

$$u'(1) \cdot (1 - x_2) = \frac{1}{2} u'(1) \cdot (1 - x_1) + \frac{1}{2} u'(1) \cdot (1 - x_2).$$

However, these equations are satisfied only when $x_1 = x_2$. This case is equivalent to the deterministic case, and in the deterministic case Equations (13) and (14) are equivalent. Once x_1 and x_2 are not the same, Equations (13) and (14) are no longer equivalent.

4.4 Numerical example

In this section, I illustrate by means of a numerical example. Let the periodic utility function be

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

where $\sigma = 0.8$. Assume that $S = \{1, 2\}$ and $\pi(1) = \pi(2) = 0.5$. Set $e^y(1) = 1.6$, $e^y(2) = 1.8$ and $e^o(1) = 0.4$ and $e^o(2) = 0.2$.

In Figure 1, the horizontal axis depicts an old agent's consumption when the shock is 1, and the vertical axis represents the old agent's consumption when the shock is 2. Notice that because in the SPE an old agent's consumption is not less than his/her endowment, both axes start $e^o(1)$ and $e^o(2)$. In the figure, a solid line expresses the binding incentive constraint when the current shock is 1, denoted by (IC1),

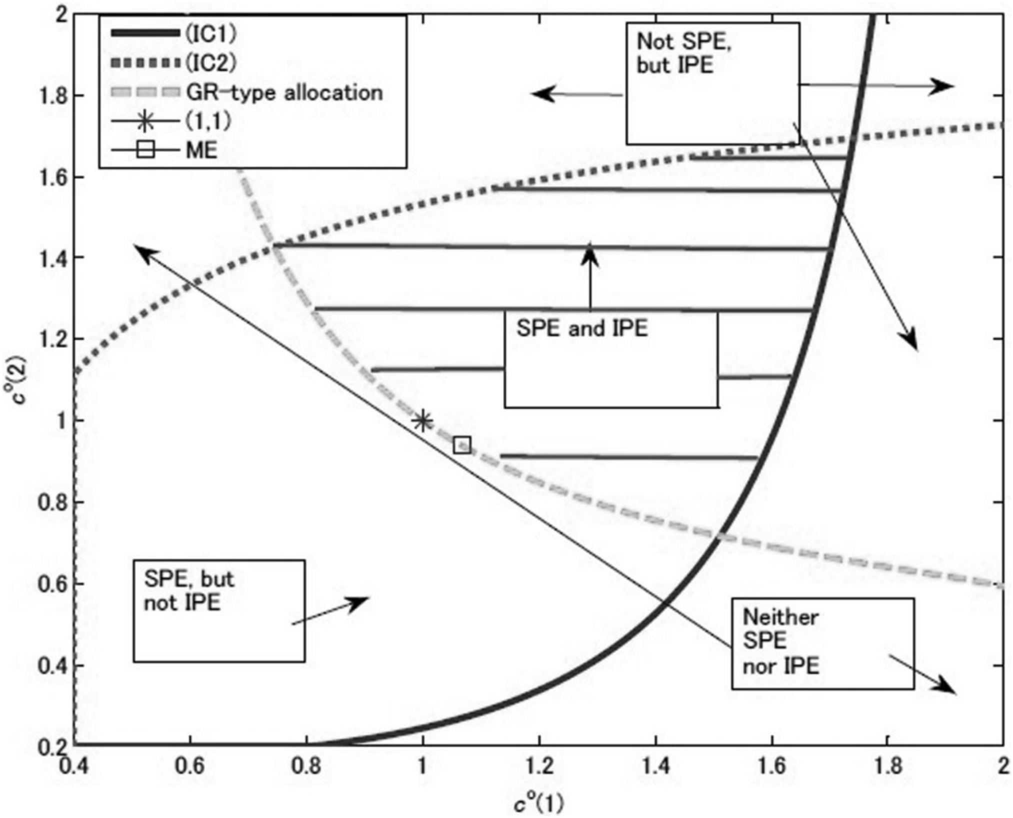

FIGURE 1. Numerical example

$$u(e(1)-c^o(1))+\sum_{s\in S}\pi(s)u(c^o(s))=u(e^y)+\sum_{s\in S}\pi(s)u(e^o(s)),\tag{IC1}$$

and a dotted line expresses the binding incentive constraint when the current shock is 2, denoted by (IC2),

$$u(e(2)-c^o(2))+\sum_{s\in S}\pi(s)u(c^o(s))=u(e^y)+\sum_{s\in S}\pi(s)u(e^o(s)).\tag{IC2}$$

Because given $c^o(1)$, a higher $c^o(2)$ relaxes the incentive constraint when the current shock is 1, all points above (IC1) satisfy the incentive constraint when the current shock is 1. Similarly, because given $c^o(2)$, a higher $c^o(1)$ relaxes the incentive constraint when the current shock is 2, all points right of (IC2) satisfy the incentive constraint when the current shock is 2. A dashed line expresses the golden-rule type allocation; that is,

$$\sum_{s\in S}\frac{\pi(s)u'(c^o(s))}{u'(e(s)-c^o(s))}=1.$$

Given $c^o(2)$, an increase in $c^o(1)$ lowers the value of $\Sigma_{s\in S}(\pi(s)u'(c^o(s))/u'(e(s)-c^o(s)))$. Hence, all points right of the dashed line satisfy $\pi(1)u'(c^o(1))/u'(c^y(1))+\pi(2)u'(c^o(2))/u'(c^o(2))\le 1$, which implies that an allocation is interim Pareto efficient. The star on the dashed line is $c^o(1) = c^o(2) = 1$, which is a stationary allocation that maximizes an agent's *ex-ante* expected utility. The square on the dashed line depicts a unique monetary equilibrium allocation ("ME" in the figure).

Figure 1 explains which allocations are interim Pareto efficient ("IPE" in the figure) or not and which allocations are SPE allocations or not. The shaded area expresses an interim Pareto efficient, subgame perfect equilibrium allocation. As the results in this paper show, when the autarkic allocation is not interim Pareto efficient, $GR \cap SPE$, which is the dashed line between the solid line and dotted line, is not empty. Moreover, it is compact and connected. A unique monetary equilibrium in this example is $c^o(1) = 1.0672$ and $c^o(2) = 0.9393$, which is different from $c^o(s) = 1$ for all $s \in S$.

5. Conclusion

This paper considers an overlapping generations model with aggregate endowment shocks and a limited commitment constraint. I characterize a stationary golden-rule type allocation that satisfies a limited commitment constraint. One implication is that history-dependent behavior improves welfare. The paper also considers the role of money and history in this environment. As for the implementation of the golden-rule type allocations, the history considered here implements more allocations than money in a stochastic environment, while they are the same in a deterministic environment. The interesting finding is that history can *ex-ante* welfare-dominate the money. This is because the condition for a stationary allocation to be a monetary equilibrium allocation and the condition for a stationary allocation to be a golden-rule type allocation are not the same in a stochastic environment, whereas they are the same in a deterministic environment.

An interesting area for further research would be to characterize efficient, self-enforcing allocations completely. This paper characterizes them partially in the sense that the paper focuses on stationary allocations. The beauty of contemporaneous, infinitely-lived agent models such as Thomas and Worrall (1988) and Kocherlakota (1996) is that they can characterize efficient, self-enforcing allocations fully. One advantage of the infinitely-lived agent model is the ease in formulating the problem for the optimality. Because of this, they use dynamic programming to characterize the constrained efficient allocations. However, a similar approach cannot be used with the overlapping generations model, and, hence, the characterization is more difficult. It would be an interesting future research agenda to characterize constrained efficient allocations completely in an overlapping generations model.

Appendix

A.1 Proof of Lemma 1

Proof: Consider a strategy profile, σ, in which no agent transfers anything to the other agent regardless of what has happened in the past. Then, σ is an SPE, because the only deviation for any agent is to transfer a positive amount to the other agent and it decreases the agent's utility.

Suppose there exists another SPE, σ', that gives lower expected lifetime utility to some agent at some date $t \geq 1$, some history of transfers, h^{t-1}, and some history of shocks, s^t, than

σ. In any SPE, no old agent transfers anything to the young agent after each history. Hence, the old agent's consumption is his or her own endowment and the transfer from the young agent. Because, in autarky, no young agent transfers anything to the old agent, the autarkic allocation provides the lowest consumption among the SPE. That is why I can focus on the conditional expected lifetime utility of the young agent. By the supposition above,

$$u(e^y(s_t)) + \sum_{s\in S}\pi(s)u(e^o(s)) > U(c_t|s^t, h^{t-1}, \sigma'), \tag{15}$$

where $U(c_t|s^t, h^{t-1}, \sigma')$ is the expected lifetime utility under a strategy profile σ' and a history of transfers h^{t-1}. Suppose that the generation-t agent deviates from σ'_t by not transferring anything to the old agent when he or she is young and by not transferring anything to the young agent when he or she is old. Then, the conditional expected lifetime utility of that young agent from this deviation is

$$u(e^y(s_t)) + E[u(c^o_{t+1})|\sigma'_{-t}].$$

Because this agent does not transfer anything to the young agent when he or she is old,

$$u(e^y(s_t)) + E[u(c^o_{t+1})|\sigma'_{-t}] \geq u(e^y(s_t)) + \sum_{s\in S}\pi(s)u(e^o(s))$$

holds. By combining this equation with Equation (15),

$$u(e^y(s_t)) + E[u(c^o_{t+1})|\sigma'_{-t}] > U(c_t|s^t, h^{t-1}, \sigma')$$

holds. This contradicts the fact that σ' is an SPE, because the generation-t agent has an incentive to deviate from σ'_t when young.

A.2 Proof of Proposition 1

Proof: For necessity, suppose that for some t and some $s^t \in S^t$, either Equation (6) or (7) is violated. When Equation (6) does not hold, an old agent strictly prefers doing nothing. Then, c is not subgame perfect. When Equation (7) does not hold, a young agent prefers the autarkic allocation to c_t. Hence, c cannot be an SPE allocation.

For sufficiency, suppose that for all t and $s^t \in S^t$, both Equations (6) and (7) hold. Let $\gamma_t(s^t) := c^y_t(s^t) - e^y(s_t)$ for all $s^t \in S^t$. Notice that $c^o_t(s^t) = e^o(s_t) + \gamma_t(s^t)$. Consider the following strategy, $\hat{\sigma}_t$:for all $t \geq 1$, all $s^t \in S^t$ and all $h^{t-1} \in H^{t-1}$,

$$\hat{\sigma}^y_t(h^{t-1}, s^t) = \begin{cases} \gamma_t(s^t) & \text{if no agents before date } t \text{ deviate from } \hat{\sigma}_\tau \text{ for } \tau < t \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\hat{\sigma}^o_{t+1}(h^t, s^t, s_{t+1}) = 0$$

for all $h^t \in H^t$ and all $s_{t+1} \in S$. For the initial old agent,

$$\hat{\sigma}_1^o(s_1) = 0$$

for all $s_1 \in S$. This strategy profile, $\hat{\sigma}$, is an *SPE* when Equations (6) and (7) hold. The outcome of $\hat{\sigma}$ is c. Therefore, Equations (6) and (7) are sufficient conditions for c being subgame perfect.

A.3 Proof of Proposition 2

Proof: Sufficiency: Proposition 1 in Magill and Quinzii (2003) implies that, under Assumption 1, there is a mapping $\gamma: S \to (0, e^y(s))$ such that

$$u'(e^y(s) - \gamma(s))\gamma(s) = \sum\nolimits_{s' \in S} \pi(s')u'(e^o(s') + \gamma(s'))\gamma(s') \tag{16}$$

for all $s \in S$. What I am going to show is that the allocation $(e^y(s) - \gamma(s), e^o(s) + \gamma(s))_{s \in S}$ is an SPE allocation. Let $c^y(s) = e^y(s) - \gamma(s)$ and $c^o(s) = e^o(s) + \gamma(s)$. Because u is strictly concave and $\gamma(s) > 0$ for all $s \in S$,

$$u(c^o(s)) - u(e^o(s)) > u'(c^o(s))\gamma(s)$$

and

$$u(e^y(s)) - u(c^y(s)) < u'(c^y(s))\gamma(s).$$

Equation (16) implies that for all $s \in S$,

$$\sum\nolimits_{s' \in S} \pi(s')[u(c^o(s')) - u(e^o(s'))] > u(e^y(s)) - u(c^y(s)),$$

which is equivalent to Equation (7).

Necessity: Suppose by way of the contradiction that Equation (8) does not hold with weak inequality; that is,

$$\sum\nolimits_{s \in S} \frac{\pi(s)u'(e^o(s))}{u'(e^y(s))} < 1.$$

Let

$$\Pi := \left[\frac{\pi(s)u'(e^o(s))}{u'(e^y(s'))} \right]_{s,s' \in S}$$

be an $\mathbf{S} \times \mathbf{S}$ matrix. Because $u' > 0$, Π is a strictly positive matrix. Thus, by the Perron–Frobenius theorem, there exists a unique eigenvalue, $\lambda > 0$, such that $\Pi x = \lambda x$ and x is a strictly positive eigenvector. Note that $\lambda < 1$. Therefore, $\Pi x < x$ holds. Moreover, for any integer n, $(\Pi x)/n < x/n$. This inequality implies

$$\sum\nolimits_{s' \in S} \pi(s')u'(e^o(s'))\frac{x(s')}{n} < u'(e^y(s))\frac{x(s)}{n}$$

for all $s \in S$. This inequality implies that for a sufficiently large n,

$$\sum_{s' \in S} \pi(s') \left[u\left(e^o(s') + \frac{x(s')}{n} \right) - u(e^o(s')) \right] < u(e^y(s)) - u\left(e^y(s) - \frac{x(s)}{n} \right)$$

for all $s \in S$. This inequality implies that a young agent does not have an incentive to transfer a positive amount of endowment to an old agent. Because incentive conditions are satisfied most easily at the autarkic allocation, this leads to the conclusion that there does not exist a stationary SPE allocation except for the autarkic allocation.

A.4 Proof of Proposition 3

Proof: (i): Proposition 1 in Magill and Quinzii (2003) implies that under Assumption 1, there is a mapping $\gamma : S \to (0, e^y(s))$ such that

$$u'(e^y(s) - \gamma(s))\gamma(s) = \sum_{s' \in S} \pi(s') u'(e^o(s') + \gamma(s'))\gamma(s')$$

for all $s \in S$, and the resulting allocation $(e^y(s) - \gamma(s), e^o(s) + \gamma(s))_{s \in S}$ satisfies $\Sigma_{s \in S} \pi(s) u'(e^o(s) + \gamma(s))/u'(e^y(s) - \gamma(s)) = 1$; that is, the allocation is a golden-rule type allocation. In Proposition 2, I showed that this allocation is an SPE allocation. Therefore, $GR \cap SPE \neq \emptyset$. As for compactness, consider a sequence $(c_n^o)_{n=1}^{\infty}$, where $c_n^o := (c_n^o(s))_{s \in S}$ and $c_n^o \in GR$ for all n. Let c^o be a limit of c_n^o. For all n,

$$\sum_{s \in S} \frac{\pi(s) u'(c_n^o(s))}{u'(e(s) - c_n^o(s))} = 1$$

is satisfied. Because u' is continuous,

$$1 = \lim_{n \to \infty} \left(\sum_{s \in S} \frac{\pi(s) u'(c_n^o(s))}{u'(e(s) - c_n^o(s))} \right) = \sum_{s \in S} \frac{\pi(s) u'(c^o(s))}{u'(e(s) - c^o(s))}.$$

Therefore, $c^o \in GR$. This confirms that GR is closed. By analogy, consider a sequence $(c_n^o)_{n=1}^{\infty}$, where $c_n^o := (c_n^o(s))_{s \in S}$ and $c_n^o \in SPE$ for all n. Let c^o be a limit of c_n^o. Because u is continuous,

$$u(e^o(s)) \leq \lim_{n \to \infty} u(c_n^o(s)) = u(c^o(s))$$

for all $s \in S$ and

$$\begin{aligned} u(e^y(s)) + \sum_{s' \in S} \pi(s') u(e^o(s')) &\leq \lim_{n \to \infty} \left[u(e(s) - c_n^o(s)) + \sum_{s' \in S} \pi(s') u(c_n^o(s')) \right] \\ &= u(e(s) - c^o(s)) + \sum_{s' \in S} \pi(s') u(c^o(s')). \end{aligned}$$

This implies that $c^o \in SPE$, and hence, SPE is closed. Therefore, $GR \cap SPE$ is also closed. Because the endowment in any state is finite, $GR \cap SPE$ is bounded in $\mathbb{R}_+^S$. Therefore, $GR \cap SPE$ is compact in $\mathbb{R}_+^S$.

(ii): First, show that *SPE* is connected in $\mathbb{R}^S_+$. To show this, I show that *SPE* is convex in $\mathbb{R}^S_+$. Pick two allocations, c and c' from *SPE*. Fix any $\mu \in [0,1]$. Let $c^\mu := \mu c + (1-\mu)c'$. Because u is strictly concave,

$$u(c^{o,\mu}(s)) \geq \mu u(c^o(s)) + (1-\mu)u(c^{o'}(s)) \geq \mu u(e^o(s)) + (1-\mu)u(e^o(s)) = u(e^o(S))$$

holds for all $s \in S$. Furthermore,

$$\begin{aligned} u(e(s) - c^{o,\mu}(s)) &= u(\mu(e(s) - c^o(s)) + (1-\mu)(e(s) - c^{o'}(s))) \\ &\geq \mu u(e(s) - c^o(s)) + (1-\mu)u(e(s) - c^{o'}(s)) \end{aligned}$$

holds. This implies that for all $s \in S$,

$$\begin{aligned} & u(e(s) - c^{o,\mu}(s)) + \sum\nolimits_{s' \in S} \pi(s')u(c^{o,\mu}(s')) \geq \mu\left[u(e(s) - c^o(s)) + \sum\nolimits_{s' \in S} \pi(s')u(c^o(s'))\right] \\ & \quad + (1-\mu)\left[u(e(s) - c^{o'}(s)) + \sum\nolimits_{s' \in S} \pi(s')u(c^{o'}(s'))\right] \\ & \quad \geq u(e(s) - e^o(s)) + \sum\nolimits_{s' \in S} \pi(s')u(e^o(s')). \end{aligned}$$

Therefore, $c^\mu \in SPE$. Because *SPE* is a convex set, it is connected in $\mathbb{R}^S_+$.

Second, prove that *GR* is connected. Because u is twice-continuously differentiable, $u' > 0$ and $u'' < 0$, from

$$\sum\nolimits_{s \in S} \frac{\pi(s)u'(c^o(s))}{u'(e(s) - c^o(s))} = 1,$$

there exists a continuous mapping f_i: $\times_{s \in S\setminus\{i\}}[e^o(s), e(s)] \to [e^o(i), e(i)]$ such that given $c^o_{-i} := (c^o(s))_{s \in S\setminus\{i\}} \in \times_{s \in S\setminus\{i\}}[e^o(s), e(s))$, $(f(c^o_{-i}), c^o_{-i})$ satisfies $\sum_{s \in S} \pi(s)u'(c^o(s))/u'(e(s) - c^o(s)) = 1$ for all $i \in S$.[18] Letting

$$F_i := \{c^o(i) \in [e^o(i), e(i)] \,|\, c^o(i) = f_i(c^o_{-i}) \text{ for } c^o_{-i} \in \times_{s \in S\setminus\{i\}}[e^o(s), e(s)]\},$$

$$GR = \bigcup\nolimits_{i \in S} F_i.$$

Because f_i is continuous, F_i is connected in $\mathbb{R}_+$. The allocation that is used in the proof of (i) is included in all F_i. Thus, for any i, j, $i \neq j$, $F_i \cap F_j \neq \emptyset$. This implies that $GR = \cup_{i \in S} F_i$ is also connected (see e.g. Munkres, (2000)).

Because *SPE* and *GR* are connected and $SPE \cap GR \neq \emptyset$, $SPE \cap GR$ is connected.

A.5 Proof of Proposition 4

Proof: Because the autarkic allocation is an SPE allocation and $(e^o(s))_{s \in S} \in \hat{\mathbf{C}}$, $(e^o(s))_{s \in S} \in$ SPE $\cap\ \hat{\mathbf{C}}$. Consider an allocation $(c^o(s))_{s \in S} \in \hat{\mathbf{C}}$ such that $c^o(s) > e^o(s)$ for some $s \in S^o$. Notice that if $c^o(s) > e^o(s)$ for $s \in S^o$, then

[18] If $\lim_{c \to 0} u'(c) = +\infty$, then instead of $[e^o(s), e(s)]$, consider an interval, $[e^o(s), e(s))$. Because in such a case any SPE allocation satisfies $c^y(s) = e(s) - c^o(s) > 0$, we can only focus on the above interval.

$$u(e^y(s)) - u(c^y(s)) > u(c^o(s)) - u(e^o(s)),$$

because u is strictly concave and $e^y(s) - c^y(s) = c^o(s) - e^o(s)$. From this, for the above allocation $(c^o(s))_{s \in S}$,

$$\sum_{s \in S} \pi(s)[u(c^o(s)) - u(e^o(s))] < \sum_{s \in S} \pi(s)[u(e^y(s)) - u(c^y(s))]$$
$$\leq \max_{s \in S} \{u(e^y(s)) - u(c^y(s))\},$$

which contradicts Equation (7). Thus, the above allocation $(c^o(s))_{s \in S}$ is not in SPE.

A.6 Proof of Lemma 3

Proof: A necessary and sufficient condition for a stationary allocation c to be an SPE allocation is that $c^o \geq e^o$ and

$$u(e - c^o) + u(c^o) \geq u(e - e^o) + u(e^o),$$

where $e := e^y + e^o$. Because $u(e - c^o) + u(c^o)$ is an inverse U-shaped concave function in c^o and the maximum is achieved at $c^o = e/2$, the right-hand side and the left-hand side of the equation again hold at $c^o = e^y$. Therefore, any stationary allocation, $c^o \in [e^o, e^y]$, is an SPE allocation. If a stationary allocation is Pareto efficient,

$$\frac{u'(c^o)}{u'(c^y)} \leq 1.$$

Hence, a stationary SPE allocation, c, satisfying $c^o \in [e^o, (e^y + e^o)/2)$, is not Pareto efficient.

A.7 Proof of Proposition 6

Proof: First, show that the stationary allocation is an SPE allocation for x sufficiently close to 0. Because $1 > x_s$ for all $s \in S$, Equation (6) is satisfied. For Equation (7), for $s = 1$,

$$u(1) + u(1) \geq u(2 - x_1) + \frac{1}{2}u(x_1) + \frac{1}{2}u(x_2),$$

and for $s = 2$,

$$u(1) + u(1) \geq u(2 - x_2) + \frac{1}{2}u(x_1) + \frac{1}{2}u(x_2).$$

When $x_s = 0$ for all $s \in S$, the right-hand sides of the above equations are $u(2) + u(0)$. Because u is strictly concave, $2u(1) > u(2) + u(0)$. Because u is continuous, for x_1 and x_2 sufficiently close to 0, Equation (7) holds. Therefore, the stationary allocation c is an SPE allocation.

Next, show that this stationary allocation is not a monetary equilibrium allocation. Under Assumptions 1 and 2, Magill and Quinzii (2003) show that there is a unique monetary equilibrium and it satisfies Equation (14). For the stationary allocation c to be a monetary equilibrium allocation, for $s = 1$,

$$u'(1)\cdot(1-x_1)=\frac{1}{2}u'(1)\cdot(1-x_1)+\frac{1}{2}u'(1)\cdot(1-x_2)$$

and for $s = 2$,

$$u'(1)\cdot(1-x_2)=\frac{1}{2}u'(1)\cdot(1-x_1)+\frac{1}{2}u'(1)\cdot(1-x_2)$$

must hold. These are satisfied only when $x_1 = x_2$. This implies that when $x_1 \neq x_2$, the stationary allocation c is not a monetary equilibrium allocation.

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