Quantum Game Theoretical Frameworks in Economics

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1 Introduction: Quantum and Quantum-Like Models

Recently the field of "Quantum social science" has emerged (Haven and Khrennikov 2013; Slanina 2014). What sort of entity is hiding behind this term? There is no simple answer to this question. It hardly corresponds to the reduction of human behavior to intermolecular interactions. Roughly speaking, the idea is to use the apparatus developed to describe quantum phenomena to analyse macroscopic complex systems (including living systems). But why? Since its beginning, the development of mathematics has mostly corresponded to practical needs. From ancient times through the Middle Ages mathematical creativity focused on arithmetic and geometry. To some extent, farther development was stimulated by new discoveries in physics. Differential geometry was used for modeling the universe as a whole, probability theory

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helped us to cope with uncertainties, and functional analysis created the mathematical apparatus of quantum theory to describe phenomena in the microworld, cf. Haven and Khrennikov (2016) or Susskind and Friedman (2014). Nowadays these tools are widely used almost everywhere in mathematics, computing, chemistry, and biology. The analysis of human decisions has revealed that the foundations of probability theory and Boolean logic are often violated in the process (Busemeyer and Bruza 2012; Moreira and Wichert 2014). The *sure-thing principle* formulated by Savage (1954) is a well known example. The mathematical apparatus of quantum theory seems to offer a solution to some problems of this kind. The simplest examples come from game theory (Osborne 2003; Osborne and Rubinstein 1994), where a general notion, mixed strategy is widely used. A mixed strategy is an assignment of a probability to each pure (basic) strategy and a random adoption of a pure strategy. In quantum theory, instead of "adding probabilities" one is allowed or even forced to use (normalized) complex linear combinations of states (amplitudes), in which only the squared modulus of such amplitudes defines the probability. This idea is at the root of quantum game theory (Meyer 1999; Eisert et al. 1999; Piotrowski and Sładkowski 2003a): the assembling of probabilities happens at the level of probability amplitudes. This trick is also used in the Fisher information approach in statistics (Frieden 2004). The most interesting fields where this approach can be applied include:

- Pricing of financial instruments. Here, the path integral (Baaquie 2004, 2009; Kleinert 2009) and quantum game theory (Haven 2005; Choustova 2006; Segal and Segal 1998; Piotrowski and Sładkowski 2004) can be used.
- Theory of decisions. Here various important aspects have been approached (Deutsch 2000; Haven and Khrennikov 2009; Asano et al. 2011; Piotrowski and Sładkowski 2003b).
- Risk theory. Here, besides problems related to decision science, the formalism of noncommutative quantum mechanics (QM) can be explored (Piotrowski and Sładkowski 2001).
- Game theory. Here a whole new subfield was developed. Quantum mechanism design seems to be a very promising field of research that has mostly been neglected up to now (Wu 2011a).
- Psychology. Here, various paradoxes can be discussed from a quantum-like point of view (Busemeyer and Bruza 2012); even problems connected with consciousness can be approached (Baaquie 2009; Miakisz et al. 2006).
- Network theory. This is quite a new development with plenty of possible applications.

All this means that the mathematical formalism of QM is not firmly adjoined with quantum physics, but can have a much wider class of applications. We are not able to review all these fields, therefore we focus on quantum game theoretical models in economics and interested readers are referred to other contributions in this volume and more specialized books and references (Haven and Khrennikov 2013; Busemeyer and Bruza 2012; Slanina 2014).

2 Quantum Game Theory

Information processing is a physical phenomenon and therefore information theory is inseparable from both applied and fundamental physics. Attention to the quantum aspects of information processing has revealed new perspectives in computation, cryptography, and communication methods. In numerous cases a quantum description of the system provides some advantages over the classical situation, at least in theory. But does QM offer more subtle mechanisms for playing games? In game theory one often has to consider strategies that are probabilistic mixtures of pure strategies (Osborne 2003; Osborne and Rubinstein 1994). Can they be intertwined in a more complicated way by exploring interference or entanglement? There certainly are situations in which it can be assumed that quantum theory can enlarge the set of possible strategies (Meyer 1999; Eisert et al. 1999; Piotrowski and Sładkowski 2003a). This is a very nontrivial issue as genuine quantum systems usually are unstable and their preparation and maintenance might be difficult, for example due to decoherence, the practically inevitable destructive interactions with the environment. We have already mentioned the astonishing fact that quantum formalism can be used in game theory in a more abstract way without any reference to physical quantum states—the decoherence is not a problem in such cases. The question is whether quantum games are of any practical value. In some sense the answer is positive: commercial cryptographical and communication methods/products are already available. The abstract field of using the quantum apparatus outside physical systems is also appealing. Here we aim at providing a theoretical explanation of decisions or behavior in quantum mechanical terms (Haven and Khrennikov 2013; Busemeyer and Bruza 2012; Slanina 2014; Miakisz et al. 2006).

By a quantum game we usually understand a quantum system that can be manipulated by at least one party and for which the utilities of moves can be reasonably defined. Here we will use the concept of quantum game in a more abstract sense.[1] Therefore, we assume that the analysed system can be with satisfactory accuracy represented by a density operator (matrix) related to a more or less abstract vector space[2] (Haven and Khrennikov 2016). We shall suppose that all players know the state of the game at the beginning and, possibly, at some crucial stages of the actual game being played.[3] We neglect the possible technical problems with actual identification of the state—we will assume that the corresponding structures are definable. Implementation of a genuine quantum game should in addition include measuring apparatuses and information channels that provide necessary information on the state of the game at crucial stages and specify the moment and methods of its termination. We will not discuss these issues here.

We will consider only two-player quantum games: the generalization for the N players case is straightforward. Therefore we will suppose that a two-player quantum game $\Gamma = (\mathcal{H}, \rho_i, S_A, S_B, P_A, P_B)$ is completely specified by the underlying Hilbert space $\mathcal{H}$ of the quantum system, the initial state given by the density matrix $\rho_i \in \mathcal{S}(\mathcal{H})$, where $\mathcal{S}(\mathcal{H})$ is the associated state space, the sets S_A and S_B of quantum operations representing moves (strategies) of the players, and the pay-off (utility) functions P_A and P_B which specify the pay-off for each player after the final measurement performed on the final state ρ_f. A *quantum strategy* $s_A \in S_A$, $s_B \in S_B$ is a collection of admissible quantum operations, that is the mappings of the space of states onto itself. One usually supposes that they are completely positive trace-preserving maps. Schematically we have:

$$\rho_i \mapsto (s_A, s_B) \mapsto \rho_f \mapsto \text{measurement} \Rightarrow (P_A, P_B)\,.$$

This scheme for a quantum two-player game can be implemented as a quantum map:

$$\rho_f = \mathbb{J}^{-1} \circ \mathbb{S} \circ \mathbb{D} \circ \mathbb{J}(\rho_i), \tag{1}$$

where initially

$$\rho_i = |00\rangle\langle 00| \tag{2}$$

[1] Quantum auctions are the only exception, as their implementation seems to be feasible.

[2] Actually a Hilbert space, though this should not bother us at the moment.

[3] Actually one can consider quantum games played against Nature. In such cases the agents might not even be aware of playing the game!

describes identical starting positions of Alice (A) and Bob (B). $\mathbb{J}$ describes the process of creation of entanglement in the system and $\mathbb{D}$ the possible destructive noise effects that will be neglected here. The use of entanglement is one of several possible ways to utilize the power of QM in quantum games. One of the possibilities is that the states of players are transformed using

$$\mathbb{J}(\rho) = J(\gamma)\rho J(\gamma)^{\dagger} \tag{3}$$

with

$$J(\gamma) = \cos(\gamma/2)\mathcal{I} \otimes \mathcal{I} + i\sin(\gamma/2)\sigma_x \otimes \sigma_x \tag{4}$$

into an entangled state. Here $\mathcal{I}$ and σ_x denote the identity operator and the Pauli matrix, respectively. The additional parameter γ describes the possible destructive role of the environment (noise).

Equation (3) has the following explicit matrix form for the initial state given by Eq. (2) (Flitney and Abbott 2003a):

$$\mathbb{J}(\rho_i) = \begin{pmatrix} \cos(\gamma/2)^2 & 0 & 0 & i\cos(\gamma/2)\sin(\gamma/2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i\cos(\gamma/2)\sin(\gamma/2) & 0 & 0 & \sin(\gamma/2)^2 \end{pmatrix}. \tag{5}$$

The individual strategies of players S_X, $X = A(\text{lice}),\ B(\text{ob})$ are implemented as unitary transformations of the form:

$$\mathbb{S}(\rho) = (S_A \otimes S_B)\rho(S_A \otimes S_B)^{\dagger}, \tag{6}$$

where the quantum strategy is realized by unitary transformations. For example, both S_A and S_B can have the general matrix form in the two-dimensional case (Flitney and Abbott 2003a) :

$$U(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix}. \tag{7}$$

The short description of quantum games presented here will be sufficient for our aims. Interested readers are referred to Piotrowski and Sładkowski (2003a) and Flitney and Abbott (2003a) for further clarification and details.

3 Quantum Approach to Risk

Let us begin with an abstract interlude. We will consider the generalization of QM, the so-called "noncommutative" QM. The adjective "noncommutative" reflects the additional assumptions that the operators $\hat{x}^i$ fulfil

$$[\hat{x}^i, \hat{x}^j] = i\,\theta^{ij},\ \ \theta^{ij} \in \mathbb{C}, \tag{1a}$$

$$[\hat{x}^i, \hat{x}^j] = i\,C^{ij}{}_k \hat{x}^k,\ \ C^{ij}{}_k \in \mathbb{C}, \tag{1b}$$

$$\hat{x}^i\hat{x}^j = \mathrm{q}^{-1}\hat{R}^{ij}{}_{kl}\hat{x}^k\hat{x}^l,\ \ \hat{R}^{ij}{}_{kl} \in \mathbb{C}.$$

Labels i, j, k, l take values from 1 to N. The parameters q, θ^{ij}, C^{ij}, and R^{ij} describe the model; their actual values are not important. Suppose that the strategies of agents are given by vectors $|\psi\rangle$ from the corresponding Hilbert space $\mathcal{H}$ (Piotrowski and Sładkowski 2003c). Let the probabilities of signaling of private values for random variables p and q by Alice and Bob using strategies $|\psi\rangle_A$ and $|\psi\rangle_B$ be given by (that is, by the corresponding probability amplitudes after normalization):

$$\frac{|\langle q|\psi\rangle_A|^2}{{}_A\langle\psi|\psi\rangle_A}\,\frac{|\langle p|\psi\rangle_B|^2}{{}_B\langle\psi|\psi\rangle_B}\,dqdp\,, \tag{8}$$

where $\langle q|\psi\rangle_A$ is the probability amplitude of Alice's bid of value q. The reverse position of Bob is represented by the amplitude $\langle p|\psi\rangle_B$ (Bob ask p). Of course, the deal is not always realized. Recall that (Elton et al. 2013; Luenberger 2009):

- In classical error theory second moments of a random variable are related to its "random" errors.
- In Markowitz's portfolio theory variance (σ) "measures" risk.
- In Bachelier's option valuation model the random variable $q^2 + p^2$ "measures" the joint risk associated with the buying-selling process.

Therefore, we are tempted to define the *operator of inclination to risk* as:

$$H(\mathcal{P}_k, \mathcal{Q}_k) := \frac{(\mathcal{P}_k - p_{k0})^2}{2\,m} + \frac{m\,\omega^2(\mathcal{Q}_k - q_{k0})^2}{2}\,,$$

where $p_{k0} := \frac{{}_k\langle\psi|\mathcal{P}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k}$, $q_{k0} := \frac{{}_k\langle\psi|\mathcal{Q}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k}$, $\omega := \frac{2\pi}{\theta}$. θ, roughly speaking, denotes the mean duration of the whole cycle of buying-selling (Piotrowski and Sładkowski 2003d). The parameter $m > 0$ is introduced to describe a possible asymmetry in risk connected with selling and buying. If you browse through any textbook on QM you will discover that the above operator H is the energy operator for harmonic oscillation, a classical issue in physics, if one notices that $\mathcal{P}_k \sim \hat{p}_k$ and $\mathcal{Q}_k \sim \hat{x}_k$ (Haven and Khrennikov 2016). This allows for the following analogy. There exists some constant h_E that characterizes the minimal inclination to taking risk (the minimal energy level in physics).[4] Here, it is equal to the product of the minimal eigenvalue of the operator $H(\mathcal{P}_k, \mathcal{Q}_k)$ and the parameter 2θ. This means that, in our interpretation, 2θ gives the minimal period when it makes sense to calculate profits. Note that, in general, $\mathcal{Q}_k$ do not commute for different k. This should not surprise us: agents observe each other and react accordingly. Any ask or bid influences the market, at least for a short period. This explains why we have used the noncommutative QM instead of the "classical" one. For example, if

$$[x^i, x^k] = i\Theta^{ik} := i\Theta\,\epsilon^{ik}$$

then the results of Hatzinikitas and Smyrnakis (2002) suggest that Θ modifies our "economic Planck constant" $\hbar_E \to \sqrt{\hbar_E^2 + \Theta^2}$ and the eigenvalues of $H(\mathcal{P}_k, \mathcal{Q}_k)$. This implies the obvious conclusion that the activity of agents modifies their attitudes toward risk. Strategies with definite values of risk are given by eigenvectors of $H(\mathcal{P}_k, \mathcal{Q}_k)$. Remember that the minimal value of risk is always greater than zero. Interesting, isn't it?

4 Quantum Approach to Market Phenomena

Let the real random variable $\mathfrak{q}$

$$\mathfrak{q} := \ln \mathfrak{c}_q - E(\ln \mathfrak{c}_q) \tag{9}$$

correspond to the logarithm of the (bid) withdrawal price $\mathfrak{c}_q$, that is the maximal price at which the agent adopting the strategy $|\psi\rangle_k$ is willing to buy

[4] In physics h_E is the Planck constant.

the good $\mathfrak{G}$. We define $\mathfrak{q}$ to ensure that its expectation value in the state $|\psi\rangle_k$ is zero, $E(\mathfrak{q}) = 0$. In turn, the random variable $\mathfrak{p}$

$$\mathfrak{p} := E(\ln \mathfrak{c}_p) - \ln \mathfrak{c}_p \tag{10}$$

corresponds to the analogous situation for the supplier of $\mathfrak{G}$ adopting strategy $|\psi\rangle_k$ (ask). Note that q and p do not depend on the units selected for $\mathfrak{G}$, and we can use units such that $E(\ln \mathfrak{c}) = 0$. Let us consider the general situation of simultaneous trading of an arbitrary amount of goods. The state of the game is given by the vector $|\Psi\rangle_{\text{in}} := \sum_k |\psi\rangle_k$ living in the direct sum of the Hilbert spaces of the agents (Haven and Khrennikov 2016) $\sum_k \oplus \mathcal{H}_k$. Hermitian operators of demand $\mathcal{Q}_k$ and supply $\mathcal{P}_k$ acting on subspaces $\mathcal{H}_k$ form canonically conjugated observables (Susskind and Friedman 2014). We will denote their eigenvalues by q and p respectively. This construction can be validated in the following way. If the unique price e^{-p} (ask p) results from the application of $\mathcal{P}_k$ there is no sense in agent k reporting bids at the same price, and the corresponding operators should not be simultaneously measurable (commuting). The corresponding capital flows are determined according to some algorithm $\mathcal{A}$ representing the clearing house. The transaction is described by the scattering operator $\mathcal{T}_\sigma$ mapping the initial state $|\Psi\rangle_{\text{in}}$ to the final state $|\Psi\rangle_{\text{out}} := \mathcal{T}_\sigma |\Psi\rangle_{\text{in}}$, where

$$\mathcal{T}_\sigma := \sum_{k_d} |q\rangle_{k_d\,k_d}\langle q| + \sum_{k_s} |p\rangle_{k_s\,k_s}\langle p|$$

is a projection operator given by the partition σ of the set of agents k into two disjoint sets $\{k\} = \{k_d\} \cup \{k_s\}$ of agents buying at prices $\mathrm{e}^{q_{k_d}}$ and selling at prices $\mathrm{e}^{-p_{k_s}}$ at this round. The algorithm $\mathcal{A}$ should determine the market partition σ, prices $\{q_{k_d}, p_{k_s}\}$, and the capital flows. Capital flows are fixed according to the probability distributions

$$\int_{-\infty}^{\ln c} \frac{|\langle q|\psi\rangle_k|^2}{{}_k\langle\psi|\psi\rangle_k}\, dq\,, \tag{11}$$

and

$$\int_{-\infty}^{\ln \frac{1}{c}} \frac{|\langle p|\psi\rangle_k|^2}{{}_k\langle\psi|\psi\rangle_k}\, dp \tag{12}$$

giving the probabilities of selling and buying $\mathfrak{G}$ at price c, respectively. These probabilities are conditioned on the partition σ. We can envisage a future market administered by a quantum computer where the above quantum computations can be implemented, though at this moment this is only a theoretical tool. More details and some simulations can be found in Piotrowski and Sładkowski (2002a,b, 2004). There are natural ways of incorporating the subjectivity of decisions to this formalism, cf. Piotrowski and Sładkowski (2009) and Piotrowski et al. (2010).

Another interesting model of a quantum market based on the second quantization method was put forward by Gonçalves and Gonçalves (2007). They introduced a "population number" $n_1, n_2, \ldots, n_m$ for all alternative combinations of strategies. This is implemented by bosonic creation and annihilation operators $a_k^\dagger$ and a_k (Susskind and Friedman 2014). The number of all possible combinations, $m = \prod_k N_k$, is unlimited (N_j is the number of alternative strategies for the jth player). The jth agent strategy profile is $|p_j\rangle = \sum_i c_i |s_i(p_j)\rangle$, where c_i is the probability amplitude of strategy s_i. The unitary evolution of the strategy state $|p_j, t_{\text{fin}}\rangle = U(t_{\text{fin}}, t_{\text{ini}})|p_j, t_{\text{ini}}\rangle$ is governed by a unitary operator of the form

$$U(t_{\text{fin}}, t_{\text{ini}}) = \prod_{k=0}^{k_{\text{fin}}} U(t_{k+1}, t_k),$$

where k parameterizes the $k_{\text{fin}} + 1$ trading rounds. In a simplified single-asset model, where there are only two strategies (buying and selling), for each agent the unitary evolution for the kth trading round can be given in the following form:

$$U(t_{k+1}, t_k) = \exp(\sum_{j=0}^{1}(\xi_j(k, \tau_k)a_j^\dagger - \xi_j(k, \tau_k)^* a_j)),$$

where τ_k is the duration of each trading round, $\xi_j(k, \tau_k) = -i\tau_k \mu_j(k)$ with $\mu_j(k)$ a game-dependent real number that incorporates the dynamics.

5 Quantum Auctions

We now discuss the concept of a quantum auction, its advantages and drawbacks. Quantum auctions are quantum games designed for various goods allocations that one should anticipate. It is well known that for some types of auctions the associated computational issues are difficult to cope with Cramton et al. (2005). There is hope that in future, due to quantum computation speed up, that some of these problems can be overcome. We envisage that the implementation might not be an easy task. Quantum information processing, in principle, can provide tools for secure transmission of bids and asks and their treatment. Such topics have been discussed in the case of sealed-bid auctions (Naseri 2009; Zhao et al. 2010; Liu et al. 2014). Here we would like to focus on more specific issues of using quantum theory for designing the very mechanisms of auctions.[5] We will begin by presenting the general idea of a quantum auction. Then we will suggest methods of gaining an advantage over "classical opponent" and describe some proposals of quantum auction. Farther we will proceed to the quantum mechanism design problems, that is the theory of construction of quantum games with equilibria implementing given social choice rules (Haven and Khrennikov 2016). Finally we will try to show some problems that should be addressed in the near future. In a discussion we will use quantum auction theory as a formal theoretical tool, though widespread opinion is that it seems probable that it will be used in the future for massive combinatorial auctions or in compound securities trading. A genuine quantum bidding language might have to be developed to this end. Encoding bids/asks in quantum states is a challenge to quantum game theory. Quantum auctions would almost always be probabilistic and may provide us with specific incentive mechanisms and so on. As the outcome may depend on amplitudes of quantum strategies, sophisticated apparatus and specialists may be necessary. Therefore, we envisage some changes in the law and in practices. Commercial implementation of quantum auctions is a demanding challenge that cannot be accomplished without a major technological breakthrough in controlling and maintaining quantum systems. Extreme security and privacy are certainly strong points of quantum auctions. Currently, it is difficult to find out if this is a feasible task, but as a theoretical tool it is also very interesting (Piotrowski and Sładkowski 2008; Patel 2007). Quantum auctions are specified by the following data.

- Auctioneer specifies conventional "classical" details of the auction such as the schedule and goods to be sold.
- Auctioneer specifies the implementation of the quantum auction.
- Auctioneer specifies the initial state distribution, implementation of strategies, and main features of the search algorithms to be used (e.g. probabilistic, deterministic).
- Search for the winners and good allocations (this process might be repeated several times).

[5] We call such auctions "genuine quantum auctions."

- Methods of goods delivery and clearing, which is a standard issue.

This scheme is consistent with our definition of a quantum game.

5.1 Examples of Quantum Auctions

In the first model all possible prices of items are encoded in strings of qubits (Hogg et al. 2007). The auctioneer wants to sell n items and to this end distributes to m bidders p qubits initially in state $|o\rangle$ ($p \cdot m$ qubits in total[6]). Each bidder can only operate on his or her qubits and encodes via unitary operation the details of the bid qubits (prices for all bundles of items). Thus each bidder has 2^p possible bid values, and can create superpositions of these bids: for multiple-item auctions the bid is a superposition $\sum_j \alpha_j|\text{bundle}_j\rangle \otimes |\text{price}_j\rangle$ for each bundle of items. A superposition of qubits specifies a set of distinct bids, with at most one allowed to win[7]; amplitudes of the superposition correspond to the likelihood of various outcomes for the auction. The protocol uses a distributed adiabatic search that guarantee that bidders' strategies remain private. The search operation, processing input from the bidders, is implemented by unitary operators, giving the overall operator $U = U_1 \otimes U_2 \otimes \ldots \otimes U_m$, where m is the number of bidders and U_i stands for the operator of the ith bidder. This "brute force" proposal seems to be the easiest to implement and especially suitable for combinatorial auctions.

Piotrowski and Sładkowski discussed an abstract model of bargaining (Piotrowski and Sładkowski 2002a). In their approach a two-dimensional complex Hilbert space is associated with two agents, Alice and Bob. The vectors (qubits) are called polarizations, which are identified with elements of a one-dimensional complex projective space $\mathbb{C}P^1$. On an orthonormal basis $(|0\rangle, |1\rangle)$, $|\xi\rangle = \xi_0|0\rangle + \xi_1|1\rangle \in \mathcal{H}_s$. The scalar product of two vectors $|\xi'\rangle, |\xi''\rangle \in \mathcal{H}_s$ is given by

$$\langle\xi'|\xi''\rangle = \bar{\xi}'_0\xi''_0 + \bar{\xi}'_1\xi''_1 = \bar{\xi} \cdot \xi \ , \tag{13}$$

where $\bar{\xi}_k$, $k = 0, 1$ denotes the complex conjugate of ξ_k. The proportional vectors $|\xi\rangle$ and $t|\xi\rangle$ ($t \in \mathbb{C} \setminus \{0\}$) are identified. The probability of measuring the strategy $|\xi''\rangle$ in the state strategy $|\xi'\rangle$ is given by the squared module of the scalar product (13) of the states. The following interpretation of Alice's

[6]To implement this model additional qubits will be necessary for error correction.

[7]This corresponds to the *XOR* bidding language. This assumption can be relaxed.

polarization state $|\xi\rangle_A \in \mathcal{H}_{sA}$ (that is of her strategy) is proposed. If she formulates the conditions of the transaction we say she has the polarization 1 (and is in the state $|\overrightarrow{r}\rangle_A = |1\rangle$). In quantum bargaining (q-bargaining) this means that she put forward the price. In the opposite case, when she decides whether the transaction is to be made or not, we say she has the polarization $|0\rangle$. (She accepts or not the conditions of the proposed transaction.) The vectors $(|0\rangle, |1\rangle)$ form an orthonormal basis in $\mathcal{H}_{sA}$, the linear hull of all possible Alice polarization states. Bob's polarization is defined in an analogous way. The states of Alice and Bob became entangled if they enter into q-bargaining. The reduction of the state $|\xi\rangle_A$ (Alice) to $|1\rangle_A$ or $|0\rangle_A$ always results in Bob finishing in the state $|0\rangle_B$ or $|1\rangle_B$, respectively. The polarizations of q-bargaining form a two-dimensional complex Hilbert space $\mathcal{H}_s \subset \mathcal{H}_{sA} \bigotimes \mathcal{H}_{sB}$ spanned by two orthonormal vectors $|10\rangle := |1\rangle_A|0\rangle_B$ and $|01\rangle := |0\rangle_A|1\rangle_B$. A market process resulting in q-bargaining is described by a projection $\mathrm{P}_{|1\rangle}$: $\mathcal{H}_{sA} \bigotimes \mathcal{H}_{sB} \to \mathcal{H}_s$. This model of bargaining can be generalized to describe quantum English auctions (Piotrowski and Sładkowski 2003e).

6 Quantum Mechanism Design and Implementation Theory

Mechanism design (reverse game theory) is a field in game theory that studies solution concepts for various classes of games (e.g. private information games) (Hurwicz and Reiter 2006; Narahari 2014). According to Leonid Hurwicz, in a design problem the goal function is the main "given," while the mechanism is the unknown. In that sense the design problem is the "inverse" of traditional game theory, which is typically devoted to the analysis of the performance of an externally given mechanism. One defines a game form as a method for modeling the rules of a game or an institution, independently of the players' utility functions. An n-agent game form $\Gamma = (\mathbf{S}, A, g)$ is defined by a set of n strategy spaces of the players, $S_1, \ldots, S_n$, a set of alternatives A, and an outcome function $g : \mathbf{S} \longrightarrow A$, where $\mathbf{S} = \prod_{i=1}^{i=n} S_i$. An ($n$-agent) mechanism is defined by n agents' message spaces $M_1, \ldots, M_n$, a set of alternatives A, and an outcome function $g : \mathbf{M} \longrightarrow A$, where $\mathbf{M} = \prod_{i=1}^{i=n} M_i$. Shortly, a game form (mechanism) maps profiles of strategies (messages) into feasible outcomes. In contrast, a game as such assigns a profile of payoffs (utilities) to each profile of strategies (messages)! The idea is to use the "invented" mechanism in practice. Implementation theory provides a systematic methodology for designing an information exchange process followed by allocation processes that are "optimal" with respect to some pre-specified performance criteria. It provides analytical frameworks for the analysis and design of allocations among agents in various information contexts e.g. as a Bayesian game). One assumes that agents behave strategically and are self-utility maximizers. Information exchange among the agents might be allowed or even necessary. Let $N = \{1, \ldots, n\}$ denote a finite set of agents, $n \geq 2$, and $A = \{a_1, \ldots, a_k\}$ be a finite set of alternatives. We assume that an agent can have private information encoded as his or her type. Next, let T_i be the finite set of agent i's types, and the private information possessed by agent i is denoted as $t_i \in T_i$. A profile of types $t = (t_1, \ldots, t_n)$ is referred to as a state and $\mathcal{T} = \prod_{i \in N} T_i$ denote the set of states. At state $t \in \mathcal{T}$, each agent $i \in N$ is assumed to have a complete and transitive preference relation $\succeq_i^t$ over the set A. Let $\succeq^t = (\succeq_1^t, \ldots, \succeq_n^t)$ denote the profile of preferences in state t. The utility of agent i for alternative a in state t is $u_i(a, t) : A \times \mathcal{T} \to R$, that is, $u_i(a, t) \geq u_i(b, t)$ if and only if $a \succeq_i^t b$. We denote by $\succ_i^t$ the strict preference part of $\succeq_i^t$. Fixing a state t, we refer to the collection $E =< N, A, (\succeq_i^t)_{i \in N} >$ as an *environment*. Let ϵ be the class of possible environments. A social choice rule F is a mapping $F : \epsilon \to 2^A \backslash \{\emptyset\}$. Finally, a mechanism $\Gamma = ((M_i)_{i \in N}, g)$ consists in prescribing a message (strategy) set M_i for agent i, and an outcome function $g : \prod_{i \in N} M_i \to A$.

The distribution of information among the agents plays the key role in determining their actions, therefore specific implementation should involve an appropriate solution concept (equilibrium), for example Nash equilibrium implementation, Bayesian implementation, and Pareto efficient implementation. If one is trying to design a mechanism to achieve, for example, a Pareto optimal solution, one needs to take into account how individuals are likely to behave if one attempts to implement the mechanism. It can be shown that even in the case of simple voting rules some of the desirable properties which they appear to have if agents vote truthfully may disappear if agents have an incentive to vote strategically (Hurwicz and Reiter 2006). Therefore an important requirement is that of universality: the mechanism should work no matter what the individual preferences happen to be. Maskin provided an almost complete characterization of social choice rules that were Nash implementable. It should come as no surprise for you to learn that one can "design" quantum games and mechanisms; but would they be any good? Haoyang Wu from Wan-Dou-Miao Research Lab, Shanghai was the first who recognized this problem (Wu 2011a, 2013). Following the general model of a quantum game a sequential (multistage) scheme (Moore and Repullo 1988) can be developed (Wu 2011a):

$$\frac{\text{type}}{\text{selection}} \Rightarrow \frac{\text{measurement}}{\text{of coin state}} \Rightarrow \frac{\text{message}}{\text{processing}} \Rightarrow \text{outcome: } g(m)\,.$$

where the agents have "quantum coins" and "classical cards." Each agent independently realizes strategies by a local unitary operation [Eq.(7)] on his or her own quantum coin:

$$\mathbb{J}^{-1} \circ \text{quantum coin} \circ \mathbb{U} \circ \mathbb{J} \to \frac{\text{measurement}}{\text{of coin state}}.$$

As usual, $\mathbb{J}$ creates (annihilates) entanglement. The designer (e.g. auctioneer) receives the overall strategy as the cards and announces the result. Details of the resulting algorithms can be found in the original papers. Note that quantum mechanisms would certainly be probabilistic in nature. Therefore quantum mechanisms are substantially nontrivial, and simple or direct extensions of "classical" mechanisms that do not involve uncertainty are not possible cf. Ieong et al. (2007). If quantum effects (i.e. strategies) are possible, the traditional sufficient conditions of no-go theorems for the implementation of some types of social choice rules may fail (Wu 2011b; Bao and Halpern 2015; Makowski and Piotrowski 2011a).

In mechanism design theory one usually supposes that agents' preferences are transitive. Various simulations show that intransitive preference relations (Makowski 2009; Makowski and Piotrowski 2006, 2011a,b; Makowski et al. 2015) form a key ingredient of quantum mechanisms. This issue certainly deserves further investigation.

7 More Specific Quantum Games

In the previous section we discussed the modeling of whole branches of economics and the social sciences in a quantum game-theoretical setting. There are a whole lot more specific situations that can be successfully described as a game. Most of them can in principle be or have already been "quantized." We have put the word quantized in inverted commas to stress that the quantization of games does not exactly correspond to its physical counterpart. By quantization of a concrete game we mean the construction of such a quantum game that, after usually drastic reduction, strategy sets reproduce the initial (classical) game. In some cases some paradoxes or conundrums can be resolved in that way, but a review of these results is beyond the scope of this text. We only mention some results that we think are representative or interesting. The prisoner's dilemma is one of the flagships of game theory. Its quantum version is usually discussed in the context of cooperation (Eisert et al. 1999; Nawaz 2013) and network games (Pawela and Sładkowski 2013a; Li and Yong

2014). Quantum games on networks, hypernetworks, and cellular automata are being intensively studied as they involve cooperation, coordination, and synchronization problems (Li et al. 2012; Pawela and Sładkowski 2013a; Miszczak et al. 2014; Alonso-Sanz 2012). Another interesting class of problems that can be studied in the quantum setting is related to the famous Parrondo paradox (Flitney and Abbott 2003b; Meyer and Blumer 2002; Pawela and Sładkowski 2013b). New light can be shed on various aspects of Bertrand duopoly analysis (Khan et al. 2013; Lo and Kiang 2004), Cournot duopoly (Sekiguchi et al. 2010), and Stackelberg duopoly (Lo and Kiang 2005; Wang et al. 2013). The ultimatum game studied in experimental economics has also been analysed from the quantum game theory point of view (Frąckiewicz and Sładkowski 2014). The quantum battle of the sexes also became a popular research topic (Frąckiewicz 2009; Nawaz and Toor 2004; Weng and Yu 2014). A lot more can be said, but we have to stop somewhere. We apologize to authors whose works have not been mentioned here.

8 Conclusions

Quantum game theory aspires to be a fruitful theoretical tool in various fields of research. Quantum auctions have potential commercial value, but their implementation is a demanding challenge that would hardly be accomplished without a major theoretical and technological breakthrough. Nevertheless, we envisage the emergence of quantum computational choice theory and related fields (Bisconi et al. 2015). Quantum-like description will remain an important theoretical tool, even if never commercially implemented.

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