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ORIGINAL ARTICLE

Performance attribution, time-weighted rate of return, and clean finite change sensitivity index

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Abstract
We propose an innovative methodology for decomposing the value added generated by a money manager within a given assessment interval into the contributions of the manager's investment decisions made in the various periods, in order to identify the most (and the least) impactful period decisions. To this end, we benchmark an actively-managed investment against a reference portfolio replicating the client's contributions and distributions and earning the benchmark returns. We apply the Clean Finite Change Sensitivity Index method (Borgonovo in Eur J Oper Res 200:127–138, 2010a, Risk Anal 30(3):385–399, 2010b; Magni et al. in J Oper Res Soc 71(12):1940–1958, 2020) to the investment's value added in order to obtain a complete decomposition of it into the contributions of the investment decisions made in the various periods; we rank the period decisions according to their contributions and show that, if the contribution-and-distribution policy changes, the effect of the investment choices made in the various periods on the value added changes as well, which testifies of the interaction between the manager's decisions and the client's decisions, the former affecting the financial efficiency and latter affecting the investment scale. In particular, neutralizing the contributions and distributions (and, therefore, the investment scale), we show that the Time-Weighted Rate of Return (TWRR) can be arithmetically obtained from the value added and can be decomposed into the same period contributions of the value added, thereby providing a reconciliation between the value added notion and the TWRR.

Keywords Value added · Performance measurement · Investment policy · Sensitivity analysis · TWRR

Mathematics Subject Classification 49M27 · 90C31 · 91-08 · 91G10 · 91G60 · 91G80

Introduction

Managerial and financial skills are at the core of the academic literature both in the field of *ex-ante portfolio optimization* (Cerny 2020; Wang and Yu Zhou 2020; Low et al. 2012; Lim et al. 2011; Jin and Yu Zhou 2008) and *ex-post performance measurement* (Elton and Gruber 2020; Andreu Sánchez et al. 2018; Spaulding 2004; Angelidis et al. 2013; Clarke et al. 2005; Binay 2005). Considering the ex-ante perspective, that is the ex-ante construction of optimal investments, among others, Battocchio and Menoncin (2004) focus on the asset allocation problem of a money manager who aims at maximizing the expected utility of his terminal wealth, taking into account salary risk and inflation risk. Colombo and Haberman (2005), dealing with pension schemes, analyze the variability of the mismatch between assets and liability and determine optimal contributions strategies. Josa-Fombellida (2001) study the optimal contribution rate of a pension plan minimizing solvency and contribution-rate.

Considering the perspective of performance measurement, scholars provide measures for assessing the performance of portfolios/funds and quantify the role played by the money manager's decisions about asset selection and allocation as opposed to the investor's decisions about contributions and distributions. Such measures may be relative metrics of worth, such as the internal rate of return and the Time-Weighted Rate of Return (TWRR) (see Dietz 1966; Fisher 1968; Gray and Dewar 1971; Feibel 2003; Bacon 2008), or absolute measures of worth, such as the net present value (also used for ex-ante purposes) or the value added

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(VA). As for the relative valuation, Magni (2013) uses the Average Internal Rate of Return for assessing funds' performance and Magni (2014) isolates the role of the manager's decisions appraising the manager's skills and showing that the manager's AIRR (MAIRR) is the annual equivalent of the TWRR. As for the absolute valuation, focusing on value added, some scholars propose a value-based approach to measure a fund's performance and the impact of the manager's decisions on the fund's VA (e.g., Bagot and Armitage 2004; Armitage and Bagot 2009).

This paper adopts an *ex-post* perspective and aims at going beyond the conflict between relative measures vs. absolute measures. We quantify the impact of a money manager's period decisions on an investment's performance and trace a possible path for a reconciliation of relative and absolute measures of worth. In particular, this work

- Quantifies the share of an investment's VA generated by the period decision made by the manager
- Ranks the single periods' decisions according to their effect on the investment's VA
- Highlights the dependence of the result on the contribution-and-distribution policy of the client and, finally,
- Provides the link of such period effects with the classical TWRR by decomposing the latter arithmetically into (the same) period effects.

To accomplish these tasks, we will use a recently-conceived technique of sensitivity analysis, so-called Clean Finite Change Sensitivity Index (CFCSI) which decomposes the finite change in a model output into the effects of its input parameters (Borgonovo 2010a, b; Magni et al. 2020).

As the first step, we consider the contribution-and-distribution policy as given and measure the effect of the investment decisions made by the manager in a given period onto the overall investment's value added, taking into account the reverberating effect of the period decisions on the following periods. In particular, we describe the investment's value added as the change in the net terminal value obtained by switching from a passive investment in a benchmark portfolio to an active investment generating returns which are different from the benchmark returns. Since the investment choices made by the manager in the period determine the investment's holding period rate, we assess the impact of each rate by making use of the above-mentioned CFCSI technique. This method enables one to decompose the investment's VA and ranking the rates according to their impact on it. Ranking the rates boils down to ranking the effects of the manager's period decisions on the investment's performance and understand in which periods the most important (and less important) decisions have been made.

As the second step, we show that, assuming a different contribution-and-distribution policy the manager's performance attribution (as measured by the CFCSI) changes, owing to the change of the investment's scale. Among the various policies, we focus on the case of zero interim cash flows (i.e., no interim contribution nor distribution). This analysis permits us to reconcile the VA, which is an absolute measure of value creation, with the Time-Weighted Rate of Return (TWRR), which is a relative performance measure expressed as per unit of initial invested capital, and, furthermore, to arithmetically decompose the TWRR according to the effects of the various period decisions, which are shown to be the same as the zero interim cash flows case previously analyzed.

The remainder of the paper is structured as follows. "Benchmark portfolio and value added" section introduces the setting and, in particular, presents the benchmark portfolio and its role in the definition of an investment's value added. "Clean finite change sensitivity indices" section introduces the CFCSI method and shows how it triggers a decomposition of the finite change of an objective function. In "Attribution of value added" section, we apportion the effects of the investment decisions in the various periods on the value added using the CFCSIs and show that different decisions by the client would trigger (not only a different VA but also) different CFCSIs. In "Worked example" section we illustrate the value added decomposition with a numerical example under two different assumptions about the client's decisions, one of which features zero interim cash flows. In "Time-Weighted Rate of Return" section, we show how to reconcile the VA with the TWRR, arithmetically decomposing the TWRR into period contributions and showing that the period effects of the TWRR are the same as the ones obtained from the VA under the zero-interim cash flow assumption. Some remarks conclude the paper.

Benchmark portfolio and value added

Following is a simple description of a model for the (discrete) evaluation of the investment, consisting of a portfolio of assets. An investor invests a capital B_0 at time $t = 0$. By selecting the assets and allocating them in every period, the portfolio's value is increased or decreased. Furthermore, the investor (client) makes decisions about capital contributions or distributions in the various periods, which increase or decrease the amount of capital invested in the portfolio.

We assume that the investment starts at time $t = 0$ and analyze its performance in the time interval $[0, n]$ where, for convenience, we assume that n is the current date.

Let E_t be the end-of-period (EOP) portfolios's value and B_t its beginning-of-period (BOP) value. Let F_t be the investor's contribution/distribution into/from the portfolio at time $t = 0, 1, \ldots, n-1$. From the point of view of the investor, a contribution is an outflow ($F_t < 0$), a distribution is an inflow ($F_t > 0$).[1] In particular, at time 0, the contributed amount is an outflow, so $F_0 = -B_0 < 0$. Then, the following relations hold:

$$\begin{aligned} B_t &= E_t - F_t \\ i_t &= \frac{E_t - B_{t-1}}{B_{t-1}} \\ E_t &= B_{t-1} \cdot (1 + i_t) \end{aligned} \tag{1}$$

where i_t denotes the rate of return in the period. The first equation says that the beginning-of-period value is obtained by deducting the capital call or adding the contribution made by the investor; the second relation says that the investment's holding period rate expresses the relative increase in the capital value; the third relation says that the ending value is obtained from the beginning value by marking it up by the return rate i_t. The selection and allocation policy affects i_t, which in turn affects E_t and, hence, B_t. The investor's choices about withdrawals and deposits affects B_t and, hence, E_t. Therefore, both types of policies affect the capital values, but only the investment policy affects i_t. The latter is then an appropriate measure of the effect on the value added of the investment policy in a given period.

Let us focus on the terminal date, $t = n$, and on its closing value, $E_n = B_{n-1}(1 + i_n)$.[2] Using (1) and solving for $t = n$, one can express E_n as a function of the return rates and the cash flows prior to n:

$$E_n = -\sum_{t=0}^{n-1} F_t(1 + i_{t+1})(1 + i_{t+2}) \dots (1 + i_n). \tag{2}$$

The above relation states that the terminal net investment's value is the compounded amount of the contributions (net of distributions) made by the investor.

Let us assume a benchmark index whose holding period rate is denoted as i_t^*, and a reference (benchmark) portfolio which acts as the opportunity cost of capital for the investment. More precisely, let us consider what would have occurred if the investor had made the same contributions/distributions in the benchmark portfolio. Under this assumption, the investor follows a passive strategy and replicates the investment's cash flows: Every contribution to the investment is matched by an equal contribution in the benchmark portfolio and every distribution from the investment is matched by an equal distribution from the benchmark. In general, the benchmark portfolio's value is different from the investment's value at every date t, which means that the holding period rates i_t and i_t^* are different. The difference between the two returns is determined by the active choices of asset selection and stock allocation in period t. In such a way, the benchmark portfolio is a replica of the investment's cash flows up to (and including) time $n - 1$. At time n, the investment's residual value will differ from the benchmark's residual value.

Formally, let $F_t^* = F_t$ be the cash flows in the reference portfolio, $t = 0, \dots, n-1$. We denote as B_t^* and E_t^* the beginning-of-period (BOP) and end-of-period (EOP) market value of this benchmark portfolio. Then, the following relations mimic the ones presented in (1):

$$\begin{aligned} B_t^* &= E_t^* - F_t^* \\ i_t^* &= \frac{E_t^* - B_{t-1}^*}{B_{t-1}^*} \\ E_t^* &= B_{t-1}^* \cdot (1 + i_t^*). \end{aligned} \tag{3}$$

In $t = n$, the net value of the benchmark portfolio is $E_n^* = B_{n-1}^*(1 + i_n^*)$. Analogously to eq. (2), the benchmark terminal net asset value E_n^* depends on the previous cash flows and the benchmark index return rates:

$$E_n^* = -\sum_{t=0}^{n-1} F_t(1 + i_{t+1}^*)(1 + i_{t+2}^*) \dots (1 + i_n^*). \tag{4}$$

As the investment and the benchmark portfolio release the same sequence of inflows and outflows up to time $n - 1$, the investment outperforms the benchmark if and only if the terminal value of the fund is greater than the terminal value of the replicating portfolio: $E_n > E_n^*$. The difference $E_n - E_n^*$ is the value added (VA):

$$\begin{aligned} \text{VA}^F = E_n - E_n^* = \sum_{t=0}^{n-1} F_t \cdot \Big(&(1 + i_{t+1}^*)(1 + i_{t+2}^*) \dots (1 + i_n^*) \\ &- (1 + i_{t+1})(1 + i_{t+2}) \dots (1 + i_n) \Big). \end{aligned} \tag{5}$$

Therefore, the investment outperforms the benchmark if and only if the value added is positive, $\text{VA}^F > 0$.

For a given sequence of injections and withdrawals $(F_0, F_1, \dots, F_{n-1})$ and a given sequence of benchmark returns $(i_1^*, i_2^*, \dots, i_n^*)$, the value added by such an investment depends on the active investment decisions, which is reflected in the return vector $(i_1, i_2, \dots, i_n)$.

Clean finite change sensitivity indices

Sensitivity analysis (SA) decomposes the variation in the output of a model $f(x)$ to the different input key parameters $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ (Saltelli et al. 2004). Among the several SA techniques defined in the literature (see (Borgonovo and Plischke 2016; Pianosi et al. 2016; Saltelli

[1] While we define the cash flows under the perspective of the investor (not of the fund), the results of the paper do not depend on this perspective.

[2] Since the investment has been liquidated at time n, then $E_n = F_n$.

Table 1 Decomposition of Δf via CFCSIs

x_j	$\Delta_j^1 f$	$\Delta_j^I f$	$\Delta_j^T f$	$\Phi_j^T f$	R_j
x_1	$\Delta_1^1 f$	$\Delta_1^I f$	$\Delta_1^T f$	$\Phi_1^T f$	R_1
x_2	$\Delta_2^1 f$	$\Delta_2^I f$	$\Delta_2^T f$	$\Phi_2^T f$	R_2
…	…	…	…	…	…
x_j	$\Delta_j^1 f$	$\Delta_j^I f$	$\Delta_j^T f$	$\Phi_j^T f$	R_j
…	…	…	…	…	…
x_n	$\Delta_n^1 f$	$\Delta_n^I f$	$\Delta_n^T f$	$\Phi_n^T f$	R_n
Total			Δf	100.00%	

et al. 2008, 2004 for reviews of SA methods)), the Finite Change Sensitivity Indices (FCSIs) (Borgonovo 2010a, b) have been recently conceived for analyzing the effect of the finite changes in the model inputs (parameters, key drivers) onto the finite change of the model output. The inputs variation from a so-called base value $x^0 = (x_1^0, \dots, x_n^0)$ to a realized value $x^1 = \left(x_1^1, x_2^1, \dots, x_n^1\right)$ brings about a model change $\Delta f = f(x^1) - f(x^0)$ from the base-case output $f(x^0)$ to the realized output $f(x^1)$. Two versions of FCSIs are defined: First Order FCSI and Total Order FCSI. The First Order FCSI of parameter x_j measures the individual effect of x_j, $\Delta_j^1 f = f(x_j^1, x_{(-j)}^0) - f(x^0)$, and, in normalized version, $\Phi_j^1 f = \frac{\Delta_j^1 f}{\Delta f}$, where $(x_j^1, x_{(-j)}^0) = (x_1^0, x_2^0, \dots, x_{j-1}^0, x_j^1, x_{j+1}^0, \dots, x_n^0)$ is the vector consisting of all the inputs set at their base value x^0 except parameter x_j which is given the realized value x_j^1 (Borgonovo 2010a). On the other side, the Total Order FCSI, $\Delta_j^{\mathcal{T}} f$, quantifies the total effect of the parameter, including both its individual contribution and its interactions with other parameters Borgonovo (2010a), (Proposition 1) shows that the total FCSI can be computed as

$$\Delta_j^{\mathcal{T}} f = f(x^1) - f(x_j^0, x_{(-j)}^1), \ \forall j = 1, 2, \dots, n, \tag{6}$$

such that in vector $(x_j^0, x_{(-j)}^1)$ each input equals the realized value x^1 except x_j, which is set equal to x_j^0. Its normalized version is $\Phi_j^{\mathcal{T}} f = \frac{\Delta_j^{\mathcal{T}} f}{\Delta f}$. It is worth noting that the difference between the Total Order FCSI and the First Order FCSI is named interation FCSI of parameter x_j, denoted as $\Delta_j^{\mathcal{I}} f = \Delta_j^{\mathcal{T}} f - \Delta_j^1 f$, implying that $\Delta_j^{\mathcal{T}} f = \Delta_j^1 f + \Delta_j^{\mathcal{I}} f$.

Magni et al. (2020) show that the Total Order FCSI does not provide a complete decomposition of the output change and introduce a duplication-clearing procedure giving rise to the Clean Finite Change Sensitivity Index (CFCSI) which eliminates the redundant, multiple interactions, allowing the exact decomposition of the finite functional variation. More precisely, they define the *Clean* Interaction FCSI of x_j, $\Delta_j^I f$, as a percentage $\alpha_j \in \mathbb{R}$ of the total interaction, $\Delta f - \sum_{l=1}^n \Delta_l^1 f$, where α_j is obtained as the ratio of Borgonovo's (Borgonovo 2010a, b) interaction FCSI and the sum of all such interaction FCSIs, $\sum_{l=1}^n \Delta_l^{\mathcal{I}} f$:

$$\Delta_j^I f = \alpha_j \cdot \left(\Delta f - \sum_{l=1}^n \Delta_l^1 f\right) \tag{7}$$

where

$$\alpha_j = \frac{\Delta_j^{\mathcal{I}} f}{\sum_{l=1}^n \Delta_l^{\mathcal{I}} f}.$$

Hence, the Clean Total Order FCSI of parameter x_j, $\Delta_j^T f$, is the sum of individual contribution and Clean Interaction FCSI of x_j, that is,

$$\Delta_j^T f = \Delta_j^1 f + \Delta_j^I f \tag{8}$$

and, in normalized version,

$$\Phi_j^T f = \frac{\Delta_j^T f}{\Delta f}. \tag{9}$$

They prove that the Clean Total FCSIs offer a complete decomposition of the output change, that is $\sum_{l=1}^n \Delta_l^T f = \Delta f$, and, in normalized version, $\sum_{l=1}^n \Phi_l^T f = 1$. The directional effect of an input change onto the objective function is signalled by the sign of the Clean Total FCSI, $\Delta_j^T f$, where a positive (negative) index suggests an increasing (a decreasing) effect on the output. The magnitude of the effect is quantified via the absolute value of the Clean Total FCSI, which determines the parameters' ranking: x_j has higher rank than x_k if and only if $|\Delta_j^T f| > |\Delta_k^T f|$. Denoting the rank of input x_j as R_j, the resulting rank vector is $R = (R_1, R_2, \dots, R_n)$. Table 1 summarizes the procedure determining the clean decomposition of the finite change in $f(x)$.

Attribution of value added

Let $x = (x_1, x_2, \dots, x_n)$ be the vector of time-varying return rates of an investment with cash flows F_t from $t = 0$ to $n-1$. Generalizing equations (2) and (4), the terminal net asset value implied by the return rates vector x, denoted as $f(x)$, is, for a given sequence of cash flows $(F_0, F_1, \dots, F_{n-1})$, equal to

$$f(x) = -\sum_{t=0}^{n-1} F_t(1+x_{t+1})(1+x_{t+2})\dots(1+x_n). \tag{10}$$

Let $x^0 = i^*$ be the stream of benchmark returns (base value). The manager's investment decisions in the various periods move the return rates from $x^0 = i^*$ to $x^1 = i$ (realized case). This in turn has the effect of changing the terminal value from $f(x^0) = f(i^*)$ to $f(x^1) = f(i)$. However,

$$f(i^*) = E_n^* \tag{11}$$

$$f(i) = E_n. \tag{12}$$

Therefore, the value added by the investment may be written as

$$\text{VA}^F = E_n - E_n^* = f(i) - f(i^*). \tag{13}$$

As a result, the value added is equal to a finite change of f. Therefore, one may apply the FCSI technique integrated by the duplication-clearing procedure for decomposing VA^F in terms of the impact of the period rates. It is then possible to identify the periods whose investment choices have most affected the investment's performance. In particular, for any given sequence of contributions and distributions, the value added may be considered as the sum of all the effects of the active selection and allocation choices made in the various periods, as opposed to a passive strategy consisting in investing in a benchmark portfolio with equal contribution-distribution policy.

For accomplishing a complete, exact decomposition of the value added, we use the CFCSI method. These indices inform about whether the investment decisions made in period t have contributed, overall, to outperform or underperform the benchmark in the time interval $[0, n]$ and quantify their contribution to the investment's value added. Note that the CFCSIs take account of the effect of the decisions made in period t onto the following periods. Indeed, the decisions made in period t determine i_t, which measures the value increase in the net asset value during period t. This implies that the manager's decisions in a given period t affect the investment's scale in the following periods: Other things unvaried, the capital invested at the beginning of the following periods will be increased (decreased) by a positive (negative) period return i_t. As such, i_t not only contributes to create (or destroy) value in period t, but its effect reverberates, via the investment scale, in the periods $t+1$, $t+2, \dots, n$ as well. The Clean Total FCSI, $\Delta_j^T f$, precisely provides the amount of value added in the assessment period $[0, n]$ that is determined by the investment policy in period t.

The analysis above assumes that the policy of contributions and distributions is fixed and equal to $(F_0, F_1, \dots, F_{n-1})$. Consider now a different sequence of contributions and distributions:

$$(G_0, G_1, \dots, G_{n-1}) \tag{14}$$

such that $G_0 = F_0$, and let

$$g(x) = -\sum_{t=0}^{n-1} G_t(1+x_{t+1})(1+x_{t+2})\dots(1+x_n) \tag{15}$$

be the investment's terminal value.

In general, the functions $f(x)$ and $g(x)$ are different and the value added is different as well: $\text{VA}^F = f(i) - f(i^*) \neq g(i) - g(i^*) = \text{VA}^G$. Futhermore, the CFCSIs of the parameters under f and g will generally be different, implying that the same choices about investments in a given period have a different impact on the value added depending on the choices about injections/withdrawals made by the investor. Therefore, it may occur the case where a given parameter x_t triggered by a given investment policy in period t has a substantial impact on value added for a contribution-and-distribution policy and a negligible impact on value added for a different contribution-and-distribution policy. The reason is that, given the manager's investment decisions in period t (i.e., given a return i_t), the client's contributions or distributions at time $t = 0, 1, \dots, n-1$ affects the BOP value (i.e., the capital invested) in period t and in the following periods $t+1, t+2, \dots, n-1$. This tends to amplify (shrink) the manager's performance *in absolute terms*. This reverberates in turn on the following periods.[3]

A policy G of particular significance is the one where the initial investment is the same as F, and the following cash flows from $t = 1$ to $t = n-1$ are zero. We denote this sequence as $\hat{F}$, such that $\hat{F}_0 = F_0 = -B_0$ and $\hat{F}_t = 0, t = 1, \dots, n-1$, and we call this the *neutral interim vector*, because it is obtained from F by neutralizing, so to say, the interim cash flows. Assuming that the manager's choices are unvaried with respect to F, the period rates do not change and the BOP values and EOP values, denoted as $\hat{B}_t$ and $\hat{E}_t$, are

[3] This phenomenon arises because VA depends on both investment scale and investment efficiency: The financial efficiency depends on the manager's investment decisions, the investment scale depends on both the client's decisions and the manager's decisions (see also "Clean finite change sensitivity indices" section).

Table 2 Investment F: Input data and resulting BOP and EOP values

Time	Input data			Resulting BOP and EOP values			
	Cash flows	Fund's returns	Benchmark's returns	Fund's BOP value	Fund's EOP value	Benchmark's BOP value	Benchmark's EOP value
t	F_t	i_t (%)	i_t^* (%)	B_t	E_t	B_t^*	E_t^*
0	− 100.00			100.00	0.00	100.00	0.00
1	30.00	4	3	74.00	104.00	73.00	103.00
2	− 20.00	5	4	97.70	77.70	95.92	75.92
3	40.00	2	3	59.65	99.65	58.80	98.80
4	10.00	4	6	52.04	62.04	52.33	62.33
5	− 30.00	3	1	83.60	53.60	82.85	52.85
6	60.00	3	2	26.11	86.11	24.51	84.51
7	20.00	5	2	7.41	27.41	5.00	25.00
8		4	5	0.00	7.71	0.00	5.25

$$\hat{B}_t = \hat{E}_t = \hat{B}_{t-1}(1+i_t) = B_0(1+i_1)(1+i_2)\dots(1+i_t),\quad t = 1, 2, \dots n-1 \tag{16}$$

and $\hat{E}_n = \hat{B}_{n-1}(1+i_n) = B_0(1+i_1)(1+i_2)\dots(1+i_n)$. The VA of such investment is

$$\text{VA}^{\hat{F}} = \hat{E}_n - \hat{E}_n^* = \hat{f}(i) - \hat{f}(i^*) \tag{17}$$

where

$$\hat{f}(x) = B_0(1+x_1)(1+x_2)\dots(1+x_n). \tag{18}$$

In the following section, we present a worked example where we measure the impact of the period investment decisions in the two investments F and $\hat{F}$. In other words, we analyze the impact of the manager's decisions on the value added on the same investment characterized by two different contribution-and-distribution policies.

Worked example

Table 2 reports the input data for an investment F (e.g., investment in a fund) along with the resultant BOP and EOP values. The initial contribution to the money manager by the client is $B_0 = -F_0 = \$100$. We assume the investor exits the investment at time $t = 8$ (therefore, $n = 8$). The contribution-and-distribution policy is under full control of the investor, who determines the timing and amount of withdrawals and deposits from $t = 1$ to $t = 7$. The investment policy of the money manager in period t brings about a return rate equal to i_t in period t, $t = 1, 2, \dots, 8$. In the same period, the benchmark index's return is i_t^*. From (2) and (4), the terminal values of the fund and of the replicating portfolio are $E_8 = 7.71$ and $E_8^* = 5.25$, respectively, (reported in Table 2), implying that, given the sequence of contributions and distributions, the value added is $\text{VA}^F = 2.47 = 7.71 - 5.25 > 0$.

We decompose the investment's value added and rank the period decisions by comparing the active and passive investment strategy. This is done by evaluating the effect of the change of the net terminal value when the return vector is changed from the benchmark return vector, i^*, to the fund's return vector, i. To this end, we consider the objective function

$$f(x) = -\sum_{t=0}^{7} F_t(1+x_{t+1})\dots(1+x_8)$$

with

$$x^0 = i^* = (3\%, 4\%, 3\%, 6\%, 1\%, 2\%, 2\%, 5\%)$$

and

$$x^1 = i = (4\%, 5\%, 2\%, 4\%, 3\%, 3\%, 5\%, 4\%).$$

The results of the analysis are reported in Table 3. The first column collects the vector of input parameters, $(x_1, x_2, \dots, x_8)$, which are determined by the investment choices made in the various periods. The second column describes the First Order FCSIs, the third column is the Clean Interaction FCSIs, which is computed as in (7); the fourth column represents the Clean Total Order FCSI as defined in (8); the fifth column reports the normalized Clean Total Order FCSI, and, finally the sixth column shows the inputs' ranking.

The most relevant input on the value added is the return rate in period 4, x_4, with $\Delta_4^T f = -\$1.25$ and $\Phi_4^T f = -50.70\%$, implying that the choices made by the manager in the fourth period have had a negative effect on the VA, with a magnitude about half of the value added.

For a better comprehension of the FCSI technique, as an example, we supplement here the computational process of the

Table 3 Investment F: Value-added decomposition and ranking of investment (period) decisions

x_j	$\Delta_j^1 f$	$\Delta_j^I f$	$\Delta_j^T f$	$\Phi_j^T f$ (%)	R_j
x_1	1.25	− 0.02	1.23	49.96	2
x_2	0.88	− 0.02	0.86	34.98	6
x_3	− 1.12	0.03	− 1.09	− 44.24	4
x_4	− 1.30	0.05	− 1.25	− 50.70	1
x_5	1.14	− 0.02	1.13	45.74	3
x_6	0.89	− 0.01	0.87	35.40	5
x_7	0.77	− 0.02	0.75	30.34	7
x_8	− 0.05	0.01	− 0.04	− 1.48	8
Total (Value added)			2.47	100.00	

individual contribution of x_4. The client invests in the benchmark from $t = 0$ until $t = 3$, switches to the active investment at $t = 3$ and then switches back to the benchmark at $t = 4$ (with cash flows equal to the investment's cash flows). This strategy results in the following terminal value:

$$\begin{aligned} &f(\overbrace{0.03, 0.04, 0.03, \mathbf{0.04}, 0.01, 0.02, 0.02, 0.05}^{(i_1^*, i_2^*, i_3^*, i_4, i_5^*, i_6^*, i_7^*, i_8^*)}) \\ &= 100(1.03)(1.04)(1.03)(1.04)(1.01)(1.02)(1.02(1.05) \\ &-30(1.04)(1.03)(1.04)(1.01)(1.02)(1.02)(1.05) \\ &+20(1.03)(1.04)(1.01)(1.02)(1.02)(1.05) \\ &-40(1.04)(1.01)(1.02)(1.02)(1.05) \\ &-10(1.01)(1.02)(1.02)(1.05) \\ &+30(1.02)(1.02)(1.05) \\ &-60(1.02)(1.05) \\ &-20(1.05) \\ &= \$3.95 \end{aligned}$$

If no switching occurs, the terminal capital value of the benchmark investment, E_n^*, is

$$\begin{aligned} &f(\overbrace{0.03, 0.04, 0.03, \mathbf{0.06}, 0.01, 0.02, 0.02, 0.05}^{(i_1^*, i_2^*, i_3^*, i_4^*, i_5^*, i_6^*, i_7^*, i_8^*)}) \\ &= 100(1.03)(1.04)(1.03)(1.06)(1.01)(1.02)(1.02(1.05) \\ &-30(1.04)(1.03)(1.06)(1.01)(1.02)(1.02)(1.05) \\ &+20(1.03)(1.06)(1.01)(1.02)(1.02)(1.05) \\ &-40(1.06)(1.01)(1.02)(1.02)(1.05) \\ &-10(1.01)(1.02)(1.02)(1.05) \\ &+30(1.02)(1.02)(1.05) \\ &-60(1.02)(1.05) \\ &-20(1.05) \\ &= \$5.25. \end{aligned}$$

The individual contribution of x_4, $\Delta_4^1 f$, is defined as the difference in terminal values, that is $\Delta_4^1 f = \$3.95 - \$5.25 = -\$1.30$, signaling value destruction. This represents the impact of the period-4 decisions on the value added, taken in isolation from the other inputs (i.e., neglecting the interaction effects). The clean interaction FCSI, calculated as in (7), is positive ($\Delta_4^I f = \$0.05$) and, therefore, supplies a partial compensating effect. Finally, the overall contribution of the investment decisions in the fourth period on the value added is $\Delta_4^T f = -\$1.30 + \$0.05 = -\$1.25$, corresponding to the relative weight $\Phi_4^T f = -50.7\%$.

The second and third most influential inputs are the decisions in periods 1 and 5, represented by x_1 and x_5, which have had a positive effect on value added. In particular, their total contributions are, respectively, $\Delta_1^T f = \$1.23$ and $\Delta_5^T f = \$1.13$. In relative terms, their weights are $\Phi_1^T f = 49.96\%$ and $\Phi_5^T f = 45.74\%$. Next come x_3 (negative impact), x_6, x_2, x_7 (positive impact) and x_8 (negative effect). The latter explains just -1.48% of VA^F. The Clean Total Order FCSIs exactly decompose the value added:

$$\overbrace{1.23 + 0.86 - 1.09 - 1.25 + 1.13 + 0.87 + 0.75 - 0.04}^{\text{sum of Clean Total FCSIs}} = 2.47.$$

Consider now a different contribution-and-distribution policy such that the investment turns to $G = \hat{F}$. This means that the same investment is analyzed but the contribution-and-distribution policy is neutral, that is, $\hat{F}_0 = -\$100$ and $\hat{F}_t = \$0$ for $t = 1, 2, \dots 7$. The input data and the resulting BOP and EOP values are reported in Table 4. Since $\hat{B}_{n-1}(1 + i_n) = B_0(1 + i_1)(1 + i_2)\dots(1 + i_n)$, the fund's and the benchmark portfolio's net terminal values at time 8 (reported in Table 4) are, respectively,

$$\hat{E}_8 = \hat{f}(i) = 100(1.04)^3(1.05)^2(1.02)(1.03)^2 = \$134.20$$

and

$$\hat{E}_8^* = \hat{f}(i^*) = 100(1.03)^2(1.04)(1.06)(1.01)(1.02)^2(1.05) = \$129.04,$$

Table 4 Investment $\hat{F}$: Input data and resulting BOP and EOP values

	Input data			Resulting BOP and EOP values			
Time	Cash flows	Fund's returns	Benchmark's returns	Fund's BOP value	Fund's EOP value	Benchmark's BOP value	Benchmark's EOP value
t	F_t	i_t (%)	i_t^* (%)	B_t	E_t	B_t^*	E_t^*
0	− 100.00			100.00	0.00	100.00	0.00
1	0.00	4	3	104.00	104.00	103.00	103.00
2	0.00	5	4	109.20	109.20	107.12	107.12
3	0.00	2	3	111.38	111.38	110.33	110.33
4	0.00	4	6	115.84	115.84	116.95	116.95
5	0.00	3	1	119.31	119.31	118.12	118.12
6	0.00	3	2	122.89	122.89	120.49	120.49
7	0.00	5	2	129.04	129.04	122.90	122.90
8		4	5	0.00	134.20	0.00	129.04

Table 5 Investment $\hat{F}$ (no interim cash flows): Value-added decomposition and ranking of investment (period) decisions

x_j	$\Delta_j^1\hat{f}$	$\Delta_j^I\hat{f}$	$\Delta_j^T\hat{f}$	$\Phi_j^T\hat{f}$ (%)	R_j
x_1	1.25	0.02	1.27	24.63	6
x_2	1.24	0.02	1.26	24.39	7
x_3	− 1.25	− 0.03	− 1.28	− 24.86	5
x_4	− 2.43	− 0.07	− 2.50	− 48.54	3
x_5	2.56	0.02	2.58	49.99	2
x_6	1.27	0.02	1.28	24.87	4
x_7	3.80	0.02	3.81	73.91	1
x_8	− 1.23	− 0.03	− 1.26	− 24.39	8
Total (Value added)			5.16	100.00	

implying that the value added is

$$\mathrm{VA}^{\hat{F}} = \hat{f}(i) - \hat{f}(i^*) = \$134.2 - \$129.04 = \$5.16,$$

which, alternatively, may be directly computed as

$$\mathrm{VA}^{\hat{F}} = 100 \cdot \Big((1.04)^3(1.05)^2(1.02)(1.03)^2 - (1.03)^2(1.04)(1.06)(1.01)(1.02)^2(1.05)\Big) = 5.16.$$

The value added has increased with respect to the previous case, since $\mathrm{VA}^{\hat{F}} = 5.16 > 2.47 = \mathrm{VA}^{F}$. The CFCSIs are reported in Table 5, showing that, in the case of no interim contributions and distributions, the same investment choices have a very different impact on the value added. The most influential investment decisions are made in period 7 ($R_7 = 1$), which have a positive effect on $\mathrm{VA}^{\hat{F}}$. As previously seen, its rank in the case where $(F_0, F_1, \dots, F_7)$ represented the choices about deposits and withdrawals was only $R_7 = 7$. This means that the (value-creating) investment decisions made by the manager in period 7 have the greatest impact if the investor does not make any interim contribution/distribution, whereas they have negligible effect in case of the timing and amounts of cash flows are $(F_0, F_1, \dots, F_7)$. This happens because in $\hat{F}$ no distribution of \$60 occurs at time 6, so the investment scale is not reduced implying a higher positive effect (\$3.81 as opposed to \$0.35 in case of distribution). Note also that the first-period rate, x_1, which reflects the investment decisions made in period 1, has rank 6 ($R_1 = 6$), whereas it represents the second most influential parameter in the previous case.

Time-Weighted Rate of Return

Consider the function $h(x) = \hat{f}(x)/B_0 - 1$. The value taken on by this function for $x = i$ is the well-known Time-Weighted Rate of Return (TWRR):

$$\mathrm{TWRR} = h(i) = \frac{\hat{f}(i)}{B_0} - 1 = (1 + i_1)(1 + i_2) \cdot \dots \cdot (1 + i_n) - 1.$$

Table 6 TWRR decomposition and ranking of investment (period) decisions

x_j	$\Delta_j^T h$ (%)	$\Phi_j^T h$ (%)	R_j
x_1	1.27	24.63	6
x_2	1.26	24.39	7
x_3	− 1.28	− 24.86	5
x_4	− 2.50	− 48.54	3
x_5	2.58	49.99	2
x_6	1.28	24.87	4
x_7	3.81	73.91	1
x_8	− 1.26	− 24.39	8
Total (Active TWRR)	5.16	100.00	

Therefore, the difference

$$\frac{\mathrm{VA}^{\hat{F}}}{B_0} = \frac{\hat{f}(i)}{B_0} - \frac{\hat{f}(i^*)}{B_0} = h(i) - h(i^*) = \prod_{t=1}^{n}(1+i_t) - \prod_{t=1}^{n}(1+i_t^*)$$

is an *active TWRR*, measuring the value added by the manager for any dollar initially contributed by the client over and above the *benchmark TWRR*, $h(i^*)$.

Lemma 1 *Given a function $f(x)$ and an affine transformation of it $g(x)$, such that $g(x) = af(x) + b$, with $a, b \in \mathbb{R}$, the Clean Total FCSIs of $g(x)$ are multiples of the Clean Total FCSIs of $f(x)$:*

$$\Delta_j^T g(x) = a\Delta_j^T f(x), \forall j = 1, \dots, n.$$

Furthermore, the normalized Clean Total FCSIs of $g(x)$ and $f(x)$ are equal, that is,

$$\Phi_j^T g(x) = \Phi_j^T f(x), \forall j = 1, \dots, n.$$

Proof See "Appendix." □

Lemma 1 provides a decomposition of the (active) TWRR into periods, as the following Proposition states.

Proposition 1 *The Clean Total FCSIs of $h(x)$ are equal to the Clean Total FCSI of $\hat{f}(x)$ divided by the initial investment, B_0, that is, $\Delta_j^T h = \Delta_j^T f / B_0$. Furthermore, the normalized Clean FCSIs (and, therefore, the ranking of the parameters) coincide*:

$$\Phi_j^T h = \Phi_j^T \hat{f}.$$

Proof The proof is straightforward from Lemma 1, considering that the TWRR is an affine transformation of the value added generated with no interim cash flows: $h(x) = \hat{f}(x)/B_0 - 1$. □

Proposition 1 refines the piece of information traditionally conveyed by the TWRR in the literature. Specifically, it decomposes arithmetically the (active) TWRR into periods. In other words, it signals how much of the (active) TWRR is generated in every period. This enables capturing the share of the manager's performance which is generated by the investment decisions made in every period.

Looking back to the example provided in the previous section, the TWRR is

$$\mathrm{TWRR} = 1.04^3 \cdot 1.05^2 \cdot 1.02 \cdot 1.03^2 - 1 = 34.2\%.$$

If the manager's investment policy had mimicked the benchmark portfolio, the TWRR would have been $1.03^2 \cdot 1.04 \cdot 1.06 \cdot 1.01 \cdot 1.02^2 \cdot 1.05 - 1 = 29.04\%$. Therefore, the active TWRR is $34.2\% - 29.04\% = 5.16\%$, representing a 5.16% return over and above the *benchmark TWRR*.

Table 6 reports the CFCSIs and the ranking for the TWRR associated with the investment illustrated in Table 2.

Whilst 5.16% quantifies the overall manager's performance, the CFCSI technique enables understanding when and how such a return is generated. Specifically, in period 3, 4, and 8 the performance is negative and, precisely, amounts to −1.28%, −2.50%, and −1.26%, respectively. The overall negative performance is $-1.28\% - 2.50\% - 1.26\% = -5.04\%$. However, the negative performance is more than compensated by the value-creating periods 1, 2, 5, 6, and 7. The overall positive performance is $1.27\% + 1.26\% + 2.58\% + 1.28\% + 3.81\% = 10.2\%$. The net effect is $10.2\% - 5.04\% = 5.16\%$, that is, the active TWRR.

Concluding remarks

This paper proposes a method for evaluating the effect of the investment policy on an investment's performance, as measured by the value added (VA). Specifically, we show

how to quantify the part of the value added generated by the investment decisions made in the various periods, given a fixed sequence of cash flows (client's contributions and distributions). We compare an active investment strategy with a passive investment strategy in a benchmark portfolio and formalize it in terms of difference between terminal values in case of active investment and passive investment, respectively. This difference, which equals the investment's value added, depends on the relations between the sequence of benchmark returns and the sequence of investment's returns. To accomplish the task, we make use of the Clean Finite Change Sensitivity Index (CFCSI) technique (Borgonovo 2010a, b; Magni et al. 2020) which quantifies and ranks the efficacy of the investment policy in the various periods. We also find that, for a given investment policy, not only different contribution-and-distribution policies give rise to different performances but also the period investment decisions have a different impact on the value added. This means that decisions about contributions and distributions and decisions about selection and allocation of assets are strictly intertwined, since both the manager's and the client's decisions affect the investment scale and the latter affects the value added along with the period returns. Finally, we apportion the Time-Weighted Rate of Return (TWRR) to periods by decomposing the manager's skills into the various decisions and reconcile the TWRR with the VA by proving that the TWRR is an affine transformation of the VA in case of zero interim cash flows. This implies that the two metrics share the same period attribution.

Appendix

Proof of Lemma 1

Marchioni and Magni (2018), (Proposition 1, part iii) prove that $\Delta g = a\Delta f$, $\Delta_j^1 g = a\Delta_j^1 f$, $\Delta_j^{\mathcal{T}} g = a\Delta_j^{\mathcal{T}} f$, $\forall j = 1, \dots, n$. Since $\Delta_j^{\mathcal{I}} f = \Delta_j^{\mathcal{T}} f - \Delta_j^1 f$, then the Interaction FCSIs of g are multiples of the Interaction FCSIs of f:

$$\Delta_j^{\mathcal{I}} g = a\Delta_j^{\mathcal{T}} f - a\Delta_j^1 f = a\Delta_j^{\mathcal{I}} f.$$

From (7), the Clean Interaction FCSIs are multiple as well:

$$\Delta_j^I g = \frac{a\Delta_j^{\mathcal{I}} f}{a\sum_{l=1}^n (\Delta_l^{\mathcal{I}} f)} \cdot a\left(\Delta f - \sum_{l=1}^n \Delta_l^1 f\right) = a\Delta_j^I f.$$

Finally, from (8), the Clean Total FCSIs are multiple, that is, $\Delta_j^T g = a\Delta_j^1 f + a\Delta_j^I f = a\Delta_j^T f$; hence, (9) implies that the normalized Clean Total FCSIs are equal:

$$\Phi_j^T g = \frac{a\Delta_j^T f}{a\Delta f} = \Phi_j^T f, \ \forall j = 1, \dots, n.$$

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