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The McDonald Extended Weibull Distribution

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Generalizing lifetime distributions is always precious for statisticians. We propose and study a new six-parameters lifetime distribution called the McDonald extended Weibull model to generalize the Weibull, extended Weibull, exponentiated Weibull, Kumaraswamy Weibull, Kumaraswamy exponential, beta Weibull, beta exponential, and McDonald extended exponential, among several others. We obtain explicit expressions for the moments, incomplete moments, generating and quantile functions, mean deviations, and Bonferroni and Lorenz curves. The method of maximum likelihood and a Bayesian procedure are adopted for estimating the model parameters. The potentiality of the new model is illustrated by means of a real data set.

AMS Subject Classification: 62N01; 62N02.

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1. Introduction

The two-parameter Weibull distribution has been the most popular model for modeling lifetime data. However, its major weakness is its inability to accommodate nonmonotone hazard rates (in particular, bathtub-shaped hazard rates). This has led us to seek generalizations of this distribution to apply to several areas. One of the first extensions allowing for nonmonotone hazard rates, including the bathtub-shaped hazard rate, is the *exponentiated Weibull* (ExpW) distribution pioneered by Mudholkar and Srivastava (1993), Mudholkar et al. (1995), and Mudholkar et al. (1996). It has been well established in the literature that the ExpW distribution provides significantly better fits than traditional models based on the exponential, gamma, Weibull, and log-normal distributions. In the last paper, the authors presented a three-parameter extended Weibull (EW) model to yield a more flexible distribution. Further, Shao et al. (2004) used this distribution to study flood frequency, and Hao and

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Singh (2008) described some of its applications in hydrology. We take the EW distribution as the baseline distribution for a further generalization proposed in this article.

The three-parameter EW distribution is defined by the density and cumulative functions (Mudholkar et al. 1996):

$$g(x;\boldsymbol{\tau}) = \begin{cases} \lambda\beta x^{\beta-1}\left(1+\varphi\lambda x^{\beta}\right)^{-\frac{1}{\varphi}-1} & \text{if} \quad \varphi > 0, \\ \lambda\beta x^{\beta-1}\mathrm{e}^{-\lambda x^{\beta}} & \text{if} \quad \varphi = 0, \end{cases} \tag{1}$$

$$G(x;\boldsymbol{\tau}) = \begin{cases} 1-\left(1+\varphi\lambda x^{\beta}\right)^{-\frac{1}{\varphi}} & \text{if} \quad \varphi > 0, \\ 1-\mathrm{e}^{-\lambda x^{\beta}} & \text{if} \quad \varphi = 0, \end{cases} \tag{2}$$

respectively, where $\boldsymbol{\tau} = (\varphi, \lambda, \beta)^T$, $\lambda > 0$ is a scale parameter and $\varphi \geq 0$ and $\beta > 0$ are shape parameters. The support of the EW distribution is $(0, \infty)$. The forms of the density and cumulative functions when $\varphi > 0$ tend to those forms of the case $\varphi = 0$ when φ goes to zero. Clearly the cumulative distribution (2) extends the Weibull cumulative distribution, and this fact justifies the denomination *extended Weibull*. Due to the shape parameter φ, more flexibility can be incorporated in Eq. (1), which will be useful for data analysis. The survival function associated with Eq. (1) is $S(x;\boldsymbol{\tau}) = 1 - G(x;\boldsymbol{\tau})$.

The statistics literature is filled with hundreds of continuous univariate distributions, and recent developments focus on new techniques for building meaningful distributions. In this note, we introduce a new six-parameter model called the *McDonald extended Weibull* (denoted by McEW) distribution, which contains several distributions as special models including the EW distribution. This distribution represents only a basic exemplar of the McEW distribution. We are motivated to study the McEW model because of the wide usage of Eq. (1) and the fact that the current extension provides means of its application to more complex data. We study some of its mathematical properties with the hope that it will attract wider applications in reliability, engineering, and other areas of research.

The article is outlined as follows. In section 2, we define the McEW distribution and some of its special models. Further, we derive useful expansions for its density and cumulative distributions. In section 3, we provide an explicit expression for the generating function. In section 4, we obtain closed-form expressions for the moments and incomplete moments. The mean deviations, Bonferroni and Lorenz curves, and Gini concentration index are determined in section 5. In section 6, we derive a power-series expansion for the quantile function (qf). The skewness and kurtosis based on quantiles are investigated in section 7. Maximum likelihood estimation and a Bayesian analysis for the model parameters are discussed in section 8. The usefulness of the new model is illustrated by means of an application to real data in section 9. Some conclusions are offered in section 10.

2. The McEW Distribution

Alexander et al. (2012) introduced a family of *generalized beta generated* (GBG) distributions with three shape parameters in the generator. Consider starting from an arbitrary baseline cumulative distribution function (cdf) $F(x;\boldsymbol{\tau})$, and probability density function (pdf) $f(x;\boldsymbol{\tau})$, with parameter vector $\boldsymbol{\tau}$; the cumulative function of the GBG distribution is defined by

$$F_{\mathcal{GBG}}(x;\boldsymbol{\tau},a,b,c) = I_{F(x;\boldsymbol{\tau})^c}(a,b) = B(a,b)^{-1}\int_0^{F(x;\boldsymbol{\tau})^c} \omega^{a-1}(1-\omega)^{b-1}d\omega, \tag{3}$$

where $B(a, b)$ is the beta function, $I_x(a,b) = B(a,b)^{-1}\int_0^x \omega^{a-1}(1-\omega)^{b-1}d\omega$ denotes the incomplete beta function ratio, and a, b, and c are additional positive shape parameters to those τ in $F(x;\, \boldsymbol{\tau})$. Here, the parameters a and b are the classical beta generator parameters to control skewness, kurtosis, and tail weights, and c aims to add entropy to the center of the transformed distribution. The random variable X having cdf (3) has a simple stochastic representation $X = Q_F(V^{1/c}; \boldsymbol{\tau})$, where $x = Q_F(u; \boldsymbol{\tau}) = F^{-1}(u; \boldsymbol{\tau})$ denotes the qf corresponding to F and V has a beta distribution with parameters a and b.

The density function corresponding to Eq. (3) can be expressed as

$$f_{\mathcal{GBG}}(x; \boldsymbol{\tau}, a, b, c) = c\, B(a,b)^{-1} f(x; \boldsymbol{\tau}) F(x; \boldsymbol{\tau})^{ac-1}\left[1 - F(x; \boldsymbol{\tau})^c\right]^{b-1}, x \in \mathcal{I}. \tag{4}$$

Two important special generators of Eq. (4) are the beta generated (BG) ($c = 1$) (Eugene et al. 2002; Jones 2004) and Kumaraswamy generated (KwG) ($a = 1$) (Cordeiro and de Castro 2011). The beta type I density function (McDonald 1984) itself arises if $F(x;\, \boldsymbol{\tau})$ is taken to be the uniform cumulative function. The BG and KwG generators have only two extra parameters, so they can add only different weights in the two tails of the transformed distribution. However, the GBG generator is a more flexible generator of distributions, since Alexander et al. (2012) demonstrated that the third extra parameter c gives a better control over both skewness and kurtosis by adding entropy to the center of Eq. (4).

It follows immediately from Eq. (4) that the GBG distribution with baseline cdf F is equal to the BG distribution with baseline exponentiated F, say, F^c. This simple transformation facilitates the computation of many of its properties. Henceforth, for an arbitrary parent F, we write $X \sim \mathrm{Exp}^c F$ if X has cdf and pdf given by

$$G_c(x) = F(x; \boldsymbol{\tau})^c \qquad \text{and} \qquad g_c(x) = cf(x; \boldsymbol{\tau})F(x; \boldsymbol{\tau})^{c-1},$$

respectively. In the last years, several properties of exponentiated distributions have been studied, such as the exponentiated Weibull (Mudholkar et al. 1995), exponentiated Pareto (Gupta et al. 1998), generalized exponential (Gupta and Kundu 2001; Nadarajah 2011), and exponentiated gamma (Nadarajah and Gupta 2007) distributions. The transformation Exp^cF is referred to the Lehmann type I distribution. There is a dual transformation defined by Exp^c(1-F) called the Lehmann type II distribution. The cdf of the last transformation corresponds to $H_c(x) = 1 - [1 - F(x; \boldsymbol{\tau})]^c$. Clearly, the double construction beta-Exp^cF yields the GBG distribution. Thus, in addition to the classical BG and KwG distributions and the parent distribution F itself, the GBG distribution encompasses the ExpF (for $b = 1$) and Exp(1 − F) (for $a = c = 1$) distributions. Thus, the GBG family (4) includes as special models the beta-generated, Kumaraswamy-generated, and exponentiated distributions.

For brevity of notation, we drop the explicit reference to the baseline parameters $\boldsymbol{\tau}$, unless otherwise stated. Expanding the binomial term in Eq. (4) gives

$$f_{\mathcal{GBG}}(x; a, b, c) = f(x; \boldsymbol{\tau}) \sum_{j=0}^{\infty} \delta_j [1 - F(x; \boldsymbol{\tau})]^j, \tag{5}$$

where

$$\delta_j = \frac{(-1)^j c}{B(a, b+1)} \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} \binom{c(a+i)-1}{j}.$$

We adopt a generalization to represent a different approach to much of the literature so far: Rather than considering the beta and Kumaraswamy generators applied to a baseline distribution, we work with a more flexible GBG generator applied to the EW distribution. The skewness parameters a and b control skewness, kurtosis, and tail weights, and the parameter c adds entropy to the center of the transformed distribution.

We derive a new distribution from the GBG model by inserting Eqs. (1) and (2) in Eq. (4). Its cumulative function (for $x > 0$) is given by

$$F_{\text{McEW}}(x;\boldsymbol{\tau},a,b,c) = \begin{cases} I_{\left[1-(1+\varphi\lambda x^{\beta})^{-\frac{1}{\varphi}}\right]^{c}}(a,b) & \text{if } \varphi > 0, \\ I_{[1-\exp(-\lambda x^{\beta})]^{c}}(a,b) & \text{if } \varphi = 0, \end{cases} \tag{6}$$

where a, b, c, φ, and β are positive shape parameters and λ is a scale parameter. Thus, $\boldsymbol{\tau} = (\varphi, \lambda, \beta)^T$. We call Eq. (6) the *McDonald extended Weibull* (McEW) distribution, since McDonald (1984) was the first to propose the GBG density function (4) when $F(x;\boldsymbol{\tau}) = x$.

The McEW density function (for $x > 0$) corresponding to (6) is given by

$$f_{\text{McEW}}(x;\boldsymbol{\tau},a,b,c) = \begin{cases} c\lambda\beta B(a,b)^{-1}x^{\beta-1}(1+\varphi\lambda x^{\beta})^{-\frac{1}{\varphi}-1}[1-(1+\varphi\lambda x^{\beta})^{-\frac{1}{\varphi}}]^{ac-1} \\ \quad\times \{1-[1-(1+\varphi\lambda x^{\beta})^{-\frac{1}{\varphi}}]^{c}\}^{b-1} & \text{if } \varphi > 0, \\ c\beta\lambda B(a,b)^{-1}x^{\beta-1}\exp(-\lambda x^{\beta})[1-\exp(-\lambda x^{\beta})]^{ac-1} \\ \quad\times \{1-[1-\exp(-\lambda x^{\beta})]^{c}\}^{b-1} & \text{if } \varphi = 0. \end{cases} \tag{7}$$

Hereafter, a random variable X is denoted by $X \sim$McEW(a, b, c, φ, λ, β) if X has pdf (7). Plots of the density (7) and its hazard rate function (hrf) for some parameter values are displayed in Figure 1. The hrf of the new distribution can have the four possible types of forms.

We study some general structural properties of the McEW distribution. Several important special models (much more than fifteen different models) can be defined from Eq. (7).

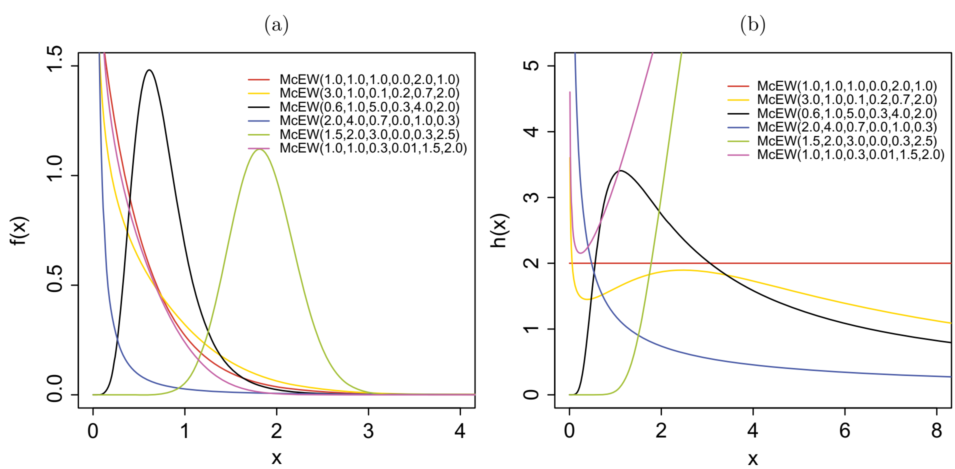

Figure 1. (a) The McEW density function for some parameter values; (b) the McEW hrf for some parameter values.

A positive point of the proposed generalization is the fact that the EW is a basic exemplar of the new model. Here, we present some examples:

i. When $\varphi = 0$, we obtain from Eq. (7) the McDonald Weibull (McW), which includes the following six special models: the McDonald exponential (McE) ($\beta = 1$), McDonald Rayleigh (McR) ($\beta = 2$), Kumaraswamy Weibull (KwW) ($a = 1$), Kumaraswamy exponential (KwE) ($a = \beta = 1$), beta Weibull (BW) ($c = 1$) (Cordeiro et al. 2012), and beta exponential (BE) (Nadarajah and Kotz 2006), among others. The McE distribution is a model studied by Barreto-Souza et al. (2010) under the name of beta generalized exponential distribution.
ii. For $\varphi \geq 0$ and $a = b = c = 1$, Eq. (7) reduces to the EW model. In addition, if $\beta = 1$, it follows the extended exponential (EE) distribution.
iii. The ExpW distribution with cdf $G(x) = (1 - e^{-\lambda x^{\beta}})^{a}$ corresponds to $\varphi = 0$ and $b = c = 1$. In addition, if $\beta = 1$, it becomes the exponentiated exponential (ExpE) distribution (Gupta and Kundu 1999) (also called the generalized exponential distribution).
iv. The beta extended exponential (BEE) distribution is a special case when $\varphi \geq 0$ and $c = \beta = 1$, whereas the beta exponential (BE) (Nadarajah and Kotz 2006) distribution corresponds to $\varphi = 0$ and $c = \beta = 1$.
v. The Kumaraswamy extended Weibull (KwEW) and KwW distributions are submodels for $\varphi \geq 0$ and $a = 1$ and $\varphi = 0$ and $a = 1$, respectively. Further, the Kumaraswamy extended exponential (KwEE) and KwE distributions are special cases when $\varphi \geq 0$ and $a = \beta = 1$ and $\varphi = 0$ and $a = \beta = 1$, respectively.
vi. For $\varphi > 0$ and $a = b = 1$, we have a new model called the exponentiated extended exponential (ExpEE) distribution.

After some algebra from the expanding form (5), we can write

$$F_{\text{McEW}}(x;\boldsymbol{\tau},a,b,c) = \begin{cases} \lambda\beta x^{\beta-1}\sum_{j=0}^{\infty}\delta_j(1+\varphi\lambda x^{\beta})^{-\frac{1}{\varphi}(j+1)-1} & \text{if} \quad \varphi > 0, \\ \sum_{j=0}^{\infty}\delta_j^{*}\lambda_{j,\beta}(x) & \text{if} \quad \varphi = 0, \end{cases} \tag{8}$$

where $\delta_j^{*} = \delta_j/(j+1)$ and $g\lambda_{j,\beta}(x)$ denotes the Weibull density with scale parameter $\lambda_j = (j+1)\lambda$ and shape parameter β. The main properties of the McEW model can be derived from Eq. (8). Alternatively, we could have used expansion (8) in Alexander et al. (2012) to provide similar results.

By integrating Eq. (8), we obtain

$$F_{\text{McEW}}(x;\boldsymbol{\tau},a,b,c) = \begin{cases} \sum_{j=0}^{\infty}\delta_j^{*}[1-S(x;\boldsymbol{\tau})^{j+1}] & \text{if} \quad \varphi > 0, \\ \sum_{j=0}^{\infty}\delta_j^{*}G_{\lambda_{j,\beta}}(x) & \text{if} \quad \varphi = 0, \end{cases}$$

where $G_{\lambda_j,\beta}(x) = 1 - e^{-\lambda_j x^{\beta}}$.

3. Generating Function

We now provide formulas for the moment-generating function (mgf) $M(t) = E[\exp(tX)]$ of X. The algebraic developments from Eq. (8) follow closely the works by Cheng et al.

(2003), Nadarajah and Kotz (2007), and Cordeiro et al. (2010). We consider two different cases. First, for $\varphi > 0$, we obtain

$$M(t) = \lambda \sum_{j=0}^{\infty} \delta_j L_j(t) \tag{9}$$

where

$$L_j(t) = \int_0^{\infty} \frac{\exp(tx)}{(1+\varphi\lambda x)^{\frac{(j+1)}{\varphi}+1}} dx.$$

This integral can be calculated from Prudnikov et al. (1986) as

$$\int_0^{\infty} \frac{x^{\alpha-1}}{(1+x)^{\rho}} \exp(-px)dx = \Gamma(\alpha)\, {}_1F_1(\alpha, \alpha+1-\rho; p),$$

where $\Gamma(\cdot)$ is the gamma function and ${}_1F_1(\alpha, \alpha+1-\rho; p)$ is the confluent hypergeometric function, defined by

$${}_1F_1(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(b+j)} \frac{x^j}{j!}.$$

For $t < 0$, using this integral in Eq. (9), we obtain

$$M(-t) = \varphi^{-1} \sum_{j=0}^{\infty} \delta_j\, {}_1F_1\left(1, 1 - \frac{(j+1)}{\varphi}; \frac{t}{\varphi\lambda}\right). \tag{10}$$

Second, for $\varphi = 0$, we have

$$M(t) = \sum_{j=0}^{\infty} \delta_j^* \lambda_j \int_0^{\infty} \exp(tx - \lambda_j x)dx = \lambda \sum_{j=0}^{\infty} \frac{(j+1)\delta_j^*}{(j+1)\lambda - t}. \tag{11}$$

Equations (10) and (11) are the main results of this section. By substituting known parameters in these equations, we can obtain from them specific formulas for the McEW special models.

4. Moments

Some of the most important features and characteristics of a distribution can be studied through moments. Consequently, we can obtain from Eq. (7)

$$\mu_r' = E(X^r) = \begin{cases} \varphi^{-(\frac{r}{\beta}+1)} \lambda^{-\frac{r}{\beta}} \sum_{j=0}^{\infty} \delta_j B\left(\frac{r}{\beta}+1, \frac{j+1}{\varphi} - \frac{r}{\beta}\right) & \text{if } \varphi > 0 \text{ and } r < \beta\varphi^{-1}, \\ \lambda^{-\frac{r}{\beta}} \Gamma\left(\frac{r}{\beta}+1\right) \sum_{j=0}^{\infty} \frac{\delta_j^*}{(j+1)^{r/\beta}} & \text{if } \varphi = 0. \end{cases} \tag{12}$$

The condition $r < \beta\varphi^{-1}$ is required for the existence of μ'_r for $\varphi > 0$. Established algebraic expansions to determine $E(X^r)$ can be more efficient than computing these moments directly by numerical integration of Eq. (4), which can be prone to rounding errors among others.

Further, the central moments (μ_r) and cumulants (κ_r) of X are immediately determined from Eq. (12) based on the well-known relationships

$$\mu_r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \mu_1^{'k} \mu'_{r-k} \quad \text{and} \quad \kappa_r = \mu'_r - \sum_{k=1}^{r-1} \binom{r-1}{k-1} \kappa_k \mu'_{r-k},$$

respectively, where $\kappa_1 = \mu'_1$. Then, $\kappa_2 = \mu'_2 - \mu_1^{'2}$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1^{'3}$, $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2^{'2} + 12\mu'_2\mu_1^{'2} - 6\mu_1^{'4}$, and so on. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ coefficients can be obtained readily from the second, third, and fourth cumulants.

Now our primary interest is to investigate the effect of the parameter c on both skewness and kurtosis. Thus, we represent these measures as functions of c assuming fixed values for the parameters a and b. In Figure 2, we plot the skewness and kurtosis ($b = 3.0$, $\varphi = 1.0$, $\lambda = 0.5$, and $\beta = 2.0$) as a function of c for some values of a. In Figure 3, we plot the skewness and kurtosis ($a = 2.0$, $\varphi = 1.0$, $\lambda = 1.5$, and $\beta = 3.0$) as a function of c for some values of b. When c increases, the behavior of the skewness is more influenced by the value of a (fixed b) than by the value of b (fixed a). On the other hand, the kurtosis of X depends heavily on the values of both a and b when c increases. This kurtosis can reach very high values such as 30 depending on both a and c. These plots indicate that the shape parameter c can yield very asymmetric heavy-tailed distributions.

The rth descending factorial moment of X is

$$\mu'_{(r)} = \mathrm{E}\left[X^{(r)}\right] = \mathrm{E}\left[X(X-1) \times \cdots \times (X-r+1)\right] = \sum_{k=0}^{r} s(r,k)\mu'_k,$$

where $s(r,k) = (k!)^{-1}\left[\frac{d^k}{dx^k}x^{(r)}\right]_{x=0}$ is the Stirling number of the first kind which counts the number of ways to permute a list of r items into k cycles. We can obtain the factorial moments from Eq. (12).

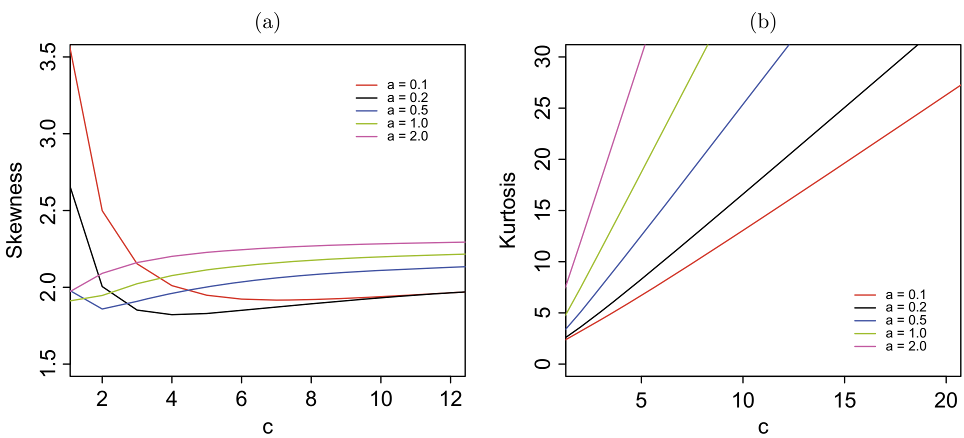

Figure 2. (a) Skewness of the McEW distribution as function of c (a fixed); (b) kurtosis of the McEW distribution as function of c (a fixed).

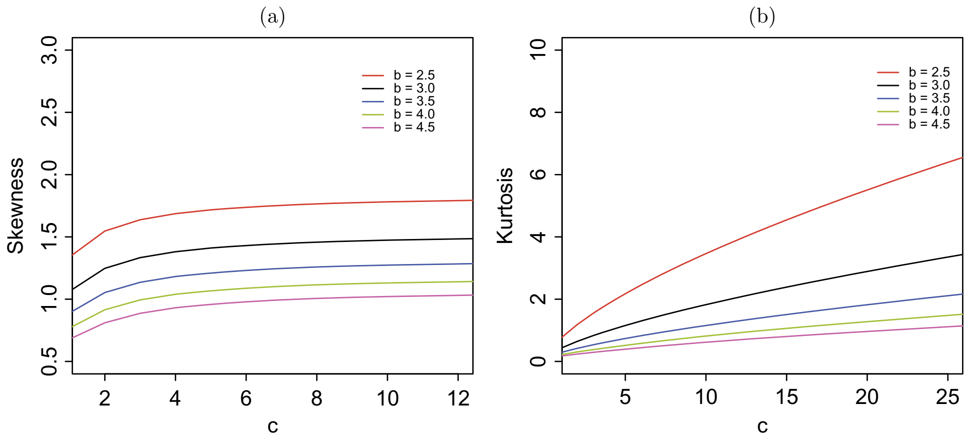

Figure 3. (a) Skewness of the McEW distribution as function of c (b fixed); (b) kurtosis of the McEW distribution as function of c (b fixed).

For lifetime models, it is of interest to know the rth incomplete moment of X defined by $T_r(y) = \int_0^y x^r f_{\text{McEW}}(x; \boldsymbol{\tau}, a, b, c)dx$. Moreover, it is simple to verify from Eq. (8) that $T_r(y)$ (when $\varphi > 0$) can be expressed as

$$T_r(y) = \lambda \sum_{j=0}^{\infty} \delta_j \rho(y; r, \varphi\lambda, \varphi^{-1}(j+1)), \tag{13}$$

where

$$\rho(y; r, p, q) = \int_0^y x^r (1+px)^{-q-1} dx$$

for $r, p, q > 0$. Using Maple, this integral can be calculated as

$$\rho(y; r, p, q) = A(r, p, q) \left\{ 2y_2^{r-q} F_1 \left(q^{-r}, q+1; q+1-r, -\frac{1}{py} \right) [B(r, p, q) + C(r, p, q)] \right\}, \tag{14}$$

where ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)} \frac{x^j}{j!},$$

$$A(r, p, q) = \{p \sin[\pi(q-r)](q-r)\Gamma(q+1)\Gamma(r+1-q)\}^{-1},$$

$$B(r, p, q) = p^{-q}\Gamma(q+1)\Gamma(r+1-q)[\cos(q\pi)\sin(\pi r) - \sin(q\pi)\cos(\pi r)],$$

and

$$C(r, p, q) = \pi p^{-r}\Gamma(r+1)(q-r).$$

Combining Eqs. (13) and (14), we obtain the incomplete moments. For $\varphi = 0$, $T_r(y)$ can be expressed as

$$T_r(y) = \lambda^{-r} \sum_{j=0}^{\infty} \delta_j^* \frac{\gamma(r+1,(j+1)\lambda y)}{(j+1)^r}, \tag{15}$$

where $\gamma(r,x) = \int_0^x t^{r-1} e^{-t} dt$ denotes the incomplete gamma function. Other kinds of moments such as L-moments may also be obtained in closed-form, but we consider only the moments already mentioned. Equations (12), (13), (14), and (15) are the main results of this section.

5. Mean Deviations

The mean deviations of X about the mean and about the median can be used as measures of spread in a population. They are given by $\delta_1 = E(|X - \mu_1'|) = 2\mu_1' F(\mu_1') - 2T_1(\mu_1')$ and $\delta_2 = E(|X - m|) = \mu_1' - 2T_1(m)$, respectively, where the mean μ_1' comes from Eq. (12), $F(q) = F_{\text{McEW}}(q; \boldsymbol{\tau}, a, b, c)$ is obtained from Eq. (6), and, for $\varphi > 0$, the median $m = Median(X)$ can be calculated from the nonlinear equation

$$I_{\left[1-(1+\varphi\lambda m)^{-\frac{1}{\varphi}}\right]^c}(a,b) = 1/2,$$

where $T_1(\cdot)$ comes from Eq. (13) with $r = 1$ as

$$T_1(y) = \lambda \sum_{j=0}^{\infty} \delta_j \rho(y; 1, \varphi\lambda, \varphi^{-1}(j+1)).$$

Here, $\rho(y; 1, p, q)$ can be reduced to

$$\rho(y; 1, p, q) = -\left[q(q-1)p^2\right]^{-1} \left[\frac{(pqy+1)}{(1+py)^q} - 1\right].$$

Further, for $\varphi = 0$, we have

$$T_1(q) = \sum_{j=0}^{\infty} \delta_j^* \lambda_j \int_0^q x \exp(-\lambda_j x) dx,$$

and then it becomes

$$T_1(q) = \sum_{j=0}^{\infty} \delta_j^* \left\{ \frac{1 - [1 + (j+1)\lambda q] \exp[-(j+1)\lambda q}{(j+1)\lambda} \right\}.$$

Both equations for $T_1(\cdot)$ can be used to determine Lorenz and Bonferroni curves defined by $L(\pi) = T_1(q)/\mu_1'$ and $B(\pi) = T_1(q)/(\pi \mu_1')$, respectively, where $q = Q(\pi)$ is the qf of X given by Eq. (16) (see section 6) at a given probability π. These curves are important in several fields such as economics, reliability, demography, insurance, and medicine. If $\pi =$

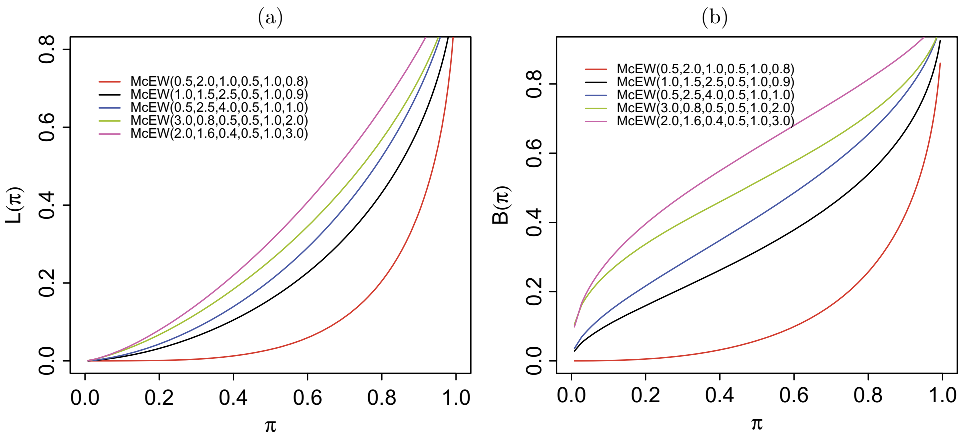

Figure 4. (a) The Lorenz curve as function of π for some parameter values; (b) the Bonferroni curve as function of π for some parameter values.

$F(q)$ is the proportion of units whose income is lower than or equal to q, the Lorenz curve $L(\pi)$ gives the proportion of total income volume accumulated by the set of units with an income lower than or equal to q. It is easy to see that $L(\pi) > \pi$, $L(0) = 0$, and $L(1) = 1$ and that $L(\pi)$ is increasing and convex. In a similar manner, the Bonferroni curve $B(\pi)$ gives the ratio between the mean income of this group and the mean income of the population. In summary, $L(\pi)$ yields fractions of the total income, whereas the values of $B(\pi)$ refer to the relative income levels. Plots of $L(\pi)$ and $B(\pi)$ for selected parameter values are displayed in Figure 4.

The Gini concentration index G is twice the area between the curve $L(\pi)$ and the straight line $L(\pi) = \pi$. Thus, $G = 1 - 2\int_0^1 L(\pi)d\pi$. A simple formula to obtain the G index is $G = \rho_1/\mu_1' - 1/2$, where $\rho_1 = E[X\,F(X)]$. For the McEW distribution, we can calculate ρ_1 (when $\varphi > 0$ and $\varphi = 0$) from the quantile expansions (21) and (22) (see section 6), respectively. We have

$$G = \begin{cases} \frac{1}{\varphi\lambda\mu_1'}\sum_{k=0}^{\infty}\frac{e_k}{(k/a+2)} - \frac{1}{2} & \text{if } \varphi > 0 \text{ and} \\ \frac{1}{\mu_1'}\sum_{k=0}^{\infty}\frac{v_k}{(k/a+2)} - \frac{1}{2} & \text{if } \varphi = 0. \end{cases}$$

6. Quantile Function

First, consider the more general case $\varphi > 0$. We can write the qf of X, say, $Q(u) = F^{-1}(u)$, in terms of the beta qf. By inverting $F(x) = I_{[1-(1+\varphi\lambda x^{\beta})]^c}(a, b) = u$, we obtain

$$x = Q(u) = \left\{\frac{\left[1 - Q_{a,b}^{1/c}(u)\right]^{-\varphi} - 1}{\varphi\lambda}\right\}^{1/\beta}, \tag{16}$$

where $Q_{a,b}(u) = I_u^{-1}(a, b)$ denotes the beta qf with parameters a and b. Then, using the binomial expansion, Eq. (16) can be expressed as

$$Q(u) = (\varphi\lambda)^{-1}\left(\sum_{j=0}^{\infty}(-1)^j\binom{-\varphi}{j}Q_{a,b}^{j/c}(u) - 1\right). \tag{17}$$

Now we obtain a power series expansion for $Q(u)$ that can be useful to derive some probability measures. First, the following expansion for the beta qf $Q_{ac^{-1},b}(u)$ can be found at the Wolfram website (http://functions.wolfram.com/06.23.06.0004.01):

$$Q_{a,b}(u) = \sum_{i=1}^{\infty} d_i u^{i/a},$$

where the coefficients are $d_i = A_i\,[a\,B(a,b)]^{i/a}$ and the quantities A_i for $i \geq 2$ can be determined from a cubic recursion of the form (with $A_1 = 1$)

$$A_i = \frac{1}{\left[i^2 + (a-2)i + (1-a)\right]}\left\{(1-\delta_{i,2})\sum_{r=2}^{i-1} A_r A_{i+1-r}\,[r(1-a)(i-r)\right.$$
$$-\,r(r-1)] + \sum_{r=1}^{i-1}\sum_{s=1}^{i-r} A_r\,A_s\,A_{i+1-r-s}\,[(r-a) + s(a+b-2)$$
$$\left.\times\,(i+1-r-s)]\right\},$$

where $\delta_{i,2} = 1$ if $i = 2$ and $\delta_{i,2} = 0$ if $i \neq 2$. In the last equation, we note that the quadratic term only contributes for $i \geq 3$. The first four quantities, for example, are

$$A_2 = \frac{b-1}{a+1}, \qquad A_3 = \frac{(b-1)\left[a^2 + a(3b-1) + 5b - 4\right]}{2(a+1)^2(a+2)}$$

and

$$A_4 = \frac{(b-1)\left[a^4 + (6b-1)a^3 + (b+2)(8b-5)a^2\right]}{(3a+1)^3(a+2)(a+3)}.$$

Second, we note that $Q_{a,b}(u) \in (0,1)$, and then we can obtain the expansion

$$Q_{a,b}(u)^{\delta} = \sum_{r=0}^{\infty} s_r(\delta) Q_{a,b}(u)^r, \tag{18}$$

where

$$s_r(\delta) = \sum_{j=r}^{\infty}(-1)^{r+j}\binom{\delta}{j}\binom{j}{r}.$$

By using Eq. (18), we can write

$$Q(u) = (\varphi\lambda)^{-1}\left(\sum_{r=0}^{\infty} q_r Q_{a,b}(u)^r\right),$$

where $q_r = \sum_{j=0}^{\infty}(-1)^j\binom{-\varphi}{j}s_r(j/c)$ for $r = 1, 2, \ldots$ and $q_0 = \sum_{j=0}^{\infty}(-1)^j\binom{-\varphi}{j}$ $s_0(j/c) - 1$. Hence, we can rewrite Eq. (17) as

$$Q(u) = (\varphi\lambda)^{-1}\left[\sum_{r=0}^{\infty} q_r\left(u^{1/a}\sum_{m=0}^{\infty} a_m u^{m/a}\right)^r\right], \tag{19}$$

where $a_m = d_{m+1} = A_{m+1}\,[a\,B(a,b)]^{(m+1)/a}$ for $m = 0, 1, \ldots$. Third, we use an equation for a power series raised to a positive integer r (Gradshteyn and Ryzhik 2000, section 0.314)

$$\left(\sum_{m=0}^{\infty} a_m z^m\right)^r = \sum_{m=0}^{\infty} p_{r,m} z^m, \tag{20}$$

where the coefficients $p_{r,m}$ (for $m = 1, 2, \ldots$) can be determined from $a_0, \ldots, a_m$ using the recurrence equation (with $p_{r,0} = a_0^r$) for $m \geq 1$

$$p_{r,m} = (m\,a_0)^{-1}\sum_{k=1}^{m}[k(r+1) - m]\,a_k p_{r,m-k}.$$

Clearly, the coefficient $p_{r,m}$ can be given explicitly in terms of the quantities a_m, although it is not necessary for programming numerically our expansions in any algebraic or numerical software. Finally, combining Eqs. (19) and (20), we obtain

$$Q(u) = (\varphi\lambda)^{-1}\sum_{k=0}^{\infty} e_k u^{k/a}, \tag{21}$$

where

$$e_k = \underbrace{\sum_{r,m=0}^{k}}_{r+m=k} q_r p_{r,m}$$

for $k = 1, 2, \ldots$ and $e_0 = q_0$.

We can derive some mathematical properties for the new distribution based on the power series expansion (21). For example, the rth ordinary moment $\mu'_r = \int_0^1 Q(u)^r du$ of X when $\varphi > 0$ can be computed from Eqs. (20) and (21) as

$$Q(u)^r = (\varphi\lambda)^{-r}\left(\sum_{k=0}^{\infty} e_k u^{k/a}\right)^r = (\varphi\lambda)^{-r}\left(\sum_{k=0}^{\infty} f_{r,k} u^{k/a}\right),$$

where $f_{r,0} = e_0^r$ and, for $k \geq 1$,

$$f_{r,k} = (k\, e_0)^{-1} \sum_{i=1}^{k} [(r+1)i - m]\, e_i f_{r,k-i}.$$

Hence, the rth moment of X becomes

$$\mu'_r = (\varphi\lambda)^{-r} \sum_{k=0}^{\infty} \frac{f_{r,k}}{\left(\frac{k}{a}+1\right)}.$$

For the case $\varphi = 0$, the algebraic calculations are simpler. It can be proved that $Q(u)$ can be expanded using Eq. (18) as

$$Q(u) = -\lambda^{-1} \log\left(1 - Q_{a,b}(u)^{1/c}\right) = -\lambda^{-1} \sum_{r=0}^{\infty} h_r\, Q_{a,b}(u)^r,$$

where $h_r = \sum_{k=1}^{\infty} k^{-1} s_r(k/c)$ for $r = 0, 1, \ldots$ Finally, following similar algebra developed from Eqs. (19) and (20) (for the case $\varphi > 0$), we obtain

$$Q(u) = \sum_{k=0}^{\infty} v_k u^{k/a}, \tag{22}$$

where

$$v_k = -\lambda^{-1} \underbrace{\sum_{r,m=0}^{k}}_{r+m=k} h_r p_{r,m}$$

for $k = 1, 2, \ldots$ and $v_0 = h_0$.

Equations (21) and (22) are the main results of this section.

7. Skewness and Kurtosis Based on Quantiles

We provide a further insight of the effect of the parameters a, b, and c on the skewness and kurtosis by considering these measures based on quantiles. The shortcomings of the classical kurtosis measure are well known. There are many heavy-tailed distributions for which this measure is infinite, so it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the nonexistence of classical kurtosis for several GBG distributions.

The Bowley skewness (see Kenney and Keeping 1962) is based on quartiles:

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

and the Moors kurtosis (see Moors 1988) is based on octiles:

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)},$$

where $Q(\cdot)$ represents the quantile function. These measures are less sensitive to outliers and they exist even for distributions without moments. For the classical Student t distribution with 10 and 5 degrees of freedom, these measures are zeroes (Bowley) and 1.27705 and 1.32688 (Moors), respectively. For the standard normal distribution, these measures are zero (Bowley) and 1.2331 (Moors).

In Figure 5, we plot B (for $a = 0.5$, $\varphi = 2.0$, $\lambda = 1.0$, and $\beta = 1.5$) as a function of c for selected values of b and B (for $b = 0.5$, $\varphi = 2.0$, $\lambda = 1.0$, and $\beta = 1.5$) as a function of c for selected values of a. In Figure 6, we plot M (for $a = 3.0$, $\varphi = 3.0$, $\lambda = 1.0$, and $\beta = 1.5$) as a function of c for selected values of b and M (for $b = 3.0$, $\varphi = 0.5$, $\lambda = 1.0$, and $\beta = 1.5$) as a function of c for some values of a. We note that the Bowley skewness for fixed a or b first decreases reaching a minimum value and then increases when c increases. A different picture emerges based on the Morris kurtosis. When c increases, for fixed a or b, the Morris kurtosis decreases sharply and then stabilizes at an asymptotic level.

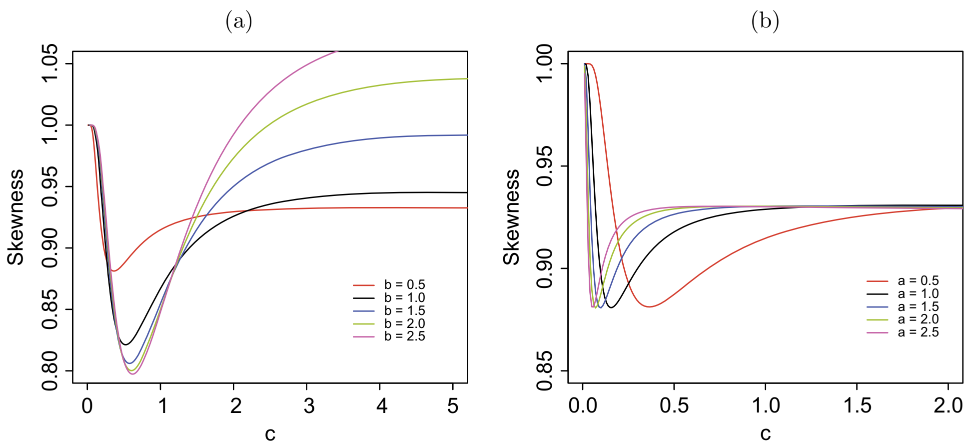

Figure 5. (a) Bowley's skewness of the McEW distribution as a function of c (b fixed); (b) Bowley's skewness of the McEW distribution as a function of c (a fixed).

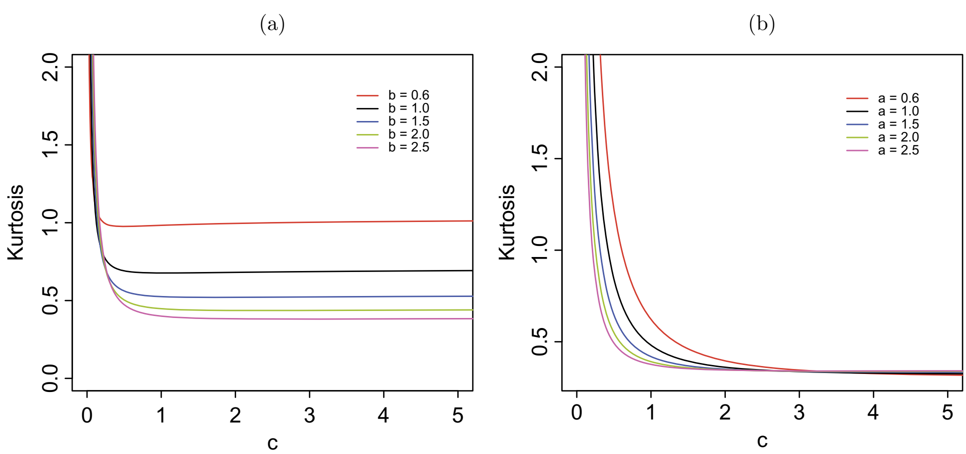

Figure 6. (a) Moors' kurtosis of the McEW distribution as a function of c (b fixed); (b) Moors' kurtosis of the McEW distribution as a function of c (a fixed).

8. Inference and Estimation

In this section, we discuss the maximum likelihood method and a Bayesian approach for inference and estimation of the McEW parameters.

8.1. Maximum Likelihood Estimation

Here, we determine the maximum likelihood estimates (MLEs) of the McEW parameters from complete samples only. Let $x_1, \ldots, x_n$ be a sample of size n from X. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (a, b, c, \varphi, \lambda, \beta)^T$ can be expressed as

$$l(\boldsymbol{\theta}) = \begin{cases} n\log(c) + n\log(\lambda) + \log(\beta) - n\log[B(a,b)] + (\beta-1)\sum_{i=1}^{n}\log(x_i) - \left(\frac{1}{\varphi}+1\right) \\ \times \sum_{i=1}^{n}\log(1+\varphi\lambda x_i^{\beta}) + (ac-1)\sum_{i=1}^{n}\log[G(x_i)] + (b-1)\sum_{i=1}^{n}\log[1-G(x_i)^c] \quad \text{if} \quad \varphi > 0 \\ n\log(c) + n\log(\beta) + n\log(\lambda) - n\log[B(a,b)] + (\beta-1)\sum_{i=1}^{n}\log(x_i) - \lambda\sum_{i=1}^{n}x_i^{\beta} \\ -(ac-1)\sum_{i=1}^{n}\log[G(x_i)] + (b-1)\sum_{i=1}^{n}\log[1-G(x_i)^c] \quad \text{if} \quad \varphi = 0, \end{cases} \tag{23}$$

where $G(x_i) = G(x_i; \boldsymbol{\tau})$is defined in Eq. (2).

The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$U_a(\boldsymbol{\theta}) = -n\psi(a) + n\psi(a+b) + c\sum_{i=1}^{n}\log[G(x_i)],$$

$$U_b(\boldsymbol{\theta}) = -n\psi(b) + n\psi(a+b) + c\sum_{i=1}^{n}\log[1-G(x_i)^c],$$

$$U_c(\boldsymbol{\theta}) = \frac{n}{c} + a\sum_{i=1}^{n}\log[G(x_i) - (b-1)\sum_{i=1}^{n}\frac{G(x_i)^c\log[G(x_i)]}{1-G(x_i)^c},$$

$$U_\varphi(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{[\dot{g}(x_i)]_\varphi}{g(x_i)} + (ac-1)\sum_{i=1}^{n}\frac{[\dot{G}(x_i)]_\varphi}{G(x_i)} + c(b-1)\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\varphi}{1-G(x_i)^c},$$

$$U_\lambda(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{[\dot{g}(x_i)]\lambda}{g(x_i)} + (ac-1)\sum_{i=1}^{n}\frac{[\dot{G}(x_i)]\lambda}{G(x_i)} + c(b-1)\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\dot{G}(x_i)]\lambda}{1-G(x_i)^c},$$

$$U_\beta(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{[\dot{g}(x_i)]\beta}{g(x_i)} + (ac-1)\sum_{i=1}^{n}\frac{[\dot{G}(x_i)]\beta}{G(x_i)} + c(b-1)\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\dot{G}(x_i)]\beta}{1-G(x_i)^c},$$

where $\psi(\cdot)$ is the digamma function and

$$[\dot{g}(x_i)]_\varphi = \frac{\partial[g(x_i)]}{\partial\varphi}, \quad [\dot{g}(x_i)]_\lambda = \frac{\partial[g(x_i)]}{\partial\lambda} \quad \text{and} \quad [\dot{g}(x_i)]_\beta = \frac{\partial[g(x_i)]}{\partial\beta}.$$

Setting these equations to zero and solving them simultaneously yields the MLEs of the six parameters. For interval estimation on the model parameters, we require the observed information matrix $J(\boldsymbol{\theta})$, namely,

$$J(\boldsymbol{\theta}) = \begin{pmatrix} J_{aa} & J_{ab} & J_{ac} & J_{a\varphi} & J_{a\varphi} & J_{a\beta} \\ \cdot & J_{bb} & J_{bc} & J_{b\varphi} & J_{b\varphi} & J_{b\beta} \\ \cdot & \cdot & J_{cc} & J_{c\varphi} & J_{c\lambda} & J_{c\beta} \\ \cdot & \cdot & \cdot & J_{\varphi\varphi} & J_{\varphi\lambda} & J_{\varphi\beta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & J_{\beta\beta} \\ & & & & & \end{pmatrix},$$

whose elements are given in the appendix.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_6(0, K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta}) = E[J(\boldsymbol{\theta})]$ is the expected information matrix. The approximate multivariate normal $N_6(0, J(\hat{\boldsymbol{\theta}})^{-1})$ distribution, where $J(\hat{\boldsymbol{\theta}})^{-1}$ is the observed information matrix evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$, can be used to construct approximate confidence intervals and confidence regions for the parameters.

The likelihood ratio (LR) statistic can be used for comparing the McEW distribution with some of its special models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain LR statistics for testing some of its submodels. In any case, hypothesis tests of the type $H_0: \psi = \psi_0$ versus $H_0: \psi \neq \psi_0$, where ψ is a vector formed with some components of $\boldsymbol{\theta}$ and ψ_0 is a specified vector, can be performed using LR statistics. For example, the test of H_0: $a = 1$ versus H:H_0 *is not true* is equivalent to compare the KwEW and KwEW distributions, and the LR statistic becomes

$$w = 2\{\ell(\hat{a}, \hat{b}, \hat{c}, \hat{\varphi}, \hat{\lambda}, \hat{\beta}) - \ell(1, \tilde{b}, \tilde{c}, \tilde{\varphi}, \tilde{\lambda}, \tilde{\beta})\},$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{\varphi}, \hat{\lambda}$, and $\hat{\beta}$ are the MLEs under H and $\tilde{b}, \tilde{c}, \tilde{\varphi}, \tilde{\lambda}$, and $\tilde{\beta}$ are the estimates under H_0.

8.2. A Bayesian Analysis

As an alternative analysis, we use the Bayesian method, which allows for the incorporation of previous knowledge of the parameters through informative prior density functions. When this information is not available, we can consider a noninformative prior. In the Bayesian approach, the information referring to the model parameters is obtained through a posterior marginal distribution. In this way, two difficulties usually arise. The first refers to attaining the marginal posterior distribution, and the second to the calculation of the moments of interest. Both cases require numerical integration that, many times, does not present an analytical solution. We use the Markov-chain Monte Carlo (MCMC) simulation method such as the Gibbs sampler and Metropolis–Hastings algorithm.

Since we have no prior information from historical data or from previous experiments, we assign conjugate but weakly informative prior distributions to the parameters. We consider informative (but weakly) prior distribution, and then the posterior distribution is a well-defined proper distribution. Further, we assume that the elements of the parameter vector are independent and consider that the joint prior distribution of the unknown parameters has a density function given by

$$\pi(a,b,c,\varphi,\lambda,\beta) \propto \pi(a)\times\pi(b)\times\pi(c)\times\pi(\lambda)\times\pi(\beta). \tag{24}$$

Here, $a \sim \Gamma(a_1,b_1)$, $b \sim \Gamma(a_2,b_2)$, $c \sim \Gamma(a_3,b_3)$, $\varphi \sim \Gamma(a_4,b_4)$, $\lambda \sim \Gamma(a_5,b_5)$, and $\beta \sim \Gamma(a_6,b_6)$, where $\Gamma(a_i,b_i)$ denotes a gamma distribution with mean $a_i\big/b_i$, variance $a_i\big/b_i^2$, and density function given by

$$f(v;a_i,b_i) = \frac{b_i^{a_i} v^{a_i-1}\exp(-vb_i)}{\Gamma(a_i)},$$

where $v > 0$, $a_i > 0$, and $b_i > 0$. All hyper-parameters are specified. Combining the likelihood function (23) and the prior distribution (24), the joint posterior distribution for a,b,c,φ,λ, and β reduces to

$$\begin{aligned}\pi(a,b,c,\varphi,\lambda,\beta|y) \propto \left(\frac{c\lambda\beta}{B(a,b)}\right)^n \prod_{i=1}^n x_i^{\beta-1} \prod_{i=1}^n \left(1+\varphi\lambda x_i^\beta\right)^{-\left(\frac{1}{\varphi}+1\right)} \prod_{i=1}^n [G(x_i)]^{ac-1} \\ \prod_{i=1}^n [1-G(x_i)^c]^{b-1} \times \pi(a,b,c,\varphi,\lambda,\beta).\end{aligned} \tag{25}$$

The joint posterior density (25) is analytically intractable because the integration of the joint posterior density is not easy to perform. Thus, the inference can be based on the MCMC simulation methods such as the Gibbs sampler and Metropolis–Hastings algorithm, which can be used to draw samples, from which features of the marginal distributions of interest can be inferred. In this direction, we obtain the full conditional distributions of the unknown quantities by

$$\pi(a|y,b,c,\varphi,\lambda,\beta) \quad\propto\quad [B(a,b)]^{-n}\prod_{i=1}^n [G(x_i)]^{ac} \times \pi(a),$$

$$\pi(b|y,a,c,\varphi,\lambda,\beta) \quad\propto\quad [B(a,b)]^{-n}\prod_{i=1}^n [1-G(x_i)^c]^{b} \times \pi(b),$$

$$\pi(c|y,a,b,\varphi,\lambda,\beta) \quad\propto\quad c^n\prod_{i=1}^n [G(x_i)]^{ac}\prod_{i=1}^n [1-G(x_i)^c]^{b-1} \times \pi(c),$$

$$\pi(\varphi|y,a,b,c,\lambda,\beta) \quad\propto\quad \prod_{i=1}^n \left(1+\varphi\lambda x_i^\beta\right)^{-\frac{1}{\varphi}}\prod_{i=1}^n [G(x_i)]^{ac-1}\prod_{i=1}^n [1-G(x_i)^c]^{b-1} \times \pi(\varphi),$$

$$\pi(\lambda|y,a,b,c,\varphi,\beta) \quad\propto\quad \lambda^n\prod_{i=1}^n \left(1+\varphi\lambda x_i^\beta\right)^{-\left(\frac{1}{\varphi}+1\right)}\prod_{i=1}^n [G(x_i)]^{ac-1}\prod_{i=1}^n [1-G(x_i)^c]^{b-1} \times \pi(\lambda),$$

and

$$\pi(\beta|y,a,b,c,\varphi,\lambda) \quad\propto\quad \beta^n\prod_{i=1}^n x_i^\beta \left(1+\varphi\lambda x_i^\beta\right)^{-\left(\frac{1}{\varphi}+1\right)}\prod_{i=1}^n [G(x_i)]^{ac-1}\prod_{i=1}^n [1-G(x_i)^c]^{b-1} \times \pi(\beta).$$

Since the full conditional distributions for a, b, c, φ, λ, and β do not have closed forms, we require the Metropolis–Hastings algorithm.

9. Application

In this section, we use a real data set ($n = 744$) related to environmental contamination previously studied by Balakrishnan et al. (2009). The McEW distribution can be applicable not only to lifetime data but also other kinds of data in sciences and technologies. Air quality gradually deteriorates in the world, according to a report released by the World Health Organization (WHO), which reveals that more than half of the world population live in cities with pollution levels at least 2.5 times higher than recommended by the agency. The most important thing to note is that the situation is worse in several places, especially in developing countries. Specifically, Santiago, Chile, is recognized as one of the most environmentally contaminated cities in the world. In order to obtain the level of air pollution and its associated adverse effects on humans in Santiago, the National Commission of Environment of the Government of Chile collects data on sulfur dioxide (SO_2) concentrations in the air. We compare the fits of the McEW distribution and three of its submodels to these data, namely, the ExpW ($\varphi = 0, b = c = 1$), BE ($\varphi = 0, c = \beta = 1$), and Weibull ($\varphi = 0, a = b = c = 1$) distributions.

First we describe the data set. Then we report the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the Akaike information criterion (AIC) and Bayesian information criterion (BIC) statistics. The lower the values of these criteria, the better is the fit. We note that overparameterization is penalized in these criteria, so that the additional parameters in the proposed distribution do not necessarily lead to lower values of the AIC or BIC statistics. In each case, the parameters are estimated by maximum likelihood (section 8.1) using the subroutine MAXBFGS in Ox. Next, we perform the LR tests (section 8.1) to verify whether the third skewness parameter is really necessary Finally, we give the histogram of these data and provide a visual comparison of the fitted density functions.

Table 1 gives a descriptive summary of the data set, which has positive skewness and kurtosis. We note that the average concentration of SO_2 (2.926 ppm) in Santigo is very high, because according to the criteria of the WHO, air quality considered good has a concentration of SO_2 less than 0.079 ppm.

Thus, we compute the MLEs of the parameters and the AIC and BIC statistics for the new model and the three submodels cited earlier fitted to these data.

The results from the fitted models are reported in Table 2. The two information criteria agree on the model's ranking in every case. The lowest values of the information criteria correspond to the McEW distribution.

Clearly, the McEW model having three skewness parameters should be chosen in this case. Formal tests for the skewness parameters in the new distribution can be based on LR statistics (section 8.1). The LR statistics for comparing the fitted models are given in

Table 1
Descriptive statistics

Mean	Median	Mode	SD	Variance	Skewness	Kurtosis	Min.	Max.
2.926	2.00	2.00	2.015	4.060	4.339	34.914	1.00	25.00

Table 2
MLEs of the model parameters for the SO_2 concentrations and information criteria

Model	a	b	c	φ	λ	β	AIC	BIC
McEW	1.2419	0.2811	0.3776	0.8603	0.00940	7.0308	2504.8	2532.5
	(0.7224)	(0.0351)	(0.2590)	(0.0800)	(0.00001)	(0.2759)		
ExpW	98.4830	1	1	0	3.33340	0.4449	2517.2	2530.9
	(16.0850)	—	—	—	(0.16040)	(0.0186)		
BE	13.5690	0.2987	1	0	2.08880	1	2538.8	2552.6
	(3.0965)	(0.0478)	—	—	(0.26830)	—		
Weibull	1	1	1	0	0.13890	1.6534	2783.6	2792.8
	—	—	—	—	(0.00980)	(0.0393)		

Table 3
LR tests

Security	Hypotheses	Statistic w	p-Value
McEW vs. ExpW	$H_0: \varphi = 0$ and $b = c = 1$ vs. H_1: H_0 is false	18.35	0.00037
McEW vs. BE	$H_0: \varphi = 0$ and $c = \beta = 1$ vs. H_1: H_0 is false	39.93	<0.00001
McEW vs. Weibull	$H_0: \varphi = 0$ and $a = b = c = 1$ vs. H_1: H_0 is false	286.75	<0.00001

Table 3. We reject the null hypotheses in the three LR tests in favor of the McEW distribution. The rejection is highly significant and it provides clear evidence of the potential need for three skewness parameters when modeling real data.

More information is provided by a visual comparison of the histogram of the data with the fitted density functions. The plots of the fitted McEw, ExpW, BE, and Weibull densities are displayed in Figure 7. The new distribution provides a closer fit to the histogram than

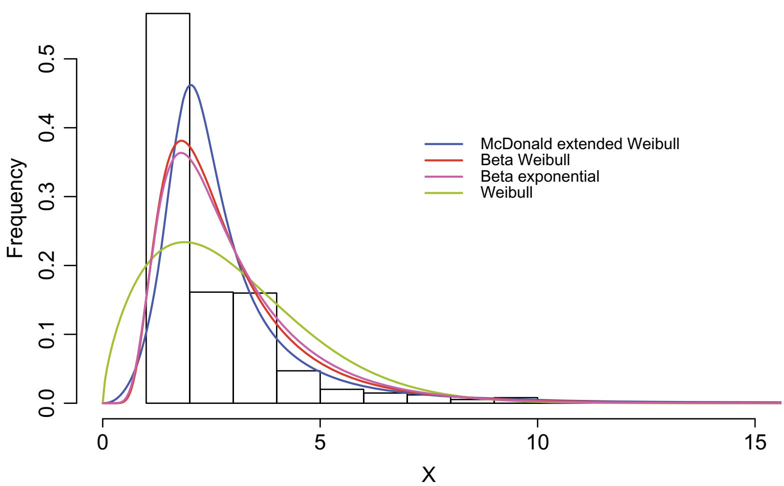

Figure 7. Fitted McEW, ExpW, BE, and Weibull densities to the SO_2 concentrations.

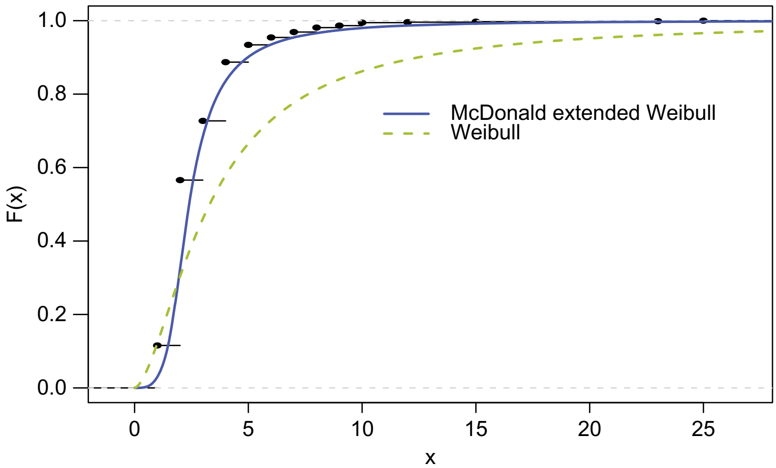

Figure 8. Estimated McEW and Weibull cdf's to the SO_2 concentrations.

the other three submodels. Evidently, besides a more adequate fit to the data, the McEW model better captures the extreme bathtub shape.

In order to assess if the model is appropriate, plots of the cdf of the McEW and Weibull distributions and the empirical cdf are displayed in Figure 8. We conclude that the McEW distribution provides the best fit to these data. Thus, based on the fitted new model, we conclude that the concentration of SO_2 increases to about 2 ppm and then immediately has a very fast decay (as illustrated by the fitted model in Figure 7). This behavior indicates that the majority of the SO_2 concentrations in Santigo are above the acceptable level and that there are possible small amounts of SO_2 concentrations above 10 ppm.

9.1. Bayesian Analysis

The following independent priors are considered to perform the Metropolis–Hastings algorithm: $a \sim \Gamma(0.01, 0.01)$, $b \sim \Gamma(0.01, 0.01)$, $c \sim \Gamma(0.01, 0.01)$, $\varphi \sim \Gamma(0.01, 0.01)$, $\lambda \sim \Gamma(0.01, 0.01)$, and $\beta \sim \Gamma(0.01, 0.01)$, so that we have a vague prior distribution. Considering these prior density functions, we generate two parallel independent runs of the Metropolis–Hastings with size 150,000 for each parameter, disregarding the first 15,000 iterations to eliminate the effect of the initial values, and, to avoid correlation problems, we consider a spacing of size 10 and thus obtain a sample of size 13,500 from each chain. To monitor the convergence of the Metropolis–Hastings, we use the methods suggested by Cowles and Carlin (1996). To monitor the convergence of the Metropolis–Hastings algorithm, we use the between and within sequence information, following the approach developed in Gelman and Rubin (1992) to obtain the potential scale reduction, $\hat{R}$. For all cases, these values are close to one, indicating the convergence of the chain. The approximate posterior marginal density functions for the parameters are presented in Figure 9. In Table 4, we report posterior summaries for the parameters of the McEW model. We note that the values for the means a posteriori (Table 4) are quite close (as expected) to the MLEs obtained for the McEW model given in Table 2. SD represents the standard deviation from the posterior distributions of the parameters and HPD represents the 95% highest posterior density (HPD) intervals.

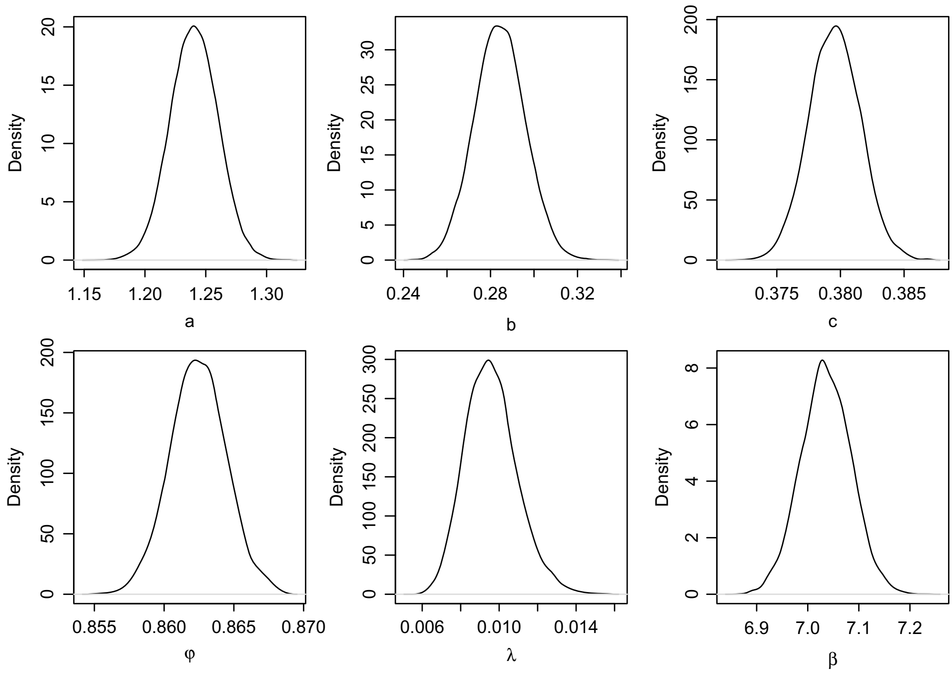

Figure 9. Approximate posterior marginal densities for the parameters of the McEW model fitted to the SO_2 concentrations.

Table 4
Posterior summaries for the parameters of the fitted McEW model for the SO_2 concentrations

Parameter	Mean	SD	HPD (95%)	$\hat{R}$
a	1.2404	0.0195	(1.2030; 1.2793)	1.0005
b	0.2843	0.0116	(0.2611; 0.3064)	1.0003
c	0.3795	0.0020	(0.3755; 0.3834)	1.0002
φ	0.8624	0.0021	(0.8582; 0.8662)	1.0027
λ	0.0096	0.0013	(0.0070; 0.0123)	1.0014
β	7.0358	0.0491	(6.9354; 7.1297)	1.0004

10. Conclusions

We introduce a new model called the McDonald extended Weibull (McEW) distribution and study some of its structural properties. It generalizes some important distributions in the literature and provides means of its continuous extension to still more complex situations. The new model contains several distributions as special models, including the generalized exponential (Gupta and Kundu 1999), extended exponential, beta exponential (Nadarajah and Kotz 2006), and beta generalized exponential (Barreto-Souza et al. 2010). Further, it includes some other new distributions as special cases, such as the Kumaraswamy extended exponential, Kumaraswamy exponential, beta extended exponential, exponentiated extended exponential, and McDonald exponential distributions. We provide explicit expressions for the density function, moments and incomplete moments, generating and quantile functions, mean deviations, Bonferroni and Lorenz curves, and Gini concentration index. The model parameters are estimated by maximum likelihood and using a Bayesian approach. The formulas derived in this article related with the McEW distribution are manageable, and, with the use of modern computer resources with analytic and numerical capabilities, may turn into adequate tools comprising the arsenal of applied statisticians. In conclusion, the McEW distribution can be used in the analysis of air quality data.

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Appendix

We can compute the elements of the observed information matrix $J(\boldsymbol{\theta})$ for the six parameters $(a, b, c, \varphi, \lambda, \beta)$. We obtain the following:

$$J_{aa}(\boldsymbol{\theta}) = -n\psi'(a) + n\psi'(a+b), \quad J_{ab}(\boldsymbol{\theta}) = n\psi'(a+b), \quad J_{ac}(\boldsymbol{\theta}) = \sum\nolimits_{i=1}^{n} \log[G(x_i)],$$

$$J_{a\varphi}(\boldsymbol{\theta}) = c\sum\nolimits_{i=1}^{n} \frac{[\dot{G}(x_i)]_\varphi}{G(x_i)}, \quad J_{a\lambda}(\boldsymbol{\theta}) = c\sum\nolimits_{i=1}^{n} \frac{[\dot{G}(x_i)]_\lambda}{G(x_i)}, \quad J_{a\beta}(\boldsymbol{\theta}) = c\sum\nolimits_{i=1}^{n} \frac{[\dot{G}(x_i)]_\beta}{G(x_i)},$$

$$J_{bb}(\boldsymbol{\theta}) = -n\psi'(b) + n\psi'(a+b), \quad J_{bc}(\boldsymbol{\theta}) = -\sum\nolimits_{i=1}^{n} \frac{G(x_i)^c \log[G(x_i)]}{1 - G(x_i)^c},$$

$$J_{b\varphi}(\boldsymbol{\theta}) = c\sum\nolimits_{i=1}^{n} \frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\varphi}{1 - G(x_i)^c}, \quad J_{b\lambda}(\boldsymbol{\theta}) = c\sum\nolimits_{i=1}^{n} \frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\lambda}{1 - G(x_i)^c},$$

$$J_{b\beta}(\boldsymbol{\theta}) = c\sum\nolimits_{i=1}^{n} \frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\beta}{1-G(x_i)^c}, \quad J_{cc}(\boldsymbol{\theta}) = -\frac{n}{c^2} - (b-1)\sum\nolimits_{i=1}^{n} \frac{G(x_i)^c \log^2[G(x_i)]}{1-G(x_i)^c}\left[1 + \frac{G(x_i)^c}{1-G(x_i)^c}\right],$$

$$J_{c\varphi}(\boldsymbol{\theta}) = a\sum_{i=1}^{n} \frac{[\dot{G}(x_i)]_\varphi}{G(x_i)} - c(b-1)\sum_{i=1}^{n} \frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\varphi \log[G(x_i)]}{1 - G(x_i)^c}\left[1 + \frac{G(x_i)^{c-1}}{1 - G(x_i)^c}\right] +$$
$$(b-1)\sum_{i=1}^{n} \frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\varphi}{1 - G(x_i)^c},$$

$$J_{c\lambda}(\boldsymbol{\theta}) = a\sum_{i=1}^{n} \frac{[\dot{G}(x_i)]_\lambda}{G(x_i)} - c(b-1)\sum_{i=1}^{n} \frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\lambda \log[G(x_i)]}{1 - G(x_i)^c}\left[1 + \frac{G(x_i)^{c-1}}{1 - G(x_i)^c}\right] +$$
$$(b-1)\sum_{i=1}^{n} \frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\lambda}{1 - G(x_i)^c},$$

$$J_{c\beta}(\boldsymbol{\theta}) = a\sum_{i=1}^{n}\frac{[\dot{G}(x_i)]_\beta}{G(x_i)} - c(b-1)\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\beta\log[G(x_i)]}{1-G(x_i)^c}\left[1+\frac{G(x_i)^{c-1}}{1-G(x_i)^c}\right]+$$
$$(b-1)\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\beta}{1-G(x_i)^c},$$

$$J_{\varphi\varphi}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{[\ddot{g}(x_i)]_{\varphi\varphi}[\dot{g}(x_i)]_\varphi-[\dot{g}(x_i)]_\varphi^2}{g(x_i)^2}+(ac-1)\sum_{i=1}^{n}\frac{[\ddot{G}(x_i)]_{\varphi\varphi}[\dot{G}(x_i)]_\varphi-[\dot{G}(x_i)]_\varphi^2}{G(x_i)^2}+$$
$$c(b-1)\left\{\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\varphi^2}{1-G(x_i)^c}\left[\frac{c-1}{G(x_i)^c}+\frac{cG(x_i)^{c-1}}{1-G(x_i)^c}\right]+\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\ddot{G}(x_i)]_{\varphi\varphi}}{1-G(x_i)^c}\right\},$$

$$J_{\varphi\lambda}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{[\ddot{g}(x_i)]_{\varphi\lambda}[\dot{g}(x_i)]_\varphi-[\dot{g}(x_i)]_\lambda[\dot{g}(x_i)]_\varphi}{g(x_i)^2}+(ac-1)\left\{\sum_{i=1}^{n}\frac{[\ddot{G}(x_i)]_{\varphi\lambda}[\dot{G}(x_i)]_\varphi}{G(x_i)^2}-\right.$$
$$\sum_{i=1}^{n}\frac{[\dot{G}(x_i)]_\lambda[\dot{G}(x_i)]_\varphi}{G(x_i)^2}+c(b-1)\left\{\sum_{i=1}^{n}\frac{[G(x_i)]^{c-1}[\dot{G}(x_i)]_\lambda[\dot{G}(x_i)]_\varphi}{1-G(x_i)^c}\left[\frac{c-1}{G(x_i)^c}+\right.\right.$$
$$\left.\left.\frac{cG(x_i)^{c-1}}{1-G(x_i)^c}\right]+\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\ddot{G}(x_i)]_{\varphi\lambda}}{1-G(x_i)^c}\right\},$$

$$J_{\varphi\beta}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{[\ddot{g}(x_i)]_{\varphi\beta}[\dot{g}(x_i)]_\varphi-[\dot{g}(x_i)]_\beta[\dot{g}(x_i)]_\varphi}{g(x_i)^2}+(ac-1)\left\{\sum_{i=1}^{n}\frac{[\ddot{G}(x_i)]_{\varphi\beta}[\dot{G}(x_i)]_\varphi}{G(x_i)^2}-\right.$$
$$\left.\sum_{i=1}^{n}\frac{[\dot{G}(x_i)]_\beta[\dot{G}(x_i)]_\varphi}{G(x_i)^2}\right\}+c(b-1)\left\{\sum_{i=1}^{n}\frac{[G(x_i)]^{c-1}[\dot{G}(x_i)]_\beta[\dot{G}(x_i)]_\varphi}{1-G(x_i)^c}\left[\frac{c-1}{G(x_i)^c}+\right.\right.$$
$$\left.\left.\frac{cG(x_i)^{c-1}}{1-G(x_i)^c}\right]+\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\ddot{G}(x_i)]_{\varphi\beta}}{1-G(x_i)^c}\right\},$$

$$J_{\lambda\lambda}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{[\ddot{g}(x_i)]_{\lambda\lambda}[\dot{g}(x_i)]_\lambda-[\dot{g}(x_i)]_\lambda^2}{g(x_i)^2}+(ac-1)\sum_{i=1}^{n}\frac{[\ddot{G}(x_i)]_{\lambda\lambda}[\dot{G}(x_i)]_\lambda-[\dot{G}(x_i)]_\lambda^2}{G(x_i)^2}+$$
$$c(b-1)\left\{\sum_{i=1}^{n}\frac{[G(x_i)]^{c-1}[\dot{G}(x_i)]_\lambda^2}{1-G(x_i)^c}\left[\frac{c-1}{G(x_i)^c}+\frac{cG(x_i)^{c-1}}{1-G(x_i)^c}\right]+\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\ddot{G}(x_i)]_{\lambda\lambda}}{1-G(x_i)^c}\right\},$$

$$J_{\lambda\beta}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{[\ddot{g}(x_i)]_{\lambda\beta}[\dot{g}(x_i)]_\lambda-[\dot{g}(x_i)]_\beta[\dot{g}(x_i)]_\lambda}{g(x_i)^2}+(ac-1)\left\{\sum_{i=1}^{n}\frac{[\ddot{G}(x_i)]_{\lambda\beta}[\dot{G}(x_i)]_\lambda}{G(x_i)^2}-\right.$$
$$\left.\sum_{i=1}^{n}\frac{[\dot{G}(x_i)]_\beta[\dot{G}(x_i)]_\lambda}{G(x_i)^2}\right\}+c(b-1)\left\{\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\dot{G}(x_i)]_\beta[\dot{G}(x_i)]_\lambda}{1-G(x_i)^c}\left[\frac{c-1}{G(x_i)^c}+\right.\right.$$
$$\left.\left.\frac{cG(x_i)^{c-1}}{1-G(x_i)^c}\right]+\sum_{i=1}^{n}\frac{G(x_i)^{c-1}[\ddot{G}(x_i)]_{\lambda\beta}}{1-G(x_i)^c}\right\},$$

$$J_{\beta\beta}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{[\ddot{g}(x_i)]_{\beta\beta}[\dot{g}(x_i)]_{\beta} - [\dot{g}(x_i)]_{\beta}^2}{g(x_i)^2} + (ac-1)\sum_{i=1}^{n} \frac{[\ddot{G}(x_i)]_{\beta\beta}[\dot{G}(x_i)]_{\beta} - [\dot{G}(x_i)]_{\beta}^2}{G(x_i)^2} +$$

$$c(b-1)\left\{\sum_{i=1}^{n} \frac{G(x_i)^{c-1}[\dot{G}(x_i)]_{\beta}^2}{1-G(x_i)^c}\left[\frac{c-1}{G(x_i)^c} + \frac{cG(x_i)^{c-1}}{1-G(x_i)^c}\right] + \sum_{i=1}^{n} \frac{G(x_i)^{c-1}[\ddot{G}(x_i)]_{\beta\beta}}{1-G(x_i)^c}\right\},$$

where $\psi'(\cdot)$ is the trigamma function and

$$[\dot{g}(x_i)]_{\varphi} = \frac{\partial[g(x_i)]}{\partial\varphi},\ [\dot{g}(x_i)]_{\lambda} = \frac{\partial[g(x_i)]}{\partial\lambda},\ [\dot{g}(x_i)]_{\beta} = \frac{\partial[g(x_i)]}{\partial\beta},\ [\dot{G}(x_i)]_{\varphi} = \frac{\partial[G(x_i)]}{\partial\varphi},$$

$$[\dot{G}(x_i)]_{\lambda} = \frac{\partial[G(x_i)]}{\partial\lambda},\ [\dot{G}(x_i)]_{\beta} = \frac{\partial[G(x_i)]}{\partial\beta},\ [\ddot{g}(x_i)]_{\varphi\varphi} = \frac{\partial^2[g(x_i)]}{\partial\varphi\partial\varphi},\ [\ddot{g}(x_i)]_{\lambda\lambda} = \frac{\partial^2[g(x_i)]}{\partial\lambda\partial\lambda},$$

$$[\ddot{g}(x_i)]_{\beta\beta} = \frac{\partial^2[g(x_i)]}{\partial\beta\partial\beta},\ [\ddot{g}(x_i)]_{\varphi\lambda} = \frac{\partial^2[g(x_i)]}{\partial\varphi\partial\lambda},\ [\ddot{g}(x_i)]_{\varphi\beta} = \frac{\partial^2[g(x_i)]}{\partial\varphi\partial\beta},\ [\ddot{g}(x_i)]_{\lambda\beta} = \frac{\partial^2[g(x_i)]}{\partial\lambda\partial\beta},$$

$$[\ddot{G}(x_i)]_{\varphi\varphi} = \frac{\partial^2[G(x_i)]}{\partial\varphi\partial\varphi},\ [\ddot{G}(x_i)]_{\lambda\lambda} = \frac{\partial^2[G(x_i)]}{\partial\lambda\partial\lambda},\ [\ddot{G}(x_i)]_{\beta\beta} = \frac{\partial^2[G(x_i)]}{\partial\beta\partial\beta},\ \left[\ddot{G}(x_i)\right]_{\varphi\lambda} = \frac{\partial^2[G(x_i)]}{\partial\varphi\partial\lambda},$$

$$[\ddot{G}(x_i)]_{\varphi\beta} = \frac{\partial^2[G(x_i)]}{\partial\varphi\partial\beta} \text{ and } [\ddot{G}(x_i)]_{\lambda\beta} = \frac{\partial^2[G(x_i)]}{\partial\lambda\partial\beta}.$$