Research Article

Dynamic pricing with time-dependent elasticities

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ABSTRACT Many stochastic dynamic sales applications are characterized by time-dependent price elasticities of demand. However, in general, such problems cannot be solved analytically. To determine smart pricing heuristics for general time-dependent dynamic pricing models, we solve a general class of deterministic dynamic pricing problems for perishable and durable goods. The continuous time model has several time-dependent parameters, for example, discount rate, marginal unit costs and price elasticity. We show how to derive the value function and optimal pricing policies. On the basis of the feedback solution to the deterministic model, we propose a method for constructing heuristics to be applied to general stochastic models. For the case of isoelastic demand, we analytically verify the excellent performance of this approach for both small and large inventory levels.

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INTRODUCTION

The traditional application of revenue management theory offers the best way to sell perishable or durable products. Many firms dynamically set prices to maximize their expected profits. Examples of such dynamic pricing problems can be found in various contexts, such as in the fashion industry, fruit markets, accommodation services and the airline industry. In many such applications, the price elasticity of demand is time-dependent. For instance, in the case of fashion goods, the demand is typically decreasing in time; in the case of airline or hotel tickets, reservation prices usually increase. However, most dynamic pricing optimization models only consider specific demand functions with constant elasticities. Thus, a dynamic pricing model with general time-varying demand is needed.

This article presents stochastic and deterministic pricing models with time-dependent demand elasticities. We consider the problem of selling a stock of items over a finite or infinite horizon, and the objective is to maximize the expected revenues. Discounting, marginal unit costs (for example, shipping costs) and inventory holding costs are included in the models and are allowed to depend on time. Unfortunately, there exist very few closed-form solutions to stochastic models. Hence, powerful heuristics and explicit special case solutions are very useful. We aim to show that such heuristics can be derived by solving deterministic versions of the problem. In accordance with these deterministic solutions, we construct dynamic pricing heuristics to apply to stochastic models. For special cases, we analytically verify the excellent performance of such heuristics. Note that the analytical solutions to the deterministic models may also provide economic insight into their stochastic counterparts.

There is extensive literature on revenue management and related practices, such as the survey articles of Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003) and Shen and Su (2007), and the books of Talluri and van Ryzin (2004) and Phillips (2005) that provide broad discussions of various aspects of revenue management. In the literature, most dynamic pricing models are time-homogeneous. However, very few articles address time-dependent demand. In some models, the arrival rate of potential customers is allowed to depend on time. In such models, the demand rate (or sales intensity) $\lambda(t, p)$ is usually of a separable form, that is, $\lambda(t, p) = u(t) \cdot \overline{F}(p)$, where u is the arrival rate and $\overline{F}$ can be interpreted as the probability that a customer's reservation price exceeds the price p. In the special cases of exponential or isoelastic demand, closed-form solutions to the corresponding stochastic dynamic pricing model can be derived (cf. McAfee and te Velde, 2008; Berman *et al*, 2013; Helmes and Schlosser, 2013). Nevertheless, the evaluations of these models show that the expected evolution of price is almost constant in time (except for a small hump at the end of the time horizon). These models – characterized by constant price elasticities – cannot explain different, significant price trends. Hence, they are not suitable for many real-life applications.

Zhao and Zheng (2000) consider more general time-dependent dynamic pricing models, where the propensity to buy $\overline{F}$ also depends on time. For the stochastic case, they analyze the structural properties of the optimal pricing policy (in feedback form) as well as those of the value function representing the expected future profits. However, they do not propose solution methods. Xu and Hopp (2009) and Berman *et al* (2013) study the price trends of generalized dynamic pricing models. They show that if a customer's willingness to pay rapidly changes over time upward or downward, price trends can be expected. In both articles, the authors apply discrete time approximations to solve the illustrating examples. Their results show that time dependencies in the price elasticity are mainly responsible for significant price trends in the stochastic dynamic pricing models. Unfortunately, due to their complexity, such models in general cannot be solved analytically. Hence, numerical methods, such as discrete time approximations, must be used. However, such numerical solution approaches rarely provide economic insight and are not appropriate for showing analytical results.

An alternative is to use heuristics. A frequently used heuristic policy is to apply an optimally chosen fixed price (see, for example, Gallego and van Ryzin, 1994). For very large inventory levels, these heuristics are shown to be asymptotically optimal if demand elasticities are constant over time. In models with time-varying demand, other heuristics should be used. In Section 7.1.4 of their paper, Gallego and van Ryzin (1997) consider a demand intensity that is piecewise constant over time and apply the following heuristic: for each of the different time intervals, a price is used that coincides with the corresponding optimal fixed price of the related deterministic model.

Following this idea, in time-dependent models, it is advisable to apply the optimal price trajectory of the deterministic model, as in the open-loop pricing policy. An even better approach is to apply the optimal feedback prices of the deterministic model as a heuristic in stochastic models. We will verify that the latter approach yields excellent results for both large and small inventories. However, the solutions to general time-dependent deterministic dynamic pricing models are needed to apply these more complex but powerful heuristics solutions. To our knowledge, no such general solutions have been analyzed in detail. Thus, our aim is to close that gap.

The main contribution of this article is that it demonstrates how to derive optimal policies of general time-dependent deterministic models. Moreover, using a special class of models, we analytically verify the excellent performance of the optimal feedback policy of the deterministic model applied to the corresponding stochastic model. Another contribution is that we describe a general transformation approach that enables us to include inventory holding costs in the model.

The article is organized as follows. The next section provides a detailed formulation of the general stochastic and deterministic time inhomogeneous dynamic pricing models. Inspired by the Lagrangian approach that Stiglitz (1976) adopts for cases with isoelastic demand, we show how to transform the deterministic dynamic optimization problem into the analysis of a single non-linear equation with one variable. Under minor assumptions, the existence and uniqueness of optimal policies will be shown. In general, we find that we must distinguish between an inventory-saturated *overage* case and an *underage* case, where it is optimal to sell the entire inventory. Finally, optimal pricing strategies are derived in feedback and open-loop forms. Furthermore, we determine the relation between the optimal Lagrange multiplier and the value function, which is characterized by the Bellman equation that is associated with the dynamic control problem.

In the section after that, we use the results of the previous section to solve the isoelastic demand model, where price elasticity, marginal unit costs, arrival rate and discount rate are time-dependent. Similarly, we solve exponential and linear demand models. For special cases, we obtain explicit feedback solution formulas for the value function and the optimal pricing policy. Moreover, we derive the optimal inventory paths and present the solution formulas as functions of time.

In the following section, we verify the quality of our heuristic approach to use the feedback version of the optimal deterministic policy in a stochastic environment. For the case with isoelastic demand, we determine the expected profits of optimal deterministic policies in stochastic models by solving the associated difference–differential equation explicitly, and we find that our heuristic solution approach yields promising results. Moreover, we illustrate how the analytical results can be used to derive simple but powerful dynamic pricing strategies for real-life applications. In the penultimate section, we introduce an adjustment of the marginal unit cost function, which in general allows us to include inventory holding costs as well as a salvage value for unsold units. The approach is based on a specific transformation of the Bellman equation. The results can also be used to derive optimal ordering decisions. In the final section, we provide our conclusions and give managerial recommendations.

STOCHASTIC AND DETERMINISTIC DYNAMIC PRICING MODELS

Description of the stochastic model

We consider the situation where a monopolist wants to sell N ($0<N<\infty$) items of a perishable or durable product over a finite or infinite time horizon T. We use a fairly general model with a time inhomogeneous non-negative discount rate $r(t)$ and non-negative time-dependent unit costs $c(t)$, $0 \leqslant t \leqslant T$. Moreover, we assume a positive revenue parameter v; in other words, if a sale occurs at time t at price p, the discounted net revenue is given by:

$$e^{-R(t)} \cdot (v \cdot p - c(t)), \text{ where} \quad R(t) := \int_0^t r(s)ds. \tag{1}$$

We assume myopic customers, who do not act strategically; that is, they do not wait or anticipate prices. The dynamic of the sales process is given by a general jump intensity $\lambda(t, p)$, $0 \leqslant t \leqslant T$, $p \geqslant 0$, where at each time t, a price p must be chosen. The rate $\lambda(t, p)$ is assumed to be differentiable in t and p. The random inventory level at time t is denoted by X_t, $0 \leqslant t \leqslant T$. The end of sale is the random time τ when all N products are sold or the horizon T is reached, that is, $\tau := \min_{0 \leqslant t \leqslant T}\{t\colon X_t = 0\} \wedge T$; for all remaining $t \geqslant \tau$, we let $\lambda(t, \cdot, \cdot) := 0$. The evolution of the profit process $(R_t)_t$ is connected to the inventory process and characterized by realized net revenues (cf. (1)). Depending on the chosen pricing strategy, the random accumulated profit up to time t amounts to, $0 \leqslant t \leqslant T$:

$$R_t := \int_0^{t \wedge \tau} e^{-R(s)} \cdot (\nu \cdot p_{s-} - c(s)) dX_s$$

We want to determine a non-anticipating Markovian feedback pricing policy $p_n(t)$ such that the expected total profit:

$$E(R_T) = E\left[\int_0^{T \wedge \tau} e^{-R(t)} \cdot (\nu \cdot p_{X_t}(t) - c(t)) \cdot \lambda(t, p_{X_t}(t)) dt | X_0 = N\right] \quad (2)$$

will be maximized. By $G_t := R_T - R_t$, we denote the random profits from time t onward ($0 \leqslant t \leqslant T$). Following the Bellman approach, the best future profits $E(G_t | X_t = n)$ describe the value function $V_n(t)$ of the stochastic control problem, which is characterized by the associated Hamilton–Jacobi–Bellman (HJB) equation, $0 \leqslant t < T$, $n = 1, 2, \ldots, N$ (see Brémaud, 1980),

$$\dot{V}_n(t) + \sup_{p \geqslant 0}\{\lambda(t, p) \cdot (\nu \cdot p - c(t) - \Delta V_n(t))\} = r(t) \cdot V_n(t), \quad (3)$$

where $\Delta V_n(t) := V_n(t) - V_{n-1}(t)$ denotes the opportunity costs. The boundary conditions are:

$$V_0(t) = 0,\ 0 \leqslant t \leqslant T, \text{ and (if } T < \infty) \quad V_n(T) = 0,\ 0 \leqslant n \leqslant N. \quad (4)$$

The optimality conditions for the feedback prices $p_n(t)$ are given by, $0 \leqslant t < T$, $n = 1, 2, \ldots, N$:

$$p_n(t) = \frac{\lambda}{\frac{-\partial \lambda}{\partial p}}(t, p_n(t)) + \frac{c(t)}{\nu} + \frac{\Delta V_n(t)}{\nu}. \quad (5)$$

For well-known, special cases with time homogeneous price elasticities, the value function can be determined explicitly (cf. Gallego and van Ryzin, 1994; McAfee and te Velde, 2006, for the exponential case, and McAfee and te Velde, 2008; Helmes and Schlosser, 2013, for the isoelastic case). Unfortunately, in general cases, an analytical solution cannot be found, and numerical or time discrete solution methods must be applied. Instead, we want to identify a general smart heuristic that is based on the solution to the corresponding deterministic model with continuous state space (see the section 'Description of the deterministic model').

Description of the deterministic model

Now, we consider the deterministic version of the dynamic pricing problem on $[0, T]$, where the time horizon T can be finite or infinite, that is, $T \leqslant \infty$, and the initial inventory is N, $0 < N < \infty$. The state space $x \in [0, N]$ is continuous, and the dynamic of the state $x(t)$, that is, the amount still to be sold at time t, is given by:

$$\dot{x}(t) = -\lambda(t, p),\ x(0) = N. \quad (6)$$

Admissible controls are assumed to be non-negative Markovian feedback controls $p(t, x) \in [0, \infty)$, $t \in [0, T)$, $x \in [0, N)$ that depend on time as well as space and are associated with a unique inventory trajectory. Specifically, this means that the differential equation $-\dot{x}(t) = \lambda(t, p(t, x(t)))$, $x(0) = N$, has a unique solution $x(t)$, and the price trajectory $p_t := p(t, x(t))$ as a function of time is piecewise continuous such that the associated profits (see below) can be evaluated. For details, see Lee and Markus (1967). As soon as the inventory process hits zero, the end of sale $\tau := \min_{0 \leqslant t \leqslant T}\{t \mid x(t)=0\} \wedge T$ is attained, and for all $t \in [\tau, T]$, we let $x(t)=0$. We want to identify an admissible feedback control such that the assigned price trajectory $\{p_t | t \in [0, \tau \wedge T]\}$ that, together with the corresponding state trajectory $\{x(t) | t \in [0, T]\}$, determined by (6), maximizes the profit function:

$$\max_{p_t \geq 0} \int_0^{T \wedge \tau} e^{-R(t)} \cdot (\nu \cdot p_t - c(t)) \cdot \lambda(t, p_t) dt \quad (7)$$

and the condition:

$$\int_0^{T \wedge \tau} \lambda(t, p_t) dt \leqslant N \quad (8)$$

for the initial inventory N is satisfied. In (7), a given time-dependent unit cost expression $c(t) \geqslant 0$, $0 \leqslant t \leqslant T$, and a time-dependent discount rate $r(t) \geqslant 0$, where $R(t) := \int_0^t r(s)ds$, are taken into account. The revenue parameter ν is a positive constant. Next, we introduce some regularity conditions for the rate of sales.

Assumption 1: *We assume that the rate of sales (cf. (6)) satisfies the following conditions.*
(A1) *For all $0 \leqslant t \leqslant T$, there is a null price $p_t^{(0)} \leqslant \infty$ such that $\lim_{p \to p_t^{(0)}} \lambda(t, p) = 0$, and for all $0 \leqslant t \leqslant T$, the rate $\lambda(t, p)$ is strictly decreasing in price p, $0 \leqslant p \leqslant p_t^{(0)}$.*
(A2) *For all $0 \leqslant t \leqslant T$, the rate of sales λ is assumed to be differentiable in p, $0 \leqslant p \leqslant p_t^{(0)}$.*
(A3) *For all $0 \leqslant t \leqslant T$, the expression $p + \lambda(t, p)/\lambda'(t, p)$ is strictly increasing in p, $0 \leqslant p \leqslant p_t^{(0)}$.*

Assumptions (A1) and (A2) are common. Assumption (A3) is satisfied if the dynamic λ is of an increasing failure rate (IFR) type and will play a prominent role. In this context, the rate of sales is often defined as $\lambda(t, p) = u(t) \cdot (1 - F(t, p))$, where the positive function $u(t)$ mirrors the arrival intensity of potential buyers, whose reservation prices are characterized by the distribution function F.

Analytical solution to the deterministic model

In this section, for general time-dependent demand intensities satisfying Assumption 1, we want to determine optimal pricing strategies for problems (7)–(8). Since no inventory costs are involved for the time being, we assume that an optimal policy will be one where the full time span is used, that is, $\tau = T$. However, at the end of this section, we will drop this assumption and allow for premature sell-outs.

Using the suboptimal policy $p_t = p_t^{(0)}$, $0 \leqslant t \leqslant T$, nothing will be sold and the optimal revenue (cf. (7)) is bounded from below by zero (cf. (A1)). We also obtain that if $c(t)/\nu \geqslant p_t^{(0)}$, $0 \leqslant t \leqslant T$, a revenue of zero is optimal. Neglecting condition (8), for example, in case of an unlimited initial inventory N, the optimal revenue (7) will not be reduced. In this overage case, the optimal prices are such that at any time t, the revenue rate $e^{-R(t)} \cdot (\nu \cdot p_t - c(t)) \cdot \lambda(t, p_t)$ is maximized. Consequently, the optimal overage prices $p_t^{OC} := \arg\max_{p_t \geqslant c(t)} \{(\nu \cdot p_t - c(t)) \cdot \lambda(t, p_t)\}$, $0 \leqslant t \leqslant T$, do not depend on the discount rate and will satisfy the optimality condition

$$\nu \cdot p_t^{OC} - c(t) = \frac{\lambda(t, p_t^{OC})}{-\lambda'(t, p_t^{OC})} \quad (9)$$

Note, the uniqueness and existence of these overage prices are guaranteed by assumption (A3). Since the associated overage sales rate is given by $\lambda^{OC}(t) := \lambda(t, p_t^{OC})$, $0 \leqslant t \leqslant T$, the total amount sold from time $t = 0$ onward is given by:

$$B(0) := \int_0^T \lambda^{OC}(s) ds \quad (10)$$

and defines a critical inventory level; in other words, if the initial inventory N is sufficiently large, that is $N \geqslant B(0)$, it is optimal to sell exactly $B(0)$ units up to time T using the overage price policy p^{OC}.

In the underage case, that is, if $N < B(0)$, it will be optimal to sell the whole amount N. Since we still assume that it is best to keep selling until $\tau = T$, an optimal sales rate should satisfy the condition

$$\int_0^T \lambda^{UC}(s)\,ds \stackrel{!}{=} N. \tag{11}$$

Moreover, due to the one-to-one correspondence between prices and sales rates (cf. (A1)), for any time t, prices p can be expressed in terms of sales rates λ via the inverse function $Q_t^{-1}(\lambda)$ such that $Q_t^{-1}(\lambda(t,p))=p$. In order to identify an optimal policy for problem (7) subject to (8), we use this bijective relation to set up the following Lagrange approach:

$$\max_{\substack{\lambda_t=\lambda(t,p_t)\\ \kappa_{0,N}\geqslant 0}} \int_0^T e^{-R(t)} \cdot \left(\nu \cdot Q_t^{-1}(\lambda_t) - c(t)\right) \cdot \lambda_t - \kappa_{0,N} \cdot \lambda_t \, dt \tag{12}$$

where $\kappa_{0,N}\geqslant 0$ is the Lagrange multiplier, which is associated with the inventory condition (11). The derivation of the integrand with respect to λ_t (see Appendix) yields the necessary optimality condition, $0<t<T$:

$$e^{-R(t)} \cdot \left(\nu \cdot \frac{\lambda(t,\,p_t)}{\lambda'(t,\,p_t)} + \nu \cdot p_t - c(t)\right) = \kappa_{0,N} \tag{13}$$

which balances the Lagrange multiplier and the discounted value of the sum of the net price and hazard rate for all t. Note, the derivative of the inverse function Q_t^{-1} must be used; afterwards, the relation for λ can be rewritten in terms of price.

As expected, we observe that $\kappa_{0,N}=0$ corresponds to the overage case and its optimal policy $(c(t)/\nu\leqslant)\ p_t^{OC}\leqslant p_t^{(0)}$, $0\leqslant t\leqslant T$. Next, we will show that there is also a unique solution in the underage case.

Lemma 2.1: *Let assumptions (A1)–(A3) be satisfied. There is a unique number $\kappa^{\star}_{0,N}\geqslant 0$ such that the associated price path $p_t(\kappa^{\star}_{0,N})$, $0\leqslant t\leqslant T$, satisfying (13) implies*

$$\int_0^T \lambda\left(t,\, p_t\left(\kappa^{\star}_{0,N}\right)\right) dt = N \text{ for all } N<B(0)$$

Proof: On the basis of the zero-revenue policy $p_t^{(0)}$, we define the parameters $k_t^{\max}:=e^{-R(t)}\cdot(\nu\cdot\lambda(t,p_t^{(0)})/\lambda'(t,p_t^{(0)})+\nu\cdot p_t^{(0)}-c(t))$, $0\leqslant t\leqslant T$ (cf. (13)). When $p_t^{(0)}$ is finite (for example, in the case of linear demand), prices that exceed $p_t^{(0)}$ are assumed to be non-admissible (cf. (A1)). To be able to treat general cases (w.l.o.g.), we consider the following technical modification of (13), $0\leqslant t\leqslant T$:

$$e^{-R(t)} \cdot \left(\nu \cdot \frac{\lambda(t,\,p_t)}{\lambda'(t,\,p_t)} + \nu \cdot p_t - c(t)\right) = k_t^{\max} \wedge \kappa_{0,N} \tag{14}$$

For any given value $\kappa_{0,N}\geqslant 0$, equation (14) implies (cf. (A3)) that for all $0\leqslant t\leqslant T$, the price $p_t=p_t(\kappa_{0,N})$ is uniquely determined. This way, each value $\kappa_{0,N}$ is associated with a specific price path. Note, if $\kappa_{0,N}\geqslant k_t^{\max}$, equation (14) implies $p_t(\kappa_{0,N})=p_t^{(0)}$. On the other hand, if $\kappa_{0,N}\geqslant k_t^{\max}$, we have $p_t(\kappa_{0,N})=p_t^{(0)}$ and positive revenues (cf. (A1)) are possible. Moreover, if $\kappa_{0,N}$ is increasing, the associated prices p_t are (strictly) continuously increasing, and the intensities $\lambda(t,p_t)$, $0\leqslant t\leqslant T$, as well as the number $\int_0^T\lambda(t,p_t)dt\leqslant B(0)$ are (strictly) continuously decreasing. By the Intermediate Value Theorem, it follows that there exists a unique $\kappa^{\star}_{0,N}$, where $\kappa^{\star}_{0,N}>0$, whose associated price path $p_t(\kappa^{\star}_{0,N})$, where $p_t^{OC}\leqslant p_t(\kappa^{\star}_{0,N})\leqslant p_t^{(0)}$, leads to:

$$\int_0^T \lambda\left(t,\,p_t^{OC}\right)dt = B(0) \geqslant \int_0^T \lambda\left(t,\,p_t\left(\kappa^{\star}_{0,N}\right)\right)dt = N \geqslant \int_0^T \lambda\left(t,\,p_t^{(0)}\right)dt = 0$$

□

By replacing the initial state $(0,N)$ with arbitrary states (t,n) (cf. starting at t with $x(t)=n$ units), the argumentation above can be generalized by considering the Lagrange multiplier $\kappa_{t,n}\geqslant 0$ and the critical inventory level $B(t)$.

Theorem 2.1: *Let assumptions (A1)–(A3) be satisfied. A unique solution to problem (7)* *subject to (8) exists and is characterized by the following:*

(i) *The optimal price trajectory (open-loop) in the overage and underage cases is uniquely determined by (5)–(10) and can be expressed as,* $0 \leqslant t \leqslant s \leqslant T$, $0 < n \leqslant N$,

$$\bar{p}(s;\ t,\ n) = \begin{cases} p^{OC}(s) & , n \geqslant B(t) \\ p\left(s; \kappa^{\star}_{t,n}\right) & , n < B(t) \end{cases}$$

(ii) *The optimal price trajectory in feedback form is given by,* $0 \leqslant t \leqslant T$, $0 < n \leqslant N$,

$$p(t,\ n) = \begin{cases} p^{OC}(t) & , n \geqslant B(t) \\ p\left(t; \kappa^{\star}_{t,n}\right) & , n < B(t) \end{cases}$$

(iii) *Starting in state* $(t, n) = (0, N)$, *the optimal inventory path is given by*

$$x^{OC}(t) := N - \int_0^t \lambda^{OC}(s)ds,\ N \geqslant B(0) \text{ and}$$

$$x^{UC}(t) := N - \int_0^t \lambda^{UC}\left(s; \kappa^{\star}_{0,N}\right)ds,\ N < B(0).$$

Remark 2.1: Following the constructive proof of Lemma 2.1, we can use the following simple (numerical) approach to determine the optimal pricing policies. For any state (t, n), we consider a positive number Z, and for all $t \leqslant s \leqslant T$, we put $e^{-R(s)} \cdot (v \cdot \lambda(s, p_s)/\lambda'(s, p_s) + v \cdot p_s - c(s)) \equiv Z$ (cf. (13)–(14)). Then, since for all s, the associated price is implicitly determined, we obtain an entire price path $p_s(Z)$, $t \leqslant s \leqslant T$. Seeking the value $Z^{\star}$ such that $\int_t^T \lambda(s, p_s(Z^{\star}))ds = n$ (cf. (11)) leads to an optimal price path.

In the following, we discuss the possibility that a sell-out before time T can also be optimal. In the overage case (cf. $\kappa_{0,N} = 0$), an additional unit is of no further value; thus, the optimal end of sale is the full time span. In the underage case, a run-out before time T, for example, at $\tilde{T} < T$, is associated with a value $\tilde{\kappa}_{0,N} = \tilde{\kappa}_{0,N}(\tilde{T})$ that is smaller than $\kappa^{\star}_{0,N}$. A Lagrange multiplier that is smaller than $\kappa^{\star}_{0,N}$ implies smaller prices, that is, $p(t; \tilde{\kappa}_{0,N}) < p(t; \kappa^{\star}_{0,N})$. The end of sale $\tau(\tilde{\kappa}_{0,N})$, where $0 \leqslant \tau(\tilde{\kappa}_{0,N}) = \tilde{T} < T$, satisfies $\int_0^{\tau(\tilde{\kappa}_{0,N})} \lambda(t,\ p_t(\tilde{\kappa}_{0,N}))dt = N$. Moreover, the end of sale $\tau(\tilde{\kappa}_{0,N}) := \min_{0 \leqslant t \leqslant T} \left\{ t \middle| \int_0^T \lambda(t,\ p_t(\tilde{\kappa}_{0,N}))dt = N \right\}$ can be defined as a continuous strictly increasing function of $\tilde{\kappa}_{0,N}$. Hence, using the optimality conditions (11) and (14), the dynamic pricing problems (7)–(8) can be translated into a simple maximization problem, where the optimal sell-out time $\tau^{\star} \leqslant T$ can be easily identified:

$$\max_{0 \leqslant \tilde{T} \leqslant T} \int_0^{\tilde{T}} e^{-R(t)} \cdot \left(v \cdot p_t\left(\tilde{\kappa}_{0,N}\left(\tilde{T}\right)\right) - c(t)\right) \cdot \lambda\left(t,\ p_t\left(\tilde{\kappa}_{0,N}\left(\tilde{T}\right)\right)\right)dt$$

Note, via the function $\tau(\tilde{\kappa}_{0,N})$ (see above), each time horizon $\tilde{T}$ is associated with a unique value $\tilde{\kappa}_{0,N}$. Hence, our problem can also be translated to:

$$\max_{0 \leqslant \tilde{\kappa}_{0,N} \leqslant \kappa^{\star}_{0,N}} \int_0^{\tau\left(\tilde{\kappa}_{0,N}\right)} e^{-R(t)} \cdot \left(v \cdot p_t\left(\tilde{\kappa}_{0,N}\right) - c(t)\right) \cdot \lambda\left(t,\ p_t\left(\tilde{\kappa}_{0,N}\right)\right)dt.$$

Further properties of the analytical solution

If the rate of sales and the unit cost function c are differentiable at time t, a further derivation of (13) with respect to time (see Appendix) yields the following:

$$r(t) \cdot \left(\frac{\lambda}{\lambda'} + p - \frac{c}{v}\right) = \left(2 - \frac{\lambda \cdot \lambda''}{\lambda'^2}\right) \cdot \dot{p} - \frac{\dot{c}}{v} + \frac{\dot{\lambda}}{\lambda'} - \frac{\lambda \cdot \dot{\lambda}'}{\lambda'^2} \quad (15)$$

The differential equation (15) yields a necessary optimality condition for the optimal open-loop pricing trajectory and can equivalently be formulated in terms of the price elasticity of demand, that is, $PE_\lambda := p \cdot \lambda'/\lambda$ and $PE_{\lambda'} := p \cdot \lambda''/\lambda'$, that is:

$$r(t) \cdot \left(\left(\frac{1}{PE_\lambda} + 1 \right) \cdot p - \frac{c}{\nu} \right) = \left(2 - \frac{PE_{\lambda'}}{PE_\lambda} \right) \cdot \dot{p} - \frac{\dot{c}}{\nu} + \frac{\dot{\lambda}}{\lambda'} - \frac{\lambda \cdot \dot{\lambda}'}{\lambda'^2}$$

Note, for separable sales rates, that is, $\lambda(t, p) = u(t) \cdot F(p)$, it directly follows that $\dot{\lambda}/\lambda' - \lambda \cdot \dot{\lambda}'/\lambda'^2 = 0$ and these terms in the equation above vanish. Moreover, if unit costs c are equal to zero the equation (15) reduces to the differential equation:

$$\frac{\dot{p}}{p} = r(t) \cdot \frac{1 + 1/PE_\lambda}{2 - PE_{\lambda'}/PE_\lambda}$$

If $r = 0$, we obtain $(2 - PE_{\lambda'}/PE_\lambda) \cdot \dot{p} = \dot{c}/\nu$; in other words, for undiscounted problems with separable sales rates and constant unit costs, the optimal price would also have to be a constant. Gallego and van Ryzin (1994) obtain the same result using time transformation arguments.

Remark 2.2: Using equation (15) allows us to examine in general whether the optimal monopoly prices coincide with (perfect) competition prices, which are characterized by the relation $\dot{p} = r \cdot p - r \cdot c/\nu + \dot{c}/\nu$ (for special cases with isoelastic demand, cf. the results of Stiglitz, 1976; McAfee and te Velde, 2008).

In the following, we examine the relation between our Lagrange approach and the Bellman approach (cf. Bertsekas, 2005). The value function for the best future profits starting from state (t, n) can be evaluated by, $0 \leqslant t \leqslant T$, $0 < n \leqslant N$:

$$V(t, n) = e^{R(t)} \cdot \int_t^T e^{-R(s)} \cdot \left(\nu \cdot p^\star(s; t, n) - c(s) \right) \cdot \lambda^\star(s; t, n)\, ds$$

If λ is assumed to be differentiable in p and t, the value function $V(t, n)$ can be expected to be continuous and differentiable in t and n. Then, the function of optimal future profits (discounted on t) satisfies the Bellman equation, $0 < t < T$, $0 < n \leqslant N$:

$$\dot{V}(t, n) + \sup_{p \geqslant 0} \{ \lambda(t, p) \cdot (\nu \cdot p - c(t) - V'(t, n)) \} = r(t) \cdot V(t, n) \qquad (16)$$

with boundary conditions $V(t, 0) = 0$ for all t, and if T is finite, $V(T, n) = 0$ for all $n \in (0, N]$, cf. (4). The first boundary condition corresponds to the case when nothing is left to sell; the second one ensures that if the end of the sales period is reached, no further profits can be made. Furthermore, equation (16) corresponds to the maximization of the integrand of (7). In this context, the optimality condition

$$\nu \cdot p - c(t) - V'(t, n) = \frac{-\nu \cdot \lambda(t, p)}{\lambda'(t, p)} \qquad (17)$$

derived from (16) corresponds to (13) for generalized initial states (t, n). Rearranging the generalized version of (13) with initial state (t, n) for $s = t$ yields the condition:

$$\nu \cdot p - c(t) - e^{R(t)} \cdot \kappa^\star_{t,n} = \frac{-\nu \cdot \lambda(t, p)}{\lambda'(t, p)}. \qquad (18)$$

Hence, with (17) and (18), we finally obtain the relation, $0 < t < T$, $0 < n \leqslant N$:

$$V'(t, n) = e^{R(t)} \cdot \kappa^\star_{t,n} \geqslant 0 \qquad (19)$$

which demonstrates the relationship between the opportunity cost term V' and the Lagrange multiplier $\kappa^\star_{t,n}$. Both expressions can be used to describe the value of an additional unit of inventory.

In the next section, we will use the results derived in the section 'Stochastic and deterministic dynamic pricing models' to explicitly solve problem (7)–(8) in the case of special demand functions with a time-dependent price impact. The solution formulas allow us to study many practical sales problems, where the demand is typically increasing or decreasing. On the basis of our results, in the section

'Deterministic feedback policies applied to stochastic models' we propose heuristic pricing strategies and describe how they can be used in various industries.

SPECIAL CASES: ISOELASTIC, EXPONENTIAL AND LINEAR DEMANDS

The isoelastic case

In this subsection, we assume isoelasic demand, that is, $\lambda(t,p)=u(t)\cdot p^{-\varepsilon(t)}$, $p\in[0,\infty)$, where $u(t)>0$, $c(t)\geqslant 0$ and $\varepsilon(t)>1$, $0\leqslant t\leqslant T$. Assumptions (A1)–(A3) are satisfied. The critical inventory level is given by the following (cf. (9) and (10)):

$$B(t):=\int_t^T u(s)\cdot\left(\frac{\varepsilon(s)}{\varepsilon(s)-1}\cdot\frac{c(s)}{\nu}\right)^{-\varepsilon(s)}ds \quad (20)$$

Following equation (13), starting in state n at time t, the optimal multiplier $\kappa^{\star}_{t,n}$ is determined by:

$$\int_t^T \lambda\left(s,\, p_s\left(\kappa^{\star}_{t,n}\right)\right)ds=\int_t^T u(s)\cdot\left(\frac{\varepsilon(s)}{\nu\cdot(\varepsilon(s)-1)}\cdot\left(e^{R(s)}\cdot\kappa^{\star}_{t,n}+c(s)\right)\right)^{-\varepsilon(s)}ds\overset{!}{=}n\wedge B(t) \quad (21)$$

(cf. the section 'Analytical solution to the deterministic model'). The next theorem summarizes the solution to the isoelastic case with time-dependent elasticities.

Theorem 3.1: *If* $\lambda(t,p)=u(t)\cdot p^{-\varepsilon(t)}$, *then the optimal prices in feedback form are given by,* $0\leqslant t\leqslant T$, $0<n\leqslant N$,

$$p(t,\,n)=\begin{cases}\frac{\varepsilon(t)}{\varepsilon(t)-1}\cdot\frac{c(t)}{\nu} & ,\, n\geqslant B(t)\\ \frac{\varepsilon(t)}{\nu\cdot(\varepsilon(t)-1)}\cdot\left(e^{R(t)}\cdot\kappa^{\star}_{t,n}+c(t)\right) & ,\, n<B(t)\end{cases} \quad (22)$$

Proof: See the derivation in the section 'Analytical solution to the deterministic model' and equations (20)–(21). □

The value function for the overage and underage cases can be easily computed using the definition given in the section 'Stochastic and deterministic dynamic pricing models'. Furthermore, the sensitivity results and the structural analysis of the impact of the various time-dependent parameters can be studied in detail. In addition, the feedback pricing policy can be applied in time-dependent stochastic models to derive excellent lower bounds for optimal profits (cf. the section 'Deterministic feedback policies applied to stochastic models').

To illustrate our findings, we consider a numerical example with increasing (isoelastic) demand, which is typical for the sale of airline tickets or hotel rooms. Figure 1a depicts the optimal feedback prices for a special parameter constellation with decreasing price elasticities. For different inventory levels, the four curves indicate which price should be chosen at a certain point of time. The price curves are

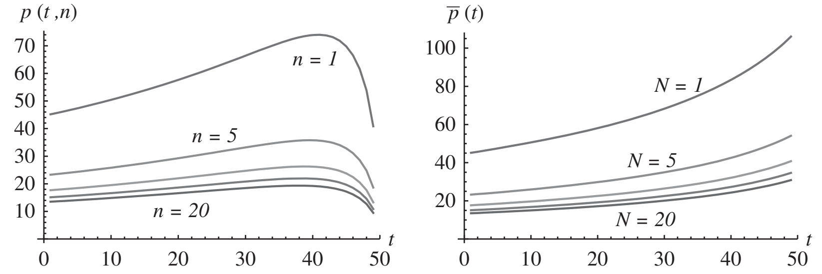

Figure 1: Optimal prices in feedback form (a) and in open-loop form (b); for N and $n\in\{1,5,10,15,20\}$, $T=50$, $\varepsilon(t)=4-2\cdot(t/T)^{1.2}$, $u=2000$, $c(t)=1$, $r=0.01$, $\nu=1$.

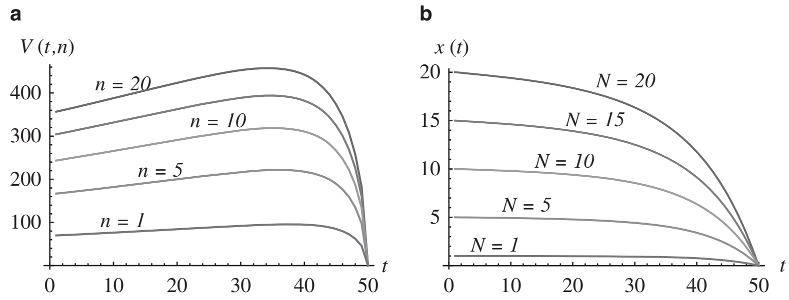

Figure 2: Value function in feedback form (a) and evolution of the inventory level (b); for N and $n \in \{1, 5, 10, 15, 20\}$, $T = 50$, $\varepsilon(t) = 4 - 2 \cdot (t/T)^{1.2}$, $u = 2000$, $c(t) = 1$, $r = 0.01$, $\nu = 1$.

increasing–decreasing; they drop at the end of the horizon. Applying the optimal feedback policy over time finally results in specific price-path evolutions. For different initial inventory levels, some open-loop prices are illustrated in Figure 1b; they are convex increasing.

For the same example in Figure 2a, we see the value function in feedback form; each of the different curves belongs to one inventory level n. Usually, these curves are decreasing; however, if discount rates are positive and demand is increasing with time, curves characterized by unimodal behavior can occur (cf. Figure 2a). The right Figure 2b illustrates how different initial inventory levels are decreasing if the optimal policy is applied. Because of decreasing price elasticity, we have an increasing demand, and thus most of the sales occur at the end.

In the special case where $c(t) \equiv 0$ and price elasticity is constant, $\varepsilon(t) \equiv \varepsilon$, we can determine $\kappa^{\star}_{t,n}$ explicitly. Since the critical inventory level B is infinity, there is only the underage case. The solution formulas for this special case are given in the following lemma.

Lemma 3.1: *Let $\lambda(t, p) = u(t) \cdot p^{-\varepsilon}$, $\varepsilon(t) \equiv \varepsilon > 1$ and $c(t) = 0$. For all $0 \leqslant t \leqslant T$, $0 \leqslant n < B(t) = \infty$, we have the following:*

(i) $\kappa^{\star}_{t,n} = (\nu \cdot (\varepsilon - 1))/(\varepsilon) \cdot ((A^{(0)}(t))/(n))^{1/\varepsilon}$, *where* $A^{(0)}(t) := \int_t^T e^{-\varepsilon \cdot R(s)} \cdot u(s) ds$;

(ii) $p(t, n) = (A^{(0)}(t)/n)^{1/\varepsilon} \cdot e^{R(t)}$ and $V(t, n) = e^{R(t)} \cdot \nu \cdot A^{(0)}(t)^{1/\varepsilon} \cdot n^{1-1/\varepsilon}$.

(iii) *Starting in $x(0) = N$, the optimal inventory path is given by $x(t) = N \cdot A^{(0)}(t)/A^{(0)}(0)$.*

We observe that this special case solution coincides with the one derived in Helmes and Schlosser (2013), Sec. 8. Using the optimal inventory trajectory, the open-loop versions of the feedback solution formulas can be obtained.

In a second special case, where $r(t) \equiv 0$, $c(t) \equiv c$ and price elasticity is constant, $\varepsilon(t) \equiv \varepsilon$, we again can determine $\kappa^{\star}_{t,n}$ explicitly ($T < \infty$). Since c is allowed to be positive, the critical inventory level $B(t)$ is finite, and there are overage (cf. Theorem 3.1) and underage case solutions. For this special case, we obtain the following solution formulas.

Lemma 3.2: *Let $\lambda(t, p) = u(t) \cdot p^{-\varepsilon}$, $\varepsilon > 1$, $r(t) \equiv 0$, $c(t) \equiv c > 0$. For all $0 \leqslant t \leqslant T < \infty$, $0 \leqslant n < \infty$, we have the following:*

(i) $\kappa^{\star}_{t,n} = \nu \cdot (\varepsilon - 1)/\varepsilon \cdot (U(t)/n)^{1/\varepsilon} - c$, $B(t) = ((\varepsilon)/(\varepsilon - 1) \cdot (c)/(\nu))^{-\varepsilon} \cdot U(t)$, *where* $U(t) := \int_t^T u(s) ds$.

(ii) *The optimal underage and overage prices are given by, cf. $n < B(t)$ and $n \geqslant B(t)$,*

$$p^{UC}(t, n) = \left(\frac{U(t)}{n}\right)^{\frac{1}{\varepsilon}} \text{ and}$$

$$p^{OC}(t) = \frac{\varepsilon}{\varepsilon - 1} \cdot \frac{c}{\nu}.$$

The associated rates of sales are

$$\lambda^{UC}(t,\ n) = u(t)\cdot\frac{n}{U(t)} \text{ and } \lambda^{OC}(t) = u(t)\cdot\left(\frac{\varepsilon}{\varepsilon-1}\cdot\frac{c}{v}\right)^{-\varepsilon}.$$

(iii) *The value function amounts to,* $0\leqslant t\leqslant T$, $0\leqslant n\leqslant N$,

$$V(t,\ n) = \begin{cases} \left(\frac{c}{(\varepsilon-1)}\right)^{-\varepsilon+1}\cdot\left(\frac{\varepsilon}{v}\right)^{-\varepsilon}\cdot U(t) & , n\geqslant B(t) \\ n^{1-\frac{1}{\varepsilon}}\cdot U(t)^{\frac{1}{\varepsilon}} - c\cdot n & , \text{else} \end{cases}.$$

(iv) *Starting in* $x(0)=N$*, the optimal inventory path is given by* $x(t)=N\cdot U(t)/U(0)$.

In contrast to many articles (cf. McAfee and te Velde, 2008; Sethi et al, 2008; Helmes and Schlosser, 2013; Helmes et al, 2013; Helmes and Schlosser, 2014), the last lemma particularly allows us to study the isoelastic case including positive marginal unit costs. In the next subsection, we analyze other important special cases of time-dependent exponential demand.

The exponential case

In this subsection, we assume exponential demand, that is, $\lambda(t,p)=u(t)\cdot e^{-\varepsilon(t)\cdot p}$, where $p\in[0,\infty)$, $u(t)>0$ and $\varepsilon(t)>0$, $0\leqslant t\leqslant T$. Assumptions (A1)–(A3) are satisfied. We obtain the following solution. The critical inventory level is given by:

$$B(t) := \int_t^T \frac{u(s)}{e}\cdot e^{\frac{-\varepsilon(s)\cdot c(s)}{v}}ds \tag{23}$$

Following equation (13), starting in state n at time t, the optimal multiplier $\kappa^\star_{t,n}$ is determined by:

$$\int_t^T \lambda\left(s,\ p_s\left(\kappa^\star_{t,n}\right)\right)ds = \int_t^T \frac{u(s)}{e}\cdot e^{\frac{-\varepsilon(s)}{v}\cdot\left(e^{R(s)}\cdot\kappa^\star_{t,n}+c(s)\right)}ds \overset{!}{=} n(t) \tag{24}$$

(cf. the section 'Analytical solution to the deterministic model'). The next theorem summarizes the solution in the exponential case with time-dependent elasticities.

Theorem 3.2: *If* $\lambda(t,p)=u(t)\cdot e^{-\varepsilon(t)\cdot p}$*, then the optimal prices in feedback form are given by,* $0\leqslant t\leqslant T$, $0<n\leqslant N$ (*cf.* (23)–(24)),

$$p(t,\ n) = \begin{cases} \frac{1}{\varepsilon(t)} + \frac{c(t)}{v} & , n\geqslant B(t) \\ \frac{1}{\varepsilon(t)} + \frac{c(t)}{v} + e^{R(t)}\cdot\frac{\kappa^\star_{t,n}}{v} & , n<B(t) \end{cases}. \tag{25}$$

Proof: See the derivation in the section 'Analytical solution to the deterministic model' and equations (23)–(24). □

Note, the optimal feedback prices are characterized by the time-dependent term $1/\varepsilon(t)+c(t)/v$, which is a combination of the hazard rate and the unit costs. The underage price consists of the overage price and the mark-up $e^{R(t)}\cdot\kappa^\star_{t,n}$, which mirrors the opportunity costs.

In the special case where $T<\infty$, $r(t)\equiv 0$ and $\varepsilon(t)\equiv\varepsilon$, we explicitly obtain $\kappa^\star_{t,n}=\varepsilon^{-1}\cdot\ln(B(t)/n)$, $B(t):=e^{-\varepsilon c/v}/e\cdot U(t)$, the underage prices $p^{UC}(t,n)=\varepsilon^{-1}\cdot\ln(U(t)/n)$ and the value function:

$$V(t,\ n) = \begin{cases} B(t)\cdot\frac{v}{\varepsilon} & , n\geqslant B(t) \\ \frac{v}{\varepsilon}\cdot n\cdot\ln\left(e\cdot\frac{B(t)}{n}\right) & , \text{else} \end{cases}$$

Starting in $x(0)=N\leqslant B(0)$, the optimal inventory path is given by $x(t)=N\cdot U(t)/U(0)$. Note, these results extend the solution formulas derived by Berman *et al* (2013) for the special case, $c=0$, $v=1$, $r=0$ and $\varepsilon(t)\equiv\varepsilon$, where static prices are optimal.

The linear case

In this subsection, we assume a linear demand, that is, $\lambda(t,p)=u(t)\cdot(K(t)-\varepsilon(t)\cdot p)$, where $\varepsilon, K>0$ and $0\leqslant p\leqslant K(t)/\varepsilon(t)$ for all $0\leqslant t\leqslant T$. Assumptions (A1)–(A3) are satisfied, and we obtain the following solution. The critical inventory level is given by:

$$B(t) := \int_t^T u(s) \cdot \left(\frac{K(s)}{2} - \varepsilon(s) \cdot \frac{c(s)}{2 \cdot \nu} \right) ds \quad (26)$$

Following equation (13), starting in state n at time t, the optimal multiplier $\kappa^{\star}_{t,n}$ is determined by:

$$\int_t^T \lambda\left(s,\, p_s\left(\kappa^{\star}_{t,n}\right)\right) ds = \int_t^T u(s) \left(\frac{K(s)}{2} - \varepsilon(s) \cdot \frac{e^{R(s)} \cdot \kappa^{\star}_{t,n} + c(s)}{2 \cdot \nu} \right) ds \overset{!}{=} n(t) \quad (27)$$

(cf. the section 'Stochastic and deterministic dynamic pricing models'. The next theorem summarizes the solution in the case of a linear demand.

Theorem 3.3: *If* $\lambda(t,p) = u(t) \cdot (K(t) - \varepsilon(t) \cdot p)$, *then the optimal prices in feedback form are given by,* $0 \leqslant t \leqslant T$, $0 < n \leqslant N$ *(cf.* (26)–(27)):

$$p(t,\, n) = \begin{cases} \frac{K(t)}{2 \cdot \varepsilon(t)} + \frac{c(t)}{2 \cdot \nu} & , n \geqslant B(t) \\ \frac{K(t)}{2 \cdot \varepsilon(t)} + \frac{c(t)}{2 \cdot \nu} + \frac{e^{R(t)} \cdot \kappa^{\star}_{t,n}}{2 \cdot \nu} & , n < B(t) \end{cases} \quad (28)$$

Proof: See the derivation in the section 'Analytical solution to the deterministic model' and (26)–(27). □

In the special case where $T < \infty$ and $r = \dot{K} = \dot{\varepsilon} = \dot{c} \equiv 0$, we explicitly obtain $\kappa^{\star}_{t,n} = \nu \cdot K/\varepsilon - \nu \cdot n/(\varepsilon \cdot U(t)) - c$, $B(t) = (K - \varepsilon \cdot c) \cdot U(t)/2$, and the underage case prices $p^{UC}(t,n) = K/\varepsilon - n/(2 \cdot \varepsilon \cdot U(t))$. Finally, the value function amounts to, $0 \leqslant n \leqslant N$:

$$V(t,\, n) = \begin{cases} \left(\frac{K}{2} - \frac{\varepsilon \cdot c}{2 \cdot \nu}\right)^2 \cdot \frac{\nu}{\varepsilon} \cdot U(t) & , n \geqslant B(t) \\ \left(\frac{\nu \cdot K}{\varepsilon} - \frac{\nu \cdot n}{\varepsilon \cdot U(t)} - c\right) \cdot n & , n < B(t) \end{cases}$$

Starting in $x(0) = N \leqslant B(0)$, the optimal inventory path is given by $x(t) = N \cdot U(t)/U(0)$. Using the optimal inventory trajectory, the open-loop versions of the feedback solution formulas can be easily obtained, and optimally controlled sales processes can be evaluated over time. The solutions provided in this section can also be used to approximate the results for other similar demand functions.

DETERMINISTIC FEEDBACK POLICIES APPLIED TO STOCHASTIC MODELS

In this section, we examine the performance of the policies derived in the section 'The isoelastic case' when applied to a stochastic framework (cf. the section 'Description of the stochastic model'). Furthermore, we compare the expected profits of such suboptimal policies to the results of the optimal policies. In the following, we consider heuristic pricing policies that are characterized by the optimal feedback prices $p^{\star}$ of the deterministic model (cf. Theorem 2.1). Motivated by the fact that the feedback prices of the deterministic model are typically dominated by the optimal prices of the stochastic model (cf. Gallego and van Ryzin, 1994), we use a positive adjustment factor γ in order to improve the $p^{\star}$-heuristic. To determine the profits that can be expected when fixed multiples of the prices $p^{\star}$ are applied to the stochastic environment, we must evaluate the expected profit (cf. (2)):

$$E\left[\int_0^{T \wedge \tau} e^{-R(t)} \cdot \left(\nu \cdot \gamma \cdot p^{\star}(t,\, X_t) - c(t)\right) \cdot \lambda\left(t,\, \gamma \cdot p^{\star}(t,\, X_t)\right) dt \,\middle|\, X_0 = N\right]. \quad (29)$$

Similar to the value function $V_n(t)$, which is associated with the HJB equation (3) with boundary conditions (4) (cf. the section 'Description of the stochastic model'), we can determine the suboptimal expected future profits $\tilde{V}_n(t)$ that are related to (29) by considering the difference–differential equation, $0 \leqslant t < T$, $n = 1, 2, \ldots, N$:

$$\dot{\tilde{V}}^{(\gamma)}_n(t) + \lambda\left(t,\, \gamma \cdot p^{\star}(t,\, n)\right) \cdot \left(\nu \cdot \gamma \cdot p^{\star}(t,\, n) - c(t) - \Delta \tilde{V}^{(\gamma)}_n(t)\right) = r(t) \cdot \tilde{V}^{(\gamma)}_n(t) \quad (30)$$

where the boundaries for $\tilde{V}_n(t)$ correspond to those given in (4).

In the following, we consider the special isoelastic case without marginal unit costs, $c(t)=0$, and jump intensity, $\lambda(t,p)=u(t)\cdot p^{-\varepsilon}$, with constant elasticity, $\varepsilon(t)\equiv\varepsilon$. On the basis of the optimal feedback prices of the deterministic model (cf. Lemma 3.1), we consider the heuristic feedback policy:

$$\tilde{p}_n^{(\gamma)}(t) := p^\star(t,\ n)\cdot\gamma = e^{R(t)}\cdot A^{(0)}(t)^{\frac{1}{\varepsilon}}\cdot n^{\frac{-1}{\varepsilon}}\cdot\gamma = A(t)^{\frac{1}{\varepsilon}}\cdot n^{\frac{-1}{\varepsilon}}\cdot\gamma$$

To determine the expected profits of such policies when applied to the stochastic model by evaluating (30) (see Appendix), we obtain, $0\leqslant t<T,\ n=1,2,\ldots,N$:

$$\dot{\tilde{V}}_n^{(\gamma)}(t)+u(t)\cdot A(t)^{-1}\cdot n\cdot\gamma^{-\varepsilon}\cdot\left(v\cdot A(t)^{\frac{1}{\varepsilon}}\cdot n^{\frac{-1}{\varepsilon}}\cdot\gamma-\Delta\tilde{V}_n^{(\gamma)}(t)\right) = r(t)\cdot\tilde{V}_n^{(\gamma)}(t) \quad (31)$$

To solve this difference–differential equation (31) with boundaries (4), we again try a separable approach, $\tilde{V}_n^{(\gamma)}(t) := e^{R(t)}\cdot A^{(0)}(t)^{1/\varepsilon}\cdot\tilde{\beta}_n^{(\gamma)} = A(t)^{1/\varepsilon}\cdot\tilde{\beta}_n^{(\gamma)}$, where the values for $\tilde{\beta}_n^{(\gamma)}$ must be determined. Using this ansatz for $\tilde{V}_n^{(\gamma)}$, equation (31) becomes:

$$\varepsilon^{-1} = n\cdot\gamma^{-\varepsilon}\cdot\frac{\left(n^{\frac{-1}{\varepsilon}}\cdot\gamma-\Delta\tilde{\beta}_n^{(\gamma)}\right)}{\tilde{\beta}_n^{(\gamma)}}$$

and we obtain the relation (see Appendix):

$$\tilde{\beta}_n^{(\gamma)} = \frac{\varepsilon\cdot n\cdot\gamma^{-\varepsilon}}{\varepsilon\cdot n\cdot\gamma^{-\varepsilon}+1}\cdot\left(n^{\frac{-1}{\varepsilon}}\cdot\gamma+\tilde{\beta}_{n-1}^{(\gamma)}\right) \quad (32)$$

which provides a simple recursion formula for the inventory effect $\tilde{\beta}_n^{(\gamma)}$ starting from $\tilde{\beta}_0^{(\gamma)}=0$. Note, the sequence $\tilde{\beta}_k^{(\gamma)},\ 1\leqslant k\leqslant n$, is determined by the parameter γ; that is for all fixed n, the value $\gamma^\star=\gamma^\star(n)$ can be chosen such that the n-th term $\tilde{\beta}_n^{(\gamma^\star)}$ is maximized.

In order to determine the quality of this suboptimal policy approach, we want to compare the suboptimal expected profits with the optimal ones. If $\lambda(t,p)=u(t)\cdot p^{-\varepsilon}$, the stochastic problem (2) can be solved by considering the Bellman equation:

$$\dot{V}_n(t)+\max_{p>0}\{u(t)\cdot p^{-\varepsilon}\cdot(v\cdot p-c(t)-\Delta V_n(t))\} = r(t)\cdot V_n(t) \quad (33)$$

cf. (3). If $c(t)\equiv 0$ and price elasticity ε is constant, the solution to (33) with boundary conditions (4) is given by (cf. Helmes and Schlosser, 2013):

$$V_n(t) = \beta_n\cdot v\cdot e^{R(t)}\cdot\left(\int_t^T e^{-\varepsilon\cdot R(s)}\cdot u(s)ds\right)^{\frac{1}{\varepsilon}} = v\cdot e^{R(t)}\cdot A^{(0)}(t)^{\frac{1}{\varepsilon}}\cdot\beta_n$$

where the implicit defined sequence β_n is such that $\beta_n\cdot(\beta_n-\beta_{n-1})^{\varepsilon-1}=(1-1/\varepsilon)^{\varepsilon-1}$, $\beta_0=0$. Moreover, the optimal feedback prices for the stochastic problem are given by:

$$p_n(t) = e^{R(t)}\cdot A^{(0)}(t)^{\frac{1}{\varepsilon}}\cdot\beta_n^{\frac{-1}{(\varepsilon-1)}} \quad (34)$$

Using (34), the suboptimal and optimal policies can be compared. In particular, the optimal feedback prices of the stochastic model dominate those of the deterministic model. We summarize our findings in the following proposition.

Proposition 4.1: *Let $\lambda(t,p)=u(t)\cdot p^{-\varepsilon}$ with constant price elasticity $\varepsilon>1$ and $c(t)\equiv 0$.*

(i) *The unique solution $\tilde{V}_n^{(\gamma)}(t)$ of (31) with (4) is given by, $0\leqslant t<T,\ n=1,2,\ldots,N,\ \gamma>0$,*

$$\tilde{V}_n^{(\gamma)}(t) = A(t)^{\frac{1}{\varepsilon}}\cdot\tilde{\beta}_n^{(\gamma)},$$

where $\tilde{\beta}_n^{(\gamma)}$ is recursively determined by (32).

(ii) *For all positive n and $\gamma>0$, the inequality $\tilde{\beta}_n^{(\gamma)}\leqslant\beta_n\leqslant\beta_n^{\det} := n^{(\varepsilon-1)/\varepsilon}$ is satisfied.*

(iii) *Asymptotically, for all $\varepsilon>1$, we have $\lim_{n\to\infty}\beta_n\cdot n^{-(\varepsilon-1)/\varepsilon}=1$.*

Proof: Since $\tilde{V}_n^{(\gamma)}(t)$ cannot exceed the optimal expected profits $V_n(t)$, inequality $\tilde{\beta}_n^{(\gamma)}\leqslant\beta_n$ follows (cf. (ii)). The relation $\beta_n\leqslant\beta_n^{\det}=n^{(\varepsilon-1)/\varepsilon}$ (cf. and Lemma 3.1) is shown by Helmes and Schlosser (2013), and property (iii) is shown by McAfee and te Velde (2008). □

Since all value functions $\tilde{V}_n^{(\gamma)}(t)$, $V_n(t)$, $V(t,n)$ are structurally identical, their differences can be measured by comparing the

Table 1: Comparison of the parameters $\tilde{\beta}_n^{(1)}$, $\tilde{\beta}_n^{(\gamma^\star)}$, β_n, β_n^{det} for different n and $\varepsilon = 2$

n	$\tilde{\beta}_n^{(1)}$	$\tilde{\beta}_n^{(1)}/\beta_n$ (%)	$\tilde{\beta}_n^{(\gamma^\star)}$	$\gamma^\star$	$\tilde{\beta}_n^{(\gamma^\star)}/\beta_n$ (%)	β_n	β_n^{det}	$\beta_n^{\text{det}}/\beta_n$ (%)
1	0.667	94.3	0.707	1.414	100.0	0.707	1.000	141.4
2	1.099	96.1	1.142	1.310	99.8	1.144	1.414	123.6
3	1.437	97.0	1.477	1.256	99.7	1.481	1.732	116.9
5	1.972	97.9	2.007	1.196	99.7	2.013	2.236	111.1
10	2.943	98.8	2.969	1.133	99.7	2.977	3.162	106.2
20	4.295	99.4	4.312	1.088	99.8	4.322	4.472	103.5
50	6.942	99.7	6.951	1.049	99.9	6.960	7.071	101.6
100	9.900	99.9	9.905	1.031	99.9	9.913	10.000	100.9

corresponding inventory β-factors. In Table 1, the different β-values are compared for different inventory levels n up to 100 in case of a price elasticity of $\varepsilon = 2$.

The numbers in Columns 2 and 3 of Table 1 show the very good performance of the (sub-optimal) deterministic feedback policy applied to the stochastic model (cf. $\gamma = 1$), when the stock of articles is large. These results can be improved to more than 99.7 per cent of the optimal expected profits by using mark-up versions of the $\tilde{p}_n^{(1)}$-heuristic, which are characterized by an optimally chosen adjustment factor, $\gamma^\star > 1$ (cf. Column 5). For example, when $n = 50$ items are left to sell, it is advisable to use a 4.9 per cent mark-up on prices $\tilde{p}_n^{(1)}$. Moreover, numerical studies show that the optimal factor $\gamma^\star$ decreases in the number of articles and in the price elasticity of demand ε. When ε is small, we also obtain that the performance of the $\tilde{p}_n^{(\gamma^\star)}$-heuristic is significantly better than the $\tilde{p}_n^{(1)}$-heuristic. For various values ε, the ratio $\tilde{\beta}_n^{(\gamma^\star)}/\beta_n = \tilde{V}_n^{(\gamma^\star)}/V_n$ is constantly very close to one (more than 99 per cent) for both small and large inventories. Hence, the performance of our adjusted heuristic is excellent and robust.

Moreover, we observe that the ratio $\beta_n^{\text{det}}/\beta_n$ exceeds one and is decreasing in the inventory level. These numbers mirror the close relationship between the deterministic and stochastic models when inventory levels are large. When inventory levels are small, our results call for the application of mark-up versions of the feedback policies of the deterministic models.

In the following we describe how simplified versions of our heuristics can be used in real-life applications. We distinguish between small and large inventory levels. When the number of articles to be sold is large the optimal policies of deterministic and stochastic models almost coincide (cf. the convergence of the optimal mark-up factor to 1, see Table 1). Hence, it is suitable to make use of the open-loop solutions of deterministic models. In accordance to our solution formulas the optimal sales path is characterized by the demand intensity which in specific applications is typically decreasing (end of season sales, cf. for example Heching *et al* (2002), Caro and Gallien (2012)) or increasing (travel industry, cf. McAfee and te Velde (2006)). However, in practical applications the permanent adjustment of prices is neither possible nor desirable. For this reason we recommend to use step functions that imitate the shape of optimal price paths. Usually already a small number of price adjustments is enough to attain near optimal results. This way suitable mark-down (skimming) or mark-up (penetration) strategies can be determined and planned in advance; that is the (expected) evolution of sales and the associated cashflows can also be anticipated. Moreover, strategies where prices are fixed over certain periods of time can be easily evaluated for deterministic models. that is the good performance of such strategies can be verified by comparing the associated profits with the optimal ones.

On the other hand, if the number of articles to be sold is small random realizations of single sales can significantly affect optimal prices. We recommend applying pricing strategies that take the realized evolution of sales into account. Our results show that in every state suitable prices can be derived by using the feedback policies of deterministic models. Mark-up versions of these prices might even be optimal. Instead of adjusting the prices in continuous time we suggest to adjust prices only from time to time. A so-called 'relaxed seller' adjusts prices in a steady manner, for instance, once or twice during any period; depending on the application, a period might be a day, a week and so on. Alternatively, it is possible to adjust prices as soon as the actual feedback price exceeds a certain threshold as compared to the last price asked. Note, the solution of the deterministic model boils down to the solution of a non-linear equation, cf. Remark 2.1, and thus can be easily/efficiently implemented in revenue management systems.

The adaptive adjustment of prices using feedback policies has a self-correcting character. Whenever the sales process deviates from its expected path, prices will be adjusted in a suitable way. Note, if these deviations become large, predetermined pricing policies will be inappropriate. Similar effects may arise when demand is significantly over- or underestimated. This may be the case if realized sales are regularly smaller or larger than expected ones. In such cases demand parameter should be adjusted. To sum up, the use of adaptive price adjustments helps to prevent substantial losses since they are robust against both stochastic effects and deviations in the demand.

MODELING TIME-DEPENDENT INVENTORY HOLDING COSTS

In this section, we introduce an approach for internalizing inventory holding costs as well as a salvage value for unsold items (if T is finite). We assume that at time t, each unsold item leads to inventory costs with the time-dependent cost rate $h(t)$ per unit of time, where $h(t)$ is an integrable function on $0 \leqslant t \leqslant T$. Furthermore, at the end of the horizon, the salvage value of each unsold item is S_E, where $S_E \geqslant 0$. While the dynamics of our sales process and the problem formulation (cf. the section 'Description of the deterministic model') remain unchanged, we now consider the generalized profit function (cf. (7)),

$$\max_{p_t \geqslant 0} \int_0^{T \wedge \tau} e^{-R(t)} \cdot \left((\nu \cdot p_t - c(t)) \cdot \lambda(t, p_t) - h(t) \cdot x(t) \right) dt + e^{-R(T)} \cdot S_E \cdot x(T) \quad (35)$$

To determine the value function $V_H(t, n)$ of the generalized model with inventory costs and salvage value, similar to (16), we consider the associated HJB-equation, $0 \leqslant t < T$, $0 < n \leqslant N$:

$$\dot{V}_H(t, n) - h(t) \cdot n + \sup_{p \geq 0} \left\{ \lambda(t, p) \cdot \left(\nu \cdot p - c(t) - V_H'(t, n) \right) \right\} = r(t) \cdot V_H(t, n) \quad (36)$$

with the boundary conditions $V_H(t, 0) = 0$ for all $0 \leqslant t \leqslant T$, and if T is finite, $V_H(T, n) = S_E \cdot n$ for all $0 \leqslant n \leqslant N$. We define the following two functions: L captures the inventory costs for one unit in the interval t to T (discounted on time t); the function S is related to the salvage value S_E. For $0 \leqslant n \leqslant N$ and $0 \leqslant t \leqslant T$, we let:

$$S(t, n) := e^{R(t) - R(T)} \cdot S_E \cdot n \text{ and } L(t, n) := n \cdot H(t),$$

$$\text{where } H(t) := e^{R(t)} \cdot \int_t^T e^{-R(s)} \cdot h(s) ds$$

In the section 'Analytical solution to the deterministic model', we derived optimal solutions for the case without inventory costs (cf. $h = S_E = 0$) and determined the associated value function $V(t, n)$ (cf. the section 'Further properties of the analytical solution'). On the basis of $V(t, n)$, we try $V_H(t, n)$ as the value function of the extended problem with holding costs and salvage value, $0 \leqslant n \leqslant N$, $0 \leqslant t \leqslant T$:

$$V_H(t, n) := V(t, n) - L(t, n) + S(t, n) \quad (37)$$

Since the auxiliary functions L and S are differentiable in t and n, from (37), we obtain:

$$V'_H(t,\ n) = V'(t,\ n) - H(t) + e^{R(t)-R(T)} \cdot S_E$$

and $\dot{V}_H(t,\ n) = \dot{V}(t,\ n) - r(t) \cdot H(t) \cdot n + h(t) \cdot n + r(t) \cdot S(t,\ n)$

Using the relation (37), the HJB-equation (36) is equivalent to, $0 \leqslant t < T$, $0 < n \leqslant N$,

$$\dot{V}(t,\ n) + \sup_{p \geqslant 0}\Big\{\lambda(t,\ p)\cdot\big(v \cdot p - c(t) - V'(t,\ n) + H(t) - e^{R(t)-R(T)} \cdot S_E\big)\Big\} = r(t) \cdot V(t,\ n) \quad (38)$$

Hence, equation (38) coincides with the HJB-equation of the basic model (cf. the section 'Stochastic and deterministic dynamic pricing models') for the case of the (adjusted) unit cost function:

$$\tilde{c}(t) := c(t) - H(t) + e^{R(t)-R(T)} \cdot S_E \quad (39)$$

Thus, we can apply the formulas of Theorem 2.1 for the adjusted cost function $\tilde{c}(t)$, instead of $c(t)$, to determine the function $V(t,\ n) = V(t,\ n;\ \tilde{c})$. Since $V(t, n)$ satisfies the boundary conditions $V(t, 0) = 0$ for all t, and (if T is finite) $V(T, n) = 0$ for all $n \in (0, N]$, we have (cf. (37)):

$$V_H(t,\ 0) := \underbrace{V(t,\ 0)}_{=0} - \underbrace{L(t,\ 0)}_{=0} + 0 \cdot S_E, \\ = 0,\ 0 \leqslant t \leqslant T, \text{ and}$$

$$V_H(T,\ n) := \underbrace{V(T,\ n)}_{=0} - \underbrace{L(T,\ n)}_{=0} + S_E \cdot n \\ = S_E \cdot n \quad ,\ 0 \leqslant n \leqslant N.$$

Hence, the function $V_H(t,\ n;\ c) := V(t,\ n;\ \tilde{c}) - n \cdot H(t) + e^{R(t)-R(T)} \cdot n \cdot S_E$ (cf. (37)) satisfies the HJB-equation (36) as well as the related boundary conditions for $V_H(t, n)$ (see above); that is, $V_H(t, n)$ coincides with the value function of the extended problem.

Following the optimality conditions of (36), which are similar to those given in the section 'Stochastic and deterministic dynamic pricing models' (cf. (16)–(17)), the optimal feedback prices are determined by (cf. (A3)), $0 \leqslant t < T$, $0 < n \leqslant N$:

$$p_H(t,\ n) + \frac{\lambda(t,\ p_H(t,\ n))}{\lambda'(t,\ p_H(t,\ n))} \overset{!}{=} \frac{c(t) + V'_H(t,\ n)}{v} \quad (40)$$

where $c(t) + V'_H(t,\ n;\ c) = \tilde{c}(t) + V'(t,\ n;\ \tilde{c}) = \tilde{c}(t) + e^{-R(t)} \cdot \kappa^{\star}_{t,n}(\tilde{c})$ (cf. (19)). Note, the controls are only admissible if they are positive. Nevertheless, in specific cases, this can easily be checked. A general condition that guarantees optimal prices to be positive is given by, $0 \leqslant t < T$:

$$c(t) + e^{R(t)-R(T)} \cdot S_E > H(t) \quad (41)$$

Condition (41) implies that the optimal solution is valid if the inventory cost rate h is sufficiently small compared to the sum of the salvage value S_E and the unit costs c. Furthermore, the impact of inventory costs is as follows. Similar to the unit costs c, a salvage value leads to a price mark-up. Equation (39) also shows that inventory costs have the same impact as an increasing unit cost function.

In addition to economic insights our results are also of practical use for various industries. The results can be used to determine optimal ordering decisions. Our results allow for selling an initial number of articles over a given period in an optimal way while taking into account discounting, unit costs and holding costs as well as time-dependent demand. Although the possibility of replenishment is not included as part of the model, our results are also applicable to inventory management models. Being able to determine an optimal dynamic pricing policy as well as an optimal sales path between two ordering decisions (cf. the time span from 0 to T) makes it possible to choose the time and the size (N) of an order such that a seller's time-discounted revenue minus ordering costs and holding costs is maximized in the long run. Equation (39) suggests that the impact of inventory holding cost is similar to unit costs that are increasing with time; that is optimal prices (on average) will increase with time.

CONCLUSION

In this article, we analyzed a general class of time-dependent deterministic dynamic pricing models. Using a Lagrangian approach, we showed how to derive the solution of the corresponding profit maximizing problem. Under some natural conditions on the sales intensity, the existence and uniqueness of optimal policies of different classes of such models are guaranteed. We derived the value function of any such problem, that is the present value of future profits, as well as the optimal pricing strategy. It turns out that the solution to the deterministic model is characterized by 'overage' and 'underage' scenarios. While in the underage case it is optimal to sell the whole amount of initial items, in the overage case it is advisable just to sell not more than a certain amount and to retain a positive leftover. For the special cases of isoelastic, exponential and linear demands with time-dependent elasticities, we derived explicit solutions for the optimal prices and for the value function.

Our analytical solutions can be used to study dynamic pricing problems in detail, specifically, the effect of (time-dependent) discount rates, arrival rates, unit costs and price elasticity. Knowing the open-loop solution trajectory of the deterministic model, that is the optimal sales path, makes it possible to evaluate optimally controlled sales processes over time and to derive sensitivity results for several quantities of interest with respect to all model parameters. These results provide economic insight into the quantitative and qualitative effects of the complex interplay between different factors. Normative results based on the analysis of the model suggest that, for instance, the unit costs lead to a mark-up on optimal prices. The evolution of optimal prices is mainly determined by the unit costs, the discount rate and the evolution of the price elasticity of demand. The evolution of sales is synchronous to the potential of remaining customers. Since it is known that if the inventory level is large on average the stochastic model will behave similarly to the deterministic model. Thus the analysis of the deterministic model offers managerial recommendations for stochastic applications.

Besides ordering (initial inventory) decisions or the choice of sales periods, cf. the end of Section 5, particularly the computation of simple but near optimal dynamic pricing strategies is beneficial for decision makers. While it is known that there are asymptotically optimal fixed-price heuristics, these are limited to time-homogeneous models. Whenever inventory stocks are high or where the problem involves inventory positions that greatly exceed demand, fixed price policies guarantee expected revenues that are close to those obtained using dynamic optimal policy. Such heuristics, however, do not perform well when inventory levels are low or when demand is time-dependent. However, in many practical applications inventory levels do not need to be large and, moreover, the sales dynamics are typically time-dependent; for instance, when the sales dynamics are characterized by time-dependent price elasticities of demand. In specific applications the demand, that is the arrival rate of remaining potential customers and their (time-dependent) reservation price distributions can be estimated. In the travel industry or accommodation services the customer's reservation prices typically increase with time, in the fashion industry demand is usually decreasing. For such models, appropriate mark up or mark down heuristics are needed, for example see Bitran and Mondschein (1997) or Valkov (2006).

In this article, we proposed heuristic pricing strategies that can be applied to very general classes of stochastic dynamic sales problems, cf. Section 4. Our heuristics are constructed on the basis of the solution to the corresponding deterministic version of the problem. Our proposed heuristics are preferable to existing heuristics because they take time-dependent demand into account and they can be applied for both small and large inventory stocks.

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APPENDIX

Proof of Equation (13) and (15)

Let $\lambda(t,p)$ be arbitrary with inverse function $Q_t^{-1}(\lambda)=p(t,\lambda)$ for all t; $p\in[0,\infty)$, $\lambda\in[0,\infty)$.

For $T\leqslant\infty$, the problem is $\max_{pt}\int_0^{T\wedge t}e^{-R(t)}\cdot(v\cdot p_t-c(t))\cdot\lambda(t,p_t)dt$.

The constraint for the initial inventory N is $\int_0^T\lambda(t,p(t))dt\leqslant N$.

Using the Lagrange ansatz in terms of the rate of sales λ, we consider the auxiliary problem:

$$\max_{\lambda_t=\lambda(t,p_t)}\int_0^T e^{-R(t)}\cdot\left(v\cdot Q_t^{-1}(\lambda_t)-c(t)\right)\cdot\lambda_t - \kappa_{0,N}\cdot\lambda_t dt$$

Via derivation with respect to $\lambda(t)$, we obtain the necessary optimality condition:

$$e^{-R(t)}\cdot\frac{v}{\lambda'\left(t,\ Q_t^{-1}(\lambda(t))\right)}\cdot\lambda(t)+e^{-R(t)}\cdot v\cdot\underbrace{Q_t^{-1}(\lambda(t))}_{p(t,\lambda(t))}-e^{-R(t)}\cdot c(t)-\kappa_{0,N}=0$$

$$\Leftrightarrow e^{-R(t)}\cdot\left(\frac{\lambda(t,\ p_t)}{\lambda'(t,\ p_t)}+p_t-\frac{c(t)}{v}\right)=\kappa_{0,N/v},$$

cf. (13). Via derivation with respect to time t, we obtain the following relation:

$$\frac{\partial}{\partial t}\left(e^{-R(t)}\cdot\frac{\lambda(t,\,p_t)}{\lambda'(t,\,p_t)}+e^{-R(t)}\cdot\left(p_t-\frac{c(t)}{\nu}\right)-\frac{\kappa_{0,N}}{\nu}\right)=0$$

$$\Leftrightarrow -r(t)\cdot e^{-R(t)}\cdot\left(\frac{\lambda}{\lambda'}+p-\frac{c}{\nu}\right)+e^{-R(t)}\cdot\left(\frac{\dot p}{\nu}\cdot\frac{\lambda}{\lambda'}+\frac{\dot\lambda+\lambda'\cdot\dot p}{\lambda'}-\lambda\cdot\frac{\dot\lambda'+\lambda''\cdot\dot p}{\lambda'^2}+\dot p-\frac{\dot c}{\nu}\right)=0$$

$$\Leftrightarrow r(t)\cdot\left(\frac{\lambda}{\lambda'}+p-\frac{c}{\nu}\right)=\left(\frac{\dot\lambda}{\lambda'}+\dot p-\frac{\lambda\cdot\dot\lambda'}{\lambda'^2}-\frac{\lambda\cdot\lambda''}{\lambda'^2}\cdot\dot p\right)+\dot p-\frac{\dot c}{\nu}$$

$$\Leftrightarrow r(t)\cdot\left(\frac{\lambda}{\lambda'}+p-\frac{c}{\nu}\right)=\left(2-\frac{\lambda\cdot\lambda''}{\lambda'^2}\right)\cdot\dot p+\frac{\dot\lambda}{\lambda'}-\frac{\lambda\cdot\dot\lambda'}{\lambda'^2}-\frac{\dot c}{\nu}$$

The last equation is equivalent to (15).

Proof of Equation (31) and (32)

We consider the suboptimal prices $\tilde p_n^{(\gamma)}(t):=p^\star(t,\,n)\cdot\gamma=e^{R(t)}\cdot A^{(0)}(t)^{1/\varepsilon}\cdot n^{-1/\varepsilon}\cdot\gamma=A(t)^{1/\varepsilon}\cdot n^{-1/\varepsilon}\cdot\gamma$

For the underage case, we try the approach $\tilde V_n^{(\gamma)}(t):=\nu\cdot e^{R(t)}\cdot A^{(0)}(t)^{1/\varepsilon}\cdot\tilde\beta_n^{(\gamma)}=\nu\cdot A(t)^{1/\varepsilon}\cdot\tilde\beta_n^{(\gamma)}$

Note that $A(t):=e^{\varepsilon\cdot R(t)}\cdot\int_t^T e^{-\varepsilon\cdot R(s)}\cdot u(s)\,ds=e^{-\varepsilon\cdot R(t)}\cdot A^{(0)}(t)$ and $\dot A(t):=\varepsilon\cdot r(t)\cdot A(t)-u(t)$.

We face the difference-differential equation, $c(t)=0$, $\varepsilon(t)=\varepsilon>1$, $0\leqslant t\leqslant T$, $0<n\leqslant N$,

$$r(t)\cdot\tilde V_n^{(\gamma)}(t)=\dot{\tilde V}_n^{(\gamma)}(t)+u(t)\cdot\left(p^\star(t,\,n)\cdot\gamma\right)^{-\varepsilon}\cdot\left(\nu\cdot p^\star(t,\,n)\cdot\gamma-\Delta\tilde V_n^{(\gamma)}(t)\right)$$

$$\Leftrightarrow\quad r(t)\cdot\tilde V_n^{(\gamma)}(t)=\dot{\tilde V}_n^{(\gamma)}(t)+u(t)\cdot A(t)^{-1}\cdot n\cdot\gamma^{-\varepsilon}\cdot\left(\nu\cdot A(t)^{\frac{1}{\varepsilon}}\cdot n^{\frac{-1}{\varepsilon}}\cdot\gamma-\Delta\tilde V_n^{(\gamma)}(t)\right)$$

$$\Leftrightarrow\quad r(t)\cdot\nu\cdot A(t)^{\frac{1}{\varepsilon}}\cdot\tilde\beta_n^{(\gamma)}=\nu\cdot\varepsilon^{-1}\cdot\dot A(t)\cdot A(t)^{\frac{1}{\varepsilon-1}}\cdot\tilde\beta_n^{(\gamma)}+u(t)\cdot A(t)^{-1}\cdot n\cdot\gamma^{-\varepsilon}\cdot\left(\nu\cdot A(t)^{\frac{1}{\varepsilon}}\cdot n^{\frac{-1}{\varepsilon}}\cdot\gamma-\nu\cdot A(t)^{\frac{1}{\varepsilon}}\cdot\Delta\tilde\beta_n^{(\gamma)}\right)$$

$$\Leftrightarrow\quad r(t)\cdot A(t)^{\frac{1}{\varepsilon}}=\varepsilon^{-1}\cdot(\varepsilon\cdot r(t)\cdot A(t)-u(t))\cdot A(t)^{\frac{1}{\varepsilon-1}}+u(t)\cdot A(t)^{\frac{1}{\varepsilon-1}}\cdot n\cdot\gamma^{-\varepsilon}\cdot\frac{\left(n^{\frac{-1}{\varepsilon}}\cdot\gamma-\Delta\tilde\beta_n^{(\gamma)}\right)}{\tilde\beta_n^{(\gamma)}}$$

$$\Leftrightarrow\quad r(t)\cdot A(t)=r(t)\cdot A(t)-\varepsilon^{-1}\cdot u(t)+u(t)\cdot n\cdot\gamma^{-\varepsilon}\cdot\frac{\left(n^{\frac{-1}{\varepsilon}}\cdot\gamma-\Delta\tilde\beta_n^{(\gamma)}\right)}{\tilde\beta_n^{(\gamma)}}$$

$$\Leftrightarrow\quad\varepsilon^{-1}=n\cdot\gamma^{-\varepsilon}\cdot\frac{\left(n^{\frac{-1}{\varepsilon}}\cdot\gamma-\Delta\tilde\beta_n^{(\gamma)}\right)}{\tilde\beta_n^{(\gamma)}}$$

$$\Leftrightarrow\quad\tilde\beta_n^{(\gamma)}=\varepsilon\cdot n\cdot\gamma^{-\varepsilon}\cdot\left(n^{\frac{-1}{\varepsilon}}\cdot\gamma-\Delta\tilde\beta_n^{(\gamma)}\right)$$

$$\Leftrightarrow\quad\tilde\beta_n^{(\gamma)}=-\varepsilon\cdot n\cdot\gamma^{-\varepsilon}\cdot\tilde\beta_n^{(\gamma)}+\varepsilon\cdot n\cdot\gamma^{-\varepsilon}\cdot\left(n^{\frac{-1}{\varepsilon}}\cdot\gamma+\tilde\beta_{n-1}^{(\gamma)}\right)$$

$$\Leftrightarrow\quad\tilde\beta_n^{(\gamma)}=\frac{\varepsilon\cdot n\cdot\gamma^{-\varepsilon}}{\varepsilon\cdot n\cdot\gamma^{-\varepsilon}+1}\cdot\left(n^{\frac{-1}{\varepsilon}}\cdot\gamma+\tilde\beta_{n-1}^{(\gamma)}\right)$$

All steps of the calculation are elementary. The last equation is equivalent to (32).