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On the likelihood ratio test for the equality of multivariate normal populations with two-step monotone missing data

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ABSTRACT
In this article, we consider the problem of testing the equality of multivariate normal populations when the data set has missing observations with a two-step monotone pattern. The likelihood ratio test (LRT) statistic for the simultaneous testing of the mean vectors and the covariance matrices is given under the condition of two-step monotone missing data. An approximate modified likelihood ratio test (MLRT) statistic is presented using linear interpolation based on the coefficients of the MLRT statistic in the case of complete data sets. As an alternative approach, we propose approximate MLRT statistics of two kinds with two-step monotone missing data using the decompositions of the likelihood ratio (LR). An approximate upper percentile of the LRT statistic with two-step monotone missing data is also derived based on an asymptotic expansion for the LRT statistic in the case of complete data sets. Finally, we investigate the accuracy of the approximations using Monte Carlo simulation.

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1. Introduction

Let $\boldsymbol{x}_1^{(\ell)}, \boldsymbol{x}_2^{(\ell)}, \ldots, \boldsymbol{x}_{N_1^{(\ell)}}^{(\ell)}$ be independent p-dimensional sample vectors and $\boldsymbol{x}_{1,N_1^{(\ell)}+1}^{(\ell)}$, $\boldsymbol{x}_{1,N_1^{(\ell)}+2}^{(\ell)}, \ldots, \boldsymbol{x}_{1N^{(\ell)}}^{(\ell)}$ be independent p_1-dimensional sample vectors from the ℓth population $(\ell = 1, 2, \ldots, m)$, where $p_1 < p < N_1^{(\ell)}$ and the $\boldsymbol{x}_j^{(\ell)}$s and $\boldsymbol{x}_{1j}^{(\ell)}$s are mutually independent. We assume that

$$\boldsymbol{x}_1^{(\ell)}, \boldsymbol{x}_2^{(\ell)}, \ldots, \boldsymbol{x}_{N_1^{(\ell)}}^{(\ell)} \sim N_p(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma}^{(\ell)}),$$

and

$$\boldsymbol{x}_{1,N_1^{(\ell)}+1}^{(\ell)}, \boldsymbol{x}_{1,N_1^{(\ell)}+2}^{(\ell)}, \ldots, \boldsymbol{x}_{1N^{(\ell)}}^{(\ell)} \sim N_{p_1}\left(\boldsymbol{\mu}_1^{(\ell)}, \boldsymbol{\Sigma}_{11}^{(\ell)}\right),$$

where

$$\boldsymbol{\mu}^{(\ell)} = \begin{pmatrix} \boldsymbol{\mu}_1^{(\ell)} \\ \boldsymbol{\mu}_2^{(\ell)} \end{pmatrix}, \quad \boldsymbol{\Sigma}^{(\ell)} = \begin{pmatrix} \boldsymbol{\Sigma}_{11}^{(\ell)} & \boldsymbol{\Sigma}_{12}^{(\ell)} \\ \boldsymbol{\Sigma}_{21}^{(\ell)} & \boldsymbol{\Sigma}_{22}^{(\ell)} \end{pmatrix}.$$

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We partition $\boldsymbol{x}_j^{(\ell)}$ into a $p_1 \times 1$ random vector and a $p_2 \times 1$ random vector, as $\boldsymbol{x}_j^{(\ell)} = (\boldsymbol{x}_{1j}^{(\ell)'}, \boldsymbol{x}_{2j}^{(\ell)'})'$, where $\boldsymbol{x}_{ij}^{(\ell)}$: $p_i \times 1$, $i = 1, 2$, $j = 1, 2, \ldots, N_1^{(\ell)}$, $\ell = 1, 2, \ldots, m$, and $p = p_1 + p_2$. That is, the data set is of the form

$$\overbrace{}^{p}$$
$$\left(\begin{array}{c|cc} \boldsymbol{x}_{11}^{(\ell)'} & \boldsymbol{x}_{21}^{(\ell)'} & \\ \vdots & \vdots & \\ \boldsymbol{x}_{1N_1^{(\ell)}}^{(\ell)'} & \boldsymbol{x}_{2N_1^{(\ell)}}^{(\ell)'} & \\ \hline \boldsymbol{x}_{1,N_1^{(\ell)}+1}^{(\ell)'} & * \cdots * & \\ \vdots & \vdots \quad \vdots & \\ \boldsymbol{x}_{1N^{(\ell)}}^{(\ell)'} & * \cdots * & \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} N_1^{(\ell)} \\ \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} N_2^{(\ell)} \end{array},$$
$$\underbrace{}_{p_1} \quad \underbrace{}_{p_2}$$

where an asterisk indicates a missing observation. Such a data set describes two-step monotone missing data from an ℓth population.

In this article, we consider the m-population problem of the simultaneous testing of the mean vectors and the covariance matrices when the data sets have the same two-step monotone missing data. For the case without missing data, the LRT statistic and the MLRT statistic have previously been derived, and an asymptotic expansion of the null distribution for MLRT statistic has been given (see Muirhead 1982; Siotani, Hayakawa, and Fujikoshi 1985; Srivastava 2002). The one-population problem under the condition of two-step monotone missing data has been discussed by Hao and Krishnamoorthy (2001) and Hosoya and Seo (2015). In particular, Hosoya and Seo (2015) derived the LRT statistic and the approximate MLRT statistics. Recently, Tsukada (2014) discussed the simultaneous testing of the mean vectors and the covariance matrices for an m-population problem under the condition of two-step monotone missing data. In this article, we present an extension of the results for a one-population problem given in Hosoya and Seo (2015) to an m-population problem. The remainder of this article is organized as follows. In section 2, we derive the MLEs using the transformation matrix, and derive the LRT statistic under the condition of two-step monotone missing data. In section 3, we propose three approximate MLRT statistics using the coefficients of the MLRT statistic for the complete data. In section 4, as an alternative approach, we propose an approximate upper percentile of the LRT statistic. Finally, in section 5 the accuracy of the approximation and the asymptotic behavior of the modified statistics are investigated using Monte Carlo simulation.

2. MLE and LRT statistic

In this section, we consider the LRT statistic for testing the equality of multivariate normal populations with two-step monotone missing data. To derive the LRT statistic for

$$H_0\colon \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} = \cdots = \boldsymbol{\mu}^{(m)}, \quad \boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(m)} \text{vs.}\, H_1\colon \text{ not}\, H_0, \tag{1}$$

we first consider the MLEs of the mean vectors and the covariance matrices under the null and nonnull hypotheses. The MLEs for a one-population problem were derived using the decomposition of the density into conditional densities by Kanda and Fujikoshi (1998). Similarly to the one-population case, we can obtain the MLEs for the m-population problem under the condition of two-step monotone missing data. Following the derivation of Kanda and Fujikoshi (1998), we use the following transformed parameters $(\boldsymbol{\eta}^{(\ell)},\ \boldsymbol{\Delta}^{(\ell)})$:

$$\boldsymbol{\eta}^{(\ell)} = \begin{pmatrix} \boldsymbol{\eta}_1^{(\ell)} \\ \boldsymbol{\eta}_2^{(\ell)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1^{(\ell)} \\ \boldsymbol{\mu}_2^{(\ell)} - \boldsymbol{\Delta}_{21}^{(\ell)}\boldsymbol{\mu}_1^{(\ell)} \end{pmatrix},$$

$$\boldsymbol{\Delta}^{(\ell)} = \begin{pmatrix} \boldsymbol{\Delta}_{11}^{(\ell)} & \boldsymbol{\Delta}_{12}^{(\ell)} \\ \boldsymbol{\Delta}_{21}^{(\ell)} & \boldsymbol{\Delta}_{22}^{(\ell)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{11}^{(\ell)} & \boldsymbol{\Sigma}_{11}^{(\ell)^{-1}}\boldsymbol{\Sigma}_{12}^{(\ell)} \\ \boldsymbol{\Sigma}_{21}^{(\ell)}\boldsymbol{\Sigma}_{11}^{(\ell)^{-1}} & \boldsymbol{\Sigma}_{22\cdot 1}^{(\ell)} \end{pmatrix},$$

where $\boldsymbol{\Sigma}_{22\cdot 1}^{(\ell)} = \boldsymbol{\Sigma}_{22}^{(\ell)} - \boldsymbol{\Sigma}_{21}^{(\ell)}\boldsymbol{\Sigma}_{11}^{(\ell)^{-1}}\boldsymbol{\Sigma}_{12}^{(\ell)}$. We note that the pair $(\boldsymbol{\eta}^{(\ell)}, \boldsymbol{\Delta}^{(\ell)})$ is in one-to-one correspondence with $(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma}^{(\ell)})$.

Under H_1, the MLEs of $\boldsymbol{\eta}^{(\ell)}$ and $\boldsymbol{\Delta}^{(\ell)}$ are given by

$$\hat{\boldsymbol{\eta}}^{(\ell)} = \begin{pmatrix} \hat{\boldsymbol{\eta}}_1^{(\ell)} \\ \hat{\boldsymbol{\eta}}_2^{(\ell)} \end{pmatrix} = \begin{pmatrix} \frac{1}{N^{(\ell)}}(N_1^{(\ell)}\bar{\boldsymbol{x}}_{(1)1}^{(\ell)} + N_2^{(\ell)}\bar{\boldsymbol{x}}_{(2)}^{(\ell)}) \\ \bar{\boldsymbol{x}}_{(1)2}^{(\ell)} - \hat{\boldsymbol{\Delta}}_{21}^{(\ell)}\bar{\boldsymbol{x}}_{(1)1}^{(\ell)} \end{pmatrix},$$

and

$$\hat{\boldsymbol{\Delta}}^{(\ell)} = \begin{pmatrix} \hat{\boldsymbol{\Delta}}_{11}^{(\ell)} & \hat{\boldsymbol{\Delta}}_{12}^{(\ell)} \\ \hat{\boldsymbol{\Delta}}_{21}^{(\ell)} & \hat{\boldsymbol{\Delta}}_{22}^{(\ell)} \end{pmatrix} = \begin{pmatrix} \frac{1}{N^{(\ell)}}(\boldsymbol{W}_{(1)11}^{(\ell)} + \boldsymbol{W}_{(2)}^{(\ell)}) & \boldsymbol{W}_{(1)11}^{(\ell)^{-1}}\boldsymbol{W}_{(1)12}^{(\ell)} \\ \boldsymbol{W}_{(1)21}^{(\ell)}\boldsymbol{W}_{(1)11}^{(\ell)^{-1}} & \frac{1}{N_1^{(\ell)}}\boldsymbol{W}_{(1)22\cdot 1}^{(\ell)} \end{pmatrix},$$

respectively, where

$$\bar{\boldsymbol{x}}_{(1)}^{(\ell)} = \begin{pmatrix} \bar{\boldsymbol{x}}_{(1)1}^{(\ell)} \\ \bar{\boldsymbol{x}}_{(1)2}^{(\ell)} \end{pmatrix}, \ \bar{\boldsymbol{x}}_{(1)1}^{(\ell)} = \frac{1}{N_1^{(\ell)}}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{x}_{1j}^{(\ell)}, \ \bar{\boldsymbol{x}}_{(1)2}^{(\ell)} = \frac{1}{N_1^{(\ell)}}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{x}_{2j}^{(\ell)},$$

$$\bar{\boldsymbol{x}}_{(2)}^{(\ell)} = \frac{1}{N_2^{(\ell)}}\sum_{j=N_1^{(\ell)}+1}^{N^{(\ell)}}\boldsymbol{x}_{1j}^{(\ell)}, \ N^{(\ell)} = N_1^{(\ell)} + N_2^{(\ell)},$$

and

$$\boldsymbol{W}_{(1)}^{(\ell)} = \begin{pmatrix} \boldsymbol{W}_{(1)11}^{(\ell)} & \boldsymbol{W}_{(1)12}^{(\ell)} \\ \boldsymbol{W}_{(1)21}^{(\ell)} & \boldsymbol{W}_{(1)22}^{(\ell)} \end{pmatrix} = \sum_{j=1}^{N_1^{(\ell)}} \left(\boldsymbol{x}_j^{(\ell)} - \bar{\boldsymbol{x}}_{(1)}^{(\ell)}\right)\left(\boldsymbol{x}_j^{(\ell)} - \bar{\boldsymbol{x}}_{(1)}^{(\ell)}\right)',$$

$$\boldsymbol{W}_{(2)}^{(\ell)} = \sum_{j=N_1^{(\ell)}+1}^{N^{(\ell)}} \left(\boldsymbol{x}_{1j}^{(\ell)} - \bar{\boldsymbol{x}}_{(2)}^{(\ell)}\right)\left(\boldsymbol{x}_{1j}^{(\ell)} - \bar{\boldsymbol{x}}_{(2)}^{(\ell)}\right)' + \frac{N_1^{(\ell)} N_2^{(\ell)}}{N^{(\ell)}} \left(\bar{\boldsymbol{x}}_{(1)1}^{(\ell)} - \bar{\boldsymbol{x}}_{(2)}^{(\ell)}\right)\left(\bar{\boldsymbol{x}}_{(1)1}^{(\ell)} - \bar{\boldsymbol{x}}_{(2)}^{(\ell)}\right)',$$

$$\boldsymbol{W}_{(1)22\cdot1}^{(\ell)} = \boldsymbol{W}_{(1)22}^{(\ell)} - \boldsymbol{W}_{(1)21}^{(\ell)} \boldsymbol{W}_{(1)11}^{(\ell)^{-1}} \boldsymbol{W}_{(1)12}^{(\ell)}.$$

On the other hand, under H_0 the MLEs of $\boldsymbol{\eta}(= \boldsymbol{\eta}^{(1)} = \boldsymbol{\eta}^{(2)} = \cdots = \boldsymbol{\eta}^{(m)})$ and $\boldsymbol{\Delta}\left(= \boldsymbol{\Delta}^{(1)} = \boldsymbol{\Delta}^{(2)} = \cdots = \boldsymbol{\Delta}^{(m)}\right)$ are given by

$$\tilde{\boldsymbol{\eta}} = \begin{pmatrix} \tilde{\boldsymbol{\eta}}_1 \\ \tilde{\boldsymbol{\eta}}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{N} \displaystyle\sum_{\ell=1}^{m} (N_1^{(\ell)} \bar{\boldsymbol{x}}_{(1)1}^{(\ell)} + N_2^{(\ell)} \bar{\boldsymbol{x}}_{(2)}^{(\ell)}) \\ \dfrac{1}{N_1} \displaystyle\sum_{\ell=1}^{m} N_1^{(\ell)} (\bar{\boldsymbol{x}}_{(1)2}^{(\ell)} - \tilde{\boldsymbol{\Delta}}_{21} \bar{\boldsymbol{x}}_{(1)1}^{(\ell)}) \end{pmatrix},$$

and

$$\tilde{\boldsymbol{\Delta}} = \begin{pmatrix} \tilde{\boldsymbol{\Delta}}_{11} & \tilde{\boldsymbol{\Delta}}_{21}' \\ \tilde{\boldsymbol{\Delta}}_{21} & \tilde{\boldsymbol{\Delta}}_{22} \end{pmatrix},$$

respectively, where

$$N = \sum_{\ell=1}^{m} N^{(\ell)}, \quad N_1 = \sum_{\ell=1}^{m} N_1^{(\ell)},$$

$$\tilde{\boldsymbol{\Delta}}_{11} = \frac{1}{N} \sum_{\ell=1}^{m} \sum_{j=1}^{N^{(\ell)}} \left(\boldsymbol{x}_{1j}^{(\ell)} - \tilde{\boldsymbol{\eta}}_1\right)\left(\boldsymbol{x}_{1j}^{(\ell)} - \tilde{\boldsymbol{\eta}}_1\right)',$$

$$\tilde{\boldsymbol{\Delta}}_{21} = \sum_{\ell=1}^{m} \left\{ \sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{x}_{2j}^{(\ell)} \boldsymbol{x}_{1j}^{(\ell)'} - N_1^{(\ell)} \left(\frac{1}{N_1} \sum_{\ell=1}^{m} N_1^{(\ell)} \bar{\boldsymbol{x}}_{(1)2}^{(\ell)} \right) \bar{\boldsymbol{x}}_{(1)1}^{(\ell)'} \right\}$$
$$\times \sum_{\ell=1}^{m} \left\{ \sum_{j=1}^{N_1(\ell)} \boldsymbol{x}_{1j}^{(\ell)} \boldsymbol{x}_{1j}^{(\ell)'} - N_1^{(\ell)} \left(\frac{1}{N_1} \sum_{\ell=1}^{m} N_1^{(\ell)} \bar{\boldsymbol{x}}_{(1)1}^{(\ell)} \right) \bar{\boldsymbol{x}}_{(1)1}^{(\ell)'} \right\}^{-1},$$

$$\tilde{\boldsymbol{\Delta}}_{22} = \frac{1}{N_1} \sum_{\ell=1}^{m} \sum_{j=1}^{N_1^{(\ell)}} \left(\boldsymbol{x}_{2j}^{(\ell)} - \tilde{\boldsymbol{\Delta}}_{21} \boldsymbol{x}_{1j}^{(\ell)} - \tilde{\boldsymbol{\eta}}_2\right)\left(\boldsymbol{x}_{2j}^{(\ell)} - \tilde{\boldsymbol{\Delta}}_{21} \boldsymbol{x}_{1j}^{(\ell)} - \tilde{\boldsymbol{\eta}}_2\right)'.$$

We can obtain the MLEs $\tilde{\boldsymbol{\eta}}$ and $\tilde{\boldsymbol{\Delta}}$ in a straightforward method using the one-population case of Kanda and Fujikoshi (1998). We note that the MLEs for the two-population case are derived by Seko, Kawasaki, and Seo (2011). For the MLEs by other notations under the k-step monotone missing data, see Jinadasa and Tracy (1992) and Yagi and Seo (2014; 2015).

From the preceding MLEs, we obtain the following theorem.

Theorem 2.1. *Suppose that the data sets have the same two-step monotone missing pattern. Then, the LR of the hypothesis test (1) is given by*

$$\lambda = \frac{\prod_{l=1}^{m} |\hat{\boldsymbol{\Delta}}_{11}^{(\ell)}|^{\frac{1}{2}N^{(\ell)}} |\hat{\boldsymbol{\Delta}}_{22}^{(\ell)}|^{\frac{1}{2}N_1^{(\ell)}}}{|\tilde{\boldsymbol{\Delta}}_{11}|^{\frac{1}{2}N} |\tilde{\boldsymbol{\Delta}}_{22}|^{\frac{1}{2}N_1}}.$$

We note that under H_0, $-2\log\lambda$ is asymptotically distributed as a χ^2 distribution with $f = p(p+3)(m-1)/2$ degrees of freedom, when $N_1^{(\ell)}$, $N^{(\ell)} \to \infty$ with $N_1^{(\ell)}/N^{(\ell)} \to \delta^{(\ell)} \in (0,1]$, $\ell = 1,2,\ldots,m$. In particular, it should be noted that a χ^2 approximation is not useful for small samples. In sections 3 and 4, we present the modified LRT statistics and an approximate upper percentile of the LRT statistic $(-2\log\lambda)$ itself.

3. Approximate MLRT statistics

In this section, we propose an approximate MLRT statistic using linear interpolation based on the MLRT statistic in the case of a complete data set. Furthermore, as an alternative approach we provide two approximate MLRT statistics using the decomposition of the LR.

First, we consider the LR in the case of a complete data set. Let $\boldsymbol{x}_1^{(\ell)}, \boldsymbol{x}_2^{(\ell)}, \ldots, \boldsymbol{x}_{N^{(\ell)}}^{(\ell)}$ be sample vectors of size $N^{(\ell)}$ from $N_p(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma}^{(\ell)})$, with $\ell = 1,2,\ldots,m$, and let $\lambda_{c,N}$ be the LR for the complete data set. Then the LR is given by

$$\lambda_{c,N} = \frac{\prod_{\ell=1}^{m} \left| \frac{1}{N^{(\ell)}} \boldsymbol{V}^{(\ell)} \right|^{\frac{1}{2}N^{(\ell)}}}{\left| \frac{1}{N}(\boldsymbol{V}+\boldsymbol{B}) \right|^{\frac{1}{2}N}},$$

where

$$\boldsymbol{V}^{(\ell)} = \sum_{i=1}^{N^{(\ell)}} (\boldsymbol{x}_i^{(\ell)} - \bar{\boldsymbol{x}}^{(\ell)})(\boldsymbol{x}_i^{(\ell)} - \bar{\boldsymbol{x}}^{(\ell)})', \quad \boldsymbol{V} = \sum_{\ell=1}^{m} \boldsymbol{V}^{(\ell)}, \quad \boldsymbol{B} = \sum_{\ell=1}^{m} N^{(\ell)} (\bar{\boldsymbol{x}}^{(\ell)} - \bar{\boldsymbol{x}})(\bar{\boldsymbol{x}}^{(\ell)} - \bar{\boldsymbol{x}})',$$

$$\bar{\boldsymbol{x}}^{(\ell)} = \frac{1}{N^{(\ell)}} \sum_{i=1}^{N^{(\ell)}} \boldsymbol{x}_i^{(\ell)}, \quad \bar{\boldsymbol{x}} = \frac{1}{N} \sum_{\ell=1}^{m} N^{(\ell)} \bar{\boldsymbol{x}}^{(\ell)}, \quad N = \sum_{\ell=1}^{m} N^{(\ell)}.$$

Furthermore, the MLRT statistic is given by $-2\rho_{c,N} \log \lambda_{c,N}$, where

$$\rho_{\mathrm{c},N}=1-\frac{2p^2+9p+11}{6N(p+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{N}{N^{(\ell)}}-1\right),$$

and its cumulative distribution function can be expanded as

$$\Pr\left(-2\rho_{\mathrm{c},N}\log\lambda_{\mathrm{c},N}\le x\right)=G_f(x)+\frac{\gamma_1}{M^2}\left\{G_{f+4}(x)-G_f(x)\right\}+\mathrm{O}\left(M^{-3}\right),$$

where

$$M=\rho_{\mathrm{c},N}N,\quad k_1^{(\ell)}=\frac{N^{(\ell)}}{N},$$

$$\gamma_1=\frac{1}{288}\left[6p(p+1)(p+2)(p+3)\left(\sum_{\ell=1}^{m}\frac{1}{\left\{k_1^{(\ell)}\right\}^2}-1\right)-\frac{(2p^2+9p+11)^2(2p-1)}{p(p+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{1}{k_1^{(\ell)}}-1\right)^2\right],$$

and $G_f(x)$ and $G_{f+4}(x)$ are the cumulative distribution functions of the χ^2 distribution with $f(=p(p+3)(m-1)/2)$ and $f+4$ degrees of freedoms, respectively. This means that if the χ^2 distribution is used as an approximation to the distribution of $-2\rho_{\mathrm{c},N}\log\lambda_{\mathrm{c},N}$, the error involved is not of order M^{-1}, but rather of order M^{-2}. The derivation of MLRT was presented by Muirhead (1982, 513).

In this section, we consider an approximation to the correction factor ρ_{miss}, because it is not easy to determine ρ_{miss} for the MLRT statistic $\left(-2\rho_{\mathrm{miss}}\log\lambda\right)$ under the condition of two-step monotone missing data. As with the one-population case considered in Hosoya and Seo (2015), we can propose an approximate MLRT statistic $-2\rho_{\mathrm{L}}\log\lambda$, where

$$\rho_{\mathrm{L}}=1-\frac{2p^2+9p+11}{6p(p+3)(m-1)}\left\{\left(\sum_{\ell=1}^{m}\frac{1}{N^{(\ell)}}-\frac{1}{N}\right)p_1+\left(\sum_{\ell=1}^{m}\frac{1}{N_1^{(\ell)}}-\frac{1}{N_1}\right)p_2\right\}.$$

Next, we provide two approximate MLRT statistics using the decompositions of LR(λ). First, the LR can be decomposed as $\lambda=\xi_1\xi_2$, where

$$\xi_1=\frac{\prod_{\ell=1}^{m}\left|\hat{\boldsymbol{\Delta}}_{11}^{(\ell)}\right|^{\frac{1}{2}N^{(\ell)}}}{\left|\hat{\boldsymbol{\Delta}}_{11}\right|^{\frac{1}{2}N}},\quad \xi_2=\frac{\prod_{\ell=1}^{m}\left|\hat{\boldsymbol{\Delta}}_{22}^{(\ell)}\right|^{\frac{1}{2}N_1^{(\ell)}}}{\left|\hat{\boldsymbol{\Delta}}_{22}\right|^{\frac{1}{2}N_1}}.$$

Because ξ_1 takes the form of the LR for H_0 in the case of without missing data, we can use the MLRT statistic $-2\rho_{\xi_1}\log\xi_1$ instead of $-2\log\xi_1$, where

$$\rho_{\xi_1}=1-\frac{2p_1^2+9p_1+11}{6N(p_1+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{N}{N^{(\ell)}}-1\right).$$

On the other hand, the quantity ξ_2 can be decomposed as $\xi_2=\xi_2^{\dagger}\xi_2^{\dagger\dagger}$, where $\xi_2^{\dagger}$ takes the form of the LR for H_0 in the case of without missing data. Therefore, we can use the

MLRT statistic $-2\rho_{\xi_2}\log\xi_2^{\dagger}$ instead of $-2\log\xi_2^{\dagger}$, where

$$\rho_{\xi_2} = 1 - \frac{2p_2^2 + 9p_2 + 11}{6N_1(p_2+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{N_1}{N_1^{(\ell)}} - 1\right),$$

$$\xi_2^{\dagger} = \frac{\prod_{\ell=1}^{m}\left|\hat{\boldsymbol{\Delta}}_{22}^{(\ell)}\right|^{\frac{1}{2}N_1^{(\ell)}}}{\left|\frac{1}{N_1}\left(\boldsymbol{V}_{p_2} + \boldsymbol{B}_{p_2}\right)\right|^{\frac{1}{2}N_1}}, \quad \xi_2^{\dagger\dagger} = \frac{\left|\frac{1}{N_1}\left(\boldsymbol{V}_{p_2} + \boldsymbol{B}_{p_2}\right)\right|^{\frac{1}{2}N_1}}{\left|\tilde{\boldsymbol{\Delta}}_{22}\right|^{\frac{1}{2}N_1}},$$

$$\boldsymbol{V}_{p_2} = \sum_{\ell=1}^{m}\boldsymbol{V}_{p_2}^{(\ell)}, \quad \boldsymbol{B}_{p_2} = \sum_{\ell=1}^{m}N_1^{(\ell)}\left(\hat{\boldsymbol{\eta}}_2^{(\ell)} - \tilde{\boldsymbol{\eta}}_2\right)\left(\hat{\boldsymbol{\eta}}_2^{(\ell)} - \tilde{\boldsymbol{\eta}}_2\right)',$$

$$\boldsymbol{V}_{p_2}^{(\ell)} = \sum_{j=1}^{N_1^{(\ell)}}\left(\boldsymbol{x}_{2j}^{(\ell)} - \hat{\boldsymbol{\Delta}}_{21}^{(\ell)}\boldsymbol{x}_{1j}^{(\ell)} - \hat{\boldsymbol{\eta}}_2^{(\ell)}\right)\left(\boldsymbol{x}_{2j}^{(\ell)} - \hat{\boldsymbol{\Delta}}_{21}^{(\ell)}\boldsymbol{x}_{1j}^{(\ell)} - \hat{\boldsymbol{\eta}}_2^{(\ell)}\right)'.$$

Thus, we propose a new approximate MLRT statistic given by $-2\log\ \tau$, where

$$\tau = (\xi_1)^{\rho_{\xi_2}}\left(\xi_2^{\dagger}\right)^{\rho_{\xi_2}}\xi_2^{\dagger\dagger}.$$

In addition, we consider the testing of the equality of covariance matrices; that is,

$$H_{01}\colon \boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(m)} \text{ vs. } H_{11}\colon \text{not } H_{01}.$$

In the case of complete data, the MLRT statistic is given by $-2\rho_{\mathrm{c}\cdot\boldsymbol{\Sigma}}^{*}\log\lambda_{\mathrm{c}\cdot\boldsymbol{\Sigma}}^{*}$, where

$$\rho_{\mathrm{c}\cdot\boldsymbol{\Sigma}}^{*} = 1 - \frac{2p^2 + 3p - 1}{6(p+1)(m-1)}\left(\sum_{\ell=1}^{m}\frac{n}{n^{(\ell)}} - 1\right), \quad \lambda_{\mathrm{c}\cdot\boldsymbol{\Sigma}}^{*} = \frac{\prod_{\ell=1}^{m}\left|\frac{1}{n^{(\ell)}}\boldsymbol{V}^{(\ell)}\right|^{\frac{1}{2}n^{(\ell)}}}{\left|\frac{1}{n}\boldsymbol{V}\right|^{\frac{1}{2}n}},$$

$n^{(\ell)} = N^{(\ell)} - 1$ and $n = \sum_{\ell=1}^{m}n^{(\ell)}$. This result can be modified to get an unbiased MLRT statistic (see Muirhead 1982, 308). Letting

$$\xi_{11}^{*} = \frac{\prod_{\ell=1}^{m}\left(\frac{1}{n^{(\ell)}}\right)^{\frac{1}{2}n^{(\ell)}p_1}\left|\boldsymbol{V}_{p_1}^{(\ell)}\right|^{\frac{1}{2}n^{(\ell)}}}{\left(\frac{1}{n}\right)^{\frac{1}{2}np_1}\left|\boldsymbol{V}_{p_1}\right|^{\frac{1}{2}n}}, \quad \xi_{21}^{*} = \frac{\prod_{\ell=1}^{m}\left(\frac{1}{n_1^{(\ell)}}\right)^{\frac{1}{2}n_1^{(\ell)}p_2}\left|\boldsymbol{V}_{p_2}^{(\ell)}\right|^{\frac{1}{2}n_1^{(\ell)}}}{\left(\frac{1}{n_1}\right)^{\frac{1}{2}n_1p_2}\left|\boldsymbol{V}_{p_2}\right|^{\frac{1}{2}n_1}},$$

ξ_{11}^{*} and ξ_{21}^{*} take the form of the LR for H_{01} in the case of without missing data. Therefore, we can give the MLRT statistics as $-2\rho_{\boldsymbol{\Sigma}\cdot p_1}^{*}\log\xi_{11}^{*}$ and $-2\rho_{\boldsymbol{\Sigma}\cdot p_2}^{*}\log\xi_{21}^{*}$, respectively, where

$$\rho_{\boldsymbol{\Sigma}\cdot p_1}^{*} = 1 - \frac{2p_1^2 + 3p_1 - 1}{6(p_1+1)(m-1)}\left(\sum_{\ell=1}^{m}\frac{n}{n^{(\ell)}} - 1\right),$$

$$\rho^*_{\Sigma \cdot p_2} = 1 - \frac{2p_2^2 + 3p_2 - 1}{6(p_2+1)(m-1)} \left(\sum_{\ell=1}^{m} \frac{n_1}{n_1^{(\ell)}} - 1 \right).$$

Then, by considering the decompositions of ξ_1 and ξ_2 using ξ^*_{11} and ξ^*_{21}, we can propose the approximate MLRT statistic $-2\log\varphi$, where

$$\varphi = \left(\xi^*_{11}\right)^{\rho^*_{\Sigma \cdot p_1}} \left(\xi^*_{21}\right)^{\rho^*_{\Sigma \cdot p_2}} \frac{\lambda}{\xi_{11}\xi_{21}},$$

$$\xi_{11} = \frac{\prod_{\ell=1}^{m} \left(\frac{1}{N^{(\ell)}}\right)^{\frac{1}{2}N^{(\ell)}p_1} \left|\boldsymbol{V}^{(\ell)}_{p_1}\right|^{\frac{1}{2}N^{(\ell)}}}{\left(\frac{1}{N}\right)^{\frac{1}{2}Np_1} \left|\boldsymbol{V}_{p_1}\right|^{\frac{1}{2}N}}, \quad \xi_{21} = \frac{\prod_{\ell=1}^{m} \left(\frac{1}{N_1^{(\ell)}}\right)^{\frac{1}{2}N_1^{(\ell)}p_2} \left|\boldsymbol{V}^{(\ell)}_{p_2}\right|^{\frac{1}{2}N_1^{(\ell)}}}{\left(\frac{1}{N_1}\right)^{\frac{1}{2}N_1p_2} \left|\boldsymbol{V}_{p_2}\right|^{\frac{1}{2}N_1}}.$$

4. An approximate upper percentile of the LRT statistic

In this section, we derive an approximate upper percentile of $-2\log\lambda$ when the data sets have the same two-step monotone missing pattern. First, in the case of complete data sets, we can obtain the following lemma by using the result of Muirhead (1982, 513).

Lemma 4.1. *Suppose that $\boldsymbol{x}_1^{(\ell)}, \boldsymbol{x}_2^{(\ell)}, \ldots, \boldsymbol{x}_{N^{(\ell)}}^{(\ell)}$ are distributed as $N_p\left(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma}^{(\ell)}\right)$, with $\ell = 1, 2, \ldots, m$. Then, under the null hypothesis H_0 given in* Eq. (1), *the upper percentile of the MLRT statistic, $-2\rho_{c,N}\log\lambda_{c,N}$, can be expanded as*

$$u_{MLR \cdot N}(\alpha) = \chi^2_f(\alpha) + \frac{1}{M^2}\frac{2\gamma_1}{f(f+2)}\chi^2_f(\alpha)\left\{\chi^2_f(\alpha) + f + 2\right\} + \mathrm{O}\left(M^{-3}\right),$$

where

$$M = \rho_{c,N}N, \ \rho_{c,N} = 1 - \frac{2p^2 + 9p + 11}{6N(p+3)(m-1)} \left(\sum_{\ell=1}^{m} \frac{N}{N^{(\ell)}} - 1 \right), \ f = \frac{1}{2}p(p+3)(m-1),$$

and $\chi^2_f(\alpha)$ is the upper percentile of the χ^2 distribution with f degrees of freedom. Furthermore, the upper percentile of $-2\log\lambda_{c,N}$ is given by

$$u_{LR \cdot c}(\alpha) = \chi^2_f(\alpha) + \frac{\nu}{N}\left(\sum_{\ell=1}^{m} \frac{1}{k_1^{(\ell)}} - 1 \right)\chi^2_f(\alpha)$$
$$+ \frac{1}{N^2}\chi^2_f(\alpha)\left\{ \nu^2 \left(\sum_{\ell=1}^{m} \frac{1}{k_1^{(\ell)}} - 1 \right)^2 + \frac{2\gamma_1}{f} + \frac{2\gamma_1}{f(f+2)}\chi^2_f(\alpha) \right\} + \mathrm{O}\left(N^{-3}\right),$$

where

$$\nu = \frac{2p^2 + 9p + 11}{6(p+3)(m-1)}, \quad k_1^{(\ell)} = \frac{N^{(\ell)}}{N},$$

$$\gamma_1 = \frac{1}{288}\left[6p(p+1)(p+2)(p+3)\left(\sum_{\ell=1}^{m}\frac{1}{\left\{k_1^{(\ell)}\right\}^2} - 1\right) - \frac{(2p^2+9p+11)^2(2p-1)}{p(p+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{1}{k_1^{(\ell)}} - 1\right)^2\right].$$

By applying a linear interpolation procedure and letting $u_{\mathrm{LR\cdot m}}(\alpha)$ be the upper percentile of $-2\log\lambda$, an approximate upper percentile of $-2\log\lambda$ can be proposed as

$$\begin{aligned}
u^*_{\mathrm{LR\cdot m}}(\alpha) &= \chi_f^2(\alpha) + \frac{\nu}{Np}\left\{p_1\left(\sum_{\ell=1}^{m}\frac{1}{k_1^{(\ell)}} - 1\right) + c_1 p_2\left(\sum_{\ell=1}^{m}\frac{1}{k_2^{(\ell)}} - 1\right)\right\}\chi_f^2(\alpha) \\
&\quad + \frac{1}{N^2 p}\left[p_1\left\{\nu^2\left(\sum_{\ell=1}^{m}\frac{1}{k_1^{(\ell)}} - 1\right)^2 + \frac{2\gamma_1}{f} + \frac{2\gamma_1}{f(f+2)}\chi_f^2(\alpha)\right\}\right. \\
&\quad \left. + c_1^2 p_2\left\{\nu^2\left(\sum_{\ell=1}^{m}\frac{1}{k_2^{(\ell)}} - 1\right)^2 + \frac{2\gamma_2}{f} + \frac{2\gamma_2}{f(f+2)}\chi_f^2(\alpha)\right\}\right]\chi_f^2(\alpha),
\end{aligned}$$

where

$$c_1 = \frac{N}{N_1}, \quad k_2^{(\ell)} = \frac{N_1^{(\ell)}}{N_1},$$

$$\gamma_i = \frac{1}{288}\left[6p(p+1)(p+2)(p+3)\left(\sum_{\ell=1}^{m}\frac{1}{\left\{k_i^{(\ell)}\right\}^2} - 1\right) - \frac{(2p^2+9p+11)^2(2p-1)}{p(p+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{1}{k_i^{(\ell)}} - 1\right)^2\right], \quad i=1,2.$$

The approximation result just given is an extension of the result for the one-population case given in Hosoya and Seo (2015, 89).

5. Accuracy of the approximations

Now we compute the upper percentiles of the null distributions of the LRT statistic and the approximate MLRT statistics with two-step monotone missing data for the m-population problem using Monte Carlo simulation (with 10^6 runs). In particular, we evaluate the asymptotic behavior of the χ^2 approximations and the accuracy of the approximate upper percentiles of $-2\log\lambda$.

Tables 1a, 1b, and 1c present the simulated upper 100α percentiles of $-2\log\lambda$, $-2\rho_{\mathrm{L}}\log\lambda$, $-2\log\tau$, and $-2\log\varphi$ and the approximate upper percentiles of $-2\log\lambda\left(u^*_{\mathrm{LR\cdot m}}(\alpha)\right)$ for $m = 2, 3, 5$; $(p_1, p_2) = (8, 4)$; $\alpha = 0.05, 0.01$; and the following three cases of $\left(N_1^{(\ell)}, N_2^{(\ell)}\right)$:

Table 1a. The simulated values for $-2\log\lambda$, $-2\rho_L\log\lambda$, $-2\log\tau$, and $-2\log\varphi$ and the approximate value for $-2\log\lambda$ when $m=2$ and $(p_1,p_2)=(8,4)$.

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{LR\cdot m}(\alpha)$
			$\alpha=0.05$			
20	20	169.56	131.22	155.65	146.73	152.45
40	40	133.19	118.13	127.85	125.20	129.37
80	80	121.97	115.07	119.53	118.42	120.40
160	160	117.32	114.01	116.14	115.63	116.56
320	320	115.17	113.54	114.59	114.35	114.80
20	10	172.56	127.04	155.74	147.09	160.27
40	20	134.40	116.68	127.83	125.30	132.39
80	40	122.53	114.45	119.50	118.44	121.69
160	80	117.61	113.73	116.16	115.66	117.15
320	160	115.31	113.41	114.59	114.36	115.08
20	40	166.76	135.34	155.44	146.26	145.64
40	80	131.93	119.50	127.82	125.04	126.60
80	160	121.35	115.64	119.49	118.32	119.17
160	320	117.06	114.31	116.17	115.64	115.99
			$\alpha=0.01$			
20	20	186.84	144.59	171.54	161.57	167.65
40	40	146.18	129.66	140.35	137.47	142.02
80	80	133.80	126.24	131.13	129.90	132.10
160	160	128.72	125.08	127.42	126.88	127.87
320	320	126.35	124.56	125.71	125.45	125.93
20	10	190.09	139.94	171.60	162.01	176.34
40	20	147.30	127.87	140.14	137.35	145.36
80	40	134.27	125.42	130.95	129.82	133.52
160	80	129.02	124.76	127.42	126.90	128.51
320	160	126.53	124.44	125.73	125.47	126.24
20	40	183.87	149.22	171.37	161.19	160.11
40	80	144.95	131.30	140.45	137.39	138.96
80	160	133.13	126.86	131.08	129.85	130.75
160	320	128.49	125.46	127.52	126.91	127.24

Note. $\chi^2_f(0.05)=113.145$, $\chi^2_f(0.01)=124.116$.

Table 1b. The simulated values for $-2\log\lambda$, $-2\rho_L\log\lambda$, $-2\log\tau$, and $-2\log\varphi$ and the approximate value for $-2\log\lambda$ when $m=3$ and $(p_1,p_2)=(8,4)$.

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{LR\cdot m}(\alpha)$
			$\alpha=0.05$			
20	20	302.09	241.37	278.46	261.58	275.55
40	40	245.06	220.43	236.15	230.90	238.78
80	80	226.93	215.53	222.85	220.62	224.26
160	160	219.18	213.67	217.21	216.20	217.96
320	320	215.63	212.92	214.66	214.16	215.05
20	10	306.92	234.95	278.53	262.10	288.00
40	20	247.14	218.17	236.15	231.05	243.67
80	40	227.89	214.53	222.83	220.69	226.37
160	80	219.69	213.25	217.25	216.26	218.93
320	160	216.04	212.87	214.83	214.34	215.52
20	40	297.60	247.76	278.33	261.13	264.63
40	80	243.03	222.68	236.11	230.65	234.27
80	160	226.00	216.54	222.88	220.55	222.24
160	320	218.74	214.16	217.23	216.17	217.01

(*Continued*)

Table 1b. (Continued).

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{LR\cdot m}(\alpha)$
			$\alpha = 0.01$			
20	20	324.15	259.00	298.77	280.46	295.13
40	40	262.26	235.90	252.68	247.03	255.48
80	80	242.83	230.63	238.46	236.11	239.87
160	160	234.46	228.57	232.37	231.25	233.11
320	320	230.73	227.83	229.70	229.16	230.00
20	10	329.41	252.16	298.96	281.14	308.56
40	20	264.40	233.40	252.66	247.18	260.74
80	40	243.72	229.43	238.35	236.03	242.13
160	80	234.97	228.09	232.38	231.34	234.15
320	160	230.79	227.41	229.51	229.01	230.49
20	40	319.28	265.81	298.41	280.04	283.37
40	80	260.11	238.33	252.72	246.91	250.64
80	160	241.54	231.43	238.18	235.72	237.71
160	320	233.97	229.07	232.38	231.22	232.10

Note. $\chi_f^2(0.05) = 212.304$, $\chi_f^2(0.01) = 227.056$.

Table 1c. The simulated values for $-2\log\lambda$, $-2\rho_L\log\lambda$, $-2\log\tau$, and $-2\log\varphi$ and the approximate value for $-2\log\lambda$ when $m = 5$ and $(p_1, p_2) = (8, 4)$.

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{LR\cdot m}(\alpha)$
			$\alpha = 0.05$			
20	20	555.35	454.89	515.93	482.05	511.13
40	40	461.01	419.31	445.84	435.14	450.04
80	80	430.13	410.68	423.15	418.61	425.61
160	160	417.11	407.68	413.75	411.64	414.92
320	320	411.11	406.47	409.46	408.44	409.95
20	10	563.70	444.74	516.24	482.79	531.80
40	20	464.50	415.49	445.76	435.32	458.26
80	40	431.93	409.14	423.26	418.86	429.19
160	80	418.00	406.97	413.81	411.77	416.58
320	160	411.46	406.03	409.39	408.39	410.75
20	40	547.54	465.00	515.50	481.16	492.88
40	80	457.47	422.99	445.68	434.79	442.42
80	160	428.54	412.39	423.19	418.55	422.18
160	320	416.32	408.48	413.76	411.61	413.29
			$\alpha = 0.01$			
20	20	584.02	478.38	542.51	506.58	536.97
40	40	484.15	440.36	468.24	456.99	472.49
80	80	451.60	431.18	444.30	439.59	446.75
160	160	437.70	427.80	434.17	431.92	435.51
320	320	431.47	426.59	429.73	428.67	430.29
20	10	592.69	467.61	542.85	507.28	558.79
40	20	487.89	436.41	468.25	457.14	481.15
80	40	453.35	429.43	444.26	439.66	450.52
160	80	438.77	427.19	434.40	432.23	437.25
320	160	431.88	426.18	429.72	428.66	431.13
20	40	576.00	489.17	542.02	505.82	517.73
40	80	480.38	444.18	467.99	456.52	464.47
80	160	449.67	432.73	444.04	439.21	443.14
160	320	437.15	428.92	434.48	432.15	433.80

Note. $\chi_f^2(0.05) = 405.244$, $\chi_f^2(0.01) = 425.347$.

$$\left(N_1^{(\ell)}, N_2^{(\ell)}\right) = \begin{cases} (t,t), & t = 20, 40, 80, 160, 320, \\ (2t,t), & t = 10, 20, 40, 80, 160, \\ (t,2t), & t = 20, 40, 80, 160, \end{cases}$$

where $N_1^{(1)} = N_1^{(2)}$ and $N_2^{(1)} = N_2^{(2)}$.

Tables 2a, 2b, and 2c present the actual type I error rates for the upper percentiles of $-2\log\lambda$, $-2\rho_L\log\lambda$, $-2\log\tau$, and $-2\log\varphi$, as well as $u^*_{\mathrm{LR\cdot m}}(\alpha)$ from Tables 1a, 1b, and 1c, which are given by

$$\alpha_{\mathrm{CHI}} = \Pr\left\{-2\log\lambda > \chi_f^2(\alpha)\right\}, \quad \alpha_{\rho_\mathrm{L}} = \Pr\left\{-2\rho_\mathrm{L}\log\lambda > \chi_f^2(\alpha)\right\},$$
$$\alpha_\tau = \Pr\left\{-2\log\tau > \chi_f^2(\alpha)\right\}, \quad \alpha_\varphi = \Pr\left\{-2\log\varphi > \chi_f^2(\alpha)\right\},$$

and

$$\alpha_{u^*_{\mathrm{LR\cdot m}}} = \Pr\left\{-2\log\lambda > u^*_{\mathrm{LR\cdot m}}(\alpha)\right\},$$

respectively.

Table 2a. The type I error rates when $m = 2$ and $(p_1, p_2) = (8, 4)$.

Sample size		Type I error rate				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	α_{CHI}	$\alpha_{\rho\mathrm{L}}$	α_τ	α_φ	$\alpha_{u^*_{\mathrm{LR\cdot m}}}$
			$\alpha = 0.05$			
20	20	0.841	0.260	0.677	0.534	**0.179**
40	40	0.307	0.091	0.216	0.176	**0.076**
80	80	0.134	0.064	0.106	0.094	**0.060**
160	160	0.083	0.056	0.073	0.068	**0.055**
320	320	0.065	**0.053**	0.060	0.059	**0.053**
20	10	0.869	0.198	0.680	0.542	**0.129**
40	20	0.331	0.077	0.217	0.179	**0.062**
80	40	0.141	0.059	0.105	0.094	**0.055**
160	80	0.086	0.054	0.072	0.068	**0.053**
320	160	0.066	**0.052**	0.060	0.058	**0.052**
20	40	0.814	0.330	0.675	0.527	**0.234**
40	80	0.285	0.105	0.216	0.174	**0.089**
80	160	0.126	0.068	0.105	0.093	**0.065**
160	320	0.081	0.058	0.073	0.068	**0.057**
			$\alpha = 0.01$			
20	20	0.663	0.104	0.447	0.302	**0.059**
40	40	0.126	0.022	0.076	0.057	**0.017**
80	80	0.038	0.014	0.027	0.023	**0.013**
160	160	0.020	0.012	0.017	0.015	**0.011**
320	320	0.014	**0.011**	0.013	0.012	**0.011**
20	10	0.708	0.069	0.451	0.310	**0.036**
40	20	0.141	0.018	0.076	0.057	**0.013**
80	40	0.041	0.012	0.027	0.023	**0.011**
160	80	0.021	**0.011**	0.017	0.015	**0.011**
320	160	0.014	0.011	0.013	0.012	**0.010**
20	40	0.622	0.146	0.445	0.296	**0.086**
40	80	0.113	0.028	0.076	0.056	**0.022**
80	160	0.035	0.015	0.027	0.023	**0.014**
160	320	0.019	**0.012**	0.017	0.015	**0.012**

Note. The closer to α in the values α_{CHI}, $\alpha_{\rho\mathrm{L}}$, α_τ, α_φ, and $\alpha_{u^*_{\mathrm{LR\cdot m}}}$ of each low is in boldface.

Table 2b. The type I error rates when m = 3 and $(p_1, p_2) = (8, 4)$.

Sample size		Type I error rate				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	α_{CHI}	$\alpha_{\rho\mathrm{L}}$	α_τ	α_φ	$\alpha_{u^*_{\mathrm{LR}\cdot m}}$
			$\alpha = 0.05$			
20	20	0.943	0.332	0.808	0.624	**0.220**
40	40	0.402	0.102	0.273	0.206	**0.083**
80	80	0.162	0.068	0.123	0.104	**0.063**
160	160	0.092	0.057	0.078	0.072	**0.056**
320	320	0.068	**0.053**	0.063	0.060	**0.053**
20	10	0.959	0.249	0.810	0.632	**0.151**
40	20	0.435	0.085	0.274	0.209	**0.067**
80	40	0.173	0.062	0.122	0.104	**0.057**
160	80	0.096	0.055	0.079	0.072	**0.054**
320	160	0.070	0.053	0.063	0.061	**0.052**
20	40	0.925	0.421	0.806	0.616	**0.295**
40	80	0.371	0.120	0.273	0.203	**0.100**
80	160	0.153	0.074	0.123	0.103	**0.070**
160	320	0.089	0.060	0.079	0.072	**0.059**
			$\alpha = 0.01$			
20	20	0.839	0.144	0.605	0.382	**0.078**
40	40	0.186	0.026	0.105	0.070	**0.020**
80	80	0.049	0.015	0.034	0.027	**0.014**
160	160	0.023	**0.012**	0.018	0.016	**0.012**
320	320	0.015	**0.011**	0.014	0.013	**0.011**
20	10	0.874	0.094	0.608	0.389	**0.045**
40	20	0.210	0.020	0.105	0.071	**0.014**
80	40	0.054	0.013	0.033	0.027	**0.012**
160	80	0.024	**0.011**	0.018	0.016	**0.011**
320	160	0.016	**0.010**	0.013	0.013	**0.010**
20	40	0.802	0.206	0.602	0.374	**0.119**
40	80	0.165	0.033	0.104	0.068	**0.026**
80	160	0.045	0.017	0.034	0.026	**0.015**
160	320	0.022	0.013	0.018	0.016	**0.012**

Note. The closer to α in the values α_{CHI}, $\alpha_{\rho\mathrm{L}}$, α_τ, α_φ, and $\alpha_{u^*_{\mathrm{LR}\cdot m}}$ of each low is in boldface.

Table 2c. The type I error rates when m = 5 and $(p_1, p_2) = (8, 4)$.

Sample size		Type I error rate				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	α_{CHI}	$\alpha_{\rho\mathrm{L}}$	α_τ	α_φ	$\alpha_{u^*_{\mathrm{LR}\cdot m}}$
			$\alpha = 0.05$			
20	20	0.992	0.452	0.933	0.742	**0.293**
40	40	0.542	0.120	0.365	0.251	**0.095**
80	80	0.207	0.072	0.148	0.116	**0.067**
160	160	0.107	0.059	0.088	0.077	**0.058**
320	320	0.074	**0.054**	0.066	0.062	**0.054**
20	10	0.995	0.342	0.935	0.751	**0.194**
40	20	0.584	0.097	0.367	0.255	**0.073**
80	40	0.222	0.065	0.149	0.118	**0.060**
160	80	0.112	0.056	0.088	0.077	**0.055**
320	160	0.076	**0.053**	0.066	0.062	**0.053**
20	40	0.986	0.563	0.931	0.736	**0.398**
40	80	0.501	0.146	0.365	0.249	**0.118**
80	160	0.192	0.080	0.148	0.116	**0.075**
160	320	0.102	0.062	0.088	0.076	**0.061**

(*Continued*)

Table 2c. (Continued).

Sample size		Type I error rate				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	α_{CHI}	$\alpha_{\rho\mathrm{L}}$	α_τ	α_φ	$\alpha_{u^*_{\mathrm{LR\cdot m}}}$
			$\alpha = 0.01$			
20	20	0.964	0.226	0.811	0.509	**0.116**
40	40	0.293	0.033	0.159	0.091	**0.023**
80	80	0.068	0.016	0.043	0.031	**0.015**
160	160	0.027	**0.012**	0.021	0.018	**0.012**
320	320	0.017	**0.011**	0.015	0.013	**0.011**
20	10	0.977	0.148	0.816	0.518	**0.063**
40	20	0.331	0.025	0.160	0.093	**0.017**
80	40	0.076	0.014	0.044	0.032	**0.013**
160	80	0.029	0.012	0.021	0.018	**0.011**
320	160	0.017	**0.011**	0.015	0.013	**0.011**
20	40	0.947	0.320	0.809	0.501	**0.183**
40	80	0.259	0.043	0.159	0.090	**0.032**
80	160	0.062	0.019	0.043	0.031	**0.017**
160	320	0.026	0.014	0.021	0.018	**0.013**

Note. The closer to α in the values α_{CHI}, $\alpha_{\rho\mathrm{L}}$, α_τ, α_φ, and $\alpha_{u^*_{\mathrm{LR\cdot m}}}$ of each low is in boldface.

It may be noted from Tables 1a, 1b, and 1c that the simulated values get closer to the upper percentiles of the χ^2 distribution when both of the sample sizes $N_1^{(\ell)}$ and $N_2^{(\ell)}$ become large. From the results presented in Tables 2a, 2b, and 2c, we note that the actual type I error rates get closer to the value of α when the sample sizes $N_1^{(\ell)}$ and $N_2^{(\ell)}$ become large, even when m is large. Furthermore, it can be seen from the tables that our approximations are accurate for the majority of cases. In particular, it should be noted that the values of $u^*_{\mathrm{LR\cdot m}}$ are highly accurate for all cases.

Tables 3a, 3b, and 3c present the same upper percentiles as those given in Tables 1a, 1b, and 1c for $m = 2, 3, 5$; $(p_1, p_2) = (8, 4)$; $\alpha = 0.05, 0.01$; and $\left(N_1^{(\ell)}, N_2^{(\ell)}\right) = (t_1, t_2)$, where $t_1 = 40, 80, 160, 320$ and $t_2 = 10, 30, 60, 120$ for Table 3a and $t_1 = 30, 60, 120$ and $t_2 = 10, 40, 80, 160, 320$ for Tables 3b and 3c, and the sets $\left(N_1^{(\ell)}, N_2^{(\ell)}\right)$ are a combination of t_1 and t_2. Tables 4a, 4b, and 4c present the actual type I error rates α_{CHI}, $\alpha_{\rho_{\mathrm{L}}}$, α_τ, and α_φ from Tables 3a, 3b, and 3c.

Table 3a. The simulated values for $-2\log\lambda$, $-2\rho_L\log\lambda$, $-2\log\tau$, and $-2\log\varphi$ and the approximate value for $-2\log\lambda$ when $m = 2$ and $(p_1, p_2) = (8, 4)$.

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{\mathrm{LR\cdot m}}(\alpha)$
			$\alpha = 0.05$			
40	10	135.56	115.63	127.93	125.49	134.99
80	10	123.37	113.68	119.54	118.58	123.50
160	10	118.07	113.26	116.13	115.70	118.15
320	10	115.63	113.22	114.65	114.45	115.60
40	30	133.69	117.50	127.84	125.25	130.63
80	30	122.82	114.30	119.57	118.54	122.17
160	30	117.90	113.43	116.14	115.69	117.78
320	30	115.55	113.24	114.62	114.42	115.50

(*Continued*)

Table 3a. (Continued).

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{\mathrm{LR}\cdot\mathrm{m}}(\alpha)$
40	60	132.50	119.01	127.89	125.15	127.68
80	60	122.14	114.74	119.44	118.36	120.94
160	60	117.70	113.62	116.13	115.67	117.36
320	60	115.45	113.26	114.58	114.37	115.38
40	120	131.35	120.21	127.80	124.94	125.31
80	120	121.74	115.54	119.64	118.50	119.66
160	120	117.45	113.89	116.15	115.65	116.81
320	120	115.37	113.37	114.60	114.37	115.18
			$\alpha = 0.01$			
40	10	148.84	126.96	140.53	137.91	148.24
80	10	135.29	124.67	131.10	130.05	135.52
160	10	129.55	124.27	127.43	126.97	129.62
320	10	126.84	124.20	125.76	125.55	126.81
40	30	146.67	128.90	140.29	137.46	143.41
80	30	134.70	125.36	131.16	130.05	134.05
160	30	129.38	124.47	127.44	126.97	129.21
320	30	126.74	124.20	125.72	125.49	126.71
40	60	145.24	130.46	140.24	137.30	140.15
80	60	134.02	125.90	131.10	129.93	132.70
160	60	128.96	124.49	127.25	126.73	128.75
320	60	126.55	124.15	125.61	125.38	126.57
40	120	144.28	132.05	140.38	137.24	137.54
80	120	133.55	126.75	131.24	130.00	131.28
160	120	128.91	125.01	127.48	126.92	128.14
320	120	126.59	124.40	125.75	125.49	126.35

Note. $\chi^2_f(0.05) = 113.145$, $\chi^2_f(0.01) = 124.116$.

Table 3b. The simulated values for $-2\log\lambda$, $-2\rho_L\log\lambda$, $-2\log\tau$, and $-2\log\varphi$ and the approximate value for $-2\log\lambda$ when $m = 3$ and $(p_1, p_2) = (8, 4)$.

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{\mathrm{LR}\cdot\mathrm{m}}(\alpha)$
			$\alpha = 0.05$			
30	10	264.73	220.39	247.48	239.81	260.82
60	10	235.41	214.01	226.93	224.01	235.37
120	10	223.36	212.71	219.08	217.81	223.46
30	40	258.81	226.61	247.17	239.07	247.06
60	40	233.33	216.13	226.95	223.90	230.57
120	40	222.67	213.35	219.06	217.72	222.00
30	80	256.29	229.75	247.16	238.88	241.29
60	80	232.00	217.57	226.96	223.78	227.60
120	80	222.11	213.92	219.09	217.70	220.78
30	160	254.50	232.07	247.12	238.62	237.42
60	160	230.72	218.78	226.90	223.61	225.05
120	160	221.40	214.52	219.02	217.59	219.43
30	320	253.44	233.55	247.15	238.56	235.15
60	320	229.84	219.71	226.87	223.51	223.26
120	320	220.86	215.15	219.07	217.59	218.24

(*Continued*)

Table 3b. (Continued).

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{\mathrm{LR}\cdot m}(\alpha)$
			$\alpha = 0.01$			
30	10	283.13	235.71	264.73	256.64	279.22
60	10	251.70	228.81	242.63	239.62	251.81
120	10	239.00	227.61	234.42	233.06	239.00
30	40	277.09	242.61	264.62	255.92	264.40
60	40	249.56	231.17	242.74	239.46	246.65
120	40	238.07	228.10	234.21	232.79	237.45
30	80	274.35	245.95	264.56	255.61	258.20
60	80	248.16	232.72	242.79	239.41	243.45
120	80	237.51	228.76	234.26	232.81	236.14
30	160	272.55	248.52	264.56	255.49	254.05
60	160	246.86	234.08	242.74	239.22	240.72
120	160	236.75	229.39	234.22	232.67	234.69
30	320	271.37	250.07	264.58	255.42	251.62
60	320	245.83	234.99	242.64	239.10	238.81
120	320	236.23	230.12	234.32	232.74	233.41

Note. $\chi_f^2(0.05) = 212.304$, $\chi_f^2(0.01) = 227.056$.

Table 3c. The simulated values for $-2\log\lambda$, $-2\rho_L\log\lambda$, $-2\log\tau$, and $-2\log\varphi$ and the approximate value for $-2\log\lambda$ when $m = 5$ and $(p_1, p_2) = (8, 4)$.

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{\mathrm{LR}\cdot m}(\alpha)$
			$\alpha = 0.05$			
30	10	493.84	419.39	464.65	448.85	486.85
60	10	444.68	408.29	430.18	424.11	444.37
120	10	424.24	406.04	416.90	414.19	424.26
30	40	483.96	429.77	464.30	447.95	463.82
60	40	441.17	411.91	430.26	423.98	436.27
120	40	422.97	407.03	416.77	414.03	421.80
30	80	479.63	434.94	464.23	447.60	454.10
60	80	438.59	414.03	429.99	423.60	431.23
120	80	422.01	408.01	416.82	414.00	419.72
30	160	476.59	438.78	464.22	447.41	447.53
60	160	436.72	416.37	430.17	423.62	426.90
120	160	420.97	409.18	416.88	413.99	417.41
30	320	474.60	441.08	464.07	447.17	443.66
60	320	435.30	418.03	430.23	423.56	423.85
120	320	419.83	410.05	416.75	413.81	415.38
			$\alpha = 0.01$			
30	10	518.77	440.57	488.06	471.55	511.30
60	10	466.70	428.51	451.53	445.18	466.50
120	10	445.25	426.15	437.53	434.73	445.33
30	40	508.28	451.37	487.57	470.46	487.03
60	40	463.14	432.42	451.69	445.11	457.98
120	40	443.95	427.22	437.46	434.59	442.74
30	80	503.78	456.83	487.52	470.06	476.79
60	80	460.41	434.63	451.36	444.58	452.68
120	80	443.01	428.32	437.56	434.57	440.56

(*Continued*)

Table 3c. (Continued).

Sample size		Upper percentile				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	$-2\log\lambda$	$-2\rho_L\log\lambda$	$-2\log\tau$	$-2\log\varphi$	$u^*_{LR\cdot m}(\alpha)$
30	160	500.65	460.93	487.56	469.82	469.88
60	160	458.28	436.93	451.41	444.54	448.12
120	160	441.91	429.54	437.60	434.56	438.13
30	320	498.54	463.33	487.35	469.69	465.82
60	320	456.90	438.77	451.55	444.53	444.92
120	320	440.73	430.46	437.51	434.39	435.99

Note. $\chi^2_f(0.05) = 405.244$, $\chi^2_f(0.01) = 425.347$.

Table 4a. The type I error rates when $m = 2$ and $(p_1, p_2) = (8, 4)$.

Sample size		Type I error rate				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	α_{CHI}	α_{ρ_L}	α_τ	α_φ	$\alpha_{u^*_{LR\cdot m}}$
			$\alpha = 0.05$			
40	10	0.351	0.068	0.216	0.180	**0.053**
80	10	0.152	0.054	0.105	0.095	**0.049**
160	10	0.090	0.051	0.073	0.069	**0.050**
320	10	0.068	0.051	0.061	0.059	**0.050**
40	30	0.317	0.084	0.217	0.177	**0.069**
80	30	0.144	0.058	0.105	0.095	**0.054**
160	30	0.089	0.052	0.072	0.069	**0.051**
320	30	0.068	0.051	0.060	0.059	**0.050**
40	60	0.295	0.100	0.217	0.175	**0.084**
80	60	0.137	0.061	0.105	0.094	**0.058**
160	60	0.087	0.053	0.073	0.069	**0.052**
320	60	0.067	0.051	0.060	0.059	**0.050**
40	120	0.275	0.113	0.216	0.173	**0.095**
80	120	0.130	0.067	0.106	0.094	**0.064**
160	120	0.085	0.055	0.073	0.068	**0.054**
320	120	0.066	**0.051**	0.060	0.059	**0.051**
			$\alpha = 0.01$			
40	10	0.153	0.015	0.076	0.059	**0.011**
80	10	0.046	0.011	0.028	0.024	**0.010**
160	10	0.022	**0.010**	0.017	0.015	**0.010**
320	10	0.015	**0.010**	0.013	0.013	**0.010**
40	30	0.132	0.020	0.076	0.057	**0.016**
80	30	0.043	0.012	0.027	0.024	**0.011**
160	30	0.022	0.011	0.017	0.015	**0.010**
320	30	0.015	**0.010**	0.013	0.012	**0.010**
40	60	0.119	0.026	0.076	0.056	**0.020**
80	60	0.039	0.013	0.027	0.023	**0.012**
160	60	0.021	0.011	0.016	0.015	**0.010**
320	60	0.015	**0.010**	0.013	0.012	**0.010**
40	120	0.107	0.030	0.076	0.055	**0.024**
80	120	0.037	0.015	0.028	0.024	**0.014**
160	120	0.021	**0.011**	0.017	0.016	**0.011**
320	120	0.015	**0.010**	0.013	0.012	**0.010**

Note. The closer to α in the values α_{CHI}, α_{ρ_L}, α_τ, α_φ, and $\alpha_{u^*_{LR\cdot m}}$ of each low is in boldface.

Table 4b. The type I error rates when $m = 3$ and $(p_1, p_2) = (8, 4)$.

Sample size		Type I error rate				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	α_{CHI}	α_{ρ_L}	α_τ	α_φ	$\alpha_{u^*_{\text{LR}\cdot\text{m}}}$
			$\alpha = 0.05$			
30	10	0.680	0.101	0.435	0.322	**0.067**
60	10	0.266	0.059	0.162	0.133	**0.050**
120	10	0.127	0.052	0.091	0.082	**0.050**
30	40	0.601	0.157	0.431	0.312	**0.117**
60	40	0.237	0.071	0.162	0.131	**0.064**
120	40	0.121	0.055	0.091	0.082	**0.053**
30	80	0.564	0.191	0.431	0.309	**0.146**
60	80	0.220	0.080	0.162	0.131	**0.073**
120	80	0.116	0.058	0.092	0.082	**0.056**
30	160	0.537	0.218	0.430	0.306	**0.167**
60	160	0.205	0.089	0.161	0.129	**0.081**
120	160	0.110	0.062	0.091	0.081	**0.060**
30	320	0.521	0.236	0.430	0.305	**0.180**
60	320	0.195	0.096	0.162	0.128	**0.088**
120	320	0.105	0.065	0.091	0.081	**0.063**
			$\alpha = 0.01$			
30	10	0.436	0.026	0.210	0.134	**0.015**
60	10	0.099	0.012	0.049	0.037	**0.010**
120	10	0.035	0.011	0.023	0.020	**0.010**
30	40	0.354	0.048	0.208	0.127	**0.032**
60	40	0.085	0.016	0.049	0.037	**0.014**
120	40	0.033	**0.011**	0.022	0.019	**0.011**
30	80	0.319	0.063	0.207	0.126	**0.043**
60	80	0.076	0.019	0.050	0.037	**0.017**
120	80	0.031	**0.012**	0.023	0.019	**0.012**
30	160	0.295	0.076	0.206	0.124	**0.052**
60	160	0.069	0.022	0.049	0.036	**0.019**
120	160	0.029	**0.013**	0.022	0.019	**0.013**
30	320	0.281	0.086	0.207	0.123	**0.058**
60	320	0.064	0.024	0.049	0.036	**0.021**
120	320	0.027	**0.014**	0.022	0.019	**0.014**

Note. The closer to α in the values α_{CHI}, α_{ρ_L}, α_τ, α_φ, and $\alpha_{u^*_{\text{LR}\cdot\text{m}}}$ of each low is in boldface.

Considering Tables 3a, 3b, and 3c, it should be noted that the behavior of the simulated values and the accuracy of the approximation exhibit the same tendencies as the corresponding results of Tables 1a, 1b, and 1c. That is, it can be observed that the proposed approximation procedures are considerably accurate, even for cases when the sample size is not large and m is moderately large.

Thus, it can be concluded that the approximation $u^*_{\text{LR}\cdot\text{m}}(\alpha)$ is highly accurate, and the accuracy of the approximation is considerably higher than that of the χ^2 approximation $(\chi^2_f(\alpha))$ in almost all cases.

Table 4c. The type I error rates when $m = 5$ and $(p_1, p_2) = (8, 4)$.

Sample size		Type I error rate				
$N_1^{(\ell)}$	$N_2^{(\ell)}$	α_{CHI}	α_{ρ_L}	α_τ	α_φ	$\alpha_{u^*_{\mathrm{LR}\cdot m}}$
			$\alpha = 0.05$			
30	10	0.843	0.121	0.581	0.399	**0.074**
60	10	0.356	0.062	0.206	0.155	**0.051**
120	10	0.156	0.053	0.105	0.090	**0.050**
30	40	0.769	0.200	0.577	0.389	**0.144**
60	40	0.315	0.078	0.206	0.154	**0.068**
120	40	0.147	0.057	0.105	0.089	**0.054**
30	80	0.731	0.248	0.576	0.385	**0.185**
60	80	0.290	0.089	0.205	0.152	**0.080**
120	80	0.139	0.060	0.105	0.089	**0.058**
30	160	0.701	0.285	0.574	0.382	**0.215**
60	160	0.268	0.102	0.205	0.151	**0.092**
120	160	0.131	0.065	0.105	0.088	**0.063**
30	320	0.682	0.310	0.573	0.380	**0.233**
60	320	0.253	0.112	0.206	0.150	**0.101**
120	320	0.124	0.069	0.105	0.088	**0.067**
			$\alpha = 0.01$			
30	10	0.645	0.033	0.329	0.182	**0.017**
60	10	0.152	0.013	0.069	0.046	**0.010**
120	10	0.046	0.011	0.027	0.022	**0.010**
30	40	0.540	0.067	0.325	0.175	**0.042**
60	40	0.127	0.018	0.069	0.045	**0.015**
120	40	0.042	0.012	0.027	0.022	**0.011**
30	80	0.491	0.090	0.324	0.172	**0.059**
60	80	0.112	0.022	0.068	0.044	**0.018**
120	80	0.040	0.013	0.027	0.021	**0.012**
30	160	0.456	0.111	0.324	0.171	**0.074**
60	160	0.101	0.026	0.068	0.044	**0.022**
120	160	0.037	**0.014**	0.027	0.021	**0.014**
30	320	0.434	0.125	0.323	0.168	**0.082**
60	320	0.092	0.030	0.069	0.044	**0.026**
120	320	0.034	0.016	0.027	0.021	**0.015**

Note. The closer to α in the values α_{CHI}, α_{ρ_L}, α_τ, α_φ, and $\alpha_{u^*_{\mathrm{LR}\cdot m}}$ of each low is in boldface.

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