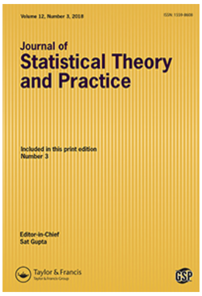

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A new class of quantile functions useful in reliability analysis

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ABSTRACT
This article introduces a new flexible family of distributions, defined by means of a quantile function. The quantile function proposed is the sum of quantile functions of the half logistic and exponential geometric distributions. Various distributional properties and reliability characteristics are discussed. The estimation of the parameters of the model using L-moments is studied. The model is applied to a real-life data set.

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1. Introduction

A probability distribution can be specified either in terms of its distribution function or by the quantile function. Although both convey the same information about the distribution with different interpretations, the concepts and methodologies based on distribution functions are more popular in most forms of theory and practice. For a nonnegative random variable X with distribution function $F(x)$, the quantile function $Q(u)$ is defined by

$$Q(u) = F^{-1}(x) = \inf\{x : F(x) \geq u\}, \ 0 \leq u \leq 1. \tag{1}$$

The derivative of $Q(u)$ is the quantile density function denoted by $q(u)$. If $F(x)$ is right continuous and strictly increasing, we have

$$F(Q(u)) = u, \tag{2}$$

so that $F(x) = u$ implies $x = Q(u)$. When $f(x)$ is the probability density function (pdf) of X, we have from Eq. (2)

$$q(u)f(Q(u)) = 1. \tag{3}$$

Quantile functions have several properties that are not shared by distribution functions. For example, the sum of two quantile functions is again a quantile function. Further, the product of two positive quantile functions is again a quantile function in the nonnegative setup. There are explicit general distribution forms for the quantile function of order statistics. It is easier to generate random numbers from the quantile function. A major development in portraying quantile functions to model statistical data is given by Hastings

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et al. (1947), who introduced a family of distributions by a quantile function. This was refined later by Tukey (1962) to form a symmetric distribution, called the Tukey lambda distribution.

This model was generalized in different ways, referred as lambda distributions. These include various forms of quantile functions discussed in Ramberg and Schmeiser (1972), Ramberg (1975), Ramberg et al. (1979), and Freimer et al. (1988). Govindarajulu (1977) introduced a new quantile function by taking the weighted sum of quantile functions of two power distributions. Hankin and Lee (2006) presented a new power-Pareto distribution by taking the product of power and Pareto quantile functions. Van Staden and Loots (2009) developed a four-parameter distribution, using a weighted sum of the generalized Pareto and its reflection quantile functions. Sankaran et al. (2016) developed a new quantile function based on the sum of quantile functions of generalized Pareto and Weibull quantile functions. The density and distribution functions for these models are not available in closed forms except for certain special cases. The great advantage of these models is that the simple forms of the quantile functions make it extremely straightforward to simulate random values, which is useful in inference problems.

The aim of the present work is to introduce a new quantile function that is useful in reliability analysis. The proposed quantile function is derived by taking the sum of quantile functions of half logistic and exponential geometric distributions. Balakrishnan (1985) considered the folded form of the standard logistic distribution and termed it the half logistic distribution. The survival function and quantile function of this distribution are respectively given by

$$\bar{G}(x) = 2\left(1 + e^{\frac{x}{\beta}}\right)^{-1}, \quad \beta > 0. \tag{4}$$

and

$$Q_1(u) = \beta \log\left(\frac{1+u}{1-u}\right), \quad \beta > 0. \tag{5}$$

Model (4) is a possible lifetime model, which has several recurrence relations for the single and the product moments of order statistics. Adamidis and Loukas (1998) introduced the exponential geometric (EG) distribution with applications to reliability modeling in the context of decreasing failure rate data. The survival function and quantile function of the EG distribution are given by

$$\bar{F}(x) = 1 - F(x) = (1-p)e^{-\frac{1}{\alpha}x}(1 - pe^{-\frac{1}{\alpha}x})^{-1}, \quad \alpha > 0 \text{ and } 0 < p < 1. \tag{6}$$

and

$$Q_2(u) = \alpha \log\left(\frac{1-pu}{1-u}\right), \quad \alpha > 0 \text{ and } 0 < p < 1. \tag{7}$$

We now propose a new class of distributions defined by a quantile function, which is the sum of quantile functions of half logistic and exponential geometric distributions. The proposed class gives a wide variety of distributional shapes for various choices of the parameters.

The rest of the article is organized as follows. In section 2 we present a family of distributions and study its basic properties. section 3 presents some well-known distributions that are either a member of the proposed class of distributions or obtained by applying some suitable transformations on the proposed quantile function. The distributional properties such as measures of location and scale, L moments, and so on are given in section 4. In section 5, we present various reliability characteristics of the class. section 6 focuses on the inference procedures. We then provide application of this class of distributions in a real life situation. Finally, section 7 provides major conclusions of the study.

2. Half logistic–exponential geometric (HLEG) quantile function

Let X and Y be two nonnegative random variables with distribution functions $F(x)$ and $G(x)$ with quantile functions $Q_1(u)$ and $Q_2(u)$, respectively. Then

$$Q(u) = Q_1(u) + Q_2(u), \tag{8}$$

is also a quantile function with quantile density function satisfying

$$(1-u)q(u) = (1-u)q_1(u) + (1-u)q_2(u). \tag{9}$$

We now introduce a class of distributions given by the quantile function

$$Q(u) = \alpha \log\left(\frac{1-pu}{1-u}\right) + \beta \log\left(\frac{u+1}{1-u}\right), \quad 0 \le p \le 1, \alpha \ge 0, \beta \ge 0. \tag{10}$$

Thus $Q(u)$ is the sum of Eqs. (5) and (7). The support of the proposed class of distributions (10) is $(0,\infty)$. The quantile density function is obtained as

$$q(u) = \frac{2\beta + \alpha((1-p))(u+1) - 2\beta pu}{(u^2-1)(pu-1)}. \tag{11}$$

The quantile function (10) represents a family of distributions with neither the density nor the distribution function is available in closed form. However, these can be calculated by numerical inversion of the quantile function. For the proposed class of distributions, the density function $f(x)$ can be written in terms of the distribution function as

$$f(x) = \frac{(1-pF(x))(1-(F(x)^2)}{\alpha(1-p)(1+F(x)) + 2(1-pF(x))\beta}. \tag{12}$$

For all values of the parameters, the density is strictly decreasing in x and it tends to zero as $x \to \infty$. Plots of the density function for different combinations of parameters are shown in Figure 1.

The mode of the distribution is at zero and the modal value is $\frac{1}{2\beta+\alpha(1-p)}$.

3. Members of the family

The proposed family of distributions (10) includes several well-known distributions for various values of the parameters. We can derive some well-known distributions from the proposed model by making use of various transformations described in Gilchrist (2000).

Case 1. $\beta = 0$, $p = 0$ and $\alpha > 0$.

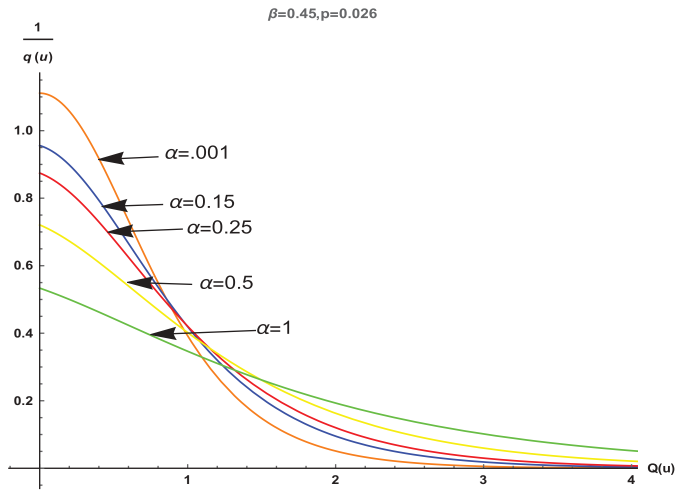


Figure 1. Plots of density function for different values of parameters.

The quantile function of the proposed class of distributions reduces to the quantile function

$$Q(u) = \alpha(-\log(1-u)), \tag{13}$$

which is the exponential distribution with mean α. We can apply the power transformation of the form $T(x) = x^K$ on Eq. (13) to form the Weibull distribution with parameters α and K.

Case 2. $\alpha = \beta$ and $p = 1$.

The quantile function of the proposed class of distributions becomes

$$Q(u) = \alpha \log\left(\frac{1+u}{1-u}\right), \tag{14}$$

which belongs to the class of distributions with linear hazard quantile functions defined by Midhu et al. (2014), with quantile function

$$Q(u) = \frac{1}{a(1+\theta)} \log\left(\frac{1+\theta u}{1-u}\right), \tag{15}$$

with $\theta = 1$ and $a = \frac{1}{2\alpha}$.

Case 3. $\beta = 0, \alpha > 0$ and $0 < p < 1$.

The quantile function of the proposed class of distributions reduces to the quantile function

$$Q(u) = \alpha \log\left(\frac{1-pu}{1-u}\right), \tag{16}$$

and this also belongs to the class of distributions (15), with parameters $\theta = -p, (-1 < \theta < 0)$ and $a = \frac{1}{\alpha(1-p)}$.

Case 4. $p = 0, \alpha > 0$ and $\beta > 0$.

The quantile function of the proposed class of distributions is obtained as

$$Q(u) = \frac{(A-B)\log(1+Au) - A(B+1)\log(1-u)}{A(A+1)K}, \tag{17}$$

where $K = \frac{1}{\alpha+2\beta}$,$A = 1$ and $B = \frac{\alpha}{\alpha+2\beta}$. The quantile function (17) corresponds to the family of distributions with bilinear hazard quantile function, given in Sankaran et al. (2015).

In the construction of our family, the sum of two quantile functions are involved. In the following theorems, we derive the random variable associated with the proposed quantile function (10).

Theorem 3.1. If $Z \sim HL(\beta)$, then the random variable $X = Z + \alpha \log\left(\frac{(1+p)+(1-p)exp\left(\frac{Z}{\beta}\right)}{2}\right)$ has $HLEG(\alpha, \beta, p)$ distribution.

Proof. Consider two random variables S and T with quantile functions $Q_S(u)$ and $Q_T(u)$ and distribution functions $F_S(x)$ and $F_T(x)$, respectively.

Now suppose $Q^*(u)$ is defined by

$$Q^*(u) = Q_S(u) + Q_T(u).$$

Then the random variable that corresponds to the quantile function $Q^*(u)$ is $S + Q_T(F_S(S))$ or $T + Q_S(F_T(T))$(Sankaran et al. 2016).

Now take $Y \sim EG(\alpha, p)$ and $Z \sim HL(\beta)$; then we $Z + Q_Y(F_Z(Z))$ has $HLEG(\alpha, \beta, p)$ distribution.

Since $Q_Y(u) = \alpha \log\left(\frac{1-pu}{1-u}\right)$ and $F_Z(Z) = 1 - 2\left(1 + exp\left(\frac{Z}{\beta}\right)\right)^{-1}$, we get

$$Z + Q_Y(F_Z(Z)) = Z + \alpha \log\left(\frac{(1+p)+(1-p)exp\left(\frac{Z}{\beta}\right)}{2}\right), \tag{18}$$

which completes the proof.

⊠

Theorem 3.2. If $Y \sim EG(\alpha, p)$, then the random variable $X = Y + \beta \log\left(\frac{p-2exp(x/\alpha)+1}{p-1}\right)$ has $HLEG(\alpha, \beta, p)$ distribution.

Proof. The proof is similar to that of Theorem 3.1, and therefore the details are omitted.

⊠

4. Distributional characteristics

The quantile based measures of the distributional characteristics of location, dispersion, skewness, and kurtosis are popular in statistical analysis. These measures are also useful for estimating parameters of the model by matching population characteristics with corresponding sample characteristics. For model (10), basic descriptive measures such as median (M), interquartile range (IQR), Galton's coefficient of skewness (S), and Moor's coefficient of kurtosis (T) are obtained as

$$M = \alpha \log(2-p) + \beta \log(3), \tag{19}$$

$$IQR = \alpha \log\left(\frac{12-9p}{4-p}\right) + \beta \log\left(\frac{21}{5}\right), \tag{20}$$

$$S = \frac{1.43\beta - 2(1.09\beta + \alpha\log(2.(1-0.5p))) - \alpha\log(1.33(1-0.25p)) + \alpha\log(4.(1-0.75p))}{1.43\beta - \alpha\log(1.33(1-0.25p)) + \alpha\log(4.(1-0.75p))}, \tag{21}$$

$$T = \frac{\alpha(-0.69\log(1.14-0.14p) + 0.7\log(1.6-0.6p) - 0.7\log(2.67-1.7p) + 0.7\log(8-7p)) + 1.24\beta}{-0.7\alpha\log(1.34-0.34p) + 0.7\alpha\log(4-3p) + \beta}. \tag{22}$$

The L-moments are often found to be more desirable than the conventional moments in describing the characteristics of the distributions as well as for inference. A unified theory and a systematic study on L-moments have been presented by Hosking (1990). The L-moments have generally lower sampling variances and are robust against outliers. See Hosking (1990) and Sankarasubramanian and Srinivasan (1999) for details.

The rth L moment is given by

$$L_r = \int_0^1 \sum_{k=0}^{r-1} (-1)^{r-1-k} \binom{r-1}{k} \binom{r-1+k}{k} u^k Q(u)du. \tag{23}$$

For the model (2.3), first four L moments are obtained as follows;

$$L_1 = \beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p}. \tag{24}$$

$$L_2 = \alpha + 2\beta - \beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p^2} - \frac{\alpha}{p}. \tag{25}$$

$$L_3 = -4\beta + \beta\log(64) - \frac{\alpha(p-2)(p-1)\log(1-p)}{p^3} + \frac{2\alpha(p-1)}{p^2}. \tag{26}$$

$$L_4 = \frac{p(4\beta p^3(23-33\log(2)) + \alpha(p-1)((p-15)p+30)) + 6\alpha(p-1)((p-5)p+5)\log(1-p)}{6p^4}. \tag{27}$$

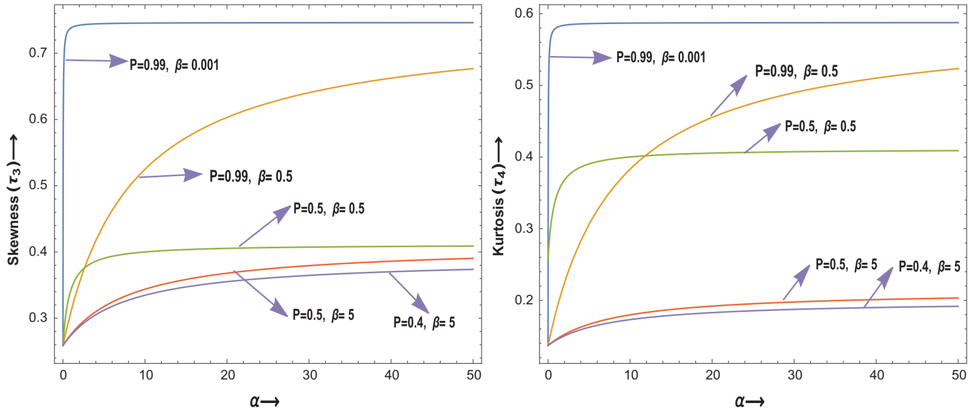
For model (10), the L-coefficient of variation (τ_2), L-coefficient of skewness (τ_3), and L-coefficient of kurtosis (τ_4) have the following expressions;

$$\tau_2 = \frac{L_2}{L_1} = \frac{\alpha + 2\beta - \beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p^2} - \frac{\alpha}{p}}{\beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p}}, \tag{28}$$

$$\tau_3 = \frac{L_3}{L_2} = \frac{-4\beta + \beta\log(64) - \frac{\alpha(p-2)(p-1)\log(1-p)}{p^3} + \frac{2\alpha(p-1)}{p^2}}{\alpha + 2\beta - \beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p^2} - \frac{\alpha}{p}}, \tag{29}$$

$$\tau_4 = \frac{L_4}{L_2} = \frac{p(4\beta p^3(23-33\log(2)) + \alpha(p-1)((p-15)p+30)) + 6\alpha(p-1)((p-5)p+5)\log(1-p)}{6p^2(p(\alpha(p-1) - \beta p(\log(4)-2)) + \alpha(p-1)\log(1-p))}. \tag{30}$$

Figures 2, 3 and 4 present skewness (τ_3) and kurtosis (τ_4) measures for various parameter values. We can show that τ_3 lies in (0.25,1) and τ_4 lies in (0.12,0.67) using numerical optimization techniques. Thus, the proposed class of distributions (10) consists of only positively skewed distributions. The curves of τ_3 and τ_4 increase with α for fixed β and p,



(a) Skewness(τ_3) (b) Kurtosis (τ_4)

Figure 2. Skewness and kurtosis of the $HLEG(\alpha, \beta, p)$ distribution for selected values of β and p as a function of the parameter α.

decrease with β for fixed α and p, and first increase and then decrease with p for fixed α and β.

4.1. Order statistics

If $X_{r:n}$ is the rth order statistic in a random sample of size n, then the density function of $X_{r:n}$ can be written as

$$f_r(x) = \frac{1}{B(r, n-r+1)} f(x) F^{r-1}(x)(1 - F(x))^{n-r}.$$

From Eq. (12), we have

$$f_r(x) = \frac{1}{B(r, n-r+1)} \frac{(1 - F(x))^{n-r}(1 - pF(x))(1 - (F(x)^2)(F(x))^{r-1}}{\alpha(1-p)(1 + F(x)) + 2(1 - pF(x))\beta}.$$

Hence,

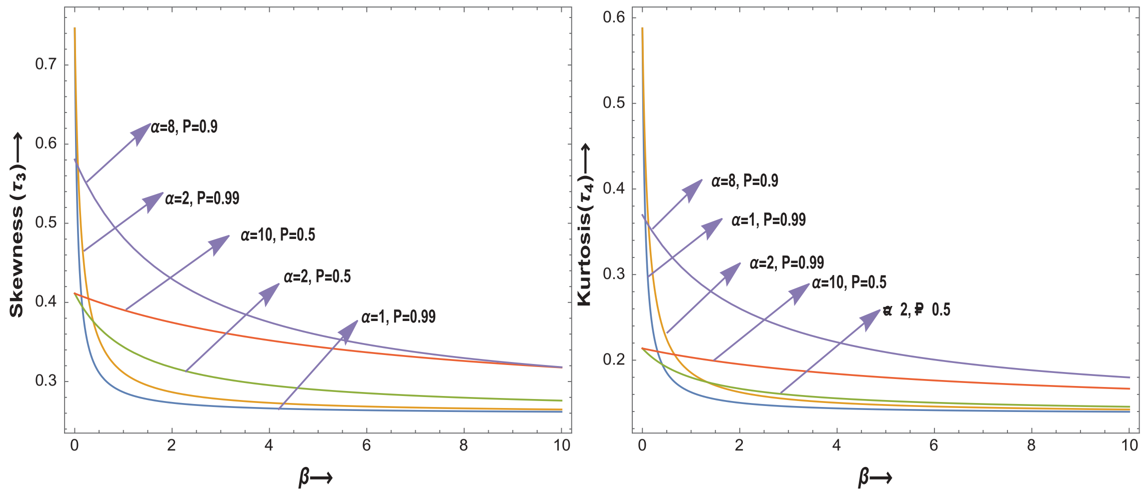
$$E(X_{r:n}) = \frac{1}{B(r, n-r+1)} \int_0^\infty x \frac{(1 - F(x))^{n-r}(1 - pF(x))(1 - (F(x)^2)(F(x))^{r-1}}{\alpha(1-p)(1 + F(x)) + 2(1 - pF(x))\beta} dx.$$

In quantile terms, we have

$$E(X_{r:n}) = \frac{1}{B(r, n-r+1)} \int_0^1 Q(u) \frac{(1-u)^{n-r}(1-pu)(1-u^2)u^{r-1}}{\alpha(1-p)(1+u) + 2(1-p+u)\beta} dx.$$

For the class of distributions (10), the first-order statistic $X_{1:n}$ has the quantile function

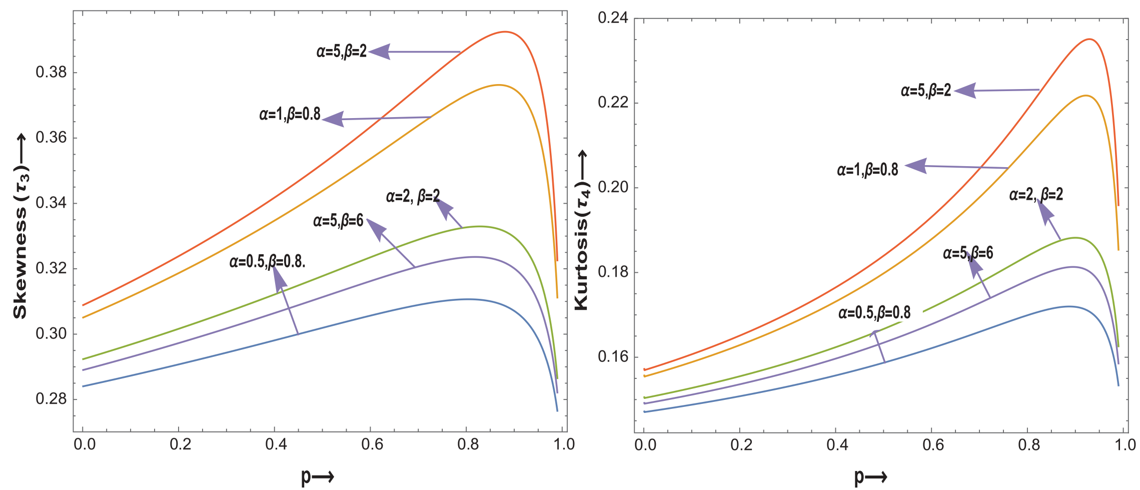
$$Q_{(1)}(u) = Q(1 - (1-u)^{\frac{1}{n}})$$



(a) Skewness(τ_3)

(b) Kurtosis (τ_4)

Figure 3. Skewness and kurtosis of the *HLEG*(α, β, p) distribution for selected values of α and p as a function of the parameter β.



(a) Skewness(τ_3) (b) Kurtosis (τ_4)

Figure 4. Skewness and kurtosis of the $HLEG(\alpha, \beta, p)$ distribution for selected values of α and β as a function of the parameter p.

$$= \alpha \log\left(p - (p-1)(1-u)^{-1/n}\right) + \beta \log\left(2(1-u)^{-1/n} - 1\right),$$

and the nth order statistic $X_{n:n}$ has the quantile function

$$Q_{(n)}(u) = Q(u^{\frac{1}{n}})$$
$$= \alpha \log\left(\frac{1 - pu^{1/n}}{1 - u^{1/n}}\right) + \beta \log\left(\frac{u^{1/n} + 1}{1 - u^{1/n}}\right).$$

5. Reliability properties

One of the basic concepts employed for modeling and analysis of lifetime data is the hazard rate. In a quantile setup, Nair and Sankaran (2009) defined the hazard quantile function, which is equivalent to the hazard rate. The hazard quantile function $H(u)$ is defined as

$$H(u) = h(Q(u)) = [(1-u)q(u)]^{-1}. \tag{31}$$

Thus, $H(u)$ can be interpreted as the conditional probability of failure of a unit in the next small interval of time given the survival of the unit until $100(1-u)\%$ point of the distribution. Note that $H(u)$ uniquely determines the distribution using the identity

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)}. \tag{32}$$

Since the proposed class of distributions is the sum of quantile functions of exponential geometric and half logistic quantile functions, Eqs. (31) and (32) give

$$\frac{1}{H(u)} = \frac{1}{H_1(u)} + \frac{1}{H_2(u)}, \tag{33}$$

where $H(u), H_1(u)$, and $H_2(u)$ are the hazard quantile functions of the proposed class of distributions, exponential geometric, and half logistic quantile functions, respectively. From Eq. (33), the proposed class of distributions (10) has hazard quantile function proportional to the harmonic average of the hazard quantile functions of exponential geometric and half logistic quantile functions. For the class of distributions (10), we have

$$H(u) = \frac{(u+1)(pu-1)}{\alpha(p-1)(u+1) + 2\beta(pu-1)}. \tag{34}$$

The shape of the hazard function is determined by the derivative of $H(u)$, which is obtained as

$$H'(u) = \frac{\alpha p(p-1)(u+1)^2 + 2\beta(pu-1)^2}{(\alpha(p-1)(u+1) + 2\beta(pu-1))^2}. \tag{35}$$

Since $(\alpha(p-1)(u+1) + 2\beta(pu-1))^2 > 0$ for all values of the parameters, the sign of $H'(u)$ depends only on

$$K(u) = \alpha p(p-1)(u+1)^2 + 2\beta(pu-1)^2. \tag{36}$$

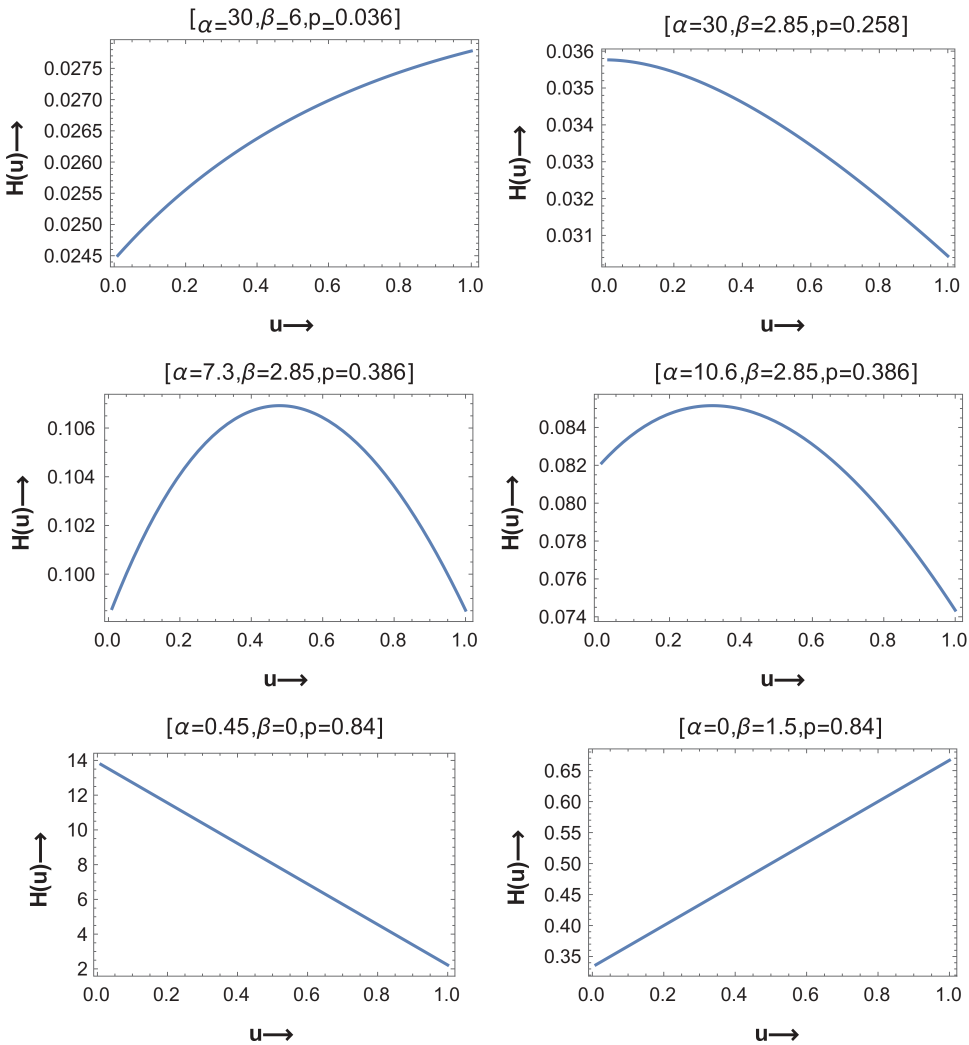


Figure 5. Plots of hazard quantile function for different values of parameters.

The hazard quantile function accommodates increasing, decreasing, linear, and upside-down bathtub shapes for different choices of parameters. Plots of hazard quantile function for different values of parameters are given in Figure 5. Now we consider the following cases.

Case 1. $p = 0, \alpha > 0$ and $\beta > 0$.

$$K(u) = 2\beta.$$

The first term in $K(u)$ is zero and the second term is positive, so that $K(u) > 0$ for all $0 < u < 1$ and the distribution has an increasing hazard rate (IHR).

Case 2. $p = 1, \alpha > 0$ and $\beta > 0$.

$$K(u) = 2\beta(u-1)^2.$$

The first term in $K(u)$ is zero and the second term is positive, so that $K(u)>0$ for all $0<u<1$ and the distribution has an increasing hazard rate (IHR).

Case 3. $p=0$, $\beta=0$ and $\alpha>0$.

$$H(u)=\frac{1}{\alpha},\ \text{aconstant}.$$

Thus, the distribution is exponential.

Case 4. $0<p\langle 1,\alpha\rangle 0$ and $\beta>\frac{2\alpha p}{(1-p)}$.

Now X is IHR if and only if $K(u)>0$ for all $u\in(0,1)$. This holds if and only if

$$p(p-1)\alpha(1+u)^2>-2\beta(pu-1)^2, \tag{37}$$

which gives

$$\frac{2\beta}{\alpha p(1-p)}>\frac{(1+u)^2}{(pu-1)^2}. \tag{38}$$

Since $(1+u)^2>(pu-1)^2$, forall $0<u<1$ and $0<p<1$, we have that the right side of Eq. (38) is increasing in u and attains its maximum when $u=1$. Now for $u=1$, the inequality (38) reduces to $\beta>\frac{2\alpha p}{(1-p)}$, and thus it is clear that $H(u)$ is increasing in this case.

Case 5. $0<p\langle 1,\alpha\rangle 0$ and $0<\beta<\frac{\alpha p(1-p)}{2}$.

Similar to Case 4, we can show that $H(u)$ has a decreasing hazard rate (DHR) if and only if

$$p(p-1)\alpha(1+u)^2<-2\beta(pu-1)^2 \tag{39}$$

or

$$\frac{2\beta}{\alpha p(1-p)}<\frac{(1+u)^2}{(pu-1)^2}. \tag{40}$$

Since right side of Eq. (40) is increasing in u and attains its minimum when $u=0$, the inequality (38) reduces to $\beta<\frac{\alpha p(1-p}{2}$. Thus, the distribution is DHR.

Case 6. $0<p\langle 1,\alpha\rangle 0$ and $\frac{\alpha p(1-p}{2}<\beta<\frac{2\alpha p}{1-p}$.

The first term of $K(u)$ is negative and the second term is positive, so that $K(u)$ attains zero in this case. This, in turn, gives $H'(u)=0$, suggesting the possibility for $H(u)$ to be nonmonotone. Let u_0 be the solution of the equation $K(u)=0$. From Eq. (36), we have that u_0 is the solution corresponding to the quadratic equation

$$u^2(\alpha p(p-1)+2\beta p^2)+u(2\alpha p(p-1)-4p\beta)+(\alpha p(p-1)+2\beta)=0, \tag{41}$$

which provides

$$u_0=\frac{-\alpha p^2-\sqrt{2}\sqrt{-\alpha\beta p^4-\alpha\beta p^3+\alpha\beta p^2+\alpha\beta p}+\alpha p+2\beta p}{\alpha p^2+2\beta p^2-\alpha p}. \tag{42}$$

For further analysis, we note that the second derivative of $H(u)$ is

$$H''(u)=\frac{4\alpha\beta(1-p)(p+1)^2}{(\alpha(p-1)(u+1)+2\beta(pu-1))^3}. \tag{43}$$

For the change point u_0 obtained in Eq. (42), we get

$$H''(u_0) = -\frac{\sqrt{2}p^2}{\sqrt{\alpha\beta(1-p)p(p+1)^2}}. \tag{44}$$

Since $H''(u_0) < 0$, we have that $H(u)$ attains a maximum at u_0. Hence X has an upside-down bathtub-shaped hazard quantile function (see Nair et al. 2013).

The aging patterns of $H(u)$ for various parameter values are summarized in Table 1.

We can easily show the following lemma, which is useful for finding bounds of $H(u)$.

Lemma 5.1. The limits of $HLEG(\alpha, \beta, p)$ hazard quantile function is given by

$$\lim_{u \to 0} H(u) = \frac{1}{\alpha(1-p) + 2\beta} \quad \text{and} \quad \lim_{u \to 1} H(u) = \frac{1}{\alpha + \beta}, \tag{45}$$

where $\alpha > 0$, $\beta > 0$, and $0 < p < 1$.

Proof. It is straightforward to show the results of Eq. (45) by taking the limit of $HLEG(\alpha, \beta, p)$ hazard quantile function, Eq. (44).

⊠

Theorem 5.1. If $X \sim HLEG(\alpha, \beta, 1)$, then the two limits of hazard quantile function are independent of the parameter α as given here:

(i) $\lim_{u\to 1} H(u) = 2 \lim_{u \to 0} H(u)$

and

(ii) $\frac{1}{2\beta} < H(u) < \frac{1}{\beta}$, for all $0 < u < 1$ and $\beta > 0$.

Proof.

(i) The proof is direct once we note that

$$\lim_{u \to 0} H(u) = \frac{1}{2\beta} \text{ and } \lim_{u \to 1} H(u) = \frac{1}{\beta}. \tag{46}$$

(ii) From Table 1, $H(u)$ is IHR for $p = 1$, $\alpha > 0$, and $\beta > 0$. Thus, lower and upper bounds for $H(u)$ exist when u approaches 0 and 1, respectively.

Now from Eq. (46), we get

$$\frac{1}{2\beta} < H(u) < \frac{1}{\beta} \quad \text{for all } 0 < u < 1 \text{ and } \beta > 0.$$

This completes the proof.

⊠

Theorem 5.2. If $X \sim HLEG(\alpha, \beta, 0)$, then the bounds of $H(u)$ are given by

Table 1. Aging behavior of the hazard quantile function for different regions of parameter space.

Serial number	Parameter region	Shape of hazard quantile function
1	$p = 0, \alpha > 0$ and $\beta > 0$	IHR
2	$p = 1, \alpha > 0$ and $\beta > 0$	IHR
3	$p = 0$, $\beta = 0$ and $\alpha > 0$	Constant
4	$0 < p\langle 1, \alpha\rangle 0$ and $\beta > \frac{2\alpha p}{(1-p)}$	IHR
5	$0 < p\langle 1, \alpha\rangle 0$ and $0 < \beta < \frac{\alpha p(1-p)}{2}$	DHR
6	$0 < p\langle 1, \alpha\rangle 0$ and $\frac{\alpha p(1-p)}{2} < \beta < \frac{2\alpha p}{1-p}$	Upside-down bathtub

$$\frac{1}{\alpha+2\beta} < H(u) < \frac{1}{\alpha+\beta}, \quad \text{for all } 0<u<1 \text{ and } \beta>0.$$

Proof.

The proof is similar to that of Theorem 5.1 once we note that

$$\lim_{u\to 0} H(u) = \frac{1}{\alpha+2\beta} \text{ and } \lim_{u\to 1} H(u) = \frac{1}{\alpha+\beta}, \tag{47}$$

and $H(u)$ is increasing for $p=0, \alpha>0$ and $\beta>0$.

☒

Theorem 5.3. Let $X \sim HLEG(\alpha,\beta,p)$. Then the hazard quantile function satisfies the following;

$$(i) \text{ If } \beta > \frac{2\alpha p}{(1-p)} \text{ then } \frac{1}{\alpha(1-p)+2\beta} < H(u) < \frac{1}{\alpha+\beta}$$

;

$$(ii) \text{ If } 0<\beta<\frac{\alpha p(1-p)}{2} \text{ then } \frac{1}{\alpha+\beta} < H(u) < \frac{1}{\alpha(1-p)+2\beta}$$

.

Proof.

From Table 1, note that X is IHR when $\beta > \frac{2\alpha p}{(1-p)}$. Now from Lemma 5.1, we get

$$\frac{1}{\alpha(1-p)+2\beta} < H(u) < \frac{1}{\alpha+\beta}. \tag{48}$$

To prove (ii), note that X is DHR for $0<\beta<\frac{\alpha p(1-p)}{2}$. Since $H(u)$ is decreasing over u, boundary values are reversed. This completes the proof.

☒

Mean residual function is a well-known measure that has been widely used for modeling lifetime data in reliability and survival analysis. For a nonnegative random variable X, the mean residual life function is defined as

$$m(x) = E(X-x|X\rangle x) = \frac{1}{1-F(x)}\int_x^\infty (1-F(t))dt. \tag{49}$$

The mean residual quantile function, which is the quantile version of the mean residual function (49), defined by Nair and Sankaran (2009), has the expression

$$M(u) = \frac{1}{1-u}\int_u^1 (Q(p)-Q(u))dp. \tag{50}$$

For the class of distributions (10), $M(u)$ has the form

$$M(u) = \frac{\beta\log(4) + \frac{\alpha(p-1)\log\left(\frac{p-1}{pu-1}\right)}{p} - 2\beta\log(u+1)}{1-u}. \tag{51}$$

It is well known that increasing (decreasing) failure rate implies decreasing (increasing) mean residual life (see Lai and Xie 2006). The aging behavior of the class of distributions (10) based on mean residual quantile function can be defined from Table 1. There exist closed-form expressions of the hazard quantile function and mean residual quantile function defined in reverse time (see Nair and Sankaran 2009) for the proposed class of distributions (10).

The total time on test transform (TTT) is a widely accepted statistical tool that has many applications in reliability analysis (see Lai and Xie 2006). The quantile-based TTT introduced in Nair et al. (2008) has the form

$$T(u) = \int_0^u (1-p)q(p)dp. \tag{52}$$

For the class of distributions (10), we obtain $T(u)$ as

$$T(u) = \frac{\alpha(p-1)\log(1-pu)}{p} + 2\beta\log(u+1). \tag{53}$$

In a fundamental paper on exploratory data analysis using quantile functions, Parzen (1979) introduced the score function, defined as

$$J(u) = \frac{q'(u)}{q^2(u)}, \tag{54}$$

where $q'(u)$ is the derivative of $q(u)$. Nair et al. (2012) studied properties of $J(u)$ in the context of lifetime data analysis. For the class of distributions (10), $J(u)$ is obtained as

$$J(u) = \frac{q'(u)}{q^2(u)} = \frac{\alpha(p-1)(u+1)^2(p(2u-1)-1) + 4\beta u(pu-1)^2}{(\alpha(p-1)(u+1) + 2\beta(pu-1))^2}. \tag{55}$$

It is customary to characterize life distributions by the relationships among reliability concepts. In the same spirit, we prove the following characterization theorem.

Theorem 5.4. A nonnegative continuous random variable X follows:

(a) HLEG$(u;\alpha,0,p)$ if and only if any one of the following properties hold.

(i) $H(u) = A_1 - A_2 u$, $0 < A_2 < A_1$

(ii) $J(u) = H(u) + C(1-u)$, $C > 0$

(iii) $T(u) = \frac{-1}{A_2}\log\left(\frac{H(u)}{A_1}\right)$

and

(b) HLEG$(u;0,\beta,p)$ if and only if any one of the following properties hold.

(i) $H(u) = K(1+u)$, $K > 0$

(ii) $J(u) = 2Ku$

(iii) $T(u) = \frac{1}{K}\log(K^K H(u))$

Proof. We prove the result for (a). The proof for (b) is similar.

(a) Suppose the identity (i) is true.

Then $H(u) = A_1 - A_2 u$, so that the corresponding quantile function is obtained as

$$Q(u) = \frac{\log\left(\frac{A_1 - A_2 u}{A_1(1-u)}\right)}{A_1 - A_2}, \tag{56}$$

which is equivalent to HLEG$(u;\alpha,0,p)$, with $\alpha = \frac{1}{A_1 - A_2} > 0$ and $0 < p = \frac{A_2}{A_1} < 1$. The converse part is direct from the definition of $H(u)$ given in section 6.

When (ii) is true, we have from Nair and Sankaran (2009)

$$(1-u)H'(u) = H(u) - J(u). \tag{57}$$

Thus, we obtain

$$(1-u)H'(u) = C(u-1). \tag{58}$$

The solution of the ordinary differential equation (58) is,

$$H(u) = D - Cu, \quad C > 0, D - C > 0, \tag{59}$$

which satisfies (i), so that proof is completed.

Suppose (iii) is true. Differentiating (iii) with respect to u we get,

$$T'(u) = \frac{-H'(u)}{A_2 H(u)}. \tag{60}$$

Differentiating Eq. (52) with respect to u we get

$$T'(u) = (1-u)q(u) = \frac{1}{H(u)}. \tag{61}$$

From Eqs. (60) and (61), we get

$$H'(u) = -A_2, \tag{62}$$

which leads to (i). Conversely, for the class of distributions HLEG$(u;\alpha,0,p)$ we obtain

$$T(u) = \frac{\alpha(p-1)\log(1-pu)}{p}, \tag{63}$$

or

$$T(u) = \frac{-1}{A_2}\log\left(\frac{H(u)}{A_1}\right), \tag{64}$$

where $A_1 = \frac{1}{\alpha(1-p)}$ and $A_2 = \frac{p}{\alpha(1-p)}$.

This completes the proof.

☒

6. Inference and applications

There are different methods for the estimation of parameters of the quantile function. The method of percentiles, method of L-moments, method of minimum absolute deviation, method of least squares, and method of maximum likelihood are commonly used techniques. To estimate the parameters of Eq. (10), we use the method of L-moments. We equate sample L-moments to corresponding population L-moments. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the population with quantile function (10); then the sample L-moments are given by

$$\begin{aligned} l_1 &= \left(\tfrac{1}{n}\right)\sum_{i=1}^{n} x_{(i)} \\ l_2 &= \left(\tfrac{1}{2}\right)\binom{n}{2}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{1}-\binom{n-i}{1}\right)x_{(i)} \\ l_3 &= \left(\tfrac{1}{3}\right)\binom{n}{3}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{2}-2\binom{i-1}{1}\binom{n-i}{1}+\binom{n-i}{2}\right)x_{(i)} \end{aligned}$$

where $x_{(i)}$ is the ith order statistic.

For estimating the parameters $\alpha, \beta,$ and σ, we equate first three sample L-moments to population L-moments given in section 4. The parameters are obtained by solving the equations

$$l_r = L_r; \quad r = 1, 2, 3. \tag{65}$$

Since L_1 is the mean of the distribution, mean survival time is estimated as l_1. Similarly, the estimate of variance is obtained as $\hat{V}(x) = \int_0^1 (\hat{Q}(u))^2 du - l_1^2,$ which can be evaluated with the help of numerical integration techniques.

Hosking (1990) has shown that the L-moment estimates are asymptotically normal and consistent. Specifically, Hosking (1990) has shown that $\sqrt{n}(l_r - L_r)$, $r = 1, 2, \ldots., m$, converges to the multivariate normal distribution $N(0, \Delta)$, where the elements $\Delta_{r,s}$ of Δ are given by,

$$\Lambda_{r,s} = \int_{0<u<v<1} \{P^*_{r-1}(u)P^*_{s-1}(v) + P^*_{s-1}(u)P^*_{r-1}(v)\} u(1-v)q(u)q(v)dudv, \tag{66}$$

where $P^*_r(x) = \sum_{k=0}^{r} (-1)^{r-k}\binom{r}{k}\binom{r+k}{k}x^k$. Since the set of equations (65) are nonlinear in α, β, and p, asymptotic variances of the L-moment estimates are difficult to compute. One can use the bootstrap method to obtain the asymptotic variance of the estimates.

To illustrate the application of the proposed class of distributions we consider a real data set reported in Zimmer et al. (1998). The data consist of times to first failure of 20 electric carts. We estimate the parameters using the method of L-moments. The sample L-moments are obtained as

$$l_1 = 12.66 \qquad l_2 = 5.91 \quad \text{and} \quad l_3 = 1.57. \tag{67}$$

We then equate these values to the corresponding population L-moments given in Eqs. (24), (25), and (26), so that we have three nonlinear equations. The Newton–Raphson method is used to find the solutions of these equations. A least-square method of estimation for quantile functions given in O¨ztu¨rk and Dale (1985) was employed for fixing the initial estimates for the Newton–Raphson iterative procedure. The estimates of the parameters are obtained as

$$\hat{\alpha} = 8.518 \qquad \hat{\beta} = 1.209 \quad \text{and} \quad \hat{p} = 0.329. \tag{68}$$

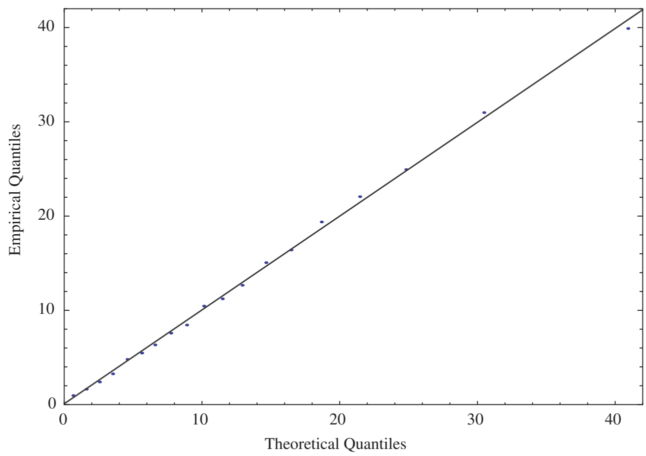


Figure 6. Q–Q plot for the data set.

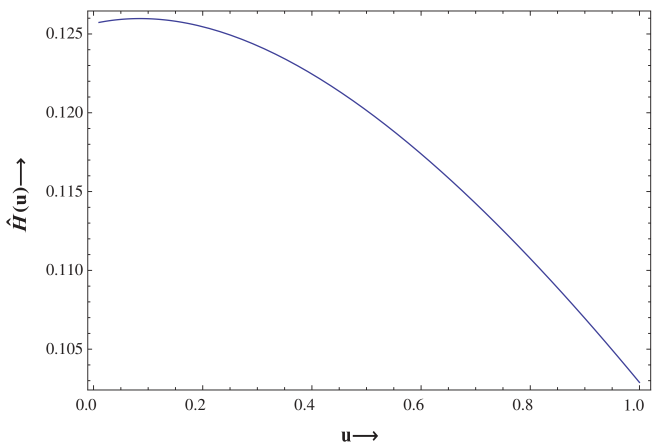


Figure 7. $H(u)$ for the data set.

To examine the adequacy of the model, two goodness-of-fit techniques are employed. The first one is the Q–Q plot, which is given in Figure 6.

The Q–Q plot shows that the proposed model is appropriate for the given data set. We also carry out the chi-squared goodness-of-fit test. The chi-squared test statistic value is 0.210, giving a p value 0.647 with one degree of freedom. This also indicates the adequacy of proposed model for the given data set. We compute the estimate of $H(u)$ by substituting the parameter values of Eq. (68) in Eq. (34), which is given in Figure 7. Note that the estimate $\hat{H}(u)$ is decreasing in u, which is consistent with our claim in Table 1.

7. Conclusion

In this article, we have introduced a class of distributions (10) that are the sum of quantile functions of the half logistic and exponential geometric quantile functions.

Various reliability properties are studied. We have identified several well-known distributions that are members of the proposed class of distributions. The estimation of the parameters of the model using L-moments was studied, and we discussed the estimation procedure with the aid of a real data set. The proposed model has several advantages over the existing quantile function models. The analysis of hazard quantile function over the whole parameter space can be done without using numerical methods. The model is useful for fitting different types of lifetime data due to the flexible behavior of the hazard quantile function. Unlike the generalized lambda distribution and generalized Tukey lambda distribution, the estimation of parameters does not involve any computational difficulties.

There are several properties and extensions of the new family of distributions not considered in this article, such as stochastic orderings and quantile-based cumulative residual entropy. Estimation using the maximum likelihood method and Bayes technique need numerical approximations. The study of multivariate generalizations of the HLEG distribution is interesting, and will be addressed later.

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