CANADIAN JOURNAL OF SCIENCE, MATHEMATICS AND TECHNOLOGY EDUCATION
2016, VOL. 16, NO. 1, 48–63
http://dx.doi.org/10.1080/14926156.2015.1119335

The Integration of Mathematics in Physics Problem Solving: A Case Study of Greek Upper Secondary School Students

Kalliopi Meli, Konstantinos Zacharos, and Dimitrios Koliopoulos

University of Patras, Rio, Greece

ABSTRACT
This article presents a case study that examines the level of integration of mathematical knowledge in physics problem solving among first grade students of upper secondary school. We explore the ways in which two specific students utilize their knowledge and we attempt to identify the epistemological framings they refer to while solving a physics problem. Participant observation was used for data collection, and the students' verbal interactions were video-recorded. The analysis shows that they tend to use a wide spectrum of epistemological framings that entangle mathematics and physics but at the same time face significant practical difficulties in modulating the two subjects.

RÉSUMÉ
Cet article présente une étude de cas qui analyse le niveau d'intégration des savoirs mathématiques appliqués à la résolution de problèmes de physique chez des élèves en première année du deuxième cycle au secondaire. Nous nous penchons sur les façons dont deux étudiants en particulier se servent de leurs connaissances et nous tentons de déterminer les cadres épistémologiques auxquels ils font référence pour la résolution d'un problème de physique. Nous avons observé les participants pour recueillir les données, et filmé sur vidéo leurs interactions verbales. L'analyse montre qu'ils utilisent un large éventail de cadres épistémologiques qui mélangent les mathématiques et la physique, mais aussi qu'ils font face à de sérieuses difficultés d'ordre pratique lorsqu'il s'agit de moduler les deux sujets.

Introduction

Mathematics is considered the spine of physics and a necessary element of physics teaching, because it provides a language for the succinct expression and application of physical laws and relations (Bing & Redish, 2009). The blending of mathematics and physics may be charming and stimulating for scientists, but the same is not true for the majority of pupils. In fact, the perplexity of mathematical relations often used in physics is considered one of the most prominent factors for discouraging pupils from choosing this subject for further studies as well as a barrier against physics instruction (Boujaoude & Jurdak, 2010; Monk, 1994; Planinic, Milin-Sipus, Katic, Susac, & Ivanjek, 2012; Viennot, 2004).

More often than not, physics teachers are unpleasantly surprised by their students' ill use of mathematics in their classroom, even though the same pupils may perform quite well in a mathematics class (Redish, 2005). In such cases, the teachers usually advise their pupils or their colleagues teaching mathematics to overcome the issue by adding more mathematics-related hours to the pupils' schedule. Even when such an option is available, it is rather unlikely would achieve satisfactory results if one does not

CONTACT Kalliopi Meli kmeli@upatras.gr Department of Educational Sciences and Early Childhood Education, University of Patras, University Campus, Rio 26504, Greece.

focus on the true causes that prevent pupils from integrating their mathematical knowledge into physics problem solving (Tuminaro, 2004).

The use of mathematics in physics differs from the application of mathematics in a mathematical context. The main cause of pupils' underperformance or even development of an aversion to physics is not related to their lack of familiarization with calculus but rather to their difficulty in comprehending the physical meaning of the entailed symbols and the relations between them (Bodin, 2012; Boujaoude & Jurdak, 2010; Monk, 1994; Viennot, 2004). One must adjust and transfer mathematics into the context of science (Gallardo, 2009); this process requires loading physical meaning onto symbols that were initially interpreted differently in the context of mathematics (Redish & Kuo, 2015). For subjects that make use of mathematics, the latter acts as an educational subject-tool with a specific functionality. The blending process of the scientific fields results in the emergence of substantial ideas and conclusions (Bing & Redish, 2007). The aim of this study is to examine the ways in which students in 10th grade use mathematics during problem solving in physics.

Theoretical background

Attributing physical meaning to mathematical relations

The use of mathematics in science, and especially physics, is not as simple as "doing mathematics" in general because it has a different goal: the representation of physical systems instead of the expression of abstract relations (Redish, 2005; Redish & Kuo, 2015). Mathematics used in physics involves the critical infusion of a vast amount of information as well as the alteration of the interpretation of equations. Additionally, problem solving in physics incorporates a metacognitive frame for deciphering and evaluating results (Bing & Redish, 2009).

The blending of mathematics and physics affects the interpretation of functional mathematical relations. This occurs because mathematical formulas in the context of physics are thought of as relations among symbols loaded with physical meaning instead of abstract relations or calculation methods (Lopez-Gay, Martinez Saez, & Martinez Torregrosa, 2015). The attribution of specific physical meaning to a symbol is considered an exceptionally convenient and powerful tool, because it allows for symbol manipulation without necessarily taking into account all the restrains imposed by mathematical formalism (Redish, 2005). Therefore, mathematical relations are "filtered" by physical restrictions; that is, they are viewed under a different light: the light shed by the specific physical context, aiming toward the description of physical systems.

In their study involving high school students aged 15 to 16 years old, Planinic et al. (2012) recognized that although mathematics and physics may use shared tools, information requires a distinctive, differentiated interpretation in the context of physics. Often enough, students encounter a concept that is common in physics and in mathematics, but the disconnected contexts of the two subjects (in terms of instruction) make them believe that what they study in one subject is completely disconnected from what they study in the other.

As an example of the above, we can consider the interpretation of an equation of the following form: $y = ax$, where x represents an independent variable, y represents a codependent variable, and a is a constant. According to the Greek mathematics syllabus, by the age of 14 students have already spent several hours working on the properties of this equation. However, when they encounter at age 15 the kinematic equation $x = ut$ for constant velocity u, where time t is the independent variable and position x is the codependent variable, they fail to realize that they deal with a kind of equation they have been working on in their mathematics class.

Problem-solving processes in physics

The heart of the difficulty in blending mathematics and physics lies in problem solving within the context of physics, and many researchers have been referring to this issue during the past decades (Bing & Redish, 2009, 2012; Blum & Niss, 1991; Bodin, 2012; Chi, 1981; Clement, 1987, 1988; Larkin, McDermott, Simon,

& Simon, 1980; Larkin & Reif, 1979; Lopez-Gay et al., 2015; Planinic et al., 2012; Sherin, 2010). These studies focus on the processes of communication and integration between the fields of mathematics and physics. A central concept that plays a significant role in bridging the gap between the two fields is *mathematization*, which, within science education, is defined as the expression of the physical analysis of a real-world problem through mathematical concepts and relationships (Lopez-Gay et al., 2015).

A mathematization process, which takes into account the particularities of physics problem solving, is the following (Gallegos, 2009). Realistic situations lead to a *pseudo-model*, namely, a simplified version of the initial situation that allows the study of some dominating parameters of the problem. The next stage includes the construction of a physical model, which directs one to the field of physics. The construction of a mathematical model that transfers us to the field of mathematics follows next. Finally, the process concludes with the evaluation of the results under the scope of physical laws and conditions governed by the initial circumstances.

Although the above-mentioned procedure describes explicitly the mathematization process of physics problem solving, one must search in a greater depth in order to discover *how* the students implement it. Bing and Redish (2009) evolved an epistemological structure in order to analyze students' reasoning when using mathematics in physics problem solving. The "epistemological framing" (p. 1), the term these researchers use for their analytical tool, refers to the students' perceptions or judgments (intentional or not) regarding the tools or the abilities they leverage to complete a specific task. The observations of Bing and Redish (2009) suggest that students are prone to be trapped in these epistemological framings for significant periods of time during problem solving. It appears that the students' reasoning changes very often and so does their interpretation of the task, as well as, evidently, their knowledge relevant to the task.

The same researchers, within the methodology they use for their analyses, examine students' declarations as the latter attempt to solve physics problems, and they then classify these statements. Their intention is to discern the common elements among the dialogues in order to form an epistemological context for the investigation of students' ideas as well as their chosen strategies. They claim that the students make use of some epistemological framings via *warrants* as justifications (*epistemological resources*), which fall under one or more of the following classes (Bing & Redish, 2009, 2012):

1. Calculation: A reliable result derives from a set of algorithmically followed computational steps.
2. Physical mapping: Some features of a physical or geometrical system can be accurately described by a mathematical symbolic representation.
3. Invoking authority: Information stemming from an authority source is trustworthy.
4. Mathematics consistency: Mathematical representations are coherent across different situations, because they are regular and reliable. The acceptance of a common mathematical structure allows the participants to trust the relations they handle as well as the consequent outcomes.

Evidently, the above epistemological categories are neither constraining nor isolated, because they can be intermingled or other can emerge, depending on the material under examination. In addition, as Bing and Redish (2009) mention: "Framing is a dynamic cognitive process. A person's mind makes an initial judgment for the situation at hand, but that judgment is continually updated and revaluated" (p. 8). However, they state that these four framing clusters adequately describe the students' trains of thought for a vast proportion of the observed situations. Although the tools the students use are usually unutterable and obscure, they may, in certain cases, appear in specific forms and become clear. This happens when the students cooperate; that is, argue on what to do next and come to an agreement on the path they will follow.

Contribution of verbal interaction

Research data indicate that cooperation between students is exceptionally creative, especially during problem solving, because it allows students to check their suggested strategies and debate them (Bing & Redish, 2009; Felder & Brent, 2009; Storm, Kemeny, Lehrer, & Forman, 2001). Furthermore, the students construct a common context of meaningful practices through verbal communication (Mercer, 1995;

Mercer & Sams, 2006; Mercer & Wegerif, 1999; Storm et al., 2001). On the other hand, students' participation in group tasks does not necessarily guarantee collaborative learning (Mercer & Sams, 2006), because constructive and successful teamwork requires the acquaintance between the members of the group and their familiarization with collaboration practices.

The significance of the use of language as a thinking tool has been studied by Teasley (1995), who analyzed 13-year-old pupils' verbal interactions while solving a mathematical problem. Findings indicate that the advantages of these interactions are more effective when the group consists of only two members. Mercer (1995) examined various forms of verbal interactions and classified them into three analytic categories. The first category involves *disputational talk*, the main characteristics of which are debate between the participants and personalized decision making. The next category is *cumulative talk*, in which partakers positively build on each other's talk but without any critical disposition against each other. Lastly, Mercer (1995) distinguished *exploratory talk*, which is believed to be the most beneficial during problem solving, because the participants discuss their ideas in a critical spirit but with an aptitude to compound their views.

Research objectives

Although the integration of mathematics in physics problem solving has been under scrutiny for many decades, many physics teachers in secondary education insist on attributing students' difficulties merely to lack of mathematics knowledge in their everyday practice (Redish, 2005; Tuminaro, 2004). The way in which the blending of mathematics and physics takes place at a school level is yet to be examined, in order for the procedure to become more explicit and practical for physics instruction. Such research may also support the need for substantial changes in the distinctive ways in which mathematics and physics have been taught for decades. Using a sample of two 15-year-old students, the present study attempts to investigate what lies underneath the students' phenomenological choices in physics problem solving, in terms of basic utilization of mathematics in physics as well as epistemological framings.

More specifically, this case study examines the following research questions:

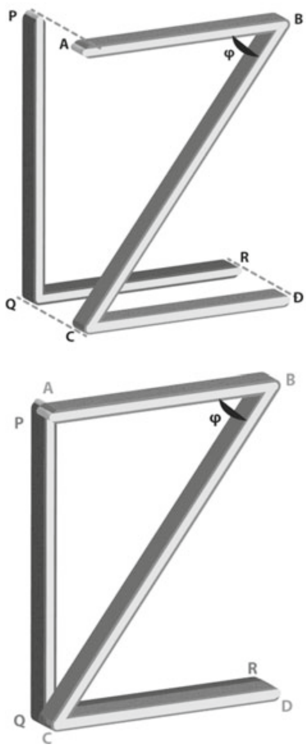
- Do pupils in our sample integrate their mathematical knowledge in physics topics and, if yes, to what extent and in which forms?
- Which epistemological framings do they use to justify their choices and what is their criterion for the selection of those framings?

Method

In order to investigate pupils' mathematical knowledge and ideas as tools for physics problem solving, a specific exercise has been created (see Figure 1); it is based on the physics textbook for the first grade of upper secondary school, equivalent to the 10th grade. Within the Greek educational system, mathematics and physics are compulsory subjects in all years of secondary education and take up a substantial part of the weekly timetable. Furthermore, there exists a syllabus that is used across the country and the textbooks are common for all students.

This problem is novel as a whole and consists of separate parts with independent forms, the contents of which are part of the units covered in classroom. The specific exercise has been created with the intention of being challenging enough to cause several debates among the pupils in order to illuminate the processes of mathematics integration and the epistemological framings they use. An optimal solution to the problem is provided in the Appendix, which is in agreement with the cognitive level of the students.

In reference to physics teaching, this problem has stemmed from the field of mechanics, which includes a vast amount of concepts and relations, out of which the students must carefully select and properly use the relevant ones in order to achieve results compatible with "reality." From a mathematical point of view, the task invites the pupils to utilize their knowledge of algebra and geometry deriving



An object of mass m is thrown at a horizontal tube AB with initial speed u_A = 5 m/s and travels through it; the length of the tube is 6m and the coefficient of dynamic friction between the tube and the object is μ=0.2. The object continues through tube BC, which has the same coefficient of dynamic friction as AB and forms an angle φ with the horizontal. The object has speed u_C = 11.7 m/s as it reaches point C. It finally travels through a horizontal frictionless tube CD and exits at point D.

2s after the initial throw of the first object, a second object of mass m is released to travel through tube PQ, which is smooth and vertical. It then travels through tube QR, which is of equal length to AB and non-smooth. The object exits the tube with speed u_R = 11 m/s.

Calculate the minimum time it takes each object to exit the tubes (at points D and R).

The following data is available: g = 10 m/s^2, sinφ = 0.8 and $\sqrt{1.6} \approx 1.3$

Answers should be rounded off to 1 decimal point.

Figure 1. The physics problem assigned to the students.

from the entire spectrum of middle school mathematics and partially from the first grade of upper secondary school. The mathematical components that have been included in the problem are mainly the following:

- First-order (linear) and second-order (quadratic) equations
- Systems of linear equations
- Basic trigonometric relations
- Relations between angles in several occasions
- The Pythagorean theorem.

As already mentioned, this is a case study in which two 15-year-old students took part: a boy (referred to as D) and a girl (referred to as S). Both students had very good or excellent grades in secondary school physics and mathematics; this choice was made in an effort to minimize the students' lack of basic knowledge or alternative perceptions of the physics and mathematics involved in the specific problem. Moreover, the students were very familiar with each other as well as with the researcher, because she had been their physics and mathematics teacher for a significant time period. The process of problem solving, which lasted for about 2½ hours, was video-recorded, and the students' verbal interactions and writings formed the research material for the qualitative analysis of the study.

The data were collected through participant observation of the group, which, as a qualitative method of research data collection, is compatible with the case study. Data collection through observation gives the researcher the chance to collect "live" data from "real" situations and to therefore examine the facts in situ rather than have to work with second-hand data (Patton, 1980). Following Gold's (1958) classification for the level of the researcher's role in observation, which places the *complete participant* at one end of the spectrum and the *complete observer* at the other end, we find ourselves at the third (intermediate) level of *observer-as-participant* (Cohen, Manion, & Morrison, 2000). This means that the researcher mainly records the facts to service her research goals but also takes part in the children's learning environment as an instructor when necessary. The students collaboratively worked toward solving the specific problem under the supervision of the researcher, who was supporting the students' endeavor whenever they asked for confirmation in order to move on.

One should note that the choice of the above method of data collection is also compatible with the *observational approach*, within which the researcher sees the students as reasonable thinkers who may make mistakes when applying their mathematical knowledge to physics problems. These mistakes occur due to a relatively small number of inappropriate interpretations coming from the students; the existence of these misconceptions leads the researcher to understand the nature of the children's mistakes. From the complete collection of the data, the objectives that have been selected for analysis are relevant to the utilization of mathematics in the physics problem and the students' epistemological framings, as described in the theoretical framework. It must be mentioned that the researcher does not intervene in the dialogues that are analyzed in the following section.

Data analysis and discussion of results

Integration of mathematical knowledge in physics problem solving

This section focuses on the way in which students utilize their mathematical knowledge for the purposes of a physics course. We investigate their chosen general strategy as well as the special components of mathematics involved.

The students in our sample started their task by reading the problem and writing down the given information and questions using mathematical symbols and relations. Later, they silently agreed on separating the problem into parts, smaller than the ones that the exercise itself indicated, but still with the intention to work on these segments collaboratively. They also wrote down those formulas they considered useful for the problem-solving process and, finally, they focused on the details of the specific problem. The following selected dialogues portray specific parts of their work.

First dialogue: Detection of variables and choice of formulas

1. **D**: Let's solve this. Where to start? We are looking for t (time) …
2. Well, we'll consider the objects one by one, let's start with object 1,
3. with the first tube. AB is a straight line … and we have …
4. **S**: Can I ask something? Since velocity is given, friction is given,
5. the coefficient of friction is given, shouldn't we write down the formulas?
6. **D**: Let's write, why not. Write T (friction) equals μ (coefficient of friction) times N (normal reaction force).

The main objective in the above dialogue (first dialogue) is the identification of the question (line 1), the information given, and the relationships between them (lines 4–6). We observe a case of disputational talk that tends to become cumulative, because it is evident that they individually initiate the process of problem solving but pretty soon they try to cooperate. This episode indicates the students' superficial approach to solving the problem, which initiated with the listing of cluttered symbols and relations; this is in agreement with similar cases described in the literature (Chi, 1981; Larkin & Reif, 1979). This tactic is severely connected to their urge for mathematical formalization, because that is the very first thing they attempt to do (lines 4–6). They are in a hurry to use symbols and involve them into relations, but all of these have not yet been assigned to specific notions regarding the situation (Clement, Lochhead, & Monk, 1981). This choice is in opposition to Arons's (1984) suggestion "idea first and name afterwards" (p. 21) and perhaps explains why the attempt of writing down formulas was straining throughout most of the procedure.

Subsequently, on the one hand, the students were having a hard time recalling the desirable formulas and, on the other hand, they were lacking a specific strategy. These remarks primarily refer to the formulas and routines the students have formerly tried to memorize without actually understanding them. They had difficulties in creating the proper mental representation of the solving process, they did not discuss the physical situation (e.g., the kind of motion) they were dealing with at the right time, and they moved at a fast pace, as one can see in the second dialogue, below. The selection of the appropriate mathematical relation required comprehension of the context; that is, the physical meaning of the formulas used for the specific exercise.

Second dialogue: Detection of variables and choice of formulas

7. **S**: $= u_o t + \frac{1}{2}at^2$.
8. **D**: Why is that?
9. **S**: Isn't this the formula? Isn't this the formula?
10. Why don't we solve this for time?
11. **D**: That's what we would do, but we don't know the acceleration.
12. **S**: So what! So what if we don't know it?
13. **D**: You have two unknown variables.
14. **S**: That's a different story. We'll deal with them within the formula,
15. we won't plug numbers in.
16. How do we solve for *t*? You know … Solve for *t*!
17. **D**: $= t\ (u_o + \frac{1}{2}at)$.
18. $t = \frac{x}{u_o + \frac{1}{2}at}$.
19. **S**: Again, you haven't solved for *t*.
20. **D**: Wait.
21. **S**: For *t*, there is another *t* (in the formula).
22. **D**: I know, wait.
23. **S**: I don't understand why we are doing this.

There is a series of incidents worth noting in the above passage (second dialogue), which is particularly rich for the observer due to the exploratory talk between the students. S suggested solving the equation for the main unknown variable, without feeling any need to substitute values for symbols (lines 10 and 14–15), in contrast to D, who hesitatingly started the process (lines 11 and 13); these diverse behaviors indicate a different way of mathematical thinking. The one student feels comfortable handling an equation without any numbers plugged in and dealing with the unknown variables later, which is more similar to an expert's approach (Redish & Kuo, 2015) in comparison to the other student, who has his mind fixed on an impending numerical result. We next notice an incorrect attempt of solving a quadratic equation (lines 17–18). This dialogue is an excellent example of the students' lack of strategy; even S, who suggested the whole solving process, could not justify her reasons after all (line 23).

Third dialogue: Quadratic equation solving, similar to one that appears in mathematics textbooks

24. **D**: $x = u_o t + \frac{1}{2}at^2$.
25. $6 = 5t + \frac{1}{2}2t^2$.
26. $6 = 5t + t^2$.
27. $t^2 + 5t - 6 = 0$.

   …

28. $t_{1,2} = \frac{-5 \pm 7}{2} = \begin{cases} t_1 = -1 \\ t_2 = 6 \end{cases}$.

The deadlock we noticed in the second dialogue (lines 18–19) drove the students to the detection of the error and to fixing it; all kinds of talks had a share in that long process. After a significant period of time and only as soon as numbers were plugged in, the physics equation transformed to a typical quadratic equation, in a way that students recognize from their mathematics textbooks. In contrast to the trouble they had earlier, they had no problem at all dealing with this stage of the process (third dialogue). Obviously, the students did actually know how to solve the equation at hand (lines 17 and 24), but they faced tremendous difficulty when the mathematical components of the problem differed slightly from those that they were used to encountering in mathematics problems.

Fourth dialogue: The deadlocks due to lack of strategy

29. **D**: $\Sigma F = ma$.

   …

30. **S**: Why don't you put a T in the ΣF …
31. **D**: A t?
32. **S**: No, the friction, since only friction is applied.
33. I have an idea, but I don't know if it's right. Substitute for everything. No?
34. I mean, let's say for the acceleration, we write u over t and we'll see.
35. It includes time. No?
36. **D**: Look, we can put μ times N equals u times t.
37. **S**: Substitute for all.
38. **D**: Should I do it?
39. **S**: Why not? Is there anything else we can do?
40. **D**: $\mu N = mut$.
41. **S**: u over t.
42. **D**: $= m\frac{u}{t}$.

In the above dialogue (fourth dialogue), exploratory talk helps students to make some progress, but it is also very revealing regarding their lack of strategy. They use a system of equations that lacks the necessary physical meaning and mathematical formalism. Because they do not know how to proceed, they keep substituting symbols with other symbols without seeing the vanity in that (lines 30 and 36). In their attempt to substitute for one of the unknown variables (acceleration a), they introduced a new unknown variable (final velocity u) to the equation. Although they solved the first-order equation they formed correctly, the number of unknown variables remained the same. This error occurred multiple times during the case study and suggests an inadequate comprehension of the physical aspect of the problem, as well as the lack of a realization that they are in fact dealing with a system of equations; it is more like a puzzle to them, playing with symbols until a relation finally pays off.

Fifth dialogue: Lack of primary mathematical knowledge

43. **D**: $6 = 13t + \frac{1}{2}\frac{10}{t_{fin}}t^2$.
44. **S**: Write 20 times final t …
45. **D**: 20?
46. **S**: 2 times 10 equals 20 …
47. **D**: Why 2 times 10?
48. **S**: What will you do? Won't you simplify it a bit?
49. **D**: 5 times final t.
50. **S**: 20 final t! Don't you do the cross product?
51. **D**: Wait … Yes.
52. $6 = 13t + 20t_{fin} \cdot t^2$.

In addition to the lack of proper mathematical formalism, in the fifth dialogue we observe errors in simple mathematical operations, such as fraction multiplication (line 50). The above dialogue is part of a rather cumulative than exploratory talk, which could explain why they were constantly directing to even more erroneous decisions. It is actually astonishing that these two students became so confused when dealing with rather simple calculations in a physics context. These errors should not be attributed to ignorance, because they both have excellent grades in the subject of mathematics and, most important, they proved on other occasions that they can fluently perform such calculations. It seems that the blending process of the mathematics and physics really tripped them up, as mentioned in the theoretical framework (e.g., Bodin, 2012; Redish, 2005). This became more obvious when they spent most of their energy tackling the difficulties raised by the physical aspect of the problem and the mathematical operations were merely minor barriers just before the finish line.

Analysis of the epistemological framing

We provide here an analysis of students' epistemological framing that is based on the types of justifications (warrants or epistemological resources), as stated in the theoretical framework, which the students offer in order to justify their choices. What becomes clear from the passages reproduced here is that any justification rarely involves a single justification type. In the following dialogues we choose to focus on the most distinctive justifications per case.

Calculations

Every time the students attempted a mathematical solution, they took for granted that the calculation steps they followed were due to lead to reliable results. It was enough for them to believe that they executed the process correctly in order to assume that they had reached a correct outcome. But if the latter was not the one they expected when they began their mathematical operations, they tended to revise the whole procedure, as shown below (sixth dialogue, calculation of discriminant). In some cases the pupils evaluated their results under the light of their own expectations, without a thorough examination of the problem (line 64). In such occasions they obviously ran the risk of being satisfied with an outcome that seemed correct according to their previous experience but could have in fact occurred due to poor choices of physical relations and/or errors in the calculations.

It must be noted that calculators are forbidden throughout the Greek school system; as a result, all pupils realize fairly early on that numbers appearing in the various steps of the solution of a physics problem are "fixed" in a way suitable for performing calculations mentally, in the majority of cases. If they arrive at a calculation that is above their mental computational abilities, they question their work, not because they are concerned about possible qualitative misjudgments or about having used the wrong formulas but merely out of a habitual uneasiness toward errors in calculations. The following dialogue confirms this well-established tactic as soon as the "unusual" root 31 (line 61) appears due to the utilization of an incorrect formula (line 55). Nevertheless, it is exploratory talk that drives them to revise and correct their solution, regardless of their reasons for doing so.

Sixth dialogue: Errors in the use of mathematical algorithms

53. **D**: $\Delta = b^2 + ac$ (the formula is incorrect—the correct one is $\Delta = b^2 - 4ac$).
54. **S**: Minus.
55. **D**: $\Delta = b^2 - ac$.
56. $\Delta = 5^2 - 1 \cdot (-6)$.
57. $\Delta = 25 + 6$.
58. $\Delta = 31$.
59. **S**: I think we'll end up with a negative answer for time and we'll have a laugh.
60. Don't you think it will be negative? What is this? Why 31? I'm scared.
61. **D**: $x_{1,2} = \frac{-b \pm \sqrt{31}}{2\alpha}$
62. **S**: b square minus 4 times a times c … (she mumbles the correct formula but she does not realize the mistake in the written formula).
63. **D**: Check if the calculations are correct.
64. **S**: Why is it like this? What is that! Root 31?
65. **D**: This is the formula … (he points at the formula $\Delta = b^2 - ac$).
66. **S**: Didn't we solve this correctly?
67. b square minus 4 times a times c … 25 and 6 … What's wrong?
68. **D**: $x_{1,2} = \frac{-5 \pm \sqrt{31}}{2}$.
69. I think we forgot 4 (he erases the previous solution).
70. $\Delta = b^2 - 4ac$.

Physical mapping

Because the students were aware of the fact that this was a physics problem and not a mathematics problem, their results were under the scope of certain rules and limitations, according to the regulations of physical laws and concepts. Therefore, they acted on the problem with partial disregard for mathematical formalism (lines 73 and 76); if they were dealing with a mathematical problem they could not have omitted the alternative results. Within a physics context, they had the chance to filter the outcomes of their calculation; however, at this stage it is not possible for us to assert whether this procedure emerges from the repetition of a well-performed routine (lines 73–74), from the guidance provided in this specific problem (lines 77–78), from a deeper understanding of the problem, or from a combination of these reasons. Typically they used cumulative or exploratory talk in order to move on, although in same cases they implicitly reached an agreement, because they considered their conclusions self-evident. The following two passages (seventh dialogue) are representative.

Seventh dialogue: Search for meaning in mathematical operations

71. **D**: $t_{1,2} = \frac{-5\pm7}{2} = \begin{cases} t_1 = 1 \\ t_2 = -6 \end{cases}$.
72. **S**: We are both amazing. 1 second.
73. **D**: We dismiss this (he points at $t_2 = -6$).
74. **S**: OK, we dismiss it.

...

75. **S**: $t_{1,2} = \frac{5\pm1}{2} = \begin{cases} t_1 = 3 \\ t_2 = 2 \end{cases}$.
76. This must be a joke … Why 2;
77. **D**: Look, the problem states that we should find the least time,
78. so we'll keep 2, not 3.

Invoking authority

The students believe that the laws and theorems they have been taught are completely trustworthy and safe to use within an appropriate context (lines 83 and 84). Of course, the process of recalling rules often comes with the "risk" of incorrectly applying them and of somehow thus disproving the invoked authority. Two relevant examples are shown below (eighth dialogue), deriving from geometry and physics respectively. In both cases, the students make progress through exploratory talk.

Eighth dialogue: Recourse to known theorems and relations

79. **D**: There's a connection here (between the tubes),
80. you can think of it as a triangle, an orthogonal one.
81. **S**: Why is it orthogonal? Are they parallel (the horizontal tubes)?
82. **D**: Look at the sketch (the representation given).
83. **S**: So, the Pythagorean (theorem)? Which Pythagorean? (they only know one)

…

84. **D**: So we've got Newton's first law, hence $\Sigma F = 0$.
85. **S**: Why Newton's first law?
86. **D**: Because it's not accelerating.
87. **S**: Since the ground is smooth, shouldn't the velocity be continuously increasing?
88. So it's accelerated, not uniform motion after all.
89. **D**: Yes, fine.
90. **S**: No, it must actually be Newton's first law.
91. **D**: It is Newton's first law. Because there is no friction applied …

Mathematics consistency

The fact that the participating students trust mathematics as a valid and reliable tool in all applications makes them resort to it even when such an action is not required by the progress of problem solving. The successful application of a method often leads to its repetition, with the hope that it will, once again, bring a correct outcome. This happened repeatedly throughout the entire problem-solving process. The above-described process includes a procedure that can be explained as "reversing a line of thought" (Arons, 1984, p. 24), as the students evidently reexamined formerly applied operations in order to reproduce them. Additionally, it embeds a reasoning process that Arons (1984) summarizes with the phrase "that is not the case" (p. 25); regarding this, the students gradually excluded all of the cases that were not suitable for repetition.

The dialogue that follows (ninth dialogue) presents one correct and one incorrect application of the strategy that stems from mathematics consistency. In the first case, the students rightfully replicate a routine they had previously used with success, because the two cases are identical (line 94). This is not true for the second case, in which the students tried to apply the same friction formulas they had earlier used (line 96); this time, though, the surface is frictionless and the mathematical process they "copy" thus led them to nonlegitimate calculations. In both cases, they used exploratory talk to reach their conclusions; even for the second example, in which they started off on the wrong foot, they realized their mistake through a productive debate.

Ninth dialogue: Use of mathematical algorithms with no specific developed strategy

92. **S**: Let's move to the next one (object).
93. **D**: How did we find the previous one? On AB (part of the tube).
94. **S**: With the formulas. ΣF for both …

. . .

95. **D**: Since it is exactly the same as AB, we need only change μ,
96. instead of 0.2 we'll use 0. It is exactly the same.

. . .

97. **D**: What is T over 0?
98. **S**: Is it legit? It's not legit.

. . .

99. **D**: Hence there is no friction, nor is there an N.
100. Then it's definitely 0. Do you understand what I'm saying?
101. **S**: $\tau = \mu\nu \Rightarrow \nu = \frac{\tau}{\mu}$.
102. It doesn't exist. Hence there is no friction.

Conclusions

This study has examined the potential of a group of two students, both in the first grade of Greek upper secondary school (equivalent to 10th grade), to solve a physics problem that required them to use mathematical knowledge appropriate for their age. The students' ability to properly handle the mathematization process, as well as their aptitude to use a variety of epistemological framings in accordance with the specific problem, was among the main interests of this study.

Throughout the problem-solving process, we observe that the participating students attempt to utilize mathematical forms before grasping the notions of physics at hand. This implicit tactic prevents them from actually dealing with the problem, because it magnifies the difficulties in selecting and using the appropriate laws and formulas of mathematics and physics. Furthermore, mathematical operations that most probably would be easily handled in the mathematics classroom are hard to tackle within the physics context. We also note that the students focus on "local" goals, in the sense that they make use of only a small amount of the data provided and in case of failure they switch to another, different task; the same facts have also been noted in the relevant literature (Redish, 2005).

A special aspect of the utilization of mathematics that has been investigated in this study is the epistemological framings that students chose when applying their knowledge, as classified in the literature and mentioned in the Theoretical Background section (Bing & Redish, 2009). The emergence of an epistemological framing pattern, consistently followed by the participants in this study, is of great value. Without aiming to apply any specific strategy, these two students did actually follow a routine: they invoked authority in the form of laws, theorems, and operational relations before beginning calculations; they took advantage of mathematical consistency whenever useful; and finally they assessed their results according to a physical mapping. One must note, however, that we are unable to know the degree to which the design of the specific problem enhanced this particular path.

Furthermore, we believe that the reported dialogues illustrate that collaborative learning proved to be exceptionally beneficial for the participating dyad. The dialogues illuminate the constant creation and reconsideration of mathematical and physical framings by the students, as well as the decisive role that verbal communication played for the emergence of these framings. The type of interaction that has been characterized as exploratory talk (Mercer, 1995) appears to be especially productive. It must be emphasized that the development of students' verbal communication skills and the students' familiarization with exploratory talk in particular must have been an aforethought didactical intervention deriving from the teacher.

Research implications

The interconnection between mathematics and physics, as already illustrated from several aspects, indicates that the separation of the instruction of these subjects suggests misleading and distortion in the knowledge production procedure, though it may serve certain practical needs. This dissection is responsible for the pedagogical and epistemological obstacles that occur during teaching (Blum & Niss, 1991). Therefore, one must revisit how mathematics and physical sciences have been taught over the years. One is thus led to discover the necessity to teach mathematics within an interdisciplinary context (Blum & Niss, 1991; Planinic et al., 2012) and, in the same time, it is believed that the instruction of mathematics will also benefit from the utilization of physics elements (Hanna et al., 2001). If such an opportunity arises, it is vital that the educational systems properly prepare teachers for such a complex instructional approach, which would be beyond the currently established practices (Ashmann, Zawojewski, & Bowman, 2006).

For further research, we believe that the above remarks indicate the need to investigate the issues involved in the integration of mathematics in physics problem solving using a larger sample of pupils as well as for a wider spectrum of problems. Such an examination could provide physics instructors at the school level with more and solid substantial insights, in order to tackle the related difficulties their students face in a physics course. The findings could be significant enough to transform the conventional, implicit way in which mathematics has been integrated in science classes and reform the instruction of the related subjects.

Acknowledgment

We thank Anna Kostouki for her insightful remarks and for her help in preparing the English version of this article.

References

Arons, A. B. (1984). Student patterns of thinking and reasoning. *The Physics Teacher, 22*(1), 21–26. doi:10.1119/1.2341444

Ashmann, S., Zawojewski, J., & Bowman, K. (2006). Integrated mathematics and science teacher education courses: A modelling perspective. *Canadian Journal of Science, Mathematics and Technology Education*, *6*(2), 189–200. doi:10.1080/14926150609556695

Bing, T. J., & Redish, E. F. (2007). The cognitive blending of mathematics and physics knowledge. In *AIP Conference Proceedings* (Vol. 883, pp. 26–29). Syracuse, NY: AIP. doi:10.1063/1.2508683

Bing, T. J., & Redish, E. F. (2009). Analyzing problem solving using math in physics: Epistemological framing via warrants. *Physics Education Research*, *5*, 1–15. doi:10.1103/PhysRevSTPER.5.020108

Bing, T. J., & Redish, E. F. (2012). Epistemic complexity and the journeyman–expert transition. *Physical Review Special Topics - Physics Education Research*, *8*(1), 1–11. doi:10.1103/PhysRevSTPER.8.010105

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects—State, trends and issues in mathematics instruction. *Education Studies in Mathematics*, *22*, 37–68.

Bodin, M. (2012). *Computational problem solving in university physics education*. Umea, Sweden: Umea University.

Boujaoude, S. B., & Jurdak, M. E. (2010). Integrating physics and math through microcomputer-based laboratories (MBL): Effects on discourse type, quality, and mathematization. *International Journal of Science and Mathematics Education*, *8*, 1019–1047.

Chi, M. T. H. (1981). Categorization and representation of physics problems by experts and novices. *Cognition Science*, *5*, 121–152.

Clement, J. (1987). Generation of spontaneous analogies by students solving science problems. In D. Topping, D. Crowell, & V. Kobayashi (Eds.), *Thinking across cultures* (pp. 303–308). Hillsdale, NJ: Lawrence Erlbaum Associates.

Clement, J. (1988). Observed methods for generating analogies in scientific problem solving. *Cognitive Science*, *12*, 563–586.

Clement, J., Lochhead, J., & Monk, G. S. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly*, *88*(4), 286–290.

Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education* (5th ed.). London, England: RoutledgeFalmer.

Felder, R. M., & Brent, R. (2009). Active learning: An introduction. *ASQ Higher Education Brief*, *2*(4), 1–6.

Gallardo, P. C. (2009). Mathematical models in the context of sciences. In M. Blomhøj & S. Carreira (Eds.), *Proceedings From Topic Study Group 21 at the 11th International Congress on Mathematical ducation* (pp. 117–131). Monterrey, Mexico: Roskilde University, Department of Science, Systems and Models, IMFUFA.

Gallegos, R. (2009). Differential equations as a tool for mathematical modelling in physics and mathematics courses—A study of high school textbooks and the modelling processes of senior high students. In M. Blomhøj & S. Carreira

(Eds.), *Proceedings From Topic Study Group 21 at the 11th International Congress on Mathematical ducation* (pp. 19–34). Monterrey, Mexico: Roskilde University, Department of Science, Systems and Models, IMFUFA.
Gold, R. (1958). Roles in sociological field observations. *Social Forces*, *36*, 217–223.
Hanna, G., Jahnke, H. N., Lomas, D., Hanna, G., Jahnke, H. N., Debruyn, Y., & Lomas, D. (2001). Teaching mathematical proofs that rely on ideas from physics. *Canadian Journal of Science, Mathematics and Technology Education*, *1*(2), 183–192. doi:10.1080/14926150109556460
Larkin, J. H., McDermott, J., Simon, D. P., & Simon, H. A. (1980). Expert and novice performance in solving physics problems. *Science*, *208*, 1335–1342.
Larkin, J. H., & Reif, F. (1979). Understanding and teaching problem—Solving in physics. *European Journal of Science Education*, *1*(2), 191–203. doi:10.1080/0140528790010208
Lopez-Gay, R., Martinez Saez, J., & Martinez Torregrosa, J. (2015). Obstacles to mathematization in physics: The case of the differential. *Science & Education*, *24*, 591–613. doi:10.1007/s11191-015-9757-7
Mercer, N. (1995). *The guided construction of knowledge: Talk amongst teachers and learners*. Clevedon, England: Multilingual Matters.
Mercer, N., & Sams, C. (2006). Teaching children how to use language to solve math problems. *Language and Education*, *20*(6), 507–528.
Mercer, N., & Wegerif, R. (1999). Is "exploratory talk" productive talk? In K. Littleton & P. Light (Eds.), *Learning with computers: Analysing productive interactions* (pp. 79–101). London, England: Routledge.
Monk, M. (1994). Mathematics in physics education: a case of more haste less speed. *Physics Education*, *28*, 209–211.
Patton, M. Q. (1980). *Qualitative evaluation methods*. Beverly Hills, CA: Sage Publications.
Planinic, M., Milin-Sipus, Z., Katic, H., Susac, A., & Ivanjek, L. (2012). Comparison of student understanding of line graph slope in physics and mathematics. *International Journal of Science and Mathematics Education*, *10*, 1393–1414.
Redish, E. F. (2005, August 21-26). Problem solving and the use of math in physics courses. In *World view on physics education in 2005: Focusing on change*. Delhi, India.
Redish, E. F., & Kuo, E. (2015). Language of physics, language of math: Disciplinary culture and dynamic epistemology. *Science & Education*, *24*, 561–590. doi:10.1007/s11191-015-9749-7
Sherin, B. L. (2010). How students understand physics equations. *Cognition and Instruction*, *19*(4), 479–541. doi:10.1207/S1532690XCI1904
Storm, D., Kemeny, V., Lehrer, R., & Forman, E. (2001). Visualising the emergent structure of children's mathematical argument. *Cognitive Science*, *25*, 733–773.
Teasley, S. D. (1995). The role of talk in children's peer collaborations. *Developmental Psychology*, *3*, 207–220.
Tuminaro, J. (2004). A cognitive framework for analyzing and describing introductory students' use and understanding of mathematics in physics. *Dissertation Abstracts International*, *65*(02B), 786.
Viennot, L. (2004). *Reasoning in physics. The part of common sense*. New York, NY: Kluwer Academic Publishers.

Appendix: Physics problem solution

First object (tube A→D):

- A→B *deceleration with friction*

$$\sum F = ma_{\mathrm{AB}} \Rightarrow \tau = ma_{\mathrm{AB}} \Rightarrow \mu N = ma_{\mathrm{AB}} \Rightarrow \mu mg = ma_{\mathrm{AB}} \Rightarrow a_{\mathrm{AB}} = 2\mathrm{m/s}^2$$

$$(\mathrm{AB}) = u_{\mathrm{A}} t_{\mathrm{AB}} - \frac{1}{2} a_{\mathrm{AB}} t_{\mathrm{AB}}^2 \Rightarrow \frac{1}{2} a_{\mathrm{AB}} t_{\mathrm{AB}}^2 - u_{\mathrm{A}} t_{\mathrm{AB}} + (\mathrm{AB}) = 0 \Rightarrow t_{\mathrm{AB}}^2 - 5t_{\mathrm{AB}} + 6 = 0$$

$$\Delta = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 6 = 1$$

$$t_{\mathrm{AB}1,2} = \frac{5 \pm \sqrt{1}}{2} \Rightarrow t_{\mathrm{AB}1,2} = \frac{5 \pm 1}{2} \Rightarrow t_{\mathrm{AB}1,2} = \begin{cases} 2 \\ 3 \end{cases} \Rightarrow t_{\mathrm{AB}} = 2\,s \text{ the least time}$$

- B→C *acceleration with friction* ($W_x > T$)

$$\sum F = ma_{\mathrm{BC}} \Rightarrow W_x - \tau = ma_{\mathrm{BC}} \tag{1}$$

$$\sin\varphi = \frac{W_x}{W} \Rightarrow W_x = W \sin\varphi \Rightarrow W_x = mg\sin\varphi \Rightarrow W_x = 8m \tag{2}$$

$$\tau = \mu N \Rightarrow \tau = \mu W_y \tag{3}$$

$$\cos\varphi = \frac{W_y}{W} \Rightarrow W_y = W\cos\varphi \Rightarrow W_y = mg\cos\varphi \tag{4}$$

$$\sin^2\varphi + \cos^2\varphi = 1 \Rightarrow \cos^2\varphi = 1 - \sin^2\varphi \Rightarrow \cos\varphi = \sqrt{1 - \sin^2\varphi} \Rightarrow$$

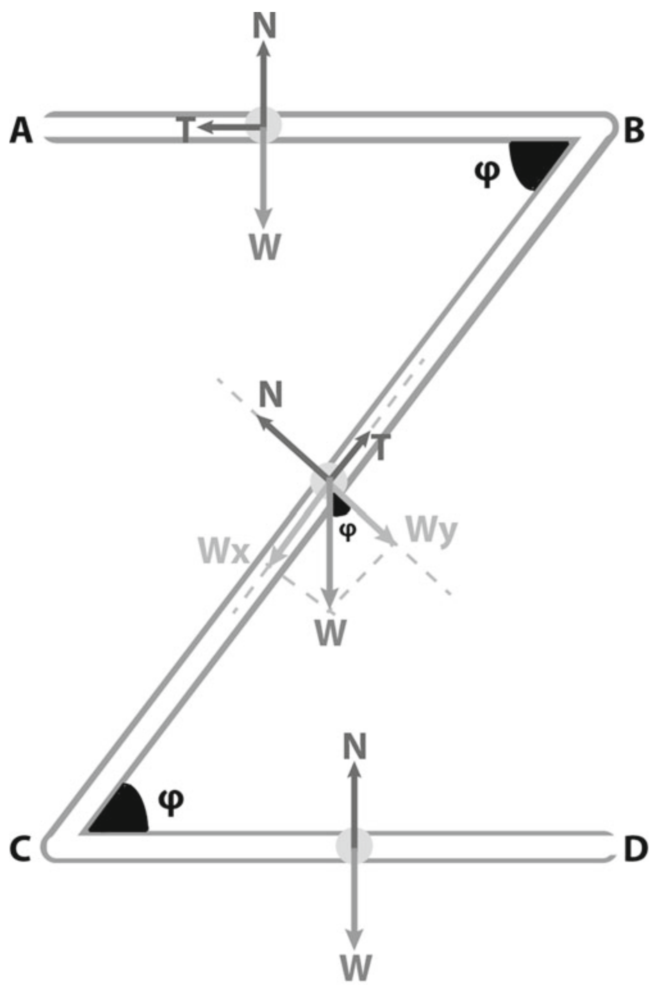


Figure A1. Representation of the forces applied to the first object (route ABCD).

$$\cos\varphi = \sqrt{1-\sin^2\varphi} \Rightarrow \cos\varphi = \sqrt{1-0.64} \Rightarrow \cos\varphi = 0,6 \quad (5)$$

$$(3) \overset{(4)}{\Rightarrow} \tau = \mu mg\cos\varphi \overset{(5)}{\Rightarrow} \tau = 0.2\cdot 10\cdot 0.6m \Rightarrow \tau = 1.2m \quad (6)$$

$$(1) \overset{(2)(6)}{\Longrightarrow} 8\text{m} - 1.2m = ma_{\text{BC}} \Rightarrow a_{\text{BC}} = 6.8\text{m/s}^2$$

$$u_{\text{C}} = u_{\text{B}} + a_{\text{BC}}t_{\text{BC}} \Rightarrow t_{\text{BC}} = \frac{u_{\text{C}} - u_{\text{B}}}{a_{\text{BC}}} \quad (7)$$

$$u_{\text{B}} = u_{\text{A}} - a_{\text{AB}}t_{\text{AB}} \Rightarrow u_{\text{B}} = 5 - 4 \Rightarrow u_{\text{B}} = 1\text{m/s for A} \rightarrow \text{B} \quad (8)$$

$$(7) \overset{(8)}{\Rightarrow} t_{\text{BC}} = \frac{11.7-1}{6.8} \Rightarrow t_{\text{BC}} = 1.6\ \text{s}$$

- C→D *uniform linear motion* ($\Sigma F = 0$)

$$(\text{CD}) = u_{\text{C}}t_{\text{CD}} \Rightarrow t_{\text{CD}} = \frac{(\text{CD})}{u_{\text{C}}} \Rightarrow t_{\text{CD}} = \frac{(\text{AB})}{u_{\text{C}}} \Rightarrow t_{\text{CD}} = 0,5\ \text{s}$$

Total time for the first object: $t_1 = t_{\text{AB}} + t_{\text{BC}} + t_{\text{CD}} = 2 + 1.6 + 0.5 \Rightarrow t_1 = 4.1\ \text{s}$
Second object (tube P→R):

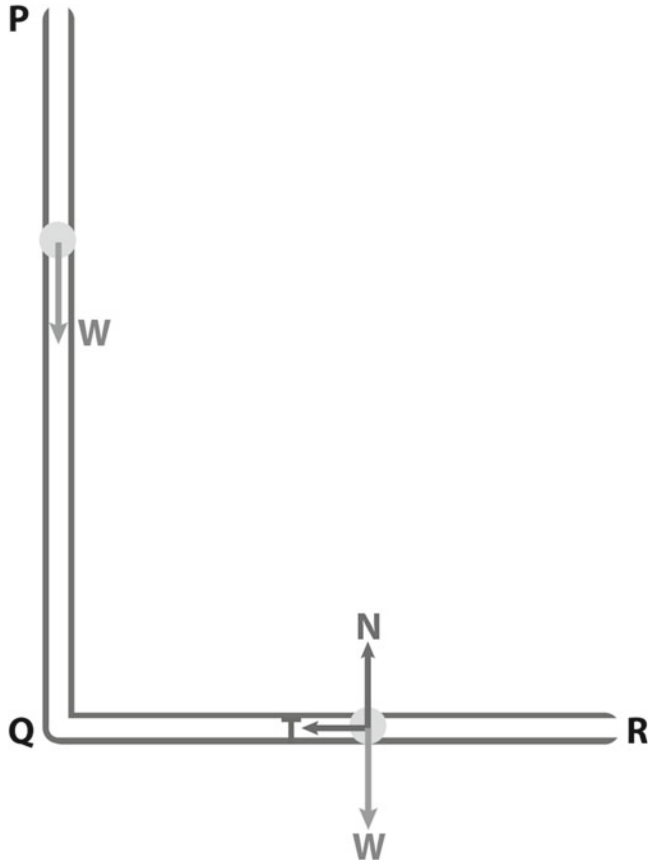


Figure A2. Representation of the forces applied to the second object (route PQR).

- P→Q *free fall*

$$(PQ) = \frac{1}{2}gt_{PQ}^2 \Rightarrow t_{PQ} = \sqrt{\frac{2\,(PQ)}{g}} \tag{9}$$

$$(BC)^2 = (AB)^2 + (PQ)^2 \Rightarrow (PQ) = \sqrt{(BC)^2 - (AB)^2} \Rightarrow$$

$$(PQ) = \sqrt{(BC)^2 - (AB)^2} \tag{10}$$

$$(BC) = \frac{(AB)}{\cos\varphi} \overset{(5)}{\Rightarrow} (BC) = \frac{6}{0.6} \Rightarrow (BC) = 10\text{ m} \tag{11}$$

$$(10) \overset{(11)}{\Rightarrow} (PQ) = \sqrt{10^2 - 6^2} \Rightarrow (PQ) = 8\text{ m} \tag{12}$$

$$(9) \overset{(12)}{\Rightarrow} t_{PQ} = \sqrt{\frac{16}{10}} \Rightarrow t_{PQ} = 1.3\text{ s}$$

- Q→R *deceleration with friction*

$$u_Q = gt_{PQ} \Rightarrow u_Q = 13\text{m/s}$$

$$\begin{cases} (QR) = u_R t_{QR} - \dfrac{1}{2}a_{QR}t_{QR}^2 \\ u_R = u_Q - a_{QR}t_{QR} \end{cases} \Rightarrow \begin{cases} (QR) = u_R t_{QR} - \dfrac{1}{2}a_{QR}t_{QR}^2 \\ a_{QR}t_{QR} = u_Q - u_R \end{cases} \Rightarrow$$

$$(QR) = u_R t_{QR} - \frac{1}{2}\left(u_Q - u_R\right) t_{QR} \Rightarrow (QR) = t_{QR}\left(\frac{3u_R - u_Q}{2}\right) \Rightarrow t_{QR} = \frac{2\,(QR)}{3u_R - u_Q} \Rightarrow$$

$$t_{QR} = \frac{2\,(AB)}{3u_R - u_Q} \Rightarrow t_{QR} = \frac{12}{20} \quad \Rightarrow t_{QR} = 0.6 \text{ s}$$

Total time for the second object: $t_2 = 2 + t_{PQ} + t_{QR} = 2 + 1.3 + 0.6 \Rightarrow t_2 = 3.9$ s