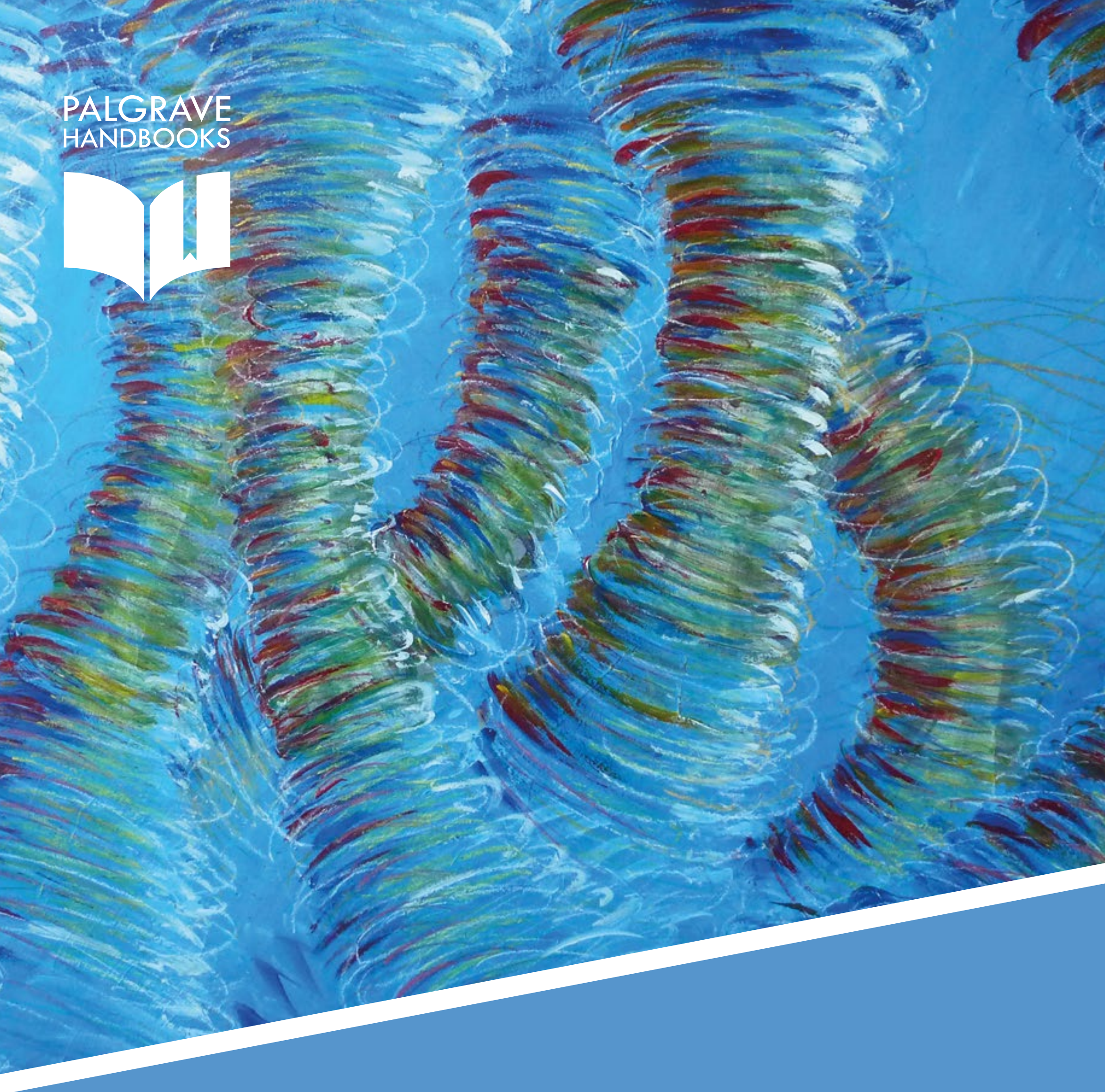

PALGRAVE HANDBOOKS

THE PALGRAVE HANDBOOK OF QUANTUM MODELS IN SOCIAL SCIENCE

Applications and Grand Challenges

Edited by Emmanuel Haven and Andrei Khrennikov

The Palgrave Handbook of Quantum Models in Social Science

Emmanuel Haven • Andrei Khrennikov
Editors

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Applications and Grand Challenges

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Preface

The recent quantum information (QI) revolution in physics not only changes essentially the interpretation of quantum physics by drifting it towards 'physics of information', but it also has a strong impact on the research outside of physics. The ideology, methodology, and the novel mathematical apparatuses of QI found applications in the cognitive and social sciences, psychology, economics, finance, and recently even in political studies. QI teaches us that the quantum wave function has a purely informational nature. Therefore, it is natural to use the formalism of quantum theory to describe the processing of information (first of all the prediction of the results of measurements) not only by quantum physical systems, but by systems of any origin, whether they are biological, social, financial (as long as there is behavior which has (at least some) distinguishing features of quantum systems). This approach to applications of the quantum formalism outside of physics is known as the *quantum-like paradigm*. By this paradigm a researcher applying the methods of quantum theory to, e.g. cognition need not search for quantum physical processes which might lead to the appearance of quantum-like features in behavior. The quantum formalism is treated as an operational formalism describing outputs of possible measurements, including the self-measurements. Of course, this approach does not exclude the possibility that quantum physics is involved in some way into the generation of quantum-like behavior. The latter possibility has been actively speculated in brain studies.[1]

[1]This quantum brain project suffers of a variety of problems, e.g. mismatching of quantum physical scales of space, time, temperature with the corresponding scales of neuronal activity. This mismatching is a subject of a great debate which has been ongoing for the last 20 years, with a variety of arguments for and against (see for more detail the article of A. Khrennikov "Why quantum?" in this handbook). For social

The most fundamental common feature of quantum physical systems and cognitive systems, including collective social systems, is that their behavior is characterized by a deeper uncertainty than uncertainty represented with the aid of the classical probability theory. In QI, such a deep uncertainty is represented by using two basic structures of the mathematical formalism of quantum theory, *superposition* and *entanglement* of states. Therefore, it is natural to explore them in applications outside of physics. These quantum structures will play a crucial role in practically all chapters of this handbook.

This deeper type of uncertainty is mathematically represented by the tools of *quantum probability theory.* This is a calculus of complex probability amplitudes which when squared, yield the probabilities of the results of measurements (this is the famous Born rule connecting the quantum wave function with probability). The laws of quantum probability theory differ fundamentally from the laws of classical probability. For example, we can mention the violation of the law of total probability and Bell's inequality. In cognitive science, psychology, social science and political studies plenty of statistical experimental data has been found which violates the laws of classical probability theory. At the same time, such data can be modeled with the aid of quantum probability by using superposition and entanglement of belief states.

The last 10 years have been characterized by a tremendous increase of research on applications of QI methods to a wide range of problems in cognition and the social sciences. The number of publications has increased dramatically. We point to four key books (van Rijsbergen 2004; Busemeyer and Bruza 2014; Khrennikov 2010; Haven and Khrennikov 2013).

The aim of this handbook is to structure the available information. We believe this will be helpful to both experts and newcomers. Experts from the QI community are looking for novel applications of their methods and 'newcomer' researchers to the QI community, whether they come from physics (especially quantum information), cognition, psychology, economics, and other social sciences may be interested in using novel methodologies and mathematical tools.

In this handbook, we have collected chapters of leading experts on the key topics. The chapters are written in an accessible style, so they can be read by both physicists and non-physicists. For the latter group of readers who are not familiar with QI and the quantum formalism in general, we wrote an introductory chapter entitled 'A Brief Introduction to Quantum Formalism'

systems, the situation is even worse: to explore quantum physics one has to accept 'mental non-locality' with natural links to parapsychology. We do not criticize this project directly and we do not say that this is completely impossible. We just argue that it is possible to proceed without such assumptions.

by E. Haven and A. Khrennikov. It contains all basic notions which will be used in the chapters of this book.

Before we describe the content of this volume in more detail, we make a last general remark. The applications of methods of classical physics in, e.g. psychology and economics have a long history. For example, in economics the methods of classical statistical physics were widely explored in *econophysics* (Mantegna and Stanley 1999). Thus, approaches presented in this handbook could be considered as extensions of such activities by using the methods of quantum physics.

In their contribution "Quantization in Financial Economics: An Information-Theoretic Approach", R. J. Hawkins and B. R. Frieden show how the formal framework of quantum mechanics arises in economics generally and in financial economics, in particular, as a natural consequence of an information-theoretic approach to these fields. Specifically, extremizing Fisher information subject to the constraint that the associated probability density reproduce observed economic phenomena, results in Schrödinger-like equations. This approach is illustrated by examples from financial economics.

The quantum approach to finance is reviewed by E. W. Piotrowski, J. Sladkowski in their chapter "Quantum Game Theoretical Framework in Economics". They start with a brief introduction to quantum game theory and then proceed to concrete applications: i.e. the quantum modeling of risks, quantum financial market, quantum auctions.

The chapter of Y. Pelosse "The Intrinsic Quantum Nature of Classical Game Theory" is devoted to the analysis of the interrelation of classical and quantum game theory and the deep connection to a number of foundational and interpretational problems of quantum mechanics.

An introduction to the general theory of quantum measurements based on quantum instruments is presented in the chapter of I. Basieva and A. Khrennikov "Decision Making and Cognition modeling from the Theory of Mental Instruments" where this theory is applied to describe the updating of belief states in the process of decision making. An even more general approach to this state update problem is presented in the chapter of M. Ohya and Y. Tanaka "Adaptive Dynamics and an Optical Illusion" devoted to the theory of quantum adaptive dynamics (generalizing the theory of open quantum systems) with applications to modeling sensation-perception dynamics in the process of recognition of ambiguous figures.

The chapter by C. Smith and C. Zorn "Strategic Choice in Hilbert Space", discusses the important argument that distance relations between response preferences (for instance) are deemed, in mainstream social science, to virtually always occur in Euclidean space. The chapter explains very well how the use of Hilbert space in decision theory does change this view quite dramatically.

In the chapter of E. Haven and P. Khrennikova "Voters Preferences in a Quantum Framework", the theory of open quantum systems is applied to describe the dynamics of voters' belief state in the process of decision making. This model is applied to model the bi-partisan behavior of voters in the US congressional and presidential elections.

D. Aerts and S. Sozzo contribute to his handbook with the chapter "Quantum Structure in Cognition: Origins, Developments, Successes and Expectations" devoted to the surprising results attained in the last decade on the identification of quantum structures in cognition and, more specifically, in the formalization and representation of natural concepts. The Fock state space is used as the basic quantum tool. The chapter contains a short introduction to the Fock space formalism which is helpful for non-physicists.

Cognition is also modeled in the chapter by J. Acacio de Barros and Gary Oas "Quantum Cognition, Neural Oscillators, and Negative Probabilities". The authors discuss a contextual neurophysiologically plausible model of neural oscillators that reproduces some of the features of quantum cognition. However, at the same time this model predicts contextual situations, where quantum cognition is inadequate. Finally, an extended probability theory based on negative probabilities is explored: it describes situations that are beyond quantum probability but also provides an advantage in terms of contextual decision making.

A. Lambert-Mogiliansky contributed to this Handbook with the chapter entitled "Quantum-like Type Indeterminacy: A Constructive Approach to Preferences à la Kahneman and Tversky." The type-Indeterminacy model proposes to use elements of the quantum formalism to model uncertain preferences. The basic idea is that the Hilbert space model of quantum mechanics can be thought of as a general contextual predictive tool particularly well suited to describing experiments in psychology or in 'revealing' preferences.

The work of J. S. Trueblood and P. K. Mistry "Quantum Models of Human Causal Reasoning" starts with an introduction to classical modeling of causal reasoning; its advances and problems. Causal graphical models (CGMs), based on Bayes' calculus, have perhaps been the most successful at explaining and predicting judgments of causal attribution. However, some recent empirical studies have reported violations of the predictions of these models, such as the local Markov condition. In this handbook the authors suggest an alternative approach to modeling human causal reasoning using quantum Bayes' nets. They show that this approach can account for a variety of behavioral phenomena including order effects, violations of the local Markov condition, anti-discounting behavior and reciprocity.

The chapter of L. C. White, E. M. Pothos, and J. R. Busemeyer "A Quantum Probability Model for the Constructive Influence of Affective Evaluation" is devoted to a concrete application of the theory of quantum decision making. This chapter starts with a brief summary of the research on constructive processes in judgment and decision making and the rationale for using quantum probability to model such processes. A quantum probabilistic model is described for the constructive role of articulating an affective impression and the empirical research that has been undertaken to support the model.

The chapter of P. Pylkkänen "Is there Room in Quantum Ontology for a Genuine Causal Role of Consciousness?" discusses the relationship between so called 'active information' and consciousness. Are there causal powers of consciousness?

The chapter of A. Khrennikov "Why Quantum?" in the section "Big Challenges" of the handbook, is devoted to the justification of the use of the quantum formalism in biology and social science, especially through the use of the theory of open quantum systems. The latter is the most general formalism which describes the interaction of the information state of a system with an environment (physical, mental, social, financial, political). The author also discusses the restrictiveness of the conventional quantum formalism and the possibilities to go beyond quantum theory in cognitive and social science applications. The idea that the wave function of quantum mechanics can be interpreted as an active information field can be found in works of D. Bohm and B. Hiley (2007) who explored such information field both in physics and cognition. Later a similar approach was developed by E. Haven and O. Choustova and A. Khrennikov who used methods of Bohmian mechanics and the information interpretation of the pilot wave in finance. The chapter of P. Pylkkänen presents an excellent introduction to this theory.

The chapter of A. Plotnitsky which also figures in the section "Big Challenges" represents a deep analysis of the possibly necessary role for quantum mathematical models to go beyond physics. This analysis is based on Einstein's distinction between 'constructive' and 'principle' scientific theories. Two types of principle thinking in quantum theory are considered here, the type defining the initial development of quantum mechanics in the 1920s and the type defining quantum information theory, a more recent and still ongoing development.

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Contents

A Brief Introduction to Quantum Formalism 1
Emmanuel Haven and Andrei Khrennikov

Quantization in Financial Economics: An Information-Theoretic Approach 19
Raymond J. Hawkins and B. Roy Frieden

Quantum Game Theoretical Frameworks in Economics 39
Edward W. Piotrowski and Jan Sładkowski

The Intrinsic Quantum Nature of Classical Game Theory 59
Y. Pelosse

Decision-Making and Cognition Modeling from the Theory of Mental Instruments 75
Irina Basieva and Andrei Khrennikov

Adaptive Dynamics and an Optical Illusion 95
Masanori Ohya and Yoshiharu Tanaka

Strategic Choice in Hilbert Space 121
Charles E. Smith Jr. and Christopher Zorn

Voters' Preferences in a Quantum Framework 141
Polina Khrennikova and Emmanuel Haven

Quantum Structure in Cognition Origins, Developments, Successes, and Expectations 157
Diederik Aerts and Sandro Sozzo

Quantum Cognition, Neural Oscillators, and Negative Probabilities 195
J. Acacio de Barros and Gary Oas

Quantum-Like Type Indeterminacy: A Constructive Approach to Preferences à la Kahneman and Tversky 229
A. Lambert-Mogiliansky

Quantum Models of Human Causal Reasoning 251
Jennifer S. Trueblood and Percy K. Mistry

A Quantum Probability Model for the Constructive Influence of Affective Evaluation 267
Lee C. White, Emmanuel M. Pothos, and Jerome R. Busemeyer

Is there Room in Quantum Ontology for a Genuine Causal Role for Consciousness? 293
Paavo Pylkkänen

Big Challenges Section 319

Why Quantum? 321
Andrei Khrennikov

Quantum Principles and Mathematical Models in Physics and Beyond 335
Arkady Plotnitsky

Index 359

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List of Figures

Fig. 1	The potential function for an expected price as a function of the level of the asset underlying the contract together with some of the associated probability amplitudes from Eq. (9)	25
Fig. 2	The function for an expected price and the payoff function for a call option as a function of the level of the asset underlying these contracts, together with a potential formed from these functions and some of the associated probability amplitudes from Eq. (9)	27
Fig. 3	The potential function for a zero-coupon bond together with the associated equilibrium probability amplitude and discount-factor function from Eq. (9)	29
Fig. 1	Schröder Stair is an ambiguous figure which has two different interpretations, "the left part (*L*) is front and the right part (*R*) is back," and its converse. Humans perceive either of them, and the tendency of the perception depends on the rotation angle θ	96
Fig. 2	Optical illusion is affected by memory bias: the subject's perception is shifted in response to the rotation direction of the figure	97
Fig. 3	We calculate the probability for clockwise rotation (B') and counterclockwise rotation (C') with our model. We also show the experimental result of random rotations (A) as a reference	117
Fig. 1	Transitions in Hilbert space	125
Fig. 2	Markovian and Hilbert transition probabilities ($\mu = 5$)	128

Fig. 1 Mach–Zehnder interferometer. A single photon state is emitted from a source (S) and impinges on the first beam splitter (BS), where it has equal probability of being in arm A or B. Upon reflection at mirrors M_A, M_B, the two paths are recombined at the second beam splitter. The probabilities for detection at detectors D_1 and D_2 are dependent upon the phase relation between the two alternatives at the second beam splitter. In the ideal case, the probability for detection at D_1 is unity. The *dashed box* in path B represents the choice of inserting a barrier, thereby changing the phase relationship and thus the detection probabilities 199

Fig. 2 Schematic representation of the SR oscillator model for two possible responses, 1 or 2, represented by the synchronization of the stimulus oscillator O_s with the response oscillators O_{r_1} or O_{r_2}. Each *circle* corresponds to groups of neurons synchronized among themselves, and the *lines* to connections between each group of neurons 211

Fig. 1 Two non-compatible measurements 238

Fig. 1 A CGM of the stomach ache scenario. There are two possible causes of a stomach ache—an unusual dinner or stress. The three variables are represented as a DAG with conditional probability tables 254

Fig. 2 Three different ways to elaborate a common cause structure with an additional variable W 255

Fig. 3 A quantum Bayes net generalization of the stomach ache scenario. The dinner node in the DAG has a thick border to indicate that it is considered before stress. The tables contain probability amplitudes rather than probabilities. These amplitudes were determined from the CGM shown in Fig. 1 257

Fig. 4 The *law of reciprocity* in quantum probability theory. In the *left panel*, $p(x_t|y_t)$ is calculated by a series of two projections. First, ψ is projected onto the y_t subspace (labelled projection 1) and then normalized to yield the output state ψ_{y_t}. Next, the output state is projected onto the x_t subspace (labelled projection 2), and the probability is calculated by squaring the length of the projection (represented by the thick black line). In the *right panel*, the series of projections is reversed to calculate $p(y_t|x_t)$ 262

Fig. 1 Sample adverts and procedure used in White et al. (2014) Experiment 2. *Source*: White et al. (2014) 273
Fig. 2 Projection: the cognitive state vector is projected onto the two-dimensional plane (indicated by the *shaded area*), corresponding to the "unhappy" possibility. The projection is denoted by the *dashed line* and its length *squared* is the probability that the hypothetical person, Bill, will decide he is unhappy. *Source*: White et al. (2014) 275
Fig. 3 Quantum Probability Model for the constructive role of measurement in the present experiments in the PN condition (**a**–**e**) and NP condition (**f**). *Source*: White et al. (2014) 277

List of Tables

Table 1	Figure 1	62
Table 2	Figure 2. Ann's equilibrium state of mind is a density matrix	66
Table 1	(Left) $(\theta_1, \theta_2, \theta_3) = (0, 10, 20)$; (Right) $(\theta_1, \theta_2, \theta_3) = (40, 45, 50)$	108
Table 2	The triple of angles $(0, 10, 20)$	108
Table 3	The triple of angles $(40, 45, 50)$	109
Table 1	The data collected in coincidence measurements on entanglement in concepts (Aerts and Sozzo 2011)	180
Table 2	List of concepts and their respective states for the psychological concept on identity and indistinguishability	186
Table 3	Results of statistical fit for the psychological experiment	187

A Brief Introduction to Quantum Formalism

Emmanuel Haven and Andrei Khrennikov

1 Introduction

In this chapter we briefly present the basic notions of the quantum formalism which will be used in specialized contributions devoted to applications of this formalism to problems of cognition, decision-making, economics, and finance and social science. See also monographs Khrennikov (2010b), Busemeyer and Bruza (2012), and Haven and Khrennikov (2012) for a 'social science' friendly presentation of the formalism of quantum mechanics (QM). For a general and condensed introduction to the field of QM, see Bowman (2008). See also Morrison (1990) and Bransden and Joachain (2000) for more elaborate textbooks.

We also present the standard formalism as it is used in quantum physics and the methodology of its applications outside of physics will be discussed in further chapters. We need to mention that in such applications quantum(-like) observables appear in the form of questions or tasks and quantum(-like) states represent belief states of people and collective social systems.

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It is well known that, although QM is established as an operational formalism, its interpretation remains a complicated foundational problem. The present situation (now about 100 years after the creation of QM) is characterized by a huge diversity of various interpretations. Nowadays the foundational debates are as hot as 100 years ago, see, for example, Jaeger et al. (2011), Accardi et al. (2009). In this chapter we are not able to discuss the problem of interpretations of QM. We proceed by considering its formalism pragmatically, merely as a tool for the calculation of probabilities.

2 States and Observables

In the quantum formalism *observables* are represented by Hermitian Matrices[1] or, in the abstract framework, by Hermitian operators. These operators act in the complex Hilbert space[2] H, that is, a complex linear space endowed with a scalar product denoted as $\langle\psi_1|\psi_2\rangle$.

Let us recall the mathematical definition and properties of a scalar product. The scalar product is a function from the Cartesian product $H \times H$ to the field of complex numbers $\mathbb{C}$; $\psi_1, \psi_2 \to \langle\psi_1|\psi_2\rangle$, having the following properties:

1. Positive definiteness: $\langle\psi|\psi\rangle \geq 0$ with $\langle\psi,\psi\rangle = 0$ if and only if $\psi = 0$.
2. Conjugate symmetry: $\langle\psi_1|\psi_2\rangle = \overline{\langle\psi_2|\psi_1\rangle}$.
3. Linearity with respect to the second argument: $\langle\phi|k_1\psi_1 + k_2\psi_2\rangle = k_1\langle\phi|\psi_1\rangle + k_2\langle\phi|\psi_2\rangle$, where k_1, k_2 are complex numbers.

Each scalar product induces the norm defined as

$$\|\psi\| = \sqrt{\langle\psi|\psi\rangle}.$$

The norm defines the metric (distance) on $H : d(\psi_1, \psi_2) = \|\psi_1 - \psi_2\|$, and hence the metric topology. The Hilbert space is complete with respect to this topology. In quantum information theory, and in applications of the quantum formalism to cognition, decision-making, and economics and social science, researchers typically use only finite dimensional Hilbert spaces as state spaces.

[1]We remind the reader that the matrix $A = (a_{ij})$ is called Hermitian if its elements satisfy the equalities $a_{ij} = \bar{a}_{ji}$, where, for the complex number $z = x + iy$, $\bar{z}$ denotes its complex conjugate, $\bar{z} = x - iy$.

[2]Note that the mathematical formalism of QM is based on complex numbers. However, this is not surprising, since QM appeared as the development of classical wave mechanics (in any event in Schrödinger's approach). Complex numbers play an important role in the mathematical formalism of the latter, e.g., in classical electromagnetism and, in particular, in electro and radio-engineering.

Here the condition of completeness is satisfied automatically. A beginner in quantum theory can ignore the presence of the topological structure and explore just the linear space structure and the geometric structure given by the scalar product.

By fixing in H an orthonormal basis (e_j), that is, $\langle e_i | e_j \rangle = \delta_{ij}$, we represent vectors by their coordinates

$$\psi_1 = (z_1, \ldots, z_n, \ldots), \ \psi_2 = (w_1, \ldots, w_n, \ldots).$$

In the coordinate representation the scalar product has the form

$$\langle \psi_1 | \psi_2 \rangle = \sum_j \bar{z}_j w_j.$$

By using this representation the reader can easily verify the aforementioned properties of the scalar product.

2.1 Pure States

Normalized vectors of H, that is, ψ such that $\langle \psi | \psi \rangle = 1$, represent a special (and the most important) class of states of quantum systems, *pure states*. In fact, a pure state is determined by a normalized vector, up to the phase factor $e^{i\theta}$, $\theta \in [0, 2\pi)$, that is, two vectors ψ_1 and

$$\psi_2 = e^{i\theta} \psi_1 \tag{1}$$

determine the same pure state. Thus, rigorously a pure state is an equivalent class of such vectors: $\psi_1 \sim \psi_2$ if (1) holds. However, it is convenient to work *not* with equivalent classes of vectors, but with concrete vectors representing such classes. Therefore the reader can simply identify pure states with their representatives, normalized vectors.

Note that each pure state (vector) can be represented as a Hermitian operator. For a pure state $|\psi\rangle$, we set $P_\psi = |\psi\rangle\langle\psi|$, the orthogonal projector on this vector, $P_\psi \phi = \langle \psi | \phi \rangle \psi$. Here $|\psi\rangle\langle\psi|$ is the Dirac notation for the projection operator P_ψ, see Sect. 5 for more details.

By fixing an orthonormal basis in H, the projector corresponding to a pure state is expressed by a matrix $\rho = (\rho_{ij})$ satisfying the following conditions:

(a) Hermitian: $\rho_{ij} = \bar{\rho}_{ij}$, in particular, the diagonal elements are real,
(b) Positive definiteness: $\langle \rho\phi, \phi \rangle \geq 0$ for any vector ϕ,

(c) Its trace equals 1: $\mathrm{tr}\rho = \sum_j \rho_{jj} = 1$.
(d) Its square equals to itself:

$$\rho^2 = \rho. \tag{2}$$

The eigenvalues of a Hermitian matrix are real numbers. In addition, a Hermitian matrix with positive definiteness has non-negative eigenvalues. Finally, from condition (c), the sum of all the eigenvalues equals 1.

As an example in the two-dimensional space $\mathbb{C}^2$, we introduce the operator given by the following matrix

$$\rho_{\text{qubit}} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \bar{\alpha}\beta \\ \alpha\bar{\beta} & |\beta|^2 \end{pmatrix} \tag{3}$$

for the corresponding pure state vector

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Here α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. The above ρ_{qubit} expresses the state of a *quantum bit* (called *qubit* hereafter), which is often seen in quantum information theory (Nielsen and Chuang 2000).

2.2 Mixed States

The next step in the development of the notion of the quantum state consists in proceeding without the constraint (2), that is, considering all possible matrices satisfying conditions (a)–(c) as representing quantum states. They are called *density matrices* and they represent the most general states of quantum systems. In the abstract framework one considers operators satisfying conditions (a)–(c) as density operators.

Each density operator can be written as a weighted sum of projection operators corresponding to pure states. If such a sum contains more than one element, then the state represented by this density operator is called a *mixed state*: the mixture of pure states with some weights.[3]

[3]Although this terminology is widely used, it is ambiguous. The representation of a density operator as a weighted sum of projectors corresponding to pure states is not unique. Thus, by using the terminology 'mixed state' one has to take into account this non-uniqueness.

3 Quantum Probability

Consider a state represented by a density operator ρ and an observable represented by a Hermitian operator $A = \sum_i a_i P_{a_i}$, where (a_i) are its eigenvalues and (P_{a_i}) are projectors onto corresponding eigen-subspaces. We note that in the finite dimensional case any Hermitian operator can be represented in this form.

Quantum theory indicates that the probability of obtaining the concrete value a_i as the result of measurement[4] is given by *Born's rule*:

$$p_\rho(a_i) \equiv p_\rho(P_{a_i}) = \mathrm{Tr}\rho P_{a_i}. \tag{4}$$

In particular, if $\rho = |\psi\rangle\langle\psi|$ is a pure state, then $p_\rho(a_i) = \langle P_{a_i}\psi|\psi\rangle = \|P_{a_i}\psi\|^2$.

Suppose now that after the measurement of the A-observable, one plans to perform a measurement of another observable B represented by a Hermitian operator $B = \sum_i b_i Q_{b_i}$, where Q_{b_i} are orthogonal projectors onto the eigen-subspaces corresponding to the eigenvalues b_i. Then one needs to know not only the result of the A-measurement, but even the output state ρ_{a_i}. This state is determined by the *projection postulate*[5]

$$\rho_{a_i} = \frac{P_{a_i}\rho P_{a_i}}{\mathrm{Tr}P_{a_i}\rho P_{a_i}}. \tag{5}$$

For the B-measurement following the A-measurement, this state plays the same role as the state ρ played for the A-measurement. In particular, by applying Born's rule once again we obtain:

$$p_{\rho_{a_i}}(b_j) \equiv p_{\rho_{a_i}}(Q_{b_j}) = \mathrm{Tr}\rho_{a_i} Q_{b_j} = \frac{\mathrm{Tr}P_{a_i}\rho P_{a_i} Q_{b_j}}{\mathrm{Tr}P_{a_i}\rho P_{a_i}}. \tag{6}$$

In quantum theory this probability is treated as the *conditional probability* $p_\rho(Q_{b_j}|P_{a_i}) \equiv p_\rho(B = b_j|A = a_i)$. The rule (6) can be considered as a quantum analog of the Bayes formula for classical conditional probability.

[4]By the spectral postulate of quantum theory, only eigenvalues can be obtained as measurement outputs.

[5]This postulate is also known as the Lüders postulate, which generalizes the von Neumann projection postulate which was formulated in this form only for observables with nondegenerate spectra.

3.1 Superposition, Measurement, and "State Collapse"

Consider a pure state ψ and an observable A. Suppose that A has a nondegenerate spectrum and denote its eigenvalues by $a_1, \ldots, a_m$ and the corresponding eigenvectors by $e_1, \ldots, e_m$ (here $a_i \neq a_j, i \neq j$.). This is an orthonormal basis in H. We expand the vector ψ on this basis:

$$\psi = c_1 e_1 + \cdots + c_m e_m, \tag{7}$$

where (c_j) are complex numbers such that

$$|c_1|^2 + \cdots + |c_m|^2 = 1. \tag{8}$$

By using the terminology of linear algebra we say that the pure state ψ is a *superposition* of the pure states e_j.

The density matrix corresponding to ψ has the elements

$$\rho_{ij} = c_i \bar{c}_j. \tag{9}$$

Hence, for the pure state ψ, the basic probabilistic postulate of QM, Born's rule, has the form

$$p(a_j) = \rho_{jj} = c_j \bar{c}_j = |c_j|^2. \tag{10}$$

This postulate can be written without using the coordinates of the state vector ψ on the basis of the eigenvectors of a quantum observable. Note that, since the basis of the eigenvectors of a Hermitian operator can always be selected as orthonormal, the coordinates c_j can be expressed in the form: $c_j = \langle \psi, e_j \rangle$. Hence, Born's rule takes the form:

$$p(a_j) = |\langle \psi, e_j \rangle|^2. \tag{11}$$

In this case (of an observer with a nondegenerate spectrum and a pure state) the projection postulate can be formulated as follows. As the result of a measurement of the A-observable with the result a_i, the superposition (7) is reduced to the basis vector e_i corresponding to this eigenvalue. This procedure can be interpreted in the following way:

> Superposition (7) encodes uncertainty in the results of measurements for the observable A. Before measurement a quantum system "does not know

how it will answer the question A." The mathematical expression (10) encodes potentialities for different answers. Thus, a quantum system in the superposition state ψ does not have any value of A as its objective property. After a measurement, the superposition is reduced to just one term in the expansion (7) corresponding to the value of A obtained in the process of measurement.

This process is often called the *quantum state collapse*.[6]

4 Heisenberg's Uncertainty Principle

In general quantum observables (represented by Hermitian operators) do not commute. This *noncommutativity* is one of the most distinguishing mathematical features of the quantum formalism. Physically it is expressed in various phenomena. The uncertainty principle is one of its basic expressions. This principle states that there is a limit on the precision with which some of the physical parameters of one system can be known simultaneously. This limit does not depend on the precision of the measurement devices used or on the level of technology; it is fundamental. The uncertainty principle was formulated for the first time by Werner Heisenberg in 1927. He pointed out that the more precisely the position of a particle can be measured, the less precisely its momentum can be determined. In 1928 the principle was stated as an inequality:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2},$$

where σ_x is the standard deviation of position, σ_p is the standard deviation of momentum, and $\hbar$ is the reduced Planck constant, $\hbar = h/2\pi$.

The most general form of the uncertainty relation is given by *the Schrödinger inequality:*

$$\sigma_A^2 \sigma_B^2 \geq \left| \frac{1}{2} \langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle \right|^2 + \left| \frac{1}{2i} \langle [A, B] \rangle \right|^2, \tag{12}$$

[6] Although, mathematically, the quantum state collapse is well defined, its proper physical interpretation is a complicated problem. There are two main viewpoints to the projection postulate: (a) this is simply the update of information about the quantum state (which is also interpreted as an information state); (b) this is a physical event happening with the state of a quantum system (which is also interpreted as a physical state). In this introductory chapter we try to avoid discussions about the foundational issues: we proceed formally with the mathematical definition of the state-projection.

where $[A, B] = AB - BA$ is the commutator operator and $\{A, B\} = AB + BA$ is the anti-commutator operator.

The derivation of the inequality is, in fact, reduced to the use of *the Cauchy–Bunyakovsky–Schwartz inequality for the scalar product*:

$$|\langle f|g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle. \tag{13}$$

In QM the position operator is defined by the formula:

$$(\hat{x}\psi)(x) = x \cdot \psi(x) \tag{14}$$

(the multiplication operator by the variable x) and the momentum operator is defined by

$$(\hat{p}\psi)(x) = \frac{\hbar}{i}\frac{d\psi}{dx}(x). \tag{15}$$

These operators act in the Hilbert space $H = L_2$ of complex valued square integrable functions, ψ (for a one-dimensional model) such that

$$\|\psi\|^2 = \int |\psi(x)|^2 dx < \infty,$$

with the scalar product

$$\langle\psi_1|\psi_2\rangle = \int \overline{\psi}_1(x)\psi_2(x)dx.$$

Note that the L_2-space is infinite-dimensional, so the real arena of quantum physics is the infinite-dimensional state space. Finite-dimensional state spaces used in quantum information are simply approximations of this infinite-dimensional state space. This point has to be taken into account even in the applications of the quantum formalism outside of physics. In such applications we typically proceed with finite-dimensional state spaces, for example, the belief-state space. We have to remember that this is not the complete state representation, but just its simplification. Of course, it is mathematically difficult to work in infinite-dimensional Hilbert spaces. For example, the operators $\hat{x}$ and $\hat{p}$ are not continuous and they are not defined on the whole L_2-space, but only on its dense subspaces. However, physicists typically totally ignore these mathematical difficulties and proceed by manipulating with formal mathematical symbols. This way of operation can also be recommended to beginners in quantum theory working in applications outside of physics.

The formulas (14), (15) give the rules of *Schrödinger's quantization* establishing the correspondence between the observables of classical mechanics and QM. We recall that in classical mechanics observables are mathematically represented by functions on the phase space with the coordinates (x, p), where x and p are the position and the momentum of a classical particle, respectively. For example, the position and momentum observables are represented simply as $f_x(x, p) = x$ and $f_p(x, p) = p$, that is, by the coordinate functions on the phase space. The energy of a classical particle is represented by the function $\mathcal{H}(x, p) = \frac{p^2}{2m} + V(x)$, where m is the mass of the particle and $V(x)$ is the potential function determining the classical force $F(x) = -\frac{dF}{dx}(x)$. Schrödinger proposed to quantize classical observables, functions $f(x, p)$, by using the rule

$$f(x, p) \to f(\hat{x}, \hat{p}), \tag{16}$$

where the operators $\hat{x}, \hat{p}$ are determined by the rules (14), (15). For example, the energy function on the phase space (the *Hamiltonian function*) is quantized as $\frac{\hat{p}^2}{2m} + V(x)$, where $V(x)$ is the multiplication operator by the potential function, $\psi(x) \to V(x)\psi(x)$. However, if a classical observable $f(x, p)$ contains the product of the phase space coordinates, for example, $f(x, p) = xp$, then the rules of Schrödinger's quantization do not determine a quantum observable uniquely, for example, you can quantize xp as $\hat{f}_1 = \hat{p}\hat{x}$, or $\hat{f}_2 = \hat{x}\hat{p}$, or $\hat{f}_3 = (\hat{p}\hat{x} + \hat{x}\hat{p})/2$. Typically the latter is the most convenient, but others are also in use. Note that there is a deep connection between Schrödinger's quantization and the theory of (in general infinite-dimensional) *pseudo-differential operators.*

Note that, as is the case with the majority of the basic elements of quantum theory, the rules of Schrödinger's quantization were simply postulated, that is, they are not derivable from heuristic principles. Nevertheless, Schrödinger's quantization plays the fundamental role in QM, since it provides the possibility to construct quantum observables from classical observables. The absence of so to say classical mental (or social) mechanics makes it impossible to apply Schrödinger's quantization in cognition, psychology, and sociology. This is an important unsolved problem.

Note that the *Robertson inequality* is a trivial consequence of the Schrödinger inequality (12):

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4} |\langle [A, B] \rangle|^2.$$

Finally, we note that the commutator of the operators of position and momentum is given by

$$[\hat{x}, \hat{p}]\psi(x) = (\hat{x}\hat{p} - \hat{p}\hat{x})\psi(x) = x\frac{\hbar}{i}\frac{d\psi(x)}{dx} - \frac{\hbar}{i}\frac{dx\psi(x)}{dx} = i\hbar I\psi(x),$$

where I is the identity operator. Then in this case the Robertson inequality becomes the Heisenberg inequality:

$$\sigma_A^2\sigma_B^2 \geq \frac{1}{4}\hbar^2.$$

Note that the Heisenberg uncertainty relation (i.e., noncommutativity of the mathematical representation of quantum observables) plays the crucial role in the formulation of the *principle of complementarity* by Bohr.

The noncommutativity of quantum observables is also responsible for special features of *quantum logic* (see Sect. 9 for an introduction) which differs crucially from classical *Boolean logic.* These differences are important for the novel approach in the modeling of cognitive and social phenomena presented in this handbook.

5 Dirac's Notation

Dirac's notation is widely used in quantum information theory. Vectors of H are called *ket-vectors* and are denoted as $|\psi\rangle$. To be completely in accordance with this notation the above formula for the projector P_ψ has to be written as

$$P_\psi|\phi\rangle = \langle\psi|\phi\rangle|\psi\rangle, \tag{17}$$

that is, formally $(|\psi\rangle\langle\psi|)|\phi\rangle = \langle\psi|\phi\rangle|\psi\rangle$. However, often states are denoted simply by letters such as ϕ, ψ and we use both symbols ψ and $|\psi\rangle$. The elements of the dual space H' of H, the space of linear continuous functionals on H, are called *bra-vectors*, which are denoted as $\langle\psi|$. Originally the expression $\langle\psi|\phi\rangle$ was used by Dirac for the duality form between H' and H, that is, $\langle\psi|\phi\rangle$ is the result of the application of the linear functional $\langle\psi|$ to the vector $|\phi\rangle$. In mathematical notation it can be written as follows. Denote the functional $\langle\psi|$ by f and the vector $|\phi\rangle$ simply by ϕ. Then $\langle\psi|\phi\rangle \equiv f(\phi)$. To simplify the model, Dirac later took the assumption that H is a Hilbert space, that is, the H' can be identified with H.

Consider an observable given by the Hermitian operator A with nondegenerate spectrum and restrict our consideration to the case of a finite dimensional H. Thus, the normalized eigenvectors e_i of A form the orthonormal basis in H. Let $Ae_i = a_i e_i$. In Dirac's notation e_i is written as $|a_i\rangle$ and, hence, any pure state can be written as

$$|\psi\rangle = \sum_i c_i |a_i\rangle, \; \sum_i |c_i|^2 = 1. \tag{18}$$

Since the projector onto $|a_i\rangle$ is denoted as $P_{a_i} = |a_i\rangle\langle a_i|$, the operator A can be written as

$$A = \sum_i a_i |a_i\rangle\langle a_i|. \tag{19}$$

Now consider two Hilbert spaces H_1 and H_2 and their tensor product $H = H_1 \otimes H_2$. Let $(|a_i\rangle)$ and $(|b_j\rangle)$ be orthonormal bases in H_1 and H_2 corresponding to the eigenvalues of two observables A and B. Then vectors $|a_i\rangle \otimes |b_j\rangle$ form the orthonormal basis in H. Typically, in physics the sign of the tensor product is omitted and these vectors are written as $|a_i\rangle|b_j\rangle$ or even as $|a_i b_j\rangle$.

6 Elements of Quantum Information Theory

In particular, in quantum information theory (Nielsen and Chuang 2000) typically qubit states are represented with the aid of observables having the eigenvalues 0, 1. Each qubit space is two dimensional:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle, \; |c_0|^2 + |c_1|^2 = 1. \tag{20}$$

A pair of qubits is represented in the tensor product of single qubit spaces, where pure states can be represented as superpositions:

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|00\rangle. \tag{21}$$

where $\sum_{ij} |c_{ij}|^2 = 1$. In the same way, the n-qubit state is represented in the tensor product of n one qubit state spaces (it has the dimension 2^n):

$$|\psi\rangle = \sum_{x_j=0,1} c_{x_1 \ldots x_n} |x_1 \ldots x_n\rangle, \tag{22}$$

where $\sum_{x_j=0,1} |c_{x_1 \ldots x_n}|^2 = 1$. Note that the dimension of the n qubit state space grows exponentially with the growth of n. The natural question about possible physical realizations of such multi-dimensional state spaces arises. The answer to it is not completely clear. It depends very much on the used interpretation of the wave function. Even such an exotic interpretation as the *many worlds interpretation* is widely used in quantum information theory, especially in quantum computing (to clarify the possible source of computational power of quantum computers). By applying the formalism of QM to cognition and decision-making one has to think about the same problem.[7]

7 State Dynamics for an Isolated Quantum System

7.1 Schrödinger's Equation: The Pure State Evolution

In QM the state dynamics of an isolated quantum system is described by Schrödinger's equation:

$$i\hbar \frac{d|\psi\rangle}{dt} = \mathcal{H}|\psi\rangle, \tag{23}$$

where $\mathcal{H}$ is the Hamiltonian which is a Hermitian positively defined operator, representing the system's energy, and $\hbar = h/2\pi$ is the reduced Planck constant. The solution to this equation uses the unitary operator. Please see below (under the Remark).

Remark. Although in this chapter we try to proceed purely formally, that is, without attempting to couple the quantum formalism to applications in

[7] In principle, one can ignore this problem completely (as the majority of the quantum information people do) and proceed in the formal mathematical framework. However, if the brain really processes the information in a quantum(-)like way, it might be useful to think about the possible sources which provide the ability to represent the quantum(-like) states of huge dimensions. One of the most natural solutions is to consider the brain as a quantum physical system, for example, Penrose (1989). Another possibility is to derive the quantum(-like) processing of information from the functioning of the macroscopic and classical physical brain, for example, on the basis of the classical electromagnetic field model (Khrennikov 2010a), or more generally the classical oscillators model (de Barros and Suppes 2009), or the model based on tree-like distributed information processing (Khrennikov 2004). In the latter model the exponential increase of the number of branches of a neuronal tree can be used to match the exponential increase of the dimension of the n-qubit space.

the social sciences, we would like to make a comment on the role of the Planck constant h in cognition and decision-making; see Khrennikov (2010b), Busemeyer and Bruza (2012) for deeper discussions. It would be naive to expect that the Planck constant h will be precisely useful in applications outside of physics. Instead of the Planck constant h, it is natural simply to use some scaling factor $\gamma > 0$:

$$i\gamma \frac{d|\psi\rangle}{dt} = \mathcal{H}|\psi\rangle. \tag{24}$$

It would be surprising if such a scaling factor was the same for all cognitive systems when the dynamics of their decision-making is considered. However, we cannot exclude its constancy for some concrete classes of cognitive systems and (or) problems. Another foundational problem related to the use of Schrödinger's dynamics in social science is to find a proper interpretation of the Hamiltonian $\mathcal{H}$. The most natural analogy with physics is to treat it as a *mental energy operator*. However, the notion of mental energy by itself generates complicated foundational issues. Therefore it may be useful to consider $\mathcal{H}$ as a dimensionless quantity (in physics it has the dimension of energy). In such a case the scaling factor γ has to have the dimension of time. It can be interpreted as the scale of the time evolution of the mental state ("belief state") of a cognitive system. This quantity can be in principle determined experimentally (although this is far from a simple problem).

The Schrödinger equation is, in fact, a system of linear differential equations with complex coefficients. In the one-dimensional case $\mathcal{H}$ is just a real number and the general solution has the form of the imaginary exponent: $\psi(t) = e^{\frac{-it\mathcal{H}}{\gamma}}\psi^0$. In the general case $\mathcal{H}$ is an operator and the solution is represented in the form of an imaginary operator-exponent (for the fixed basis it is simply the exponent of the matrix):

$$\psi(t) = U_t\psi^0, \; U_t = e^{\frac{-it\mathcal{H}}{\gamma}}. \tag{25}$$

As well as the one-dimensional imaginary exponent, the operator-exponent describes *oscillating dynamics*. This is more complicated than in the one-dimensional case and is a mixture of many oscillating imaginary exponents.

Note the following fundamental property of Schrödinger dynamics. The evolution operator U_t, see (25), is a *unitary operator*, that is, it preserves the scalar product:

$$\langle U_t\psi|U_t\psi\rangle = \langle\psi|\psi\rangle. \tag{26}$$

Thus, the dynamics transfer a pure quantum state to another pure quantum state.

7.2 Von Neumann's Equation: The Dynamics of a Mixed State

Since any general quantum state, a density operator, can be represented as a mixture of density operators corresponding to pure states, the Schrödinger dynamics for pure states imply the following dynamics for density operators:

$$\gamma\frac{d\rho(t)}{dt} = -i[H, \rho(t)], \quad \rho(0) = \rho^0. \tag{27}$$

In quantum physics the Planck constant h is used instead of the scaling factor γ. This equation is known as the *von Neumann equation* (von Neuman 1955). By using representation (25) of the Schrödinger evolution for the pure state we represent the evolution of the density operator ("mixed state") in the form

$$\rho(t) = U_t^*\rho^0 U_t, \tag{28}$$

where, for an operator W, the symbol W^* denotes its *adjoint operator*. The latter is defined by the equality

$$\langle W\psi_1|\psi_2\rangle = \langle\psi_1|W^*\psi_2\rangle; \tag{29}$$

by denoting the matrix elements of these operators as w_{ij} and w_{ij}^* we have $w_{ij}^* = \bar{w}_{ji}$.

8 Positive Operator Valued Measures

One widely used generalization of conventional QM involves replacing the projection operators with *positive-operator-valued measures.*

A positive operator valued measure (POVM) is a family of positive Hermitian operators $\{M_j\}$ *("effects") such that* $\sum_{j=1}^m M_j = I$, *where* I *is the unit operator.*

We consider the simplest case: a discrete operator valued measure on the set of indexes $J = \{1, 2, \ldots, m\}$. See, for example, Busch et al. (1995) for POVMs on continuous sets. A POVM can be considered as a random observable. Take any set of labels $\alpha_1, \ldots, \alpha_m$, for example, for $m = 2, \alpha_1 = \text{yes}, \alpha_2 = \text{no}$. Then the corresponding observable takes these values (for systems in the state ρ) with the probabilities

$$p_\rho(\alpha_j) \equiv p_\rho(M_j) = \mathrm{Tr}\rho M_j. \tag{30}$$

It is convenient to use the following representation of POVMs:

$$M_j = V_j^\star V_j, \tag{31}$$

where $V_j : H \to H$ are linear operators. Note that such a representation is non-unique. By using the representation (31) the probability $p_\rho(\alpha_j)$ can be represented as

$$p_\rho(\alpha_j) = \mathrm{Tr} V_j \rho V_j^\star. \tag{32}$$

We are also interested in the *post-measurement states.* Assume the state ρ was given, a generalized observable was measured, and the value α_j was obtained. Then in the simplest (but most useful in applications) case the output state after this measurement has the form

$$\rho_j = \frac{V_j \rho V_j^\star}{\mathrm{Tr} V_j \rho V_j^\star}. \tag{33}$$

Since the representation (31) is not unique, the state transformer is not uniquely determined by a POVM. This is an additional element of the measurement theory, see Busch et al. (1995), Nielsen and Chuang (2000) for details, as well as for the general theory of state transformers corresponding to generalized observables.

9 Quantum Logic

Following von Neuman (1955) and Birkhoff and von Neumann (1936) we represent *events, propositions,* as orthogonal projectors in complex Hilbert space H. This is the standard definition of an event which is used in quantum logic.

For an orthogonal projector P, we set $H_P = P(H)$, its image, and vice versa, for subspace L of H, the corresponding orthogonal projector is denoted by the symbol P_L.

The set of orthogonal projectors is a *lattice* with the order structure: $P \leq Q$ iff $H_P \subset H_Q$ or equivalently, for any $\psi \in H,\ \langle \psi | P\psi \rangle \leq \langle \psi | Q\psi \rangle$.

We recall that the lattice of projectors is endowed with operations "and" ($\wedge$) and "or" ($\vee$). For two projectors P_1, P_2, the projector $R = P_1 \wedge P_2$ is defined as the projector onto the subspace $H_R = H_{P_1} \cap H_{P_2}$ and the projector $S = P_1 \vee P_2$ is defined as the projector onto the subspace H_R defined as the minimal linear subspace containing the set-theoretic union $H_{P_1} \cup H_{P_2}$ of subspaces H_{P_1}, H_{P_2} : this is the space of all linear combinations of vectors belonging to these subspaces. The operation of negation is defined as the orthogonal complement: $P^{\perp} = \{y \in H : \langle y|x \rangle = 0 : \forall x \in H_P\}$.

The lattice of orthogonal projectors in complex Hilbert space is called a *quantum logic,* see Beltrametti and Cassinelli (1979) for details.

In the language of subspaces the operation 'and' coincides with the usual set-theoretic intersection, but the operations 'or' and 'not' are nontrivial deformations of the corresponding set-theoretic operations. It is natural to expect that such deformations can induce deviations from classical Boolean logic.

Consider the following simple example. Let H be a two-dimensional Hilbert space with the orthonormal basis (e_1, e_2) and let $v = (e_1 + e_2)/\sqrt{2}$. Then $P_v \wedge P_{e_1} = 0$ and $P_v \wedge P_{e_2} = 0$, but $P_v \wedge (P_{e_1} \vee P_{e_2}) = P_v$. Hence, for quantum events, in general the distributivity law is violated:

$$P \wedge (P_1 \vee P_2) \neq (P \wedge P_1) \vee (P \wedge P_2). \tag{34}$$

As can be seen from our example, even mutual orthogonality of the events P_1 and P_2 does not help to save the Boolean laws.

At first sight, the representation of events by projectors/linear subspaces might look as exotic. However, this is simply a prejudice of the common use of the set-theoretic representation of events in modern classical probability theory. The tradition of representing events by subsets was firmly established by Kolmogorov (1933).[8]

[8]Note that before him the basic classical probabilistic models were not of a set-theoretic nature. For example, the main competitor of the Kolmogorov model, the von Mises frequency model, was based on the notion of a collective. One of the first rigorous mathematical models of classical probability was invented by Boole (1854) and it was of an algebraic nature, based on Boolean algebras. In short, from the operational viewpoint both mathematical representations of events, by sets or by projectors/linear subspaces, have the same degree of justification.

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Quantization in Financial Economics: An Information-Theoretic Approach

Raymond J. Hawkins and B. Roy Frieden

1 Introduction

A remarkable feature of quantum mechanics (QM) is its ubiquity beyond the natural sciences (Khrennikov 2010), its explanatory power in the social sciences generally (Haven and Khrennikov 2013), and its reach in economics in particular.[1] Given the introduction of risk and uncertainty into economics by Knight (1921) and Keynes (1936) and the well-known relationship between stochastic and quantum representations of dynamic phenomena[2] it is perhaps not surprising that the methodology of QM would be found to work in economics. A key question, however, is whether quantization in economics

[1] See, for example, references Baaquie (2004, 2009), Choustova (2007a,b, 2008, 2009a,b), Haven (2005, 2008a,b,c, 2010), Haven and Khrennikov (2013), Ilinski (2001), Khrennikov (1999, 2003, 2010), and Kleinert (2009).

[2] See, for example, references Gardiner (2009), van Kampen (1977), Kaniadakis (2001), Klein (2010), Nelson (1966, 1967), and Risken (1996).

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is simply a convenient analytical consequence of the well-known stochastic representation of financial economics or whether the link between QM and economics has an alternative origin. Our work suggests a more fundamental information-theoretic basis for quantization in economics, and the purpose of this chapter is to present the case for this position.[3]

It is our view that all things economic are information-theoretic in origin: economies are participatory, observer participancy gives rise to information, and information gives rise to economics. Dynamical laws follow from a perturbation of information flow which arises from the asymmetry between J, the information that is intrinsic to the system, and I, the measured Fisher information of the system: a natural consequence of the notion that any observation is a result of the $J \to I$ information-flow process. In this manner financial-economic dynamics are a natural consequence of our information-theoretic approach. These dynamics have a direct link to the formalism of QM in that the equation of motion that follows from the extremization of Fisher information is isomorphic to the Schrödinger equation. As we shall see, this link enables both the interpretation of the coefficients of this equation in terms of economics and the leveraging of the considerable literature of quantum and stochastic dynamics in the interpretation of economic phenomena. We consider this latter feature to be of particular importance as the field of economics grapples with the aftermath of the recent financial crisis and works towards a reconstruction of macroeconomics that embraces these dynamics (Aoki and Yoshikawa 2007).

To this end we continue in Sect. 2 with an introduction of the Fisher-information approach to constructing Lagrangians in economics. In this section we show how a time-independent Schrödinger-like equation arises as a straightforward result of extremizing Fisher information subject to the constraints of observed market prices. From this one can obtain the ground state (or, as it is usually referred to in economics, the equilibrium state) most commonly associated with economic problems as well as the excited states which are less commonly used in economics. With this *complete* set of states one can construct the dynamics of an economy. An alternative and recent approach to information-induced quantum-like dynamics in economics is explored in Sect. 3 where we present a direct derivation of a time-dependent Schrödinger-like equation, again based on minimization of Fisher information but now incorporating the continuity equation for probability. In this derivation we use Nelson's stochastic mechanics to link the model with traditional

[3]We will draw on references Frieden and Hawkins (2010), Frieden et al. (2007), Hawkins et al. (2010, 2005), and Hawkins and Frieden (2004a,b, 2012).

financial economics and to provide an economic interpretation of all elements in the model. We close with a discussion and summary in Sect. 4.

2 Fisher Information, Financial Economics, and the Time-Independent Schrödinger Equation

2.1 General Derivation

Our object of consideration is the ideal, or 'true value,' price of a security which for a given instant in time t_0 we write as x_0. At each point in time, however, the measured price is x_{obs} which is necessarily imperfect due to fluctuations x, or

$$x_{obs} = x_0 + x \ . \tag{1}$$

This fluctuation arises from both the inevitable uncertainty in the *effective* present time t_0, and the fact that the economy undergoes persistent random change. The greater the fluctuation x the greater is one's ignorance of the price. Notice that Eq. (1) indicates that, for zero-mean noise x, the average observed price equals the ideal price. This defines an 'unbiased economy.'

To represent this fluctuation we introduce the probability amplitude $\psi(x)$ for the price fluctuation x, with the associated probability $P(x, t)$ given by

$$P(x, t) \equiv \psi(x, t)\psi^*(x, t) = |\psi(x, t)|^2 \ , \tag{2}$$

where the asterisk denotes the complex conjugate. While the use of probability amplitudes is commonly associated with QM, amplitudes have been used by others—most notably Fisher and Mather (1943)—as we shall see presently, as a convenient means for simplifying statistical calculations. If we assume that the statistics of x_{obs} are independent of the price level x_0, the likelihood law for the process obeys[4]

$$P(x_{obs}|x_0, t) \equiv P(x_{obs} - x_0, t) = P(x, t) \tag{3}$$

[4]If scaling invariance holds in an economic system rather than the translation invariance assumed here a different Schrödinger equation emerges where the equilibrium density in the absence of observed prices is Zipf's law (Hernando et al. 2009, 2010).

by Eq. (1). The classical Fisher information I for this problem is

$$I \equiv \left\langle \left[\frac{\partial \ln \left(P(x_{\text{obs}}|x_0, t) \right)}{\partial x_0} \right]^2 \right\rangle , \tag{4}$$

where the angle bracket denotes an expectation over all possible data x_{obs} (Frieden 1998, 2004). This is a universal form that applies to all data acquisition problems and measures the information in the data *irrespective of the intrinsic nature of that data*. For our problem this can be reduced to

$$I = \int \frac{1}{P} \left(\frac{\partial P}{\partial x} \right)^2 dxdt = 4 \int \ |\psi'(x)|^2 dxdt \ , \tag{5}$$

where $|\psi'|^2 = \psi' \psi'^*$ and $\psi' = d\psi/dx$. Fisher information corresponds well to our intuition regarding price discovery given in Eq. (1): if the price fluctuation x is small then the level of information in the observation should be high; whereas if x is large the information should be low. This intuition is expressed in Eq. (5) by the shape of the probability density P since for small price fluctuations P must be narrowly peaked at about $x = 0$ implying high gradient dP/dx, a consequent high gradient $d\psi/dx$, and a large value for I. Similarly, if x has many large values then P will be broad, it and ψ will have low gradients, and there will be a small value for I.

The intrinsic information J, mentioned above, is the present, most complete, and perfectly knowable collection of information concerning the system that is relevant to the measurement exercise. Systems have widely varying levels of intrinsic information which map largely into three categories. The first is exact knowledge characterized by a known unitary transformation between the observation space and some conjugate space. Our derivation of Tobin's Q theory employed this approach using a Fourier transform between conjugate spaces (Frieden et al. 2007; Frieden and Hawkins 2010). Knowledge of a phenomenon requires its observation, and this must be in a given state. A necessary condition for maximum acquired knowledge of the state is its observation with maximum accuracy. This, in turn, requires—via the Cramer–Rao inequality—a maximum level of Fisher information I. But, in fact, all statistically repeatable phenomena, *whether physical or social in nature*, obey, a priori, maximum Fisher information (Frieden and Gatenby 2013). If observation of economic phenomena are required to convey maximum information, and knowledge, they must likewise obey unitary transformations. The second category is the use of induction by way of an invariance principle such as Noether's theorem. As this reflects less knowledge about the system the laws that come therefrom are not exact, but can be highly accurate (Frieden 2007). The third category, and the focus of this chapter, is the use of empirical data for J in the manner commonly employed in statistical mechanics, as a constraint on the implied probability density.

We begin with the simplest case when the extent of one's information J about the probability density $P(x, t)$ is that (1) the density is normalized and (2) there exist observed data $d_1, \ldots, d_M = \{d_m\}$ that can be expressed as averages of known functions $\{f_m(x, t)\}$

$$\int f_m(x, t)P(x, t)dxdt = d_m, \qquad m = 1, \ldots, M . \tag{6}$$

In this situation the probability density one seeks is that which is (1) normalized and (2) consistent with the observed data with a minimum of structure in the associated probability density function. From this perspective Fisher information is a regularizer, and ensuring that normalization and observation are achieved in a manner consistent with minimal structure is obtained by minimizing the Fisher information subject to the constraints that normalization be achieved and observed data be recovered. This suggests a well-known variational approach wherein one forms the Lagrangian $\mathcal{L} = I - J$

$$\mathcal{L} = \int \frac{1}{P}\left(\frac{\partial P}{\partial x}\right)^2 dx + \lambda_0\left[1 - \int Pdx\right] + \sum_{m=1}^{M}\lambda_m\left[d_m - \int f_m P dx\right] , \tag{7}$$

from which—employing standard variational calculus—one obtains (Frieden 1998, 2004)

$$P(x) = \psi(x)\psi^*(x) , \tag{8}$$

where

$$\frac{d^2\psi(x)}{dx^2} = -\frac{1}{4}\left[\lambda_0 + \sum_{m=1}^{M}\lambda_m f_m(x)\right]\psi(x) , \tag{9}$$

the Lagrange multiplier λ_0 corresponds to the normalization condition for the probability density ($\int p(x)dx = 1$), and the remaining Lagrange multipliers $\{\lambda_m\}$ ensure that the associated observed data $\{d_m\}$ are recovered by Eq. (6). Equation (9) has the same form as the well-known Schrödinger equation[5]:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = \left[E + V(x)\right]\psi(x) . \tag{10}$$

[5] See, for example, Chap. III of Landau and Lifshitz (1977).

2.2 Financial-Economic Interpretation of the Schrödinger Equation

Comparing our Schrödinger-like equation (9) with Eq. (10) of QM we can make some very straightforward identifications. First, our Eq. (9) corresponds to a Schrödinger equation in units such that $\hbar^2 = 8m$ where $\hbar$ is Planck's constant and m is the mass. Second, as mentioned above, the Lagrange multiplier responsible for probability normalization replaces the term associated with energy E. Third, the potential energy term in the Schrödinger equation $V(x)$ is represented by a linear combination of the functions from which the observed data were generated. The data force the solution $\psi(x)$ in financial economics just as a physical potential forces a solution in physics, chemistry, and engineering. Included in this data are structural details of securities that determine, among other things, the range of support for the probability density. Equity (common stock), for example, is a claim without liability on the residual value of a firm. Consequently, the value of common stock cannot be negative and the range of support for the probability density associated with equity is $[0, +\infty)$ (Choustova 2007a,b, 2008, 2009a,b; Khrennikov 1999, 2003, 2010). Finally, in addition to a ground-state solution $(\lambda_0^{(0)}, \ \psi_0(x))$ commonly associated with equilibrium, our Schrödinger-like Equation (9) also has excited-state solutions $(\lambda_0^{(j)}, \ \psi_j(x); \ j > 0)$ from which one can construct dynamics.[6]

One of the simplest functions from which an expectation value can be obtained is a linear function $f_m(x) = x$, for which we have explored two economic interpretations. First, if x is interpreted as the asset price, the expectation corresponds to the *expected or current price* of that asset (Hawkins and Frieden 2004a,b). Second, if x is interpreted as the current level of production, the expectation corresponds to aggregate demand (Hawkins et al. 2010). In both cases the solutions to Eq. (9) are known to be Airy functions, the lowest order of which corresponds to what is generally referred to in economics as *equilibrium* and the higher order being the set of functions onto which one can project *disequilibrium* economic states. The potential and probability amplitudes associated with the expected price are shown in Fig. 1. The function $f_1(x)$ that enters into Eq. (9) is $f_1(x) = x$, and together with the associated

[6] See, for examples, references Flego et al. (2003), Frieden et al. (2002a,b, 2007), Hawkins et al. (2005), van Kampen (1977), Plastino (2004), Plastino and Plastino (2007), and Risken (1996).

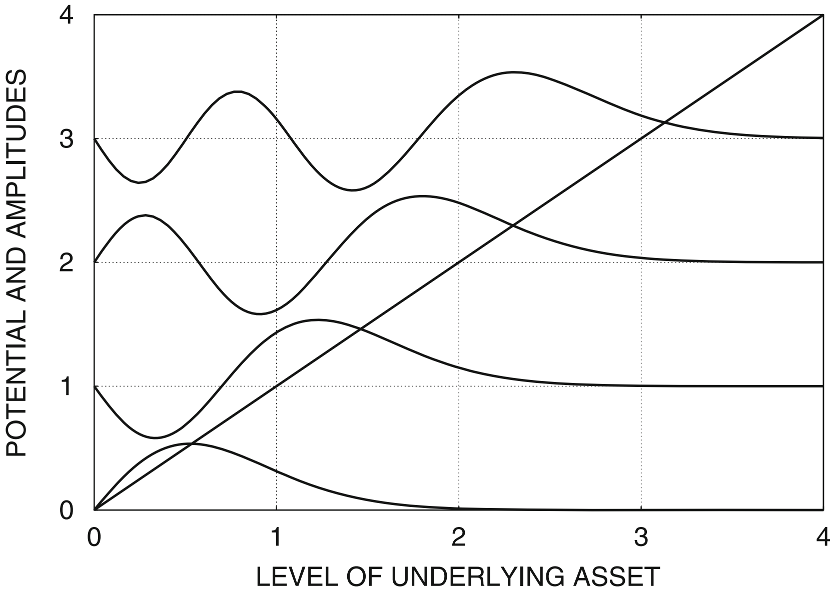

Fig. 1 The potential function for an expected price as a function of the level of the asset underlying the contract together with some of the associated probability amplitudes from Eq. (9)

Lagrange multiplier λ_1 forms the potential function. Given this potential our Schrödinger-like equation (9) has a number of solutions $(\lambda_0^{(j)},\ \psi_j(x))$ which are also shown in Fig. 1. The equilibrium state is the lognormal-like amplitude shown in the lower portion of the graph with an indicated expected price near 0.5. Were the economy to be in an equilibrium this is the density we would associate with this contract. The lognormal-like shape of this amplitude carries over into the associated equilibrium probability density and since the lognormal density is often used to represent asset prices it follows that the equilibrium results of our Schrödinger-like equation (9) would be similar to those of standard financial-economic analysis. The *disequilibrium solutions* $(\lambda_0^{(j)},\ \psi_j(x);\ j > 0)$, of which three are shown centered about the y-levels of 1–3 in Fig. 1, represent a new contribution of quantized financial economics: the states onto which departures from equilibrium can be projected and, as we shall see below, with which dynamics can be constructed.

A somewhat more complex but equally ubiquitous function of which expectation values are observed are those associated with option prices. A *call option* gives the holder the right—but not the obligation—to buy an asset at a predetermined price known as the strike price. If we denote the strike price by

k, the function corresponding to an observed call price is the payoff of the call: $f_m(x) = \max(x - k_m, 0)$. A closely related option known as a *put option* gives the holder the right—but not the obligation—to sell an asset at the strike price and has the corresponding function $f_m(x) = \max(k_m - x, 0)$. The associated observable option prices can be represented by

$$\text{Call}(k_m) = e^{-rt} \int_0^\infty \max(x - k_m, 0) P(x) dx \ , \tag{11}$$

and

$$\text{Put}(k_m) = e^{-rt} \int_0^\infty \max(k_m - x, 0) P(x) dx \ , \tag{12}$$

where r is the continuously compounded risk-free rate of interest and t is the time frame over which the option contract exists. With a collection of observed option prices one is able to reconstruct potential functions that are familiar in QM (Frieden et al. 2007; Hawkins and Frieden 2004a,b). For example, if the observed options were only a call and a put with the same strike price one would have a potential that looked like a V. With more observed option prices (and one does typically have more) a wide range of potential wells can be constructed.

An example of this is shown below in Fig. 2 where we see the functions of an expected price and a call option, the resulting potential, and some of the solutions from Eq. (9). In panel (a) of Fig. 2 we see the expected price function which we saw in the previous example. The payoff of a long call-option contract is shown in panel (b) of Fig. 2. In this case the strike price is 1.5. Since the call is the *right* to buy the asset, a holder of a call will only exercise that right if it is advantageous to do so, and in this case the notion of advantageous corresponds to the level of the underlying asset being greater than 1.5. With the right to buy the asset at 1.5 when the then market price of the asset is greater than 1.5 the holder of the call can make the difference between the market price and 1.5 by exercising his or her option to buy the asset at 1.5 and immediately selling the asset at the higher market price. Conversely, if the market price of the asset is less than 1.5 the holder of the option would lose money by exercising the option and will, consequently, allow the option to expire as worthless. This *optionality* is illustrated in the "hockey-stick" diagram shown in panel (b) of Fig. 2. Taking the simple case of the expected price and option contract, the two functions in the sum of Eq. (9) are $f_1(x) = x$ and $f_2(x) = \max(x - 1.5, 0)$. The formation of a potential well can be seen by

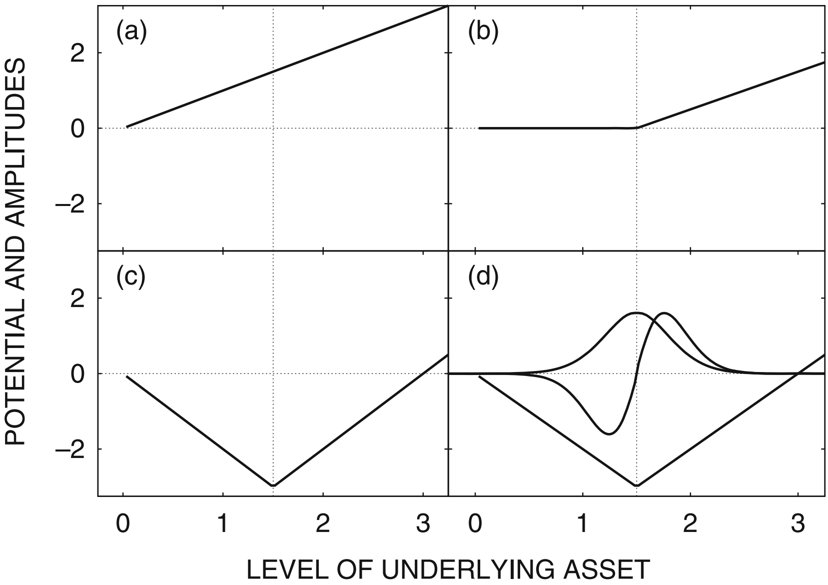

Fig. 2 The function for an expected price and the payoff function for a call option as a function of the level of the asset underlying these contracts, together with a potential formed from these functions and some of the associated probability amplitudes from Eq. (9)

considering what would result if $\lambda_1 = -2$ and $\lambda_2 = 4$, as shown in panel (c) of Fig. 2. Over the range $x < 1.5$ where the call-option payoff is zero the product of the expected price function and its Lagrange multiplier is the sole contributor to the potential. For $x > 1.5$ the product of the long call and its Lagrange multiplier offsets the negative-sloped function and the net potential increases. By varying the Lagrange multipliers one can obtain the equilibrium solution $(\lambda_0^{(0)}, \ \psi_0(x))$ shown as the Gaussian-like amplitude in panel (d) of Fig. 2; a solution that recovers the option price and which is normalized. As in the case of the expected price discussed above, however, the quantum approach to financial economics naturally results in disequilibrium solutions, one of which is the dual-lobed amplitude shown in the panel. Other disequilibrium solutions with more lobes exist and represent the basis upon which a complete description of both equilibrium and disequilibrium states of an economy can be expressed.

Finally, if one interprets x as the time t until a cash flow c_t associated with a bond (e.g. coupon and/or principal payment) is due, the observed price of a coupon bond can be written as the sum of the present value of each cash flow where the present value is obtained by multiplying each cash flow by its associated discount factor DF(t). The discount factor represents the amount of money one would need to invest now in a risk-free account to have one unit of currency at time t and which can be represented as the integral (Brody and Hughston 2001, 2002)

$$\mathrm{DF}(t) = \int_t^\infty p(\tau)d\tau = \int_0^\infty \Theta(\tau - t)p(\tau)d\tau \ , \tag{13}$$

where $\Theta(x)$ is the Heaviside step function. The intuition behind this representation is that: (1) a unit of currency (e.g. a dollar) to be received immediately is worth one unit of currency, consequently the discount factor for $t = 0$ is one; (2) a unit of currency to be received in the future is determined by the rate at which interest compounds and is generally less than one, and correspondingly the discount factor represents the amount by which this is less than one; (3) as the time between now and when the unit of currency is to be received increases, the amount needed to be deposited in an interest bearing account to, through compounding, equal ultimately the unit of currency decreases. Collectively this implies that the discount factor is a cumulative density function as represented by Eq. (13) above (Brody and Hughston 2001, 2002). From this perspective the price of a bond—which like all assets is the present value of its expected future cash flows—is the average of the function $\sum_{n=1}^{N} c(t_n)\Theta(x - t_i)$ where $c(t_n)$ for $n < N$ are the coupon payments of the bond and $c(t_N)$ is the sum of the final coupon payment and principal payment (Frieden et al. 2007; Hawkins and Frieden 2004a; Hawkins et al. 2005). Thus, for a single cash flow, or zero-coupon bond, the associated potential is the simple step-function barrier. For a complete coupon bond the potential is a stepped ramp with each step corresponding to each coupon to be paid together with a final large step associated with the final coupon and principal payment.

The solution of Eq. (9) for a zero-coupon bond corresponds to the well-known problem in QM and quantum electronics of a finite-height square potential illustrated in Fig. 3. In this example we consider a zero-coupon bond with a 10-year maturity, the associated potential implied from Eq. (13) being the step function shown in the figure. Varying the associated Lagrange multiplier until the observed discount factor is recovered in Eq. (13) leads to the equilibrium amplitude—a cosine function within the potential well and an exponential function outside of the potential well—shown in Fig. 3. From

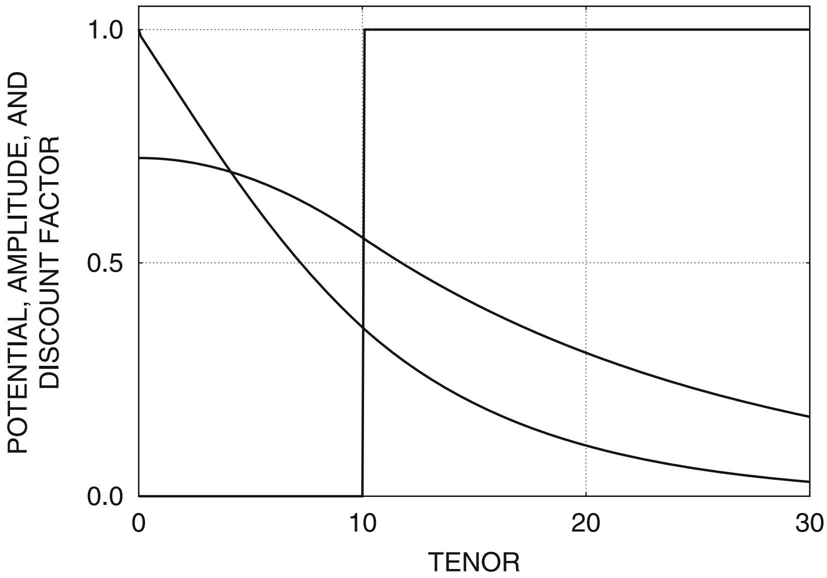

Fig. 3 The potential function for a zero-coupon bond together with the associated equilibrium probability amplitude and discount-factor function from Eq. (9)

this amplitude one can also obtain the implied discount factors for all tenors illustrated in Fig. 3 by the decaying curve with the value of one for zero tenor; from the discount-factor curve the interest rates commonly used in applied financial economics (e.g. spot and forward) are easily obtained (Galitz 2013).

2.3 Segue to Dynamics

The temporal evolution of economic observables described by our time-independent Schrödinger equation [Eq. (9)] can be derived from the relationship between the solutions of Eq. (9) and those of the Fokker–Planck equation.[7] The solutions of Eq. (9) $\{\lambda_0^{(j)}, \psi_j(x)\}$ form a general solution

$$P(x, t) = \sum_{j=0}^{\infty} a_j \psi_0(x) \psi_j(x) e^{-\vartheta(\lambda_0^{(j)} - \lambda_0^{(0)})t/4} \tag{14}$$

[7] See references Frieden et al. (2007), Hawkins et al. (2005), van Kampen (1977), and Risken (1996).

to the Fokker–Planck equation

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial U(x)}{\partial x} + \vartheta \frac{\partial}{\partial x}\right] P(x,t) \ , \tag{15}$$

where the potential function $U(x)$ is related to the equilibrium amplitude $\psi_0(x)$ via

$$U(x) = -2\vartheta \ln\left(\psi_0(x)\right) \ , \tag{16}$$

and to the observed functions by

$$-\frac{1}{4}\sum_{m=1}^{M} \lambda_m f_m(x) = U'^2(x)/4\vartheta^2 - U''(x)/2\vartheta + \text{const} \ . \tag{17}$$

Together, Eqs. (8)–(17) provide a practical information-theoretic basis for economic dynamics. One can, however, directly introduce time dependence into the Fisher-information approach to financial economics which leads to the *time-dependent* Schrödinger equation of financial economics; and it is to this that we now turn.

3 Fisher Information, Financial Economics, and the Time-Dependent Schrödinger Equation

In a recent communication (Hawkins and Frieden 2012) we introduced time dependence directly by employing the assumption that price trajectories form a coherent system (Reginatto 1998; Synge 1960). This complements Haven's (Haven 2005, 2008a,b,c, 2010; Ishio and Haven 2009) analysis of derivatives from a de Broglie–Bohm perspective—by providing (as we shall see presently) an information-theoretic basis for the de Broglie–Bohm approach in financial economics—and generalizes the work of Choustova (2007a,b, 2008, 2009a,b), Khrennikov (1999, 2003, 2010) on equity prices in the de Broglie–Bohm framework as equity can be viewed as a derivative security, namely a call option on the assets of the issuing firm (Black and Cox 1976; Black and Scholes 1973; Merton 1974). The assumptions of time dependence and coherence imply that the rate of change (or velocity) v of a cash flow at price point x can be related to a real function $S(x,t)$ by an expression of the form (Reginatto 1998)

$$v = \frac{1}{m_e}\frac{\partial S}{\partial x} , \tag{18}$$

where m_e is the effective mass of the price representing the ease with which a security traverses price space and is a function of the economy.[8] The probability distribution then satisfies a conservation law of the form

$$\frac{\partial P}{\partial t} + \frac{1}{m_e}\frac{\partial}{\partial x}\left(P\frac{\partial S}{\partial x}\right) = 0 , \tag{19}$$

and, as discussed by Reginatto (1998), Eq. (19) can be derived from a variational principle, by minimization of the expression

$$\int P\left(\frac{\partial S}{\partial t} + \frac{1}{2m_e}\left(\frac{\partial S}{\partial x}\right)^2\right) dxdt , \tag{20}$$

with respect to S.

To minimize the Fisher information I in a manner consistent with our information J we again form the Lagrangian $\mathcal{L} = I - J$ which, with the addition of the conservation law, takes the form (Reginatto 1998)

$$\begin{aligned}\mathcal{L} = \frac{\lambda_0}{m_e}\int \frac{1}{P}\left(\frac{\partial P}{\partial t}\right)^2 dxdt - \int P\left(\frac{\partial S}{\partial t} + \frac{1}{2m_e}\left(\frac{\partial S}{\partial x}\right)^2\right) dxdt \\ - \sum_{n=1}^{N}\lambda_n\left[d_n - \int f_n(x,t)P\, dxdt\right] .\end{aligned} \tag{21}$$

Variation of the information asymmetry $\mathcal{L}$ with respect to S and P yields

$$\frac{\partial P}{\partial t} + \frac{1}{m_e}\frac{\partial}{\partial x}\left(P\frac{\partial S}{\partial x}\right) = 0 , \tag{22}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m_e}\left(\frac{\partial S}{\partial x}\right)^2 + V(x,t) - \frac{\lambda_0}{m_e}\left(\frac{2}{P}\frac{\partial^2 P}{\partial x^2} - \frac{1}{P^2}\left(\frac{\partial P}{\partial x}\right)^2\right) = 0 , \tag{23}$$

[8]This economic concept of economic mass is closely related to the physical notion articulated by Synge (1960, p. 4) as "a measure of the reluctance of a body to change its velocity" and presented in terms of market turnover which, as discussed in reference Hawkins and Frieden (2012), builds on a considerable literature in both econophysics (Ausloos and Ivanova 2002; Choustova 2007a,b, 2008, 2009a,b; Khrennikov 1999, 2003, 2010) and financial economics (Blume et al. 1994; Karpoff 1987; Lee and Swaminathan 2000).

where

$$V(x,t) = \sum_{n=1}^{N} \lambda_n f_n(x,t) \; . \tag{24}$$

Equations (22) and (23) are, conveniently, the Madelung hydrodynamic equations (Madelung 1927) that, via the Madelung transform $\psi = \sqrt{P}\exp\left(iS/\sqrt{8\lambda_0}\right)$, where $i \equiv \sqrt{-1}$, are the real and imaginary parts of

$$i\frac{\partial \psi}{\partial t} = -\frac{\sqrt{2\lambda_0}}{m_e}\left[\frac{\partial^2 \psi}{\partial x^2} - V(x,t)\right]\psi \;\; : \tag{25}$$

a Schrödinger-like wave equation with a potential function V that, as we saw in Sect. 2, is a linear combination of the associated functions of the observed security prices in the economy.

As in the time-independent case above in Sect. 2, the Lagrange multipliers are chosen to be consistent with our information J. One can resolve the Lagrange multiplier λ_0 in a manner consistent with the traditional stochastic representation of financial economics by using Nelson's stochastic mechanics (Nelson 1966, 1967) in which the evolution of the price is represented by the stochastic differential equation

$$dx(t) = b(x(t),t)dt + dw(t) \; , \tag{26}$$

where

$$b = \frac{\sqrt{8\lambda_0}}{m_e}\frac{\partial}{\partial x}\left(\frac{1}{2}\ln P + S\right) \; , \tag{27}$$

$w(t)$ is a Wiener process, and $dw(t)$ is Gaussian with zero mean and product expectation $Edw_i(t)dw_j(t) = 2\nu\delta_{i,j}dt$, where $\nu = \sqrt{2\lambda_0}/m_e$ is the diffusion coefficient and $\delta_{i,j}$ is the Kronecker delta function.[9] So doing, one finds that the Lagrange multiplier λ_0 is a simple function of the turnover and diffusivity of the economy: $\lambda_0 = m_e^2\nu^2/2$. The remaining Lagrange multipliers $\lambda_1, \ldots, \lambda_N$ are, as before, determined by the requirement that the corresponding observed prices as expressed in Eq. (6) be recovered.

[9]The identity of the Lagrange multiplier λ_0 depends on the context of the associated derivation. In QM, Reginatto identified λ_0 in terms of Plank's constant as $\hbar = \sqrt{8\lambda_0}$ (Reginatto 1998). Derivations in optics or acoustics often identify λ_0 in terms of a wavelength (see, for example, Schulman 1981 and references therein). In this chapter we have identified it as a function of economic variables.

4 Discussion and Summary

Our assertion in the Introduction that all things economic are information-theoretic in origin is an adaptation to the social sciences of the Wheeler program of physics (Wheeler 1990a,b, 1994). Though controversial in some quarters of the physical sciences, the Wheeler program has gained considerable traction through the success of information theory as the basis of a wide range of problems in the physical sciences generally (Frieden 1998, 2004; Frieden and Gatenby 2007; Plastino 2004; Plastino and Plastino 2007) and in statistical and thermal physics in particular.[10]

From the perspective of information theory, the quantum representation of economics is a natural outcome of the process of inference using Fisher information. A convenient consequence of this approach is that the calculated $P(x, t)$ is consistent with all observed security prices (i.e. is arbitrage free) by construction. A measure of the price uncertainty in the economy σ^2 can be had through the use of the Cramer–Rao inequality $\sigma^2 \geq 1/I$ (Cramér 1946; Rao 1945) with which a lower bound of the price variance, or the notion of implied volatility employed widely in derivatives trading, is seen to be the inverse of the Fisher information (Hawkins and Frieden 2004a,b). At a deeper level, Bohm's interpretation of the final term on the left-hand side of Eq. (23) as a quantum potential (Bohm 1952a,b) provides, as discussed by Reginatto (1998), unique insight into the ontological and epistemological content of this theory: the epistemological content being the use of the minimization of Fisher information to choose the probability distribution that describes prices in an economy; the economic content being the assumption that the price-space motion of securities is a coherent structure and the existence of observed prices (Hawkins and Frieden 2012). Extending the insights of Reginatto and Bohm regarding the quantum potential to financial economics—most notably that (1) "the average value of the quantum potential is proportional to the Fisher information" and (2) that the quantum potential acts together with the potential function V on prices—it follows that the forces acting on prices "depend on the probability assignment used to infer their" level and are "a consequence of the inference process": both of which are fundamental tenets of behavioral financial economics (Keynes 1936, 1937; Minsky 1977, 2008; Soros 2003).

[10] See, for example, references Balian (1982), Ben-Naim (2008), Flego et al. (2003), Frieden et al. (1999, 2002a,b), Hernando et al. (2009, 2010), Katz (1967), Plastino (2004), and Plastino and Plastino (2007).

Quantization arises naturally in economics due to the manner in which uncertainty arises.[11] In this chapter we have shown how the use of information theory to infer probability in a manner common to both the natural and social sciences yields a canonical result of QM: the Schrödinger equation. The ubiquity of information generally, and of Fisher information in particular, as a fundamental theoretical framework throughout the sciences, can thus be seen as a source of the ubiquity of quantum structure across the sciences, be they natural or social.

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[11] See references Choustova (2007a,b, 2008, 2009a,b), Ishio and Haven (2009), Haven (2005, 2008a,b,c, 2010), Khrennikov (1999, 2003, 2010), Kleinert (2009), and Wright (2007).

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Quantum Game Theoretical Frameworks in Economics

Edward W. Piotrowski and Jan Sładkowski

1 Introduction: Quantum and Quantum-Like Models

Recently the field of "Quantum social science" has emerged (Haven and Khrennikov 2013; Slanina 2014). What sort of entity is hiding behind this term? There is no simple answer to this question. It hardly corresponds to the reduction of human behavior to intermolecular interactions. Roughly speaking, the idea is to use the apparatus developed to describe quantum phenomena to analyse macroscopic complex systems (including living systems). But why? Since its beginning, the development of mathematics has mostly corresponded to practical needs. From ancient times through the Middle Ages mathematical creativity focused on arithmetic and geometry. To some extent, farther development was stimulated by new discoveries in physics. Differential geometry was used for modeling the universe as a whole, probability theory

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helped us to cope with uncertainties, and functional analysis created the mathematical apparatus of quantum theory to describe phenomena in the microworld, cf. Haven and Khrennikov (2016) or Susskind and Friedman (2014). Nowadays these tools are widely used almost everywhere in mathematics, computing, chemistry, and biology. The analysis of human decisions has revealed that the foundations of probability theory and Boolean logic are often violated in the process (Busemeyer and Bruza 2012; Moreira and Wichert 2014). The *sure-thing principle* formulated by Savage (1954) is a well known example. The mathematical apparatus of quantum theory seems to offer a solution to some problems of this kind. The simplest examples come from game theory (Osborne 2003; Osborne and Rubinstein 1994), where a general notion, mixed strategy is widely used. A mixed strategy is an assignment of a probability to each pure (basic) strategy and a random adoption of a pure strategy. In quantum theory, instead of "adding probabilities" one is allowed or even forced to use (normalized) complex linear combinations of states (amplitudes), in which only the squared modulus of such amplitudes defines the probability. This idea is at the root of quantum game theory (Meyer 1999; Eisert et al. 1999; Piotrowski and Sładkowski 2003a): the assembling of probabilities happens at the level of probability amplitudes. This trick is also used in the Fisher information approach in statistics (Frieden 2004). The most interesting fields where this approach can be applied include:

- Pricing of financial instruments. Here, the path integral (Baaquie 2004, 2009; Kleinert 2009) and quantum game theory (Haven 2005; Choustova 2006; Segal and Segal 1998; Piotrowski and Sładkowski 2004) can be used.
- Theory of decisions. Here various important aspects have been approached (Deutsch 2000; Haven and Khrennikov 2009; Asano et al. 2011; Piotrowski and Sładkowski 2003b).
- Risk theory. Here, besides problems related to decision science, the formalism of noncommutative quantum mechanics (QM) can be explored (Piotrowski and Sładkowski 2001).
- Game theory. Here a whole new subfield was developed. Quantum mechanism design seems to be a very promising field of research that has mostly been neglected up to now (Wu 2011a).
- Psychology. Here, various paradoxes can be discussed from a quantum-like point of view (Busemeyer and Bruza 2012); even problems connected with consciousness can be approached (Baaquie 2009; Miakisz et al. 2006).
- Network theory. This is quite a new development with plenty of possible applications.

All this means that the mathematical formalism of QM is not firmly adjoined with quantum physics, but can have a much wider class of applications. We are not able to review all these fields, therefore we focus on quantum game theoretical models in economics and interested readers are referred to other contributions in this volume and more specialized books and references (Haven and Khrennikov 2013; Busemeyer and Bruza 2012; Slanina 2014).

2 Quantum Game Theory

Information processing is a physical phenomenon and therefore information theory is inseparable from both applied and fundamental physics. Attention to the quantum aspects of information processing has revealed new perspectives in computation, cryptography, and communication methods. In numerous cases a quantum description of the system provides some advantages over the classical situation, at least in theory. But does QM offer more subtle mechanisms for playing games? In game theory one often has to consider strategies that are probabilistic mixtures of pure strategies (Osborne 2003; Osborne and Rubinstein 1994). Can they be intertwined in a more complicated way by exploring interference or entanglement? There certainly are situations in which it can be assumed that quantum theory can enlarge the set of possible strategies (Meyer 1999; Eisert et al. 1999; Piotrowski and Sładkowski 2003a). This is a very nontrivial issue as genuine quantum systems usually are unstable and their preparation and maintenance might be difficult, for example due to decoherence, the practically inevitable destructive interactions with the environment. We have already mentioned the astonishing fact that quantum formalism can be used in game theory in a more abstract way without any reference to physical quantum states—the decoherence is not a problem in such cases. The question is whether quantum games are of any practical value. In some sense the answer is positive: commercial cryptographical and communication methods/products are already available. The abstract field of using the quantum apparatus outside physical systems is also appealing. Here we aim at providing a theoretical explanation of decisions or behavior in quantum mechanical terms (Haven and Khrennikov 2013; Busemeyer and Bruza 2012; Slanina 2014; Miakisz et al. 2006).

By a quantum game we usually understand a quantum system that can be manipulated by at least one party and for which the utilities of moves can be reasonably defined. Here we will use the concept of quantum game in a more abstract sense.[1] Therefore, we assume that the analysed system can be with satisfactory accuracy represented by a density operator (matrix) related to a more or less abstract vector space[2] (Haven and Khrennikov 2016). We shall suppose that all players know the state of the game at the beginning and, possibly, at some crucial stages of the actual game being played.[3] We neglect the possible technical problems with actual identification of the state—we will assume that the corresponding structures are definable. Implementation of a genuine quantum game should in addition include measuring apparatuses and information channels that provide necessary information on the state of the game at crucial stages and specify the moment and methods of its termination. We will not discuss these issues here.

We will consider only two-player quantum games: the generalization for the N players case is straightforward. Therefore we will suppose that a two-player quantum game $\Gamma = (\mathcal{H}, \rho_i, S_A, S_B, P_A, P_B)$ is completely specified by the underlying Hilbert space $\mathcal{H}$ of the quantum system, the initial state given by the density matrix $\rho_i \in \mathcal{S}(\mathcal{H})$, where $\mathcal{S}(\mathcal{H})$ is the associated state space, the sets S_A and S_B of quantum operations representing moves (strategies) of the players, and the pay-off (utility) functions P_A and P_B which specify the pay-off for each player after the final measurement performed on the final state ρ_f. A *quantum strategy* $s_A \in S_A$, $s_B \in S_B$ is a collection of admissible quantum operations, that is the mappings of the space of states onto itself. One usually supposes that they are completely positive trace-preserving maps. Schematically we have:

$$\rho_i \mapsto (s_A, s_B) \mapsto \rho_f \mapsto \text{measurement} \Rightarrow (P_A, P_B)\,.$$

This scheme for a quantum two-player game can be implemented as a quantum map:

$$\rho_f = \mathbb{J}^{-1} \circ \mathbb{S} \circ \mathbb{D} \circ \mathbb{J}(\rho_i), \tag{1}$$

where initially

$$\rho_i = |00\rangle\langle 00| \tag{2}$$

[1]Quantum auctions are the only exception, as their implementation seems to be feasible.

[2]Actually a Hilbert space, though this should not bother us at the moment.

[3]Actually one can consider quantum games played against Nature. In such cases the agents might not even be aware of playing the game!

describes identical starting positions of Alice (A) and Bob (B). $\mathbb{J}$ describes the process of creation of entanglement in the system and $\mathbb{D}$ the possible destructive noise effects that will be neglected here. The use of entanglement is one of several possible ways to utilize the power of QM in quantum games. One of the possibilities is that the states of players are transformed using

$$\mathbb{J}(\rho) = J(\gamma)\rho J(\gamma)^{\dagger} \tag{3}$$

with

$$J(\gamma) = \cos(\gamma/2)\mathcal{I} \otimes \mathcal{I} + i\sin(\gamma/2)\sigma_x \otimes \sigma_x \tag{4}$$

into an entangled state. Here $\mathcal{I}$ and σ_x denote the identity operator and the Pauli matrix, respectively. The additional parameter γ describes the possible destructive role of the environment (noise).

Equation (3) has the following explicit matrix form for the initial state given by Eq. (2) (Flitney and Abbott 2003a):

$$\mathbb{J}(\rho_i) = \begin{pmatrix} \cos(\gamma/2)^2 & 0 & 0 & i\cos(\gamma/2)\sin(\gamma/2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i\cos(\gamma/2)\sin(\gamma/2) & 0 & 0 & \sin(\gamma/2)^2 \end{pmatrix}. \tag{5}$$

The individual strategies of players S_X, $X = A(\text{lice}),\ B(\text{ob})$ are implemented as unitary transformations of the form:

$$\mathbb{S}(\rho) = (S_A \otimes S_B)\rho(S_A \otimes S_B)^{\dagger}, \tag{6}$$

where the quantum strategy is realized by unitary transformations. For example, both S_A and S_B can have the general matrix form in the two-dimensional case (Flitney and Abbott 2003a) :

$$U(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix}. \tag{7}$$

The short description of quantum games presented here will be sufficient for our aims. Interested readers are referred to Piotrowski and Sładkowski (2003a) and Flitney and Abbott (2003a) for further clarification and details.

3 Quantum Approach to Risk

Let us begin with an abstract interlude. We will consider the generalization of QM, the so-called "noncommutative" QM. The adjective "noncommutative" reflects the additional assumptions that the operators $\hat{x}^i$ fulfil

$$[\hat{x}^i, \hat{x}^j] = i\,\theta^{ij},\ \ \theta^{ij} \in \mathbb{C}, \tag{1a}$$

$$[\hat{x}^i, \hat{x}^j] = i\,C^{ij}{}_k \hat{x}^k,\ \ C^{ij}{}_k \in \mathbb{C}, \tag{1b}$$

$$\hat{x}^i \hat{x}^j = \mathrm{q}^{-1} \hat{R}^{ij}{}_{kl} \hat{x}^k \hat{x}^l,\ \ \hat{R}^{ij}{}_{kl} \in \mathbb{C}.$$

Labels i, j, k, l take values from 1 to N. The parameters q, θ^{ij}, C^{ij}, and R^{ij} describe the model; their actual values are not important. Suppose that the strategies of agents are given by vectors $|\psi\rangle$ from the corresponding Hilbert space $\mathcal{H}$ (Piotrowski and Sładkowski 2003c). Let the probabilities of signaling of private values for random variables p and q by Alice and Bob using strategies $|\psi\rangle_A$ and $|\psi\rangle_B$ be given by (that is, by the corresponding probability amplitudes after normalization):

$$\frac{|\langle q|\psi\rangle_A|^2}{{}_A\langle\psi|\psi\rangle_A}\,\frac{|\langle p|\psi\rangle_B|^2}{{}_B\langle\psi|\psi\rangle_B}\,dqdp\,, \tag{8}$$

where $\langle q|\psi\rangle_A$ is the probability amplitude of Alice's bid of value q. The reverse position of Bob is represented by the amplitude $\langle p|\psi\rangle_B$ (Bob ask p). Of course, the deal is not always realized. Recall that (Elton et al. 2013; Luenberger 2009):

- In classical error theory second moments of a random variable are related to its "random" errors.
- In Markowitz's portfolio theory variance (σ) "measures" risk.
- In Bachelier's option valuation model the random variable $q^2 + p^2$ "measures" the joint risk associated with the buying-selling process.

Therefore, we are tempted to define the *operator of inclination to risk* as:

$$H(\mathcal{P}_k, \mathcal{Q}_k) := \frac{(\mathcal{P}_k - p_{k0})^2}{2\,m} + \frac{m\,\omega^2(\mathcal{Q}_k - q_{k0})^2}{2}\,,$$

where $p_{k0} := \frac{{}_k\langle\psi|\mathcal{P}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k}$, $q_{k0} := \frac{{}_k\langle\psi|\mathcal{Q}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k}$, $\omega := \frac{2\pi}{\theta}$. θ, roughly speaking, denotes the mean duration of the whole cycle of buying-selling (Piotrowski and Sładkowski 2003d). The parameter $m > 0$ is introduced to describe a possible asymmetry in risk connected with selling and buying. If you browse through any textbook on QM you will discover that the above operator H is the energy operator for harmonic oscillation, a classical issue in physics, if one notices that $\mathcal{P}_k \sim \hat{p}_k$ and $\mathcal{Q}_k \sim \hat{x}_k$ (Haven and Khrennikov 2016). This allows for the following analogy. There exists some constant h_E that characterizes the minimal inclination to taking risk (the minimal energy level in physics).[4] Here, it is equal to the product of the minimal eigenvalue of the operator $H(\mathcal{P}_k, \mathcal{Q}_k)$ and the parameter 2θ. This means that, in our interpretation, 2θ gives the minimal period when it makes sense to calculate profits. Note that, in general, $\mathcal{Q}_k$ do not commute for different k. This should not surprise us: agents observe each other and react accordingly. Any ask or bid influences the market, at least for a short period. This explains why we have used the noncommutative QM instead of the "classical" one. For example, if

$$[x^i, x^k] = i\Theta^{ik} := i\Theta\, \epsilon^{ik}$$

then the results of Hatzinikitas and Smyrnakis (2002) suggest that Θ modifies our "economic Planck constant" $\hbar_E \to \sqrt{\hbar_E^2 + \Theta^2}$ and the eigenvalues of $H(\mathcal{P}_k, \mathcal{Q}_k)$. This implies the obvious conclusion that the activity of agents modifies their attitudes toward risk. Strategies with definite values of risk are given by eigenvectors of $H(\mathcal{P}_k, \mathcal{Q}_k)$. Remember that the minimal value of risk is always greater than zero. Interesting, isn't it?

4 Quantum Approach to Market Phenomena

Let the real random variable $\mathfrak{q}$

$$\mathfrak{q} := \ln \mathfrak{c}_q - E(\ln \mathfrak{c}_q) \tag{9}$$

correspond to the logarithm of the (bid) withdrawal price $\mathfrak{c}_q$, that is the maximal price at which the agent adopting the strategy $|\psi\rangle_k$ is willing to buy

[4] In physics h_E is the Planck constant.

the good $\mathfrak{G}$. We define $\mathfrak{q}$ to ensure that its expectation value in the state $|\psi\rangle_k$ is zero, $E(\mathfrak{q}) = 0$. In turn, the random variable $\mathfrak{p}$

$$\mathfrak{p} := E(\ln \mathfrak{c}_p) - \ln \mathfrak{c}_p \tag{10}$$

corresponds to the analogous situation for the supplier of $\mathfrak{G}$ adopting strategy $|\psi\rangle_k$ (ask). Note that q and p do not depend on the units selected for $\mathfrak{G}$, and we can use units such that $E(\ln \mathfrak{c}) = 0$. Let us consider the general situation of simultaneous trading of an arbitrary amount of goods. The state of the game is given by the vector $|\Psi\rangle_{\text{in}} := \sum_k |\psi\rangle_k$ living in the direct sum of the Hilbert spaces of the agents (Haven and Khrennikov 2016) $\sum_k \oplus \mathcal{H}_k$. Hermitian operators of demand $\mathcal{Q}_k$ and supply $\mathcal{P}_k$ acting on subspaces $\mathcal{H}_k$ form canonically conjugated observables (Susskind and Friedman 2014). We will denote their eigenvalues by q and p respectively. This construction can be validated in the following way. If the unique price e^{-p} (ask p) results from the application of $\mathcal{P}_k$ there is no sense in agent k reporting bids at the same price, and the corresponding operators should not be simultaneously measurable (commuting). The corresponding capital flows are determined according to some algorithm $\mathcal{A}$ representing the clearing house. The transaction is described by the scattering operator $\mathcal{T}_\sigma$ mapping the initial state $|\Psi\rangle_{\text{in}}$ to the final state $|\Psi\rangle_{\text{out}} := \mathcal{T}_\sigma |\Psi\rangle_{\text{in}}$, where

$$\mathcal{T}_\sigma := \sum_{k_d} |q\rangle_{k_d\,k_d}\langle q| + \sum_{k_s} |p\rangle_{k_s\,k_s}\langle p|$$

is a projection operator given by the partition σ of the set of agents k into two disjoint sets $\{k\} = \{k_d\} \cup \{k_s\}$ of agents buying at prices $\mathrm{e}^{q_{k_d}}$ and selling at prices $\mathrm{e}^{-p_{k_s}}$ at this round. The algorithm $\mathcal{A}$ should determine the market partition σ, prices $\{q_{k_d}, p_{k_s}\}$, and the capital flows. Capital flows are fixed according to the probability distributions

$$\int_{-\infty}^{\ln c} \frac{|\langle q|\psi\rangle_k|^2}{{}_k\langle\psi|\psi\rangle_k}\, dq\,, \tag{11}$$

and

$$\int_{-\infty}^{\ln \frac{1}{c}} \frac{|\langle p|\psi\rangle_k|^2}{{}_k\langle\psi|\psi\rangle_k}\, dp \tag{12}$$

giving the probabilities of selling and buying $\mathfrak{G}$ at price c, respectively. These probabilities are conditioned on the partition σ. We can envisage a future market administered by a quantum computer where the above quantum computations can be implemented, though at this moment this is only a theoretical tool. More details and some simulations can be found in Piotrowski and Sładkowski (2002a,b, 2004). There are natural ways of incorporating the subjectivity of decisions to this formalism, cf. Piotrowski and Sładkowski (2009) and Piotrowski et al. (2010).

Another interesting model of a quantum market based on the second quantization method was put forward by Gonçalves and Gonçalves (2007). They introduced a "population number" $n_1, n_2, \ldots, n_m$ for all alternative combinations of strategies. This is implemented by bosonic creation and annihilation operators $a_k^\dagger$ and a_k (Susskind and Friedman 2014). The number of all possible combinations, $m = \prod_k N_k$, is unlimited (N_j is the number of alternative strategies for the jth player). The jth agent strategy profile is $|p_j\rangle = \sum_i c_i |s_i(p_j)\rangle$, where c_i is the probability amplitude of strategy s_i. The unitary evolution of the strategy state $|p_j, t_{\text{fin}}\rangle = U(t_{\text{fin}}, t_{\text{ini}})|p_j, t_{\text{ini}}\rangle$ is governed by a unitary operator of the form

$$U(t_{\text{fin}}, t_{\text{ini}}) = \prod_{k=0}^{k_{\text{fin}}} U(t_{k+1}, t_k),$$

where k parameterizes the $k_{\text{fin}} + 1$ trading rounds. In a simplified single-asset model, where there are only two strategies (buying and selling), for each agent the unitary evolution for the kth trading round can be given in the following form:

$$U(t_{k+1}, t_k) = \exp(\sum_{j=0}^{1} (\xi_j(k, \tau_k) a_j^\dagger - \xi_j(k, \tau_k)^* a_j)),$$

where τ_k is the duration of each trading round, $\xi_j(k, \tau_k) = -i\tau_k \mu_j(k)$ with $\mu_j(k)$ a game-dependent real number that incorporates the dynamics.

5 Quantum Auctions

We now discuss the concept of a quantum auction, its advantages and drawbacks. Quantum auctions are quantum games designed for various goods allocations that one should anticipate. It is well known that for some types of auctions the associated computational issues are difficult to cope with Cramton et al. (2005). There is hope that in future, due to quantum computation speed up, that some of these problems can be overcome. We envisage that the implementation might not be an easy task. Quantum information processing, in principle, can provide tools for secure transmission of bids and asks and their treatment. Such topics have been discussed in the case of sealed-bid auctions (Naseri 2009; Zhao et al. 2010; Liu et al. 2014). Here we would like to focus on more specific issues of using quantum theory for designing the very mechanisms of auctions.[5] We will begin by presenting the general idea of a quantum auction. Then we will suggest methods of gaining an advantage over "classical opponent" and describe some proposals of quantum auction. Farther we will proceed to the quantum mechanism design problems, that is the theory of construction of quantum games with equilibria implementing given social choice rules (Haven and Khrennikov 2016). Finally we will try to show some problems that should be addressed in the near future. In a discussion we will use quantum auction theory as a formal theoretical tool, though widespread opinion is that it seems probable that it will be used in the future for massive combinatorial auctions or in compound securities trading. A genuine quantum bidding language might have to be developed to this end. Encoding bids/asks in quantum states is a challenge to quantum game theory. Quantum auctions would almost always be probabilistic and may provide us with specific incentive mechanisms and so on. As the outcome may depend on amplitudes of quantum strategies, sophisticated apparatus and specialists may be necessary. Therefore, we envisage some changes in the law and in practices. Commercial implementation of quantum auctions is a demanding challenge that cannot be accomplished without a major technological breakthrough in controlling and maintaining quantum systems. Extreme security and privacy are certainly strong points of quantum auctions. Currently, it is difficult to find out if this is a feasible task, but as a theoretical tool it is also very interesting (Piotrowski and Sładkowski 2008; Patel 2007). Quantum auctions are specified by the following data.

- Auctioneer specifies conventional "classical" details of the auction such as the schedule and goods to be sold.
- Auctioneer specifies the implementation of the quantum auction.
- Auctioneer specifies the initial state distribution, implementation of strategies, and main features of the search algorithms to be used (e.g. probabilistic, deterministic).
- Search for the winners and good allocations (this process might be repeated several times).

[5] We call such auctions "genuine quantum auctions."

- Methods of goods delivery and clearing, which is a standard issue.

This scheme is consistent with our definition of a quantum game.

5.1 Examples of Quantum Auctions

In the first model all possible prices of items are encoded in strings of qubits (Hogg et al. 2007). The auctioneer wants to sell n items and to this end distributes to m bidders p qubits initially in state $|o\rangle$ ($p \cdot m$ qubits in total[6]). Each bidder can only operate on his or her qubits and encodes via unitary operation the details of the bid qubits (prices for all bundles of items). Thus each bidder has 2^p possible bid values, and can create superpositions of these bids: for multiple-item auctions the bid is a superposition $\sum_j \alpha_j |\text{bundle}_j\rangle \otimes |\text{price}_j\rangle$ for each bundle of items. A superposition of qubits specifies a set of distinct bids, with at most one allowed to win[7]; amplitudes of the superposition correspond to the likelihood of various outcomes for the auction. The protocol uses a distributed adiabatic search that guarantee that bidders' strategies remain private. The search operation, processing input from the bidders, is implemented by unitary operators, giving the overall operator $U = U_1 \otimes U_2 \otimes \ldots \otimes U_m$, where m is the number of bidders and U_i stands for the operator of the ith bidder. This "brute force" proposal seems to be the easiest to implement and especially suitable for combinatorial auctions.

Piotrowski and Sładkowski discussed an abstract model of bargaining (Piotrowski and Sładkowski 2002a). In their approach a two-dimensional complex Hilbert space is associated with two agents, Alice and Bob. The vectors (qubits) are called polarizations, which are identified with elements of a one-dimensional complex projective space $\mathbb{C}P^1$. On an orthonormal basis $(|0\rangle, |1\rangle)$, $|\xi\rangle = \xi_0|0\rangle + \xi_1|1\rangle \in \mathcal{H}_s$. The scalar product of two vectors $|\xi'\rangle, |\xi''\rangle \in \mathcal{H}_s$ is given by

$$\langle \xi' | \xi'' \rangle = \bar{\xi}'_0 \xi''_0 + \bar{\xi}'_1 \xi''_1 = \bar{\xi} \cdot \xi \ , \tag{13}$$

where $\bar{\xi}_k$, $k = 0, 1$ denotes the complex conjugate of ξ_k. The proportional vectors $|\xi\rangle$ and $t|\xi\rangle$ ($t \in \mathbb{C} \setminus \{0\}$) are identified. The probability of measuring the strategy $|\xi''\rangle$ in the state strategy $|\xi'\rangle$ is given by the squared module of the scalar product (13) of the states. The following interpretation of Alice's

[6]To implement this model additional qubits will be necessary for error correction.

[7]This corresponds to the *XOR* bidding language. This assumption can be relaxed.

polarization state $|\xi\rangle_A \in \mathcal{H}_{sA}$ (that is of her strategy) is proposed. If she formulates the conditions of the transaction we say she has the polarization 1 (and is in the state $|\overrightarrow{r}\rangle_A = |1\rangle$). In quantum bargaining (q-bargaining) this means that she put forward the price. In the opposite case, when she decides whether the transaction is to be made or not, we say she has the polarization $|0\rangle$. (She accepts or not the conditions of the proposed transaction.) The vectors $(|0\rangle, |1\rangle)$ form an orthonormal basis in $\mathcal{H}_{sA}$, the linear hull of all possible Alice polarization states. Bob's polarization is defined in an analogous way. The states of Alice and Bob became entangled if they enter into q-bargaining. The reduction of the state $|\xi\rangle_A$ (Alice) to $|1\rangle_A$ or $|0\rangle_A$ always results in Bob finishing in the state $|0\rangle_B$ or $|1\rangle_B$, respectively. The polarizations of q-bargaining form a two-dimensional complex Hilbert space $\mathcal{H}_s \subset \mathcal{H}_{sA} \bigotimes \mathcal{H}_{sB}$ spanned by two orthonormal vectors $|10\rangle := |1\rangle_A|0\rangle_B$ and $|01\rangle := |0\rangle_A|1\rangle_B$. A market process resulting in q-bargaining is described by a projection $\mathrm{P}_{|1\rangle} : \mathcal{H}_{sA} \bigotimes \mathcal{H}_{sB} \rightarrow \mathcal{H}_s$. This model of bargaining can be generalized to describe quantum English auctions (Piotrowski and Sładkowski 2003e).

6 Quantum Mechanism Design and Implementation Theory

Mechanism design (reverse game theory) is a field in game theory that studies solution concepts for various classes of games (e.g. private information games) (Hurwicz and Reiter 2006; Narahari 2014). According to Leonid Hurwicz, in a design problem the goal function is the main "given," while the mechanism is the unknown. In that sense the design problem is the "inverse" of traditional game theory, which is typically devoted to the analysis of the performance of an externally given mechanism. One defines a game form as a method for modeling the rules of a game or an institution, independently of the players' utility functions. An n-agent game form $\Gamma = (\mathbf{S}, A, g)$ is defined by a set of n strategy spaces of the players, $S_1, \ldots, S_n$, a set of alternatives A, and an outcome function $g : \mathbf{S} \longrightarrow A$, where $\mathbf{S} = \prod_{i=1}^{i=n} S_i$. An ($n$-agent) mechanism is defined by n agents' message spaces $M_1, \ldots, M_n$, a set of alternatives A, and an outcome function $g : \mathbf{M} \longrightarrow A$, where $\mathbf{M} = \prod_{i=1}^{i=n} M_i$. Shortly, a game form (mechanism) maps profiles of strategies (messages) into feasible outcomes. In contrast, a game as such assigns a profile of payoffs (utilities) to each profile of strategies (messages)! The idea is to use the "invented" mechanism in practice. Implementation theory provides a systematic methodology for designing an information exchange process followed by allocation processes that are "optimal" with respect to some pre-specified performance criteria. It provides analytical frameworks for the analysis and design of allocations among agents in various information contexts e.g. as a Bayesian game). One assumes that agents behave strategically and are self-utility maximizers. Information exchange among the agents might be allowed or even necessary. Let $N = \{1, \ldots, n\}$ denote a finite set of agents, $n \geq 2$, and $A = \{a_1, \ldots, a_k\}$ be a finite set of alternatives. We assume that an agent can have private information encoded as his or her type. Next, let T_i be the finite set of agent i's types, and the private information possessed by agent i is denoted as $t_i \in T_i$. A profile of types $t = (t_1, \ldots, t_n)$ is referred to as a state and $\mathcal{T} = \prod_{i \in N} T_i$ denote the set of states. At state $t \in \mathcal{T}$, each agent $i \in N$ is assumed to have a complete and transitive preference relation $\succeq_i^t$ over the set A. Let $\succeq^t = (\succeq_1^t, \ldots, \succeq_n^t)$ denote the profile of preferences in state t. The utility of agent i for alternative a in state t is $u_i(a, t) : A \times \mathcal{T} \to R$, that is, $u_i(a, t) \geq u_i(b, t)$ if and only if $a \succeq_i^t b$. We denote by $\succ_i^t$ the strict preference part of $\succeq_i^t$. Fixing a state t, we refer to the collection $E =< N, A, (\succeq_i^t)_{i \in N} >$ as an *environment*. Let ϵ be the class of possible environments. A social choice rule F is a mapping $F : \epsilon \to 2^A \backslash \{\emptyset\}$. Finally, a mechanism $\Gamma = ((M_i)_{i \in N}, g)$ consists in prescribing a message (strategy) set M_i for agent i, and an outcome function $g : \prod_{i \in N} M_i \to A$.

The distribution of information among the agents plays the key role in determining their actions, therefore specific implementation should involve an appropriate solution concept (equilibrium), for example Nash equilibrium implementation, Bayesian implementation, and Pareto efficient implementation. If one is trying to design a mechanism to achieve, for example, a Pareto optimal solution, one needs to take into account how individuals are likely to behave if one attempts to implement the mechanism. It can be shown that even in the case of simple voting rules some of the desirable properties which they appear to have if agents vote truthfully may disappear if agents have an incentive to vote strategically (Hurwicz and Reiter 2006). Therefore an important requirement is that of universality: the mechanism should work no matter what the individual preferences happen to be. Maskin provided an almost complete characterization of social choice rules that were Nash implementable. It should come as no surprise for you to learn that one can "design" quantum games and mechanisms; but would they be any good? Haoyang Wu from Wan-Dou-Miao Research Lab, Shanghai was the first who recognized this problem (Wu 2011a, 2013). Following the general model of a quantum game a sequential (multistage) scheme (Moore and Repullo 1988) can be developed (Wu 2011a):

$$\frac{\text{type}}{\text{selection}} \Rightarrow \frac{\text{measurement}}{\text{of coin state}} \Rightarrow \frac{\text{message}}{\text{processing}} \Rightarrow \text{outcome: } g(m)\,.$$

where the agents have "quantum coins" and "classical cards." Each agent independently realizes strategies by a local unitary operation [Eq.(7)] on his or her own quantum coin:

$$\mathbb{J}^{-1} \circ \text{quantum coin} \circ \mathbb{U} \circ \mathbb{J} \to \frac{\text{measurement}}{\text{of coin state}}.$$

As usual, $\mathbb{J}$ creates (annihilates) entanglement. The designer (e.g. auctioneer) receives the overall strategy as the cards and announces the result. Details of the resulting algorithms can be found in the original papers. Note that quantum mechanisms would certainly be probabilistic in nature. Therefore quantum mechanisms are substantially nontrivial, and simple or direct extensions of "classical" mechanisms that do not involve uncertainty are not possible cf. Ieong et al. (2007). If quantum effects (i.e. strategies) are possible, the traditional sufficient conditions of no-go theorems for the implementation of some types of social choice rules may fail (Wu 2011b; Bao and Halpern 2015; Makowski and Piotrowski 2011a).

In mechanism design theory one usually supposes that agents' preferences are transitive. Various simulations show that intransitive preference relations (Makowski 2009; Makowski and Piotrowski 2006, 2011a,b; Makowski et al. 2015) form a key ingredient of quantum mechanisms. This issue certainly deserves further investigation.

7 More Specific Quantum Games

In the previous section we discussed the modeling of whole branches of economics and the social sciences in a quantum game-theoretical setting. There are a whole lot more specific situations that can be successfully described as a game. Most of them can in principle be or have already been "quantized." We have put the word quantized in inverted commas to stress that the quantization of games does not exactly correspond to its physical counterpart. By quantization of a concrete game we mean the construction of such a quantum game that, after usually drastic reduction, strategy sets reproduce the initial (classical) game. In some cases some paradoxes or conundrums can be resolved in that way, but a review of these results is beyond the scope of this text. We only mention some results that we think are representative or interesting. The prisoner's dilemma is one of the flagships of game theory. Its quantum version is usually discussed in the context of cooperation (Eisert et al. 1999; Nawaz 2013) and network games (Pawela and Sładkowski 2013a; Li and Yong

2014). Quantum games on networks, hypernetworks, and cellular automata are being intensively studied as they involve cooperation, coordination, and synchronization problems (Li et al. 2012; Pawela and Sładkowski 2013a; Miszczak et al. 2014; Alonso-Sanz 2012). Another interesting class of problems that can be studied in the quantum setting is related to the famous Parrondo paradox (Flitney and Abbott 2003b; Meyer and Blumer 2002; Pawela and Sładkowski 2013b). New light can be shed on various aspects of Bertrand duopoly analysis (Khan et al. 2013; Lo and Kiang 2004), Cournot duopoly (Sekiguchi et al. 2010), and Stackelberg duopoly (Lo and Kiang 2005; Wang et al. 2013). The ultimatum game studied in experimental economics has also been analysed from the quantum game theory point of view (Frąckiewicz and Sładkowski 2014). The quantum battle of the sexes also became a popular research topic (Frąckiewicz 2009; Nawaz and Toor 2004; Weng and Yu 2014). A lot more can be said, but we have to stop somewhere. We apologize to authors whose works have not been mentioned here.

8 Conclusions

Quantum game theory aspires to be a fruitful theoretical tool in various fields of research. Quantum auctions have potential commercial value, but their implementation is a demanding challenge that would hardly be accomplished without a major theoretical and technological breakthrough. Nevertheless, we envisage the emergence of quantum computational choice theory and related fields (Bisconi et al. 2015). Quantum-like description will remain an important theoretical tool, even if never commercially implemented.

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The Intrinsic Quantum Nature of Classical Game Theory

Y. Pelosse

1 Introduction

The notion of mixed strategy, as originally introduced by Von Neumann and Morgenstern (1944) is a basic ingredient of classical game theory. Yet, as pointed out by von Neumann and Morgenstern themselves, the idea that a rational player may have to use a randomizing device, such as a coin flip, to decide on their actions poses some insuperable conceptual difficulties:

> This is certainly no maximization problem, but a peculiar and disconcerting mixture of several conflicting maximum problems (. . .) we face here and now a really conceptual—and not merely technical—difficulty. And it is this problem which the theory of "games of strategy" is mainly devised to meet.

As acknowledged in the subsequent game-theoretic literature (see, e.g. Aumann 1987), the introduction of the Nash equilibrium (Nash 1950, 1951) has just rendered this puzzle even more unpalatable. A Nash equilibrium is defined as a n-tuple of strategies or strategy profile (one strategy for each player) if each player's strategy is optimal against the others' strategies. According to Nash's theorem (Nash 1950, 1951), the very existence of such "equilibrium points" relies on the use of such "randomized strategies," which leads to

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the following conundrum. In equilibrium, each player has to be completely indifferent to the different actions of his or her mixed strategy. Moreover, the idea that decisions must appear indeterminate in order to be rational is rather troubling. Although there are many attempts to explain the underpinnings of mixed strategies, none has been unanimously accepted as satisfactory (see, e.g. Aumann 1987).[1] In the present chapter, we shall review a recent result proven in Pelosse (2016) which establishes the intrinsically quantum-mechanical nature of mixed Nash equilibria. It turns out that this eliminates all the aforementioned conceptual difficulties associated with such randomized strategies. So this raises an immediate question: why is this foundational problem of game theory connected to the apparently far removed field of quantum physics?

In a nutshell, the thread that connects the two theories can be traced back from the initial under-determination of the game model. On the one hand, the game model does not initially endow players of any pre-existing strategies and beliefs, that is statements about whether a strategy or belief is rational are neither true, nor false, but indeterminate in the sense of the three-valued logic of Lukasiewicz (1930). This means that the origin and genesis of a rational choice of a player in the classical game model is *ex nihilo*, depending of its own deliberation process, rather than the mere elaboration of the past. On the other hand, the crucial distinction between classical and quantum physics is that the behavior (e.g. the position and momentum of a particle) of a quantum system (an electron, an atom, etc.) is *not* a function of the past.[2] In fact, the orthodox view—the so-called "Copenhagen interpretation"— is that the hallmark of quantum mechanics (QM) is its "irreducible indeterminism," also called "*indefiniteness*," suggesting that we cannot attribute values to a quantum system until the occurrence of a measurement. In plain terms, this means that a question like "Where was a particle before the measurement?" is "the particle wasn't really anywhere." As well put by Jordan [the citation is taken from Kochen and Specker (1967)], "*observations not only disturb what is to be measured, they produce it ... We compel the particle to assume a definite position.*" This interpretation of QM is notably supported by the fact that alternative interpretations of quantum theory, based on the assumption of "hidden variables theories," have failed; a well-known series of results in QM—

[1]The alternative foundation—the so-called mass action approach—avoids becoming entangled in such philosophical issues. Rather than considering randomizations implemented by individual players, it addresses the question of how evolutionary selection processes or social learning allow us to understand an equilibrium as an aggregate statistical behavior, in the spirit of Harsanyi's purification method (Harsanyi 1973).

[2]A standard textbook on quantum mechanics is Mermin (1985).

the so-called "no-go theorems"—entail that measurements do *not* reveal the pre-existing properties of a quantum system (see Sakurai 1994).[3] Thus, our main result states that the classical von Neumann–Morgenstern paradigm is not only compatible with a quantum-mechanical description of players but is in fact its inevitable consequence, once we take into account the inherent initial indeterminism of the game model.

Our starting point is an equivalence result established in Pelosse (2016), which shows that the initial under-determination of the game model, that is each player has free choice and free belief, leads each player to "self-interact" into a Nash equilibrium.[4] This ontological foundation of the Nash equilibrium concept proves that, unlike conventional wisdom, an "equilibrium point" is *not* an interactive solution concept, but a process of "rational deliberation" followed by each player in order to unravel the initial indeterminism of the game model.

We shall use this equivalence result in order to uncover formally a structural connection—an isomorphism—between the knowledge structure of a player, without any initial strategy and beliefs in his or her mind, and the algebraic structure of an individual quantum system. The implications of the quantum nature of equilibrium mixtures for our understanding of players' behavior in a game are multifold. We shall illustrate and discuss some of its most important consequences by way of the following "gedankenexperiment." The proofs of all the claims stated in this chapter can be found in Pelosse (2016).

2 A Game-Theoretic Gedankenexperiment

Consider the game of Figure 1 (Table 1). In this game, Ann chooses the row, Bob chooses the column, Charlie chooses the matrix. For Ann and Bob, this is simply the "Battle of the Sexes"; their payoffs are not affected by Charlie's choice. We first need to explore the implication of formally incorporating the absence of pre-existing beliefs in the game of Figure 1. Let us focus our discussion on Ann and on the following Nash equilibrium of this game $(\frac{2}{3}\mathtt{N} \oplus \frac{1}{3}\mathtt{S}, \frac{1}{3}\mathtt{n} \oplus \frac{2}{3}\mathtt{s}, N)$. In the classical model, the absence of pre-existing

[3]Briefly stated, the Kochen–Specker theorem is a mathematical result about the nature of Hilbert spaces (the special type of vector spaces that are the most general representation of the state space for a quantum system). It states that if properties are represented as operators on a Hilbert space in a $1-1$ fashion (i.e., each property is represented by a unique operator), then these properties cannot all be said simultaneously to have values.

[4]Brandenburger (2008) provides an insightful discussion on the notion of "indeterminism" in the classical game model.

Table 1 Figure 1

	n	s
N	2,1,0	0,0,0
S	0,0,0	1,2,0

N

	n	s
N	2,1,0	0,0,0
S	0,0,0	1,2,0

S

rational strategies and beliefs for Ann implies that she must literally "create" what she deems as being rational. Formally, this means that when she has to choose a destination all statements like "*The North Pole is rational*," "*The South Pole for Charlie and the North for Bob are rational beliefs for Ann*," or "*The South Pole for Charlie and the North Pole for Bob are rational*," and so on, are all indeterminate statements; these statements are neither true, nor false, but indeterminate in the sense of the three-valued logic of Pelosse (2016). This is the initial *lack* of sharp truth values that leads Ann to " create" the truth value of statements like "*The (mixed) strategy* σ_A *is rational*" is true because the statement "*The (mixed) strategy* σ_C *for Charlie and the (mixed) strategy* σ_B *for Bob are rational strategies*" is true. Hence, this in turn requires that Ann determines whether statements like "*The (mixed) strategy* σ_C *for Charlie and the (mixed) strategy* σ_B *for Bob are rational strategies*" are true or not. Doing so means that she must *simultaneously* "put herself in the shoes" of Bob and Charlie while she is considering her own decision problem. Of course, Ann must also be rational when she adopts the decision problems of others, that is when she adopts Bob and Charlie's decision problems at $A_{B\otimes C}$. This raises the question: What are the "mental states" that Ann has to take in order to break the above initial under-determination of the game of Figure 1?

The quick answer is that the determination of a rational strategy is equivalent to Ann "self-interacting" in a mixed Nash equilibrium of the game of Figure 1. Here is a visual way to grasp this situation more clearly. Imagine that Ann is sitting simultaneously in two different *transparent* "cubicles" (the term is taken from Kohlberg and Mertens (1986, p. 1005)) A_A and $A_{B\otimes C}$. In order to deliberate, Ann must *simultaneously* put herself "in the shoes" of Bob and Charlie (the meta-perspective $A_{B\otimes C}$) *and* in her own shoes (her own perspective A_A). This deliberation process can be identified by the directed graph,

$$\longleftarrow\!\!\circ \quad A_A$$

describing the informational process "input-output" by which Ann can determine her rational strategy (the output) given that she has determined her beliefs over Bob and Charlie (the input). For short, let $\multimap$ denote this mental process. The converse process is Ann determining her beliefs, that is a profile of a mutually rational profile of strategies for Bob and Charlie. Hence, the "dual" directed graph

$$A_{B\otimes C} \circ\!\!\longrightarrow$$

corresponds to Ann determining the truth value of the *input* of the first process. For short, we represent this "dual" mental process by $(\multimap)^{\dagger}$, that is the "dual" directed graph, representing the converse direction, output-input. Note that $\dagger$ is the operation which changes the direction of the mental process that would yield an involution in the sense that $(\multimap)^{\dagger\dagger} = \multimap$. This "duality property" of the graph is not accidental; it is directly linked to the dual algebraic structure between vector and co-vectors (or linear mappings) of the Hilbert space structure of QM. The formal definition of the classical game model implies that Ann's beliefs are on a par with her (rational) strategies: both are initially nonexistent in Ann's mind. This implies that each process $\multimap$ (or $(\multimap)^{\dagger}$) alone cannot yield anything; Ann cannot turn some nonexistent beliefs (resp. rational strategy) into a rational strategy (resp. beliefs). This simple observation leads us to conclude that the only possible process for Ann to create a rational strategy in her mind is to combine the two processes *simultaneously*,

$$A_{B\otimes C} \circ\!\!\longrightarrow \quad \wedge \quad \longleftarrow\!\!\circ\ A_A \equiv$$

$$A_{B\otimes C} \circ\!\!-\!\!\!-\!\!\circ\ A_A.$$

The result of this deliberational process allows Ann to determine that statement "$\frac{1}{3}\mathtt{n} \oplus \frac{2}{3}\mathtt{s}$ *is rational for Bob and the South Pole is rational for Charlie*" is true in her cubicle $A_{B\otimes C}$, because Ann can check—by taking a look at cubicle A_A—that the statement "$\frac{2}{3}\mathtt{N} \oplus \frac{1}{3}\mathtt{S}$ *is rational for Ann*" is indeed true in the corresponding cubicle, A_A, relative to the statement that "$\frac{2}{3}\mathtt{N} \oplus \frac{1}{3}\mathtt{S}$ *is rational for Bob and the South Pole is rational for Charlie*" is true in cubicle $A_{B\otimes C}$, which she knows to be true by looking from her cubicle A_A, and so on. The bottom line of this story is thus that the mere determination of a rational strategy induces Ann to find a Nash equilibrium of the game, that is she must at least adopt two perspectives.

Of course, Ann could arrive at a similar result by sitting in the three distinct glass cubicles, A_A, A_B, and A_C, instead of the big cubicle $A_{B\otimes C}$. However, as shown in Pelosse (2016), models with *more* than two perspectives are in fact not well-defined. The reason is that these models would not induce Ann to play a well-defined probability measure—her equilibrium mixture $\frac{2}{3}\mathtt{N} \oplus \frac{1}{3}\mathtt{S}$—in an experiment. This result allows us to streamline the theory considerably and to focus attention on the class of canonical models with exactly two perspectives.[5]

Note that the above "self-interaction" process is reminiscent of the "dynamic models of deliberation" and very much in line with the "deliberational equilibrium" of Skyrms (11, 12). It turns out that this inevitable deliberation process leads to a view of mixed strategies which differs radically from the usual interpretations. Consider the above mixed Nash equilibrium wherein Ann chooses the North Pole ($\mathtt{N}$) with a probability $\frac{2}{3}$ and the South Pole ($\mathtt{S}$) with probability $\frac{1}{3}$. Suppose that an outside observer can read the mind of Ann. What would he or she observe *before* Ann chooses a destination (before she makes her actual choice)?

According to the classical view, a mixed strategy represents deliberate randomizations on the part of players. In this case a player commits to a randomization device and delegates the play to a trustworthy party. Thus, everyone, including the player *him or herself*, is just ignorant about the *actual* choice made by the random device. Hence, in this approach an observer would realize that Ann believes she will go to the North Pole with probability $\frac{2}{3}$ and to the South Pole with probability $\frac{1}{3}$.

The epistemic approach would suggest a different answer. Under this approach, a mixed-strategy equilibrium is interpreted as an expression of what each player believes his or her opponent will do. Therefore, in the present thought experiment, an observer reading Ann's mind would realize that she is completely *determined*.[6]

The result of Ann's deliberation required to break the initial under-determination of the game model results in the determination of a Nash equilibrium. So, by looking simultaneously through the transparent walls of her two cubicles—during her introspection—Ann has been able to determine that:

(1) "*Strategy* $\frac{2}{3}N \oplus \frac{1}{3}S$ *is rational for Ann*" is true at A_A and;
(2) "*Strategy* $\frac{2}{3}N \oplus \frac{1}{3}S$ *is rational for Ann*" is true at $A_{B\otimes C}$.

[5] Of course, we could also examine the situations of a player adopting (simultaneously) the perspectives of all players. However, it can be shown [see Pelosse (2011, 2016)] that a player cannot simultaneously assign truth values to statements for all players, without "checking" that a given profile is indeed a fixed point of the combined best-response mapping.

[6] If a player does have the option of making a randomized choice, this can be added to the (pure) strategy set. Of course a similar interpretation follows in the "mass-action" interpretation of Nash (1950), as a mixed-strategy profile is formally identical with a population distribution over the pure strategies.

In other words, Ann thinks that strategy $\frac{2}{3}\mathtt{N} \oplus \frac{1}{3}\mathtt{S}$ is rational for her in *each* of her two cubicles ((meta)-perspectives). Of course, this is just the usual definition of an equilibrium: The common belief of Bob and Charlie at $A_{B\otimes C}$ meets the mixed strategy of Ann. Now we ask: How would an outside observer describe the equilibrium state of mind of Ann?

Let us first point out that Ann's state of mind should respect the following minimal reasonable properties:

[Indivisibility] Ann is a *single* player (she is in the flesh!);

[Independence] A_A and $A_{B\otimes C}$ are mutually *independent* perspectives (in her mind, Ann determines simultaneously each statement in two *different* "cubicles"). Moreover, at A_A, Ann determines her *mixed strategy*, while she determines the *common belief* of Bob and Charlie at $A_{B\otimes C}$;

[Rationality] Operationally, if an outside observer runs an experiment with many copies of Ann, the empirical frequencies of outcomes N and S must agree with what each copy of Ann has determined in her mind as a rational strategy.

The axiom of indivisibility is just the fact that, in an experiment, Ann has to choose a *unique* action. Transported *in the mind* of Ann, this means that a choice occurs if and only if the pole chosen by Ann at one of her two cubicles matches the other. So we can visually describe a choice as the situation where Ann picks a choice in each of her two cubicles, A_A and $A_{B\otimes C}$, while looking simultaneously through the windows of her cubicles.

The axiom of independence simply states that Ann remains seated in her two cubicles, A_A and $A_{B\otimes C}$, which corresponds to the "non-cooperative" assumption of the game model. The last axiom amounts to stating that, in a series of experiments, the empirical probability measure obtained by an observer would exactly reflect Ann's rational mixed strategy, $\frac{2}{3}\mathtt{N} \oplus \frac{1}{3}\mathtt{S}$.

Now we return to the description of the state of mind of Ann. First, recall that, in (any) mixed equilibrium, Ann is indifferent to going either to the North Pole or to the South Pole.[7] Taken together, the axioms entail that Ann may think in her two cubicles that the choice of the North Pole and the South Pole are rational. The upshot is that the state of mind of Ann will be given by a 2×2 matrix (see Figure 2), which is the simplest example of the so-called "density matrices" (in fact an orthogonal projector describing a pure quantum state) of QM. What are the entries of this matrix?

[7] It is well known that if a strategy profile, $(\sigma^i)_{i\in N}$, is a Nash equilibrium, then every pure strategy in the support of each strategy σ^i is a best reply to i's belief σ^{-i}.

The faithful property implies that the diagonal elements, $(\mathtt{N},\mathtt{N})$ and $(\mathtt{S},\mathtt{S})$, take on values $\frac{2}{3}$ and $\frac{1}{3}$, respectively, together with the property that the vector representing Ann when she is in cubicle A_A can be distinguished from her vector representing her cubicle $A_{B\otimes C}$. Thus, by the axiom of independence, we conclude that complex vectors like, for example, $(\frac{1}{\sqrt{6}} \pm i\frac{1}{\sqrt{2}})\mathtt{N} \oplus (\frac{1}{2\sqrt{3}} \pm i\frac{1}{2})\mathtt{S} := z_1\mathtt{N} \oplus z_2\mathtt{S}$ characterize the state of mind of Ann (her mixed strategy) when she is in perspective A_A, while its conjugate, $(\frac{1}{\sqrt{6}} \mp i\frac{1}{\sqrt{2}})\mathtt{N} \oplus (\frac{1}{2\sqrt{3}} \mp i\frac{1}{2})\mathtt{S} := \bar{z}_1\mathtt{N} \oplus \bar{z}_2\mathtt{S}$ reflects her state of mind (the common belief of Bob and Charlie) at $A_{B\otimes C}$. Why do we have such complex vectors? Intuitively, we are looking for a faithful algebraic characterization of a pair of Ann's states of mind. Such a faithful representation is an isomorphism, which explains the occurrence of such complex vectors. How do we interpret these vectors?

Table 2 Figure 2. Ann's equilibrium state of mind is a density matrix

	N	S
N	$\frac{2}{3}$	$z_1\bar{z}_2$
S	$z_2\bar{z}_1$	$\frac{1}{3}$

The matrix of Figure 2 (Table 2) represents the global structure of knowledge of Ann i.e., her equilibrium state of mind. For example, the complex weight $z_1\bar{z}_2$ assigned to entry $(\mathtt{N},\mathtt{S})$ represents the fact that Ann has not yet fixed her mind on N or S: When sitting in cubicle A_A she thinks that going to the North Pole is rational but she simultaneously thinks in the other cubicle, $A_{B\otimes C}$, that the South pole is also rational. Alternatively put, the matrix of Figure 2 is the expression of the ignorance of Ann about her own choice of a definite destination during a measurement. We can now answer the question: What is the nature of equilibrium mixtures in games?

The quick answer is that the nature of these game-theoretic probabilities are epistemic *and* ontic. On the one hand, in a mixed Nash equilibrium, the state of mind of a player is a state of knowledge. This state is thus epistemic in nature since it is only expressible in terms of the player's knowledge. On the other hand, this epistemic state is also ontic, that is, it is a state of reality, because it provides the actual complete specification of all the properties of the player. The upshot is thus that these probabilities have an "ontological status"—a player has not singled out a particular pure action in his mind prior to his being asked to do so—rather than reflecting the ignorance of an outside observer—a player has already fixed his mind on a particular action, but the other players cannot read his mind. Alternatively put, we could also say that these equilibrium mixtures are maximally informative, in the sense that they describe the fact that the state of knowledge of a rational player is on a par with the state of knowledge of an ideal observer (or any other player in the game).

The above Gedankenexperiment is the illustration of our main finding—that each player is characterized by a quantum state of mind in a Nash equilibrium. Loosely stated, the main result established in Pelosse (2016) is as follows.

Suppose we have a finite n-person game in strategic form where each player is rational and has no extra piece of information than the payoff matrix itself, that is the axiom of `no-supplementary data` *holds. Then, in any mixed Nash equilibrium profile* $(\sigma^i)_{i\in N}$ *of the game being played, each player i is described by a superposition of states of mind over the pure strategies* s^i *in the support of* σ^i*, that is a pure quantum state representing the global state of mind of player i is given by an orthogonal projector (or dyad) with complex off-diagonal terms, and the empirical probability of having a pure strategy* s^i *in an experiment is given by the Born rule, as in QM.*

2.1 "Deliberational Interference"

Let us now see how non-additive probabilities enter the picture of the classical game model. We consider the following extension of the previous thought experiment. As before, Ann is still playing the game illustrated by Table 1. But now we append a hypothetical experimental set-up in which an experimenter can read the mind of Ann *before* she makes a definite choice. More precisely, this "mind reading machine" translates Ann's states of mind onto a fluorescent sphere representing the Earth, so that each time she thinks she will go to one of the two poles, her thought is registered as a flash of light on the sphere at one of the continents of the hemisphere containing the pole chosen by her. For instance, a flash of light on Europe is the sign that Ann has fixed her mind on `N`. Note that if we force Ann to say the South Pole, then we oblige her to fix her mind on one of the continents of the southern hemisphere, and we see one flash on one of the continents of this hemisphere. That is, we do not observe a flash on Europe.

We suppose that the occurrence of a flash on a particular continent lying in the hemisphere is random. This randomness has nothing to do with the choice of Ann in the game, it simply reflects some additional characteristics of the players, like nationality, or the culture of Ann, which are not under the control of the experimenter. From an *operational* viewpoint, we can construct an empirical probability distribution by counting up the number of flashes on a continent to obtain the fraction of flashes that occur on this continent when a large number of copies of Ann play the game. Let $P_{\mathtt{N}}(x)$ (resp. $P_{\mathtt{S}}(y)$) be the probability that we observe a flash of light on a continent $x \in \{\mathtt{Europe}, \mathtt{Africa}, \mathtt{Asia}, \mathtt{America}\}$ (resp. $y \in \{\mathtt{Antarctica}, \mathtt{Australia}, \mathtt{Africa}, \mathtt{Asia}, \mathtt{America}\}$) of the hemisphere when Ann's state of mind is fixed on the North Pole (resp. South Pole). Note that we *cannot directly observe* the state of mind of Ann, because if we knew that her state of mind was fixed on, say, the North Pole, then this would imply that she knows it too, which would mean that we have forced her to make up her mind in some way. Otherwise stated, the state of mind of Ann must be treated as a *latent variable*. Now, note that if Ann had fixed her mind on one of the two poles prior to making a choice, we would observe for each copy of her a single flash of light, either on a region of the continent of the northern hemisphere or of the southern hemisphere. In this case, we would then expect that the probability of observing a flash on a continent z overlapping the two hemispheres is $P_{\mathtt{NS}}(z) = P_{\mathtt{N}}(z) + P_{\mathtt{S}}(z)$, for $z \in \{\mathtt{Africa}, \mathtt{Asia}, \mathtt{South\ America}\}$, in conformity with the classical probability calculus. What are the formulae of these probabilities in this case?

If Ann has fixed her mind on one of the poles during an experiment, this is tantamount to saying that she has to pick the *same* destination d in her two perspectives A_A and $A_{B\otimes C}$. Let $\Psi_d(z)$ for $d = \mathtt{N}, \mathtt{S}$, (resp. $\overline{\Psi_{\mathtt{d}}(z)}$) be the complex weight (resp. its conjugate) assigned to the directed graph representing the knowledge structure of Ann at perspective A_A (resp. $A_{B\otimes C}$) when she picks continent z, given that she knows that destination d is rational. The case when Ann views the same destination d as rational, simultaneously, in her two perspectives, and picks a continent z, is illustrated below.

$$\xleftarrow{\Psi_{\mathtt{N}}(z)}\!\circ \quad A_A \qquad \bigwedge \qquad A_{B\otimes C} \xleftarrow{\overline{\Psi_{\mathtt{S}}(z)}}\!\circ$$

From the above illustration we can therefore conclude that if these weights induce the probabilities that, given a choice of d, Ann picks a continent z, then $P_{\mathtt{d}}(z) = \Psi_{\mathtt{d}}(z)\overline{\Psi_{\mathtt{d}}(z)}$, which is the Born rule of QM, and the total probability is such that

$$P_{\mathtt{NS}}(z) = |\Psi_{\mathtt{N}}(z)|^2 + |\Psi_{\mathtt{S}}(z)|^2 .$$

Now, we ask: What is the probability of a flash of light on a continent z both in the southern and northern hemispheres, if we *do not ask* each copy of Ann to choose one of the two poles in the game, so that we *only* observe the mental state of mind of her determining her rational mixed strategy in the game?

In this case, we can only indirectly observe the state of mind of Ann via a flash of light on the sphere. According to Theorem 1, the state of mind of Ann prior to a measurement has to be described by a wave, as a consequence of the "indifference condition," since her state of mind is not yet settled. Formally, this means that Ann knows *simultaneously* that $\mathtt{N}$ *or* $\mathtt{S}$ (the strategies of Ann, when viewed from her own perspective) are rational at A_A, on the one hand, *and* that $\mathtt{N}$ *or* $\mathtt{S}$ (recall that the strategies of Ann coincide with the beliefs of Bob and Charlie when she contemplates the perspective of these players) are rational at $A_{A\otimes B}$, on the other hand. Here is a visual way to derive the probability formula in this situation, in the light of the knowledge structure of Ann.

$$
\begin{array}{c}
\left(A_{B\otimes C} \xleftarrow{\Psi_{\mathtt{N}}(z)} A_A \quad \bigvee \quad A_{B\otimes C} \xleftarrow{\Psi_{\mathtt{S}}(z)} A_A\right) \\
\bigwedge \\
\left(A_{B\otimes C} \xrightarrow{\overline{\Psi_{\mathtt{N}}(z)}} A_A \quad \bigvee \quad A_{B\otimes C} \xrightarrow{\overline{\Psi_{\mathtt{S}}(z)}} A_A\right).
\end{array}
$$

This set of weighted directed graphs depicts—formally defined in Pelosse (2016)—the compositions of the different parts of the knowledge structure of Ann. Here, the operation $\bigvee$ corresponds to the usual logical connective "or." Given a perspective, Ann can consider $\mathtt{N}$ *or* $\mathtt{S}$. The operation $\bigwedge$ coincides with the logical connective "and," since Ann has to adopt the two perspectives A_A and $A_{A\otimes B}$, simultaneously. As already discussed, we have to assign a complex number $\Psi_d(z)$, for $d = \mathtt{N}, \mathtt{S}$, to each directed graph in order to have a faithful algebraic description of Ann's knowledge structure (an isomorphism). From this, we conclude that the total probability $P(z)$ of a flash of light on a continent z both in the southern and northern hemispheres is given by $P(z) = |\Psi_{\mathtt{N}}(z) + \Psi_{\mathtt{S}}(z)|^2$, which coincides with the theory of interference of waves. In fact, we have just derived the formulae used in the classical two-slit experiment in physics. The first formula, $P_{\mathtt{NS}}(z)$, corresponds to the situation where the experimental set-up allows us to monitor which slit the electron passed through (we force Ann to make a choice), while $P(z)$ is the correct formula when the experimenter does not devise an experiment to determine which slit an electron passes through (we do not oblige Ann to "make up her mind" on one of the two poles). The upshot is thus, as in this classical physics experiment, that the predictions on the future behavior of Ann require the use of non-additive probabilities since $P(z) \neq P_{\mathtt{N}}(z) + P_{\mathtt{S}}(z)$. This inequality is due to the interference term $2\overline{\Psi_{\mathtt{S}}(z)}\Psi_{\mathtt{N}}(z)$. This non-zero additional term is easily derived from the above picture and its meaning is crystal clear: In addition to the cases of perfect correlation between the two perspectives of Ann—that is given that she considers the same pole, she picks the same continent z—we must also add the cases where she picks the same continent z, while she is considering N at one perspective and S at the other. The twist is thus that the seemingly exotic non-classical probability calculus of QM arises in the classical game model as the *consequence* of the rationality of a player, via his or her (necessary) deliberation in a Nash equilibrium.

3 Discussion

What does the description of the mental state of a player through the wave functions of QM tells us about the behavior of a rational player? First, the presence of wave probability amplitudes allows us to understand why equilibrium mixtures have been so difficult to interpret. These probabilities reflect the real physical "act of creation" of the future states of mind of a player (with itself), while classical probabilities proceed from the mere ignorance of the actual state of mind of the player. This is the fact that the classical game model does not presume any pre-determined beliefs that lead inevitably to this peculiar form of probabilities.

Second, the wave description of a player's mental state prior to his or her definite choice solves the so-called "mixing problem." Indeed, it is well-known that equilibrium mixed strategies raise the question of why we should expect players to randomize; see, for example, Aumann (1987). More precisely, this puzzle goes like this:

> In equilibrium, each player i is required to randomize in exactly the way that leaves the other players indifferent between the elements in the support of their equilibrium strategies. But i has no reason to randomize in this way, precisely because i too is indifferent between the elements in the support of his equilibrium strategies.

The quantum-mechanical nature of mixed strategies offers a natural resolution of this puzzle. It states that the "indifference condition" is not the sign of a randomization, but the reflection of the indefinite state of mind of the player prior to his or her making a definite choice in an experiment, that is, the player's state of mind is not yet fixed on a particular pure action of the support. Hence, the prima facie exotic wave function of QM has a clear-cut game-theoretic interpretation: mathematically a wave, that is a unit vector of a complex Hilbert space, implies that the linear combination of any states of mind corresponding to the choices of any pure actions in the support of the determined (rational) mixed strategy of the player is itself a possible state of mind for this player prior to his or her making an actual choice. Thus, the so-called "*quantum superposition*" is nothing but the expression of the fact that the player has not "made up his or her mind" until he or she is asked to do so in an experiment. Indeed, this property is nothing but the informational or epistemic characterization of the "indifference condition." Equivalently put, the wave function represents the real mental state of the player, in the sense that the relative height of the graphical representation of the probability distribution induced in equilibrium does *not* represent the relative likelihood that some agents are assigned to the two definite pure actions (definite states of mind) of the player. Rather, the relative height of the probability distribution is a *property* of the player representing his or her unsettled state of mind, before he or she makes an actual choice in an experiment. The twist is thus that the "indifference condition," which has long been regarded as a serious defect of the Nash equilibrium, has in fact a very intuitive meaning. It is indeed almost tautological to say that, prior to his or her definite choice in an experiment, a player must not have settled his or her mind on a particular action of support.

The upshot is thus that the "indifference condition" should *not* be interpreted as reflecting a randomization on the part of the players. A Nash equilibrium is a *self-interactive* solution concept, wherein no player expects any other player to randomize since they are all in a quantum superposition, as a result of their own determination of a strategy–belief pair. Third, as illustrated above in our "game-theoretic two-slit experiment," non-additive probabilities—which have often been regarded as the hallmark of quantum phenomena (see, e.g. Feynman et al. (1965))—enter the picture of the classical game model.

Only a few papers have tried to make a connection between classical game theory and QM. Pietarinen (2002) is the first to raise explicitly the possibility of some fundamental connections between extensive form games of imperfect information, quantum logic, and QM. More recently, Brandenburger (2010) establishes a formal connection between classical game theory with the non-local correlations arising in QM. Here, we have pointed out a fundamental connection between the two theories. This suggests the potential existence of an underlying principle behind QM, the notion of free choice in physics, and the ultimate nature of consciousness. (see, e.g. Conway and Kochen 2006, 2009).

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Decision-Making and Cognition Modeling from the Theory of Mental Instruments

Irina Basieva and Andrei Khrennikov

1 Introduction

The measurement theory plays a crucial role in quantum mechanics (QM). According to Bohr, QM is not about the microworld as it is, (Bohr 1987) (see also Plotnitsky 2006, 2009). It is about the results of our measurements performed for ensembles of microsystems (Bohr 1987):

> There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how Nature is. Physics concerns what we can say about Nature.

Everything that can be said about nature is obtained from measurements. Therefore creation of the formalism describing quantum measurements was one of the most important steps in the development of QM, see Von Neuman (1955) and Dirac (1995) for the first crucial contributions, and the later works of Davies and Lewis (1970), Holevo (2001), Ozawa (1984, 1997), and Busch et al. (1995). Note that all these contributions to the quantum measurement theory were based on advanced mathematics. And one of the aims of this chapter is to present this theory in a humanities friendly way.

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We emphasize that the cognitive and social sciences can also be treated as theories of measurements. Here a great deal of effort has been put into the development of measurement formalisms, compare with, for example, the time-honored *Stimulus-Organism-Response scheme* for explaining cognitive behavior (Woodworth 1921). As well as in quantum physics, cognitive and social scientists cannot approach the mental world directly; they work with the results of observations. Both quantum physics and the cognitive and social sciences are fundamentally based on the operational formalisms for observations. As we have emphasized from the start, the operational viewpoint regarding QM has already been presented by Bohr; see also the aforementioned papers (Davies and Lewis 1970; Holevo 2001; Ozawa 1984, 1997; Busch et al. 1995). Recent years have been characterized by attempts to derive the quantum theory from a few natural operational principles (D' Ariano 2007). The operational viewpoint can be found in works on cognition, decision-making, and psychology, for example, Machina (2003):

> While advances in neuroscience may ultimately do for decision theory what vivisection did for anatomy, decision theory currently remains very much a "black box" science. Although decision theorists can (and do) use introspection to suggest theories and hypotheses, the rigorous science consists of specifying mutually observable independent variables (in particular, the objects of choice available for selection), mutually observable dependent variables (the selected alternative), and refutable hypotheses linking the two.

From this viewpoint, QM is similar to the cognitive and social sciences, but with one crucial proviso: cognitive and social systems are able to perform *self-measurements* (Khrennikov 2010)—contrary to physical systems. In quantum physics an observer is always external with respect to a physical system exposed to measurement. This separation between an observer and a system plays an important role in quantum foundations, at least for physicists using the conventional Copenhagen interpretation of QM. However, human beings or even animals (or maybe even cells (Asano et al. 2012)) are able to ask questions to themselves and in this way to perform self-measurements. An ability for self-measurement makes an important difference between physics and cognition. Therefore by applying the quantum formalism to the cognitive and social sciences (as we plan to do) we have to reconsider the principles of quantum measurement theory. We have to understand whether it can be treated in such a way that it even covers self-measurement. We shall see that, in principle, this is possible by using the quantum model of *indirect measurements,* see Sect. 3 (although we understand well that our solution is formal and the problem of embedding self-measurements into quantum measurement theory is of huge complexity and in future will have to be studied in more detail).

The basic notion of the operational formalism for quantum measurement theory is *quantum instrument* (Davies and Lewis 1970; Ozawa 1984, 1997). Quantum instruments are mathematical structures representing at a high level of abstraction the physical apparatuses used for measurements. They encode the probabilities of the results of observations and the back-actions of measurements to the states of physical systems. Such back-actions are mathematically represented with the aid of an important mathematical structure, a quantum operation (state transformation). Our aim is to explore the theory of quantum instruments and especially the part devoted to *indirect measurement* in the cognitive and social sciences. The scheme of indirect measurement is especially useful in applications: both in quantum physics and the humanities.

In this scheme, apart from the "principal system" S, there is considered to be a probe system S' and a measurement on S which is composed of the unitary interaction with S' and measurement on the latter. This approach provides the possibility to extend the class of quantum measurements which originally were only von Neumann–Lüders measurements of the projection type. The aim of such an extension was not only the natural desire for generality. Generalized quantum measurements have some new features which we plan to explore in the cognitive and social sciences, and we will concentrate only on those features. There are other physically important features which we do not discuss in this chapter, because they are not relevant to our project on quantum-like cognition.

For us, one of the main problems of exploring solely projective (direct) measurements is their fundamentally invasive nature: as the feedback of a measurement, the quantum state is "aggressively modified"—it is projected onto the subspace corresponding to the result of this measurement. This feature of the projection measurement is often referred to as the *collapse of the wave function*. The notion of collapse is very controversial and the use of the projection postulate is still actively debated in quantum foundations (Ballentine 1990; Khrennikov and Basieva 2014). In any event, this feature is not so natural for the evolution of the mental state. Its "collapse" as the back-reaction to answering any question, solving any problem, and decision-making in general would make its evolution extremely discontinuous. This picture does not match (first of all) with the self-inspection of the evolution of our own mental states. Roughly speaking the use of the collapse model implies a collapse of the self.

In the scheme of indirect measurements, the state transformation ("quantum instrument") induced by a measurement can be essentially less invasive than the projective transformation. And we plan to explore this feature of generalized quantum measurements in the cognitive and social sciences.

In this chapter social systems will be treated as special cognitive systems. Therefore we will not put special effort into specifying the features of "social instruments," although we understand that there are differences in transformations of the belief-state of a brain and the belief-state of a social system. However, we proceed at such a level of abstraction that these differences are not so important. Therefore in this presentation we shall write about transformations of the belief-states of cognitive systems (resulting from measurements, including self-measurements), but we shall bear in mind the possibility of applications to transformations of the belief-states of social systems.

Another important issue for our applications feature of quantum instruments is that a variety of different quantum instruments describing back-reaction transformations resulting from measurements can correspond to the same observable on the principal system S. Thus the same statistics of measurement can correspond to very different state transformations (very different types of interaction between the principal and probe systems). In QM (as M. Ozawa emphasized (Ozawa 1997)) the same observable can be measured by different apparatuses having different state-transforming quantum instruments. This is a very important characteristic of the theory of generalized quantum measurements. It is also very useful for cognitive modeling, since it reflects the individuality of measurement apparatuses/instruments which are used by cognitive systems (e.g., human beings) to answer the same problem/question. Roughly speaking this individuality cannot be discovered from the statistics of answers to one question, because it is encoded in the post-answering states which can be very different for the same probability distribution of answers.

We would point out that the scheme of indirect measurements represents the state dynamics in the process of measurement and not just the "yes"/"no" collapse as in the original von Neumann–Lüders approach. The possibility of describing mathematically the dynamics of the mental state in the process of decision-making by using the quantum formalism is very attractive from the viewpoint of cognitive science and psychology. A study in this direction has already been presented in the work of (Pothos and Busemeyer 2013), although without appealing to the operational approach to QM. In the series of works of Asano et al. (2010a,b, 2011) and Ohya and Tanaka (2016) the process of decision-making was described by a novel scheme of measurements generalizing the standard theory of quantum instruments (Asano et al. 2010a,b, 2011; Ohya and Tanaka 2016).

As general references to studies in quantum-like modeling of cognition, decision-making, economics, and finance we may mention the monographs (Khrennikov 2010; Haven and Khrennikov 2012; Busemeyer and Bruza 2012). Note that nowadays generalized quantum observables represented by POVMs are widely used in quantum-like modeling, see, for example, Khrennikov (2010), Asano et al. (2010a), Khrennikov and Basieva (2014), and Khrennikov et al. (2014).

We proceed formally and avoid deep discussions about the self (of an individual and of a social system) and the interrelation of the unconscious and conscious, though we shall make a few remarks about them in Sect. 6.

2 Mental Instruments

The basics of the quantum formalism are presented in chapter "A Brief Introduction to Quantum Formalism" of this handbook (Khrennikov and Haven 2016); we shall use the notions of pure and mixed states, unitary transformations, partial trace, tensor product, and POVMs. We shall proceed with finite dimensional state spaces by making remarks on the corresponding modifications in the infinite dimensional case. As in Khrennikov and Haven (2016), $D(H)$ denotes the space of density operators in the complex Hilbert space H; $L(H)$ the space of all linear operators in H (bounded operators in the infinite dimensional case).

The space $L(H)$ can itself be endowed with the structure of the linear space. We also have to consider linear operators from $L(H)$ into itself; such maps, $T : L(H) \to L(H)$, are called *superoperators.* We shall use this notion only in Sect. 4. Thus, for the moment, the reader can proceed without it.

On the space $L(H)$ it is possible to introduce the structure of Hilbert space with the scalar product

$$\langle A|B\rangle = \mathrm{Tr} A^\star B.$$

Therefore, for each superoperator $T : L(H) \to L(H)$, there is defined its adjoint (super)operator $T^\star : L(H) \to L(H)$, $\langle T(A)|B\rangle = \langle A|T^\star(B)\rangle, A, B \in L(H)$.

Consider a cognitive system, specifically a human individual. She fronts some questions/problems, that is, she has to make decisions. In the quantum(-like) model the space of her mental states is represented by complex Hilbert space $\mathcal{H}$ (pure states are represented by normalized vectors and mixed states by density operators). In the model under construction $\mathcal{H}$ is tensor-factorized into two components, namely, $\mathcal{H} = H \otimes K$, where H is the space of *belief-states* and K is the space of *decision-states*. The states of the latter are open for *conscious introspection*, but the states of the former are in general not approachable consciously.

Suppose that this individual fronts some concrete question A with possible answers (decisions) labeled $a_i, i = 1, 2, \ldots, m$. We denote the set of possible values of A by the symbol O, that is, $O = \{a_1, .., a_m\}$. At this instance of time, she has the belief-state ρ (e.g., a pure state, i.e., $\rho = |\psi\rangle\langle\psi|, \psi \in H, \|\psi\| = 1$). To answer A, the individual will use a "mental instrument," denoted $\mathcal{A}$, which will produce the results (answers) a_i randomly with the probabilities $p(a_i|\rho)$, the output probabilities.[1] An instrument represents not only decisions and the corresponding probabilities, but also the results of the evolution of the initial belief-state ρ as induced by the back-reaction to the concrete decision a_i. This is a sort of state reduction, a "belief-state collapse," as the result of the concrete decision a_i; thus ρ is transformed into the output state ρ_{a_i}. However, as we shall see, in general this belief-state update can be sufficiently harmonious, so our model differs crucially from orthodox quantum models of cognition (Busemeyer and Bruza 2012) based on the projection-type state update. Thus each mental instrument $\mathcal{A}$ corresponding to the question/problem A is mathematically represented by

- probabilities for concrete decisions $p(a_i|\rho)$;
- transformations of the initial belief-state corresponding to the concrete results of measurements,

$$\rho \to \rho_{a_i}. \tag{1}$$

The rigorous mathematical description of such state transformations leads to the notion of a quantum instrument, see Sect. 4.

2.1 Mixing Law

In the quantum operational formalism it is assumed that these probabilities, $p(a_i|\rho)$, satisfy the *mixing law*. Note that, for any pair of states (density operators) ρ_1, ρ_2 and any pair of probability weights $q_1, q_2 \geq 0, q_1 + q_2 = 1$,

[1]We are moving toward the creation of a cognitive analog of the quantum operational model of measurements with the aid of physical apparatuses.

the convex combination $\rho = q_1\rho_1 + q_2\rho_2$ is again a state (density operator). In accordance with the mixing law any instrument produces probabilities such that

$$p(a_i|q_1\rho_1 + q_2\rho_2) = q_1 p(a_i|\rho_1) + q_2 p(a_i|\rho_2). \tag{2}$$

A probabilistic mixture of beliefs produces the mixture of probabilities for decision outputs. This is a very natural assumption, although an additional analysis of its validity in cognitive science and psychology has to be done.

2.2 Composition of the Instruments

It is natural to assume that after answering some question A a person is ready to answer another question B; such a sequence of decision-making is represented as a new mental instrument, the composition of the instruments $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{BA}$. Its outputs are ordered pairs of decisions (a_i, b_j). It is postulated that the corresponding output probabilities and states are determined as

$$p((a_i, b_j)|\rho) = p(b_j|\rho_{a_i})p(a_i|\rho); \tag{3}$$

$$\rho_{(a_i,b_j)} = (\rho_{a_i})_{b_j}. \tag{4}$$

The law (3) can be considered as the quantum generalization of Bayes' rule. The law (4) is the natural composition law.

3 Decision-Making Through Unitary Interaction Between the Belief and Decision-States

The above operational description of decision-making was formulated solely in terms of belief-states. However, a belief-state is a complex informational state which is in general unapproachable for conscious introspection. The operational representation of observables in the space of belief-states is not straightforward and in general it cannot be formulated in terms of mutually exclusive decisions. Later we shall consider this problem in more detail.

It is more fruitful to define the observable decision directly in terms of the question/problem A by using an additional state space, the space of the decision-states K. *In the decision space a question-observable can be defined as the standard von Neumann–Lüders projection observable.*

Example 1

Consider the simplest case. There is just one question A, it is dichotomous, that is, there are two possible outcomes of "mental measurement," "no"=0 and "yes"=1. This question can be represented by the pair of projectors (P_0, P_1) onto the subspaces K_0 and K_1 of the decision space K. Since the answers $a_0 = 0$ and $a_1 = 1$ are mutually exclusive, and sharply exclusive (later we shall discuss the meaning of sharp/unsharp in the measurement context), the subspaces K_0 and K_1 are orthogonal. Hence, the projectors P_0 and P_1 can be selected as orthogonal. The question A can be represented by the conventional von Neumann–Lüders observable $\hat{A} = a_0P_0 + a_1P_1(= P_1)$. However, we emphasize that this representation is valid only in the decision-state space K. It is often (but not always!) possible to proceed with one-dimensional projectors, that is, to represent possible decisions just by the basis vectors in the two-dimensional decision-state space, $(|0\rangle, |1\rangle)$. Here each decision-state can be represented as a superposition

$$\phi = c_0|0\rangle + c_1|1\rangle, \ |c_0|^2 + |c_1|^2 = 1. \tag{5}$$

Measurement of A leads to probabilities given by squared coefficients, $p_0 = |c_0|^2, p_1 = |c_1|^2$.

In general (in the case of the finite-dimensional decision-state) a question-observable A can be represented as

$$A = \sum_i a_i P_i, \tag{6}$$

where (P_i) is the family of mutually orthogonal projectors in the space of decision-states K, and (a_i) are real numbers encoding possible answers (decisions).

Now we shall explore the cognitive analog of the standard scheme of indirect measurements. Here "indirectness" means that the belief-states are in general unapproachable for conscious introspection. Therefore it is impossible to perform the direct measurement on the belief-state ρ (in particular, on a pure state $\rho = |\psi\rangle\langle\psi|$). Moreover, in the belief-state the alternatives, say "no"/"yes," encoded in a question-observer A, are not represented exclusively, for they can have overlap. Mathematically this situation is described as follows. In the belief space an observable A is represented as an unsharp observable of the POVM type. Roughly speaking, in the H-representation the A-"no" contains partially the A-"yes" and vice versa. The latter is simply a consequence of interpretation of POVM observables as unsharp observables.

Remark 1. To map the quantum physics scheme (Ozawa 1984, 1997) of indirect measurements onto the quantum(-like) cognition scheme, one has to associate the state of the principal physical system S with the belief-state and the state of the probe physical system S' with the decision-state. Note that in the cognitive framework we do not consider analogs of systems. In principle, one can consider the belief-system S as a part of the neuronal system representing human beliefs and the decision system S' as another part of the neuronal system representing possible decisions. The latter can be specified: different question-observables can be associated with different neuronal networks responsible for the corresponding decisions. However, in principle we need not associate belief and decision-states with special physical neuronal networks. Moreover, in the case of cognition, the usage of isolated physical systems as carriers of the corresponding information states might be ambiguous. The interconnectivity of neuronal networks is very high. (Of course, even in physics the notion of an isolated system is just an idealization of the real situation.) Therefore it is useful to proceed with the purely information approach by operating solely with states, without coupling them to bio-physical systems. This is, in fact, the quantum information approach, where systems play the secondary role, and one operates with states, especially for the information interpretation of QM (Zeilinger 2010).

Typically, at the beginning of the process of decision-making, the belief and decision-states, ρ and σ, are not entangled.[2] Thus mathematically (in accordance with the quantum formalism) the integral belief-decision-state, the complete mental state (corresponding to the problem under consideration), can be represented as the tensor product $R = \rho \otimes \sigma$. It is natural to suppose that the initial decision-state σ is, in fact, a pure state represented by a superposition of possible decisions, see, for example, (5): the situation of complete uncertainty is represented by superposition

$$\phi = (|0\rangle + |1\rangle)/\sqrt{2}. \tag{7}$$

In the process of decision-making, the belief and decision-states (cf. Remark 1) "interact" and the evolution of the belief-decision-state R is mathematically represented by a unitary operator[3] $U : \mathcal{H} \to \mathcal{H}$:

$$R \to R_{\text{out}} \equiv URU^{\star}. \tag{8}$$

[2]One can say that they are independent. But one needs to use this terminology carefully, since the notion of quantum independence is more complicated than the classical one and is characterized by a diversity of approaches.

[3]As was mentioned, in the works of Asano et al. (2010a,b, 2011) and Ohya and Tanaka (2016) even non-unitary evolutions were in charge.

In the space of belief-decision-states $\mathcal{H}$ the question-observer A is represented by the operator $I \otimes A$. Thus the probabilities of decisions are given by

$$p_{a_i}^{A\otimes I} = \mathrm{Tr} R_{\mathrm{out}}(I \otimes P_i) = \mathrm{Tr} URU^{\star}(I \otimes P_i), \tag{9}$$

where the projectors (P_i) form the spectral decomposition of the Hermitian observable A in K; see (6). Since only the decision-state belonging to K is a subject of conscious introspection, at the conscious level the decision process can be represented solely in the state space K. The post-interaction decision-state σ_{out} can be extracted from the integral state R_{out} with the aid of the operation of the partial trace:

$$\sigma_{\mathrm{out}} = \mathrm{Tr}_H R_{\mathrm{out}}. \tag{10}$$

Then the question answering can be represented as the result of the A-measurement (measurement of the projection type) in the decision space, the measurement on the output state σ_{out}. The probabilities of the answers (a_i) are given by the standard Born rule:

$$p_{a_i}^{A} = \mathrm{Tr}_K \sigma_{\mathrm{out}} P_i = \mathrm{Tr}_K(\mathrm{Tr}_H URU^{\star})P_i = \mathrm{Tr} URU^{\star}(I \otimes P_i) = p_{a_i}^{A\otimes I}. \tag{11}$$

Thus (9) and (11) match each other.

What happens in the belief space? The expression (9) for the probability of the decision a_i can be represented as

$$p(a_i|\rho) = p_{a_i}^{A\otimes I} = \mathrm{Tr} U\rho \otimes \sigma U^{\star}(I \otimes P_i) = \mathrm{Tr}\rho \otimes \sigma U^{\star}(I \otimes P_i)U = \mathrm{Tr}_H \rho M_{a_i}, \tag{12}$$

where

$$M_{a_i} = \mathrm{Tr}_K(I \otimes \sigma)U^{\star}(I \otimes P_i)U. \tag{13}$$

The operator $M_i; H \to H$ can also be represented in the following useful form (a consequence of the cyclic property of the trace operation):

$$M_{a_i} = \mathrm{Tr}_K U^{\star}(I \otimes P_i)U(I \otimes \sigma) \tag{14}$$

Note that (13) implies:

$$\sum_i M_{a_i} = \mathrm{Tr}_K(I \otimes \sigma)U^{\star}\left(I \otimes \sum_i P_i\right)U = \mathrm{Tr}_K I \otimes \sigma = (\mathrm{Tr}_K \sigma)I.$$

Note also that each operator M_{a_i} is positively defined and Hermitian. Thus in the belief space the decision-observable of the projection-type A (acting in K) with the spectral family (P_i) is represented as POVM $M = (M_i)$. Note that in general the operators M_i are not projectors. Such measurement cannot separate sharply the answers (decisions) (a_i) for different i. In the decision a_i there are nontrivially present beliefs that the decision might even be $a_j, j \neq i$.

The operational formalism also gives the post-decision belief-state

$$\rho_{a_i} = \frac{\mathrm{Tr}_K U(\rho \otimes \sigma)U^\star(I \otimes P_i)}{\mathrm{Tr}U(\rho \otimes \sigma)U^\star(I \otimes P_i)}. \tag{15}$$

The output belief-state depends not only on the initial belief-state ρ, but also on the initial decision-state σ, interaction between beliefs and possible decisions given by U, and the question-observable A acting in K.

4 General Viewpoint Regarding State Transformers: Quantum Instruments

The considered model of decision-making, as the result of unitary interaction between the belief-state and the decision-state, describes an important class of transformations of the belief-state, see (15). We now turn to the general case which was considered in Sect. 2, see (1). Set

$$\mathcal{E}(a_i)\rho = p(a_i|\rho)\rho_{a_i} \tag{16}$$

and, for a subset Γ of O, where $O = \{a_1, \ldots, a_m\}$ is the set of all possible answers (decisions), we set

$$\mathcal{E}(\Gamma)\rho = \sum_{a_i \in \Gamma} \mathcal{E}(a_i)\rho = \sum_{a_i \in \Gamma} p(a_i|\rho)\rho_{a_i}. \tag{17}$$

We point to the basic feature of this map:

$$\mathrm{Tr}\mathcal{E}(O)\rho = \sum_{a_i \in O} p(a_i|\rho)\mathrm{Tr}\rho_{a_i} = 1. \tag{18}$$

For each concrete decision a_i, $\mathcal{E}(a_i)$ maps density operators to linear operators (in the infinite dimensional case, these are trace-class operators, but we proceed in the finite dimensional case, where all operators have finite traces). The mixing law implies that, for any $\Gamma \subset O$,

$$\mathcal{E}(\Gamma)(q_1\rho_1 + q_2\rho_2) = q_1\mathcal{E}(\Gamma)\rho_1 + q_2\mathcal{E}(\Gamma)\rho_2. \quad (19)$$

As was shown by Ozawa (1997), under the assumption of the existence of composition of the instruments, any such map $\mathcal{E}(\Gamma) : D(H) \to L(H)$ can be extended to a linear map (superoperator)

$$\mathcal{E}(\Gamma) : L(H) \to L(H) \quad (20)$$

such that:

- each $\mathcal{E}(\Gamma)$ is positive, i.e., it transfers the set of positively defined operators into itself;
- $\mathcal{E}(O) = \sum_i \mathcal{E}(a_i)$ is trace preserving:

$$\mathrm{Tr}\mathcal{E}(O)\rho = \mathrm{Tr}\rho. \quad (21)$$

The latter property is a consequence of (18).[4]

Thus the two very natural and simple assumptions, the mixing law for probabilities and the existence of composite instruments, have the fundamental mathematical consequence of being the representation of the evolution of the state by a superoperator (20).

In quantum physics such maps are known as state transformers (Busch et al. 1995) or DL (Davies and Lewis 1970) quantum operations.[5] Thus each decision induces the back-reaction which can be formally represented as a state transformer. In these terms

$$\rho_{a_i} = \frac{\mathcal{E}(a_i)\rho}{\mathrm{Tr}\mathcal{E}(a_i)\rho} \quad (22)$$

[4]If one wants to continue $\mathcal{E}(\Gamma)$ from the set of density operators to the set of all linear operators (in the infinite dimensional case it has to be the set of finite-trace operators) by linearity then it has to be set as $\mathcal{E}(\Gamma)\mu = \mathcal{E}(\Gamma)\mathrm{Tr}\mu(\mu/\mathrm{Tr}\mu) = \mathrm{Tr}\mu\ \mathcal{E}(\Gamma)(\mu/\mathrm{Tr}\mu)$ and, in particular, $\mathcal{E}(O)\mu = \mathrm{Tr}\mu\ \mathcal{E}(O)(\mu/\mathrm{Tr}\mu) = \mathrm{Tr}\mu$.

[5]DL notion of the quantum operation is more general than the notion used nowadays. The latter is based on complete positivity, instead of simply positivity as the DL notion, see Sect. 5 for the corresponding definition and a discussion on whether the reasons used in physics to restrict the class of state transformers can be automatically used in cognitive science.

Note that the map $\Gamma \rightarrow L(L(H))$, from subsets of the set of possible decisions O into the space of superoperators, is additive:

$$\mathcal{E}(\Gamma_1 \cup \Gamma_2) = \mathcal{E}(\Gamma_1) + \mathcal{E}(\Gamma_2), \;\; \Gamma_1 \cap \Gamma_2 = \emptyset. \tag{23}$$

This is a measure with values in the space $L(L(H))$. Such measures are called (DL) instruments (Davies and Lewis 1970). To specify the domain of applications in our case, we shall call them *mental instruments.*

The class of such instruments is essentially wider than the class of instruments based on the unitary interaction between belief and the decision components of the mental state, see (15). The evident generalization of the scheme of Sect. 3 is to consider nonunitary interactions between the components of the mental state. Another assumption which can evidently be violated in the modeling of cognition is that the initial belief and decision states are not entangled ("independent"), see Asano et al. (2010a,b, 2011) and Ohya and Tanaka (2016) for generalizations of the aforementioned scheme.

We start with a discussion on the possible *nonunitarity of interaction between the belief and the decision states.* In quantum physics the assumption of unitarity of interaction between the principal system S and the probe system S' (representing a part of the measurement apparatus interacting with S) is justified, because the compound system $S + \tilde{S}'$ can be considered (with a high degree of approximation) as an isolated quantum system, and its evolution can be described (at least approximately) by the Schrödinger equation. The latter induces the unitary evolution of a state.

In cognition the situation is totally different. The main scene of cognition is not the physical space–time, but the brain. And it is characterized by huge interconnectivity and parallelism of information processing. Therefore it is more natural to consider the belief and decision states corresponding to different questions (problems) as interacting, especially at the level of the belief-states. Thus the decision-making model based on the assumption of isolation of different decision-making processes one from another seems to be too idealized, although it can be used in many applications, where the concentration on one fixed problem may diminish the influence of other decision-making processes.

In physics the assumption that the initial state of the system $S + \tilde{S}'$ is factorized is also justified, since the exclusion of the influence of the state of the measurement device to the state of a system S prepared for measurement (and vice versa) is the experimental routine. In cognition the situation is more complicated. One cannot exclude that in some situations *the initial belief and decision state are entangled.*

The representation of probabilities with the aid of POVMs is not a feature of only the unitary interaction representation of instruments, see (12). In general any DL instrument generates such a representation. Take an instrument $\mathcal{E}$, where, for each $a_i \in O, \mathcal{E}(a_i) : L(H) \to L(H)$ is a superoperator. Then we can define the adjoint operator $\mathcal{E}^\star(a_i) : L(H) \to L(H)$. Set $M_{a_i} = \mathcal{E}^\star(a_i)I$, where $I : H \to H$ is the unit operator. Then, since $p_{a_i} = \mathrm{Tr}\mathcal{E}(a_i)\rho = \mathrm{Tr}\ I; \mathcal{E}(a_i)\rho = \langle I|\mathcal{E}(a_i)\rho\rangle == \langle \mathcal{E}^\star(a_i)I|\rho\rangle = \mathrm{Tr}(\mathcal{E}^\star(a_i)I)\rho = \mathrm{Tr}M_{a_i}\rho$. By using the properties of an instrument it is easy to show that M_{a_i} is a POVM. Thus each mental instrument can be represented by a POVM. We interpret this POVM as the mathematical representation of "unconscious measurements" which the brain performs on the belief states as the back-reactions to decisions which are taken at the conscious level. Such "unconscious measurements" are not sharp, they cannot separate completely different decisions a_i which are mutually exclusive at the conscious level.

5 On the Role of the Condition of Complete Positivity in the Operational Approach to Cognition

Nowadays the theory of DL instruments is considered old-fashioned; the class of such instruments is regarded as too general: it contains mathematical artifacts which have no relation to real physical measurements and state transformations as back-reactions to these measurements. The modern theory of instruments is based on the extendability postulate invented by Ozawa (1984), see also the textbook of Nielsen and Chuang (2000):

For any instrument $\mathcal{A}_S$ corresponding to the measurement of observable A on a system S and any system $\tilde{S}$ non-interacting with S there exists an instrument $\mathcal{A}_{S+\tilde{S}}$ representing measurement on the compound system $S + \tilde{S}$ such that

- $p(a_i|\rho \otimes r) = p(a_i|\rho)$;
- $(\rho \otimes r)_{a_i} = \rho_{a_i} \otimes r$

for any state ρ of S and any state r of $\tilde{S}$.

This postulate is very natural: if, besides the quantum system S which is the object of measurement, there is (somewhere in the universe) another system $\tilde{S}$ which is not entangled with S, that is, their joint pre-measurement state has the form $\rho \otimes r$, then the measurement on S with the result a_i can be considered as a measurement on $S + \tilde{S}$ as well with the same result a_i. It is clear that the back-reaction cannot change the state of $\tilde{S}$. Surprisingly this very trivial assumption has tremendous mathematical implications.

Since we proceed only in the finite dimensional case, the corresponding mathematical considerations are simplified. Consider an instrument $\mathcal{E}_S$ representing the state update as the result of the back-reaction from measurement on S. For each Γ, this is a linear map from $L(H) \to L(H)$, where H is the state space of S. Let W be the state space of the system $\tilde{S}$. Then the state space of the compound system $S + \tilde{S}$ is given by the tensor product $H \otimes W$. Note that the space of linear operators in this state space can be represented as $L(H \otimes W) = L(H) \otimes L(W)$. Then the superoperator $\mathcal{E}_S(\Gamma) : L(H) \to L(H)$ can be trivially extended to the superoperator $\mathcal{E}_S(\Gamma) \otimes I : L(H \otimes W) \to L(H \otimes W)$. It is easy to prove that the state transformer corresponding to the instrument for measurements on $S + \tilde{S}$ has to have this form $\mathcal{E}_{S+\tilde{S}}(a_i) = \mathcal{E}_S(a_i) \otimes I$. Hence, this operator also has to be positively defined. Note that if the state space W has the dimension k, then the space of linear operators $L(W)$ can be represented as the space of $k \times k$ matrices which is further denoted as $\mathbf{C}^{k\times k}$.

Formally, a superoperator $T : L(H) \to L(H)$ is called *completely positive* if it is positive and its trivial extension $T \otimes I : L(H) \otimes \mathbf{C}^{k\times k} \to L(H) \otimes \mathbf{C}^{k\times k}$ is also positive. There are natural examples of positive maps which are not completely positive (Nielsen and Chuang 2000).

A CP quantum operation is a DL quantum operation which is additionally completely positive; a CP instrument is based on CP quantum operations representing back-reactions to measurement. As was pointed out, in modern literature only CP quantum operations and instruments are in use, so they are called simply quantum operations and instruments.

The main mathematical feature of (CP) quantum operations is that the class of such operations can be described in a simple way, namely, with the aid of the Kraus representation (Ozawa 1984, 1997):

$$T\rho = \sum_j V_j^{\star} \rho V_j, \tag{24}$$

where (V_j) are some operators acting in H. Hence, for a (CP) instrument, we have: for each $a_i \in O$, there exist operators (V_{a_ij}) such that

$$\mathcal{E}(a_i)\rho = \sum_j V_{a_ij}^{\star} \rho V_{a_ij}. \tag{25}$$

Thus

$$\rho_{a_i} = \frac{\sum_j V^\star_{a_ij} \rho V_{a_ij}}{\sum_j V^\star_{a_ij} \rho V_{a_ij}}, \tag{26}$$

where the trace one condition (18) implies that

$$\sum_i \sum_j V^\star_{a_ij} V_{a_ij} = I. \tag{27}$$

The corresponding POVMs M_{a_i} can be represented as

$$M_{a_i} = \sum_j V^\star_{a_ij} V_{a_ij}. \tag{28}$$

This is really elegant mathematical representation. However, it might be that this mathematical elegance and not the real physical situation has contributed to the widespread adoption of CP in quantum information theory, cf. Shaji and Sudarshan (2005).

Is the use of the extendability postulate justified in the operational approach to cognition?

It seems not (although this is a complex problem and further analysis has to be done). Any concrete decision is made at the conscious level and it is based on interaction with beliefs related to the problem under consideration (question-observable A). The state of these beliefs corresponds to the state of the system S in the above considerations. To be able to consider the state of another group of beliefs, the analog of the state of the system $\tilde{S}$, the brain has to *activate* these beliefs. Thus we cannot simply consider all possible beliefs as existing in some kind of mental universe simultaneously and, hence, we cannot interpret a mental measurement based on one special group of beliefs, with the state space H, as extended to a mental measurement in combination with another group of beliefs.

It seems more natural to develop a theory of mental instruments as a theory of DL instruments and not CP instruments. In particular, although the Kraus representation can be used as a powerful analytic tool, one has not to overestimate the possibility of its use in the modeling of cognition.

6 Concluding Remarks

In this chapter we have presented a mental version of the quantum operational approach. This approach can be applied for a large range of the problems in cognitive science, psychology, social science, politics, economics, and finance. In the operational approach, quantum systems are considered as black boxes; compare with the citation from Machina (2003) in the introduction to this chapter. In quantum physics the Copenhagen position is commonly accepted: it is in principle impossible (forbidden) to open these black boxes (Zeilinger 2010). To support this position, physicists use various no-go theorems, for example, von Neumann's theorem or Bell's theorem. In this, note that we did not question this position; compare with, however, Khrennikov (2010). The self is defined operationally as a system performing sequential self-measurements which produce probabilistically and in accordance with the laws of quantum probability and information theory updates of mental states (in particular, belief-states).

Finally, we would remark that in quantum physics the operational theory of measurements (Davies and Lewis 1970; Holevo 2001; Ozawa 1984, 1997; Busch et al. 1995) describes all possible measurements. It is not self-evident that the same operational model (based on the same mathematical structures) would describe completely, for example, cognitive measurements. Although the quantum-like modeling of cognition and social behavior is very successful, there are some pitfalls which can make the realization of the project "quantum-like cognition" more difficult than was expected at the beginning. In particular, the possibility of covering all measurements in cognitive psychology by the operational quantum formalism was questioned in Khrennikov et al. (2014).

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Adaptive Dynamics and an Optical Illusion

Masanori Ohya and Yoshiharu Tanaka

1 Introduction

Optical illusion is one of the fundamental phenomena depending on experimental contexts (settings). Figure 1 is called a Schröder stair (Schröder 1858). We can see stairs in the middle of the figure, and it has two possible ways of observing it: One way is that "the left part (L) is front and the right part (R) is back," and another way is its converse. Such visual perception is called *bistable perception* (Atmanspacher et al. 2004). Experimentally, it is confirmed that the tendency of the perception depends on the angle θ. In $\theta = 0$, most of us see that (L) is front. Oppositely, in the $\theta = 90$, most of us see that (R) is front.

Such biased perception is induced due to our experience that we have not seen the top-heavy stairs. We can naturally select the most reasonable perception. The bistable perceptions are considered as the result of decision-making in the visual perception process for ambiguous stimulus. When angle θ is around 45°, we feel more ambiguity, and such fluctuation of bistable perceptions is seen in the probability of an answer by a subject.

In this study, we explain the context dependency of the optical illusion within non-classical probability theory (non-Kolmogorov probability theory).

Recently, it has been reported that there exist the experimentally obtained data which violate the laws of classical (Kolmogorov) probability theory. Such

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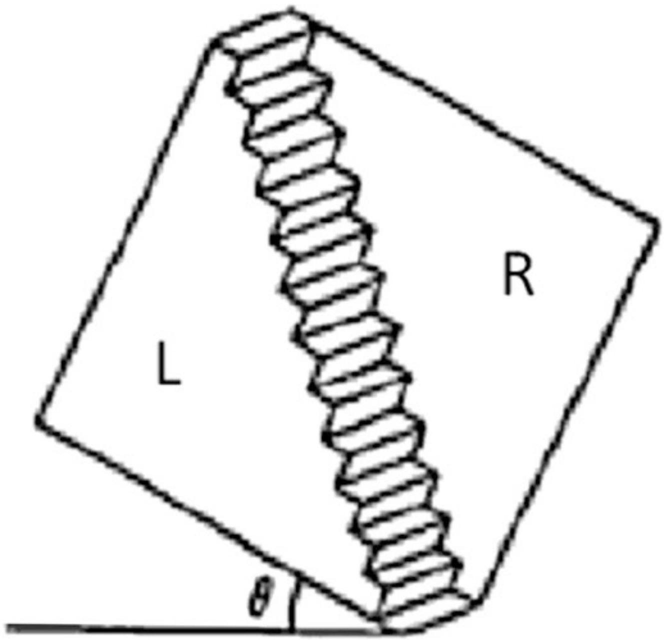

Fig. 1 Schröder Stair is an ambiguous figure which has two different interpretations, "the left part (*L*) is front and the right part (*R*) is back," and its converse. Humans perceive either of them, and the tendency of the perception depends on the rotation angle θ

violation is often seen in the experiment of context dependent phenomena, such as *Escherichia coli*'s selective metabolism of sugar (Asano et al. 2012c,d). So far, mathematical treatment of such experimental data has been investigated.

First of all, in Sects. 2–4 we introduce the idea of adaptive dynamics (AD) (Ohya and Volovich 2011; Ohya 2008; Asano et al. 2013a), which has been used to study the context dependent phenomena (Asano et al. 2014). The concept of AD is used for the study of chaos (Kossakowski et al. 2003; Ohya and Volovich 2011), the satisfiability problem (SAT) (Ohya and Volovich 2003, 2011; Accardi and Ohya 2004), and so on. In Sect. 2 we explain the conceptual meaning of AD. In Sect. 3 we explain the mathematical framework of AD in order to discuss the definition of non-Kolmogorov joint probability for the study of context dependent phenomena (Asano et al. 2013a, 2012c,d). In Sect. 4, we briefly introduce the applications of AD.

Further, in Sect. 5, we demonstrate the experimental data of the optical illusion of the Schröder stair. In our tests, we rotate the figure, that is, change the angle θ, gradually, and ask how the perception is changed, (L)

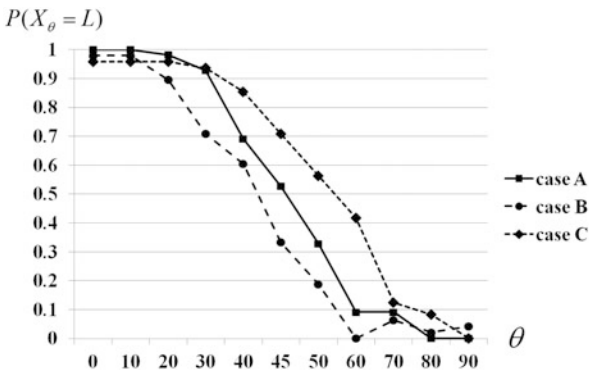

Fig. 2 Optical illusion is affected by memory bias: the subject's perception is shifted in response to the rotation direction of the figure

or (R). We found that the way of seeing the Schröder stair depends on the direction of rotation. Remarkably, there is a difference between perceptions in clockwise rotation and in counterclockwise rotation. We will show the experimentally observed difference in Fig. 2 in Sect. 5. Note that the direction of rotation is considered as the context of this cognitive experiment. The context dependency of the visual perception process was observed in the result of our tests.

In Sect. 6, we quantitatively explain the context dependency within the framework of non-Kolmogorov probability theory, which contains the usual quantum probability theory. We have the statistical probability of the data of our tests, for example joint probability of the answers for the figures at the several different angles. We check if the probabilistic data of our test violate the inequality which is derived under the assumption that there exists a common joint probability regardless of rotation directions. Such inequality is obtained by generalizing the Leggett–Garg inequality (Leggett and Garg 1985) from the view point of contextuality.

In Sect. 7, we propose the non-Kolmogorov probabilistic model of bistable perceptions. This model is based on the framework of AD which is explained in Sect. 3.2. From a mathematical point of view, the map of lifting and the operator function are used for modeling the process of visual perception.

2 Conceptual Meaning of Adaptive Dynamics

In physics the mathematical formalization of the adaptive dynamics (AD) has implicitly appeared in a series of papers (Ohya 1983b, 1998; Accardi and Ohya 1999; Inoue et al. 2000; Ohya and Volovich 2003, 2011; Inoue et al. 2002;

Kossakowski et al. 2003; Accardi and Ohya 2004; Ohya 2008) for the study of compound quantum dynamics, chaos, and the quantum realization of the algorithm on the satisfiability problem (*SAT algorithm*). The name of AD was deliberately used in Ohya (2008). AD has two aspects, one of which is the "observable-adaptive," another is the "state-adaptive." We now present very general statements about these two types of adaptivity.

The observable-adaptive dynamics is a dynamics characterized as follows:

1. *Measurement depends on how to see an observable to be measured.*
2. *The interaction between two systems depends on how a fixed observable exists, that is, the interaction is related to some aspects of observables to be measured or prepared.*

The state-adaptive dynamics is characterized as follows:

1. *Measurement depends on how the state and observable to be used exist.*
2. *The correlation between two systems depends on how the state of at least one of the systems exists, e.g., the interaction Hamiltonian depends on the state.*

3 Mathematical Representation of Adaptive Dynamics

3.1 Lifting and Channels

In this section, we present the notions of *lifting* (Accardi and Ohya 2004) and *channels* which are advanced tools of quantum information (Accardi and Ohya 1999). They will play a basic role in AD. For simplicity, we proceed in the finite dimensional case.

Let $\mathcal{H}$ be a complex Hilbert space. Denote the spaces of linear operators acting in this space by the symbol $\mathcal{O}(\mathcal{H})$. (Thus $A : \mathcal{H} \to \mathcal{H}$, linearity means that A transforms linear combinations of vectors into linear combinations. By fixing an orthonormal basis in $\mathcal{H}$ we can represent it as $\mathbb{C}^n$ and $\mathcal{O}(\mathcal{H})$ as the space of all $n \times n$ complex matrices. However, we prefer to use the operator terminology, which can be applied even to infinite dimensional Hilbert spaces.) The space of all quantum states, that is, density operators in $\mathcal{H}$, is denoted by the symbol $S(\mathcal{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces and let $\mathcal{H} \otimes \mathcal{K}$ be their tensor product. We shall also use $\mathcal{O}(\mathcal{H} \otimes \mathcal{K})$ as the space of observables for the compound system.

Definition. Quantum-like lifting is a map

$$\mathcal{E}^* : S(\mathcal{H}) \to S(\mathcal{H} \otimes \mathcal{K}).$$

A Lifting is called pure if it maps pure states onto pure states.

We shall also use a very general definition of a quantum channel:

Definition. A quantum-like channel is a map

$$\Lambda^* : S(\mathcal{H}) \to S(\mathcal{H}).$$

A channel is called pure if it maps pure states onto pure states.

In the same way, we define a channel from $S(\mathcal{H})$ to $S(\mathcal{K})$. Note also that, in fact, each channel can be represented as a lifting by selecting one of the Hilbert spaces as the one-dimensional complex space $\mathbb{C}$. The reader can check this by taking into account that, for an arbitrary Hilbert space $\mathcal{H}$, the tensor product $\mathcal{H} \otimes \mathbb{C}$ can be identified with $\mathcal{H}$.

We now present some important examples of liftings.

Example 1

Isometric lifting: Let $V : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ be an isometry, that is, a linear map satisfying the restriction

$$V^* V = I,$$

where $I \equiv I_{\mathcal{H}}$ is the unit map in $\mathcal{H}$ and V^* denotes the adjoint operator. Then the map

$$\mathcal{E}^* \rho = V \rho V^*$$

is a lifting. Liftings of this type are called isometric. Every isometric lifting is a pure lifting.

Example 2

Reduction (open system dynamics): If a system Σ_1 interacts with an external system Σ_2 described by another Hilbert space $\mathcal{K}$ and the initial states of Σ_1 and Σ_2 are ρ_1 and ρ_2, respectively, then the combined state θ_t of Σ_1 and Σ_2 at time t after the interaction between the two systems is given by

$$\theta_t \equiv U_t(\rho_1 \otimes \rho_2)U_t^*,$$

where $U_t = \exp(-itH)$ with the total Hamiltonian H of Σ_1 and Σ_2. A channel is obtained by taking the partial trace w.r.t. $\mathcal{K}$ such as

$$\rho_1 \to \Lambda^*\rho_1 \equiv \mathrm{tr}_{\mathcal{K}}\theta_t.$$

Example 3

The projected quantum measurement process is written, in the terminology of lifting, as follows. Any observable can be represented as $A = \sum_n a_n\pi_n$, where $\pi_n, n = 1, \ldots, m$, are the projectors on the eigensubspaces corresponding to the eigenvalues a_n (we state again that we work in the finite dimensional case). In accordance with the projection postulate, the postmeasurement state (if the result of measurement is not specified) will be

$$\Lambda^*\rho = \sum_n \pi_n\rho\pi_n,$$

where ρ is the input state. This is a channel. Now we fix a discrete probability μ. This is given by a sequence of weights, say $\mu_n \geq 0, \sum_n \mu_n = 1$. The lifting $\mathcal{E}^*$ associated with this channel Λ^* and with a fixed decomposition of ρ as

$$\rho = \sum_n \mu_n\rho_n$$

where $\rho_n \in S(\mathcal{H})$ is given by

$$\mathcal{E}^*\rho = \sum_n \mu_n\rho_n \otimes \Lambda^*\rho_n.$$

3.2 Joint Probability

In this section, we will discuss how to use the concept of lifting to explain phenomena which break the usual probability law.

The adaptive dynamics implies that the dynamics of a state or an observable after an instant (say the time t_0) attached to a system of interest is affected by the existence of some other observable and state at that instant. Let $\rho \in \mathcal{S}(\mathcal{H})$ and $A \in \mathcal{O}(\mathcal{H})$ be a state and an observable before t_0, and let $\sigma \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ and $Q \in \mathcal{O}(\mathcal{H} \otimes \mathcal{K})$ be a state and an observable to give an effect to the state ρ and the observable A. In many cases, the effect to the state is dual with that to the observable, so that we will only discuss the effect to the state. This effect is described by a lifting $\mathcal{E}^*_{\sigma Q}$, so that the state ρ first becomes $\mathcal{E}^*_{\sigma Q}\rho$, before becoming $\mathrm{tr}_{\mathcal{K}}\mathcal{E}^*_{\sigma Q}\rho \equiv \rho_{\sigma Q}$. The adaptive dynamics here is the whole process such as

$$\text{Adaptive Dynamics}: \ \rho \Rightarrow \mathcal{E}^*_{\sigma Q}\rho \Rightarrow \rho_{\sigma Q} = \mathrm{tr}_{\mathcal{K}}\mathcal{E}^*_{\sigma Q}\rho.$$

That is, what we need to know is how to construct the lifting for each problem to be studied, that is, we properly construct the lifting $\mathcal{E}^*_{\sigma Q}$ by choosing σ and Q properly. The expectation value of another observable $B \in \mathcal{O}(\mathcal{H})$ in the adaptive state $\rho_{\sigma Q}$ is

$$\mathrm{tr}\rho_{\sigma Q}B = \mathrm{tr}_{\mathcal{H}}\mathrm{tr}_{\mathcal{K}}B \otimes I\mathcal{E}^*_{\sigma Q}\rho.$$

Now suppose that there are two quantum event systems

$$\mathcal{A} = \{a_k \in \mathbb{R}, F_k \in \mathcal{P}(\mathcal{H})\},$$
$$\mathcal{B} = \left\{b_j \in \mathbb{R}, E_j \in \mathcal{P}(\mathcal{K})\right\},$$

where we do not assume F_k, E_j are projections, but that they satisfy the conditions $\sum_k F_k = I$, $\sum_j E_j = I$ as positive operator valued measures corresponding to the partition of a probability space in the classical system. Then the "joint-like" probability obtaining a_k and b_j might be given by the *formula*

$$P(a_k, b_j) = \mathrm{tr}E_j \boxdot F_k\mathcal{E}^*_{\sigma Q}\rho, \tag{1}$$

where $\boxdot$ is a certain operation (relation) between A and B. More generally, one can take a certain operator function $f(E_j, F_k)$ instead of $E_j \boxdot F_k$. If σ, Q are independent of any F_k, E_j and the operation $\boxdot$ is the usual tensor product $\otimes$, so that A and B can be considered as two independent systems or to be commutative. The above "joint-like" probability then becomes the joint probability. However, if this is not the case, for example, Q is related to A and B, the situation will be more subtle. Therefore, the problem is how to set the operation $\boxdot$ and how to construct the lifting $\mathcal{E}^*_{\sigma Q}$ in order to describe the particular problems associated with the systems of interest. In the following, we will discuss this problem in the context-dependent systems of visual perception in optical illusions.

4 Applications of Adaptive Dynamics

As shown in Sect. 2, the adaptive dynamics has two aspects, one of which is the "observable-adaptivity" and the other is the "state-adaptivity."

The idea of observable-adaptivity comes from studying chaos. We have claimed that any observation will be unrelated to or even contradicted by mathematical universalities such as taking limits, supremum, and infimum. Observation of chaos is a result due to taking suitable scales of, for example, time, distance, or domain, though it will not be possible in the limiting cases. Examples of observable-adaptivity are used to understand chaos (Ohya 1998; Kossakowski et al. 2003) and examine the violation of Bell's inequality, namely the chameleon dynamics of Accardi (1997). The idea of state-adaptivity is implicitly initiated when constructing a compound state for quantum communication (Ohya 1983a,b, 1989; Accardi and Ohya 1999) Examples can be seen in the algorithm for solving NP-complete problems, which is an issue that has been pending for more than 30 years, and which asks whether there exists an algorithm that will solve an NP-complete problem in polynomial time, as discussed (Ohya and Volovich 2003, 2011; Accardi and Ohya 2004).

We will discuss, in Sect. 7, how we can apply adaptive dynamics to the cognitive system of visual perception. The concept of adaptivity naturally exists in such systems. Our formulation here contains some treatments shown in the book (Ohya and Volovich 2011) to understand the evolution of HIV-1, the brain function, and the irrational behavior of prisoners.

5 Method and Results of Experiment

In this section, we explain the tests of optical illusion of rotating the Schröder stair. The subjects in the test were 151 students of the Department of Information Science, Tokyo University of Science. They were divided into three groups ($n_A = 55, n_B = 48, n_C = 48$). For all three, we showed 11 pictures which were leaning at different angles. Subjects answered $L =$"I can see that the left side is front," or they answered $R =$"I can see that the right side is front" for each picture. For the first group (A), the order of showing the pictures was randomly selected for each subject. For the second group (B), the angle θ changed from 0 to 90 as if the picture was rotating clockwise. Inversely, for the third group (C), the angle θ was changed from 90 to 0. As a result, we obtained a tendency of perceptions with respect to angles; see Fig. 2.

We denote the probability that a subject answers "Left side is front" by $P(X_\theta = L)$. We can say that the probability $P(X_0 = L)$ is approximately equal to one. That is, practically every subject says "Left side is front"at the angle $\theta = 0$. Conversely, the probability $P(X_{90} = L)$ is approximately equal to zero, that is, practically every subject says "Right side is front" at the angle $\theta = 90$. We can interpret the result as the fact that subjects feel little ambiguity regarding the picture for $\theta = 0$ and 90. In fact, it takes less than 1 s for a subject to make her or his decision when the angle θ is around 0 or 90°. However, in the middle range of the angle, we can see the difference among the three groups.

6 How to Find the Non-Kolmogorov Property of Collected Data: The Violation of Classical Probabilistic Law

In this section, we show the non-Kolmogorov property of the data. As shown in the previous section, we have 11 observables (random variables) X_θ. Now we check whether we have found a common joint probability of X_θs.

Here we introduce the inequality of Leggett and Garg (LG) (Leggett and Garg 1985). The LG inequality is classical (or Kolmogorov axiomatic) probabilistic law with respect to the correlation functions of time sequential measurements, which we will explain shortly in Sect. 6.1. However, to be matched in our test, we need some generalization of the LG inequality from the viewpoint of contextuality. In Sect. 6.2, we will discuss the generalization to check the existence of the joint probability for our tests. In Sect. 6.3, we confirm that the statistical data of our tests violate the contextually generalized LG inequality. Finally, we discuss the statistical significance of the violation with a non-parametric method.

6.1 Origin of LG Inequality

Setting aside bistable perception, let us begin the derivation of the LG inequality. In the paper Leggett and Garg (1985), LG postulated the following two assumptions:

(A1) **Macroscopic realism**: *A macroscopic system with two or more macroscopically distinct states available to it will at all times be in one or the other of these states.*

(A2) **Noninvasive measurability**: *It is possible, in principle, to determine the state of the system without arbitrarily small perturbation on its subsequent dynamics.*

In the original version of the inequality, LG consider three pairs of instances of time, $(t_1, t_2), (t_2, t_3), (t_3, t_4)$, where $t_1 < t_2 < t_3$. They then discuss the correlation function of two measurements which are performed at two of those times. Quantum mechanics violates the LG inequality as well as the same analog of Bell's inequality or the Clauser-Horne-Shimony-Holt (CHSH) inequality. Therefore this violation means that at least one of the two assumptions fails for quantum systems.

Let Q be an observable quantity which takes either $+1$ or -1. In the original discussion by LG, Q is the observable of the position of a particle in the two potential wells. However, we can discuss other two-level systems, for example, a spin-$\frac{1}{2}$ system.

The measurement of the two-level system is performed on a single system at different times $t_1 < t_2 < t_3$. We denote the observable at time t_k by Q_k ($k = 1, 2, 3$). By repeating a series of three measurements, we can estimate the value of the correlation functions by

$$C_{ij} = \frac{1}{N} \sum_{n=1}^{N} q_i^{(n)} q_j^{(n)},$$

where $q_i^{(n)}$ (or $q_j^{(n)}$) is a result of the nth measurement of Q_i (or Q_j). Note that the correlation between Q_i and Q_j takes the maximum value $C_{ij} = 1$ when $q_i^{(n)} q_j^{(n)}$ equals 1 for all the repeated trials. Here, consider the assumption A1; then the state of the system is determined at all times, even when the measurement does not perform on the system. Therefore, the values of the joint probabilities of Q_1, Q_2, and Q_3 are determined a priori at initial time t_0. We denote this by the symbol $P_{i,j}\,(Q_1, Q_2, Q_3)$. Note that the pairs of indexes i, j encode the situation that only two observables Q_i and Q_j are measured. In other words, the joint probability depends on the situations in which pairs of observables are measured. (We can consider pairs of indexes, instances of time, as parameters encoding three temporal contexts, $\mathcal{C}_{t_1 t_2}, \mathcal{C}_{t_1 t_3}, \mathcal{C}_{t_2 t_3}$, cf. Sect. 6.2.) However, if one considers (A2), then the joint probabilities do not depend on temporal contexts:

$$P_{i,j}(Q_1, Q_2, Q_3) = P(Q_1, Q_2, Q_3) \quad \forall i, j$$

Then we have the following equalities:

$$P(Q_1, Q_2) = \sum_{Q_3=\pm 1} P(Q_1, Q_2, Q_3),$$
$$P(Q_2, Q_3) = \sum_{Q_1=\pm 1} P(Q_1, Q_2, Q_3),$$
$$P(Q_1, Q_3) = \sum_{Q_2=\pm 1} P(Q_1, Q_2, Q_3)$$

which are consequences of the additivity of classical (Kolmogorov) probability. Thus pairwise joint probability distributions are context independent [as a consequence of (A2)]. We also have

$$P(Q_1) = \sum_{Q_2=\pm 1} P(Q_1, Q_2) = \sum_{Q_3=\pm 1} P(Q_1, Q_3); \tag{2}$$

$$P(Q_2) = \sum_{Q_1=\pm 1} P(Q_1, Q_2) = \sum_{Q_3=\pm 1} P(Q_2, Q_3); \tag{3}$$

$$P(Q_3) = \sum_{Q_1=\pm 1} P(Q_1, Q_3) = \sum_{Q_2=\pm 1} P(Q_2, Q_3); \tag{4}$$

Thus, for each observable, its probability distribution is also context independent (as a consequence of (A2)). Violation of these equalities is interpreted as exhibition of contextuality. In psychology and cognitive science the equalities (2)–(4) represent the special case of so-called *marginal selectivity* (Dzhafarov 2003; Dzhafarov and Kujala 2013, 2014). It is clear that if at least one of these equalities is violated then one cannot assume the existence of context-independent, joint-probability distribution.

Under the assumption of existence of the joint (triple) probability Distribution, the correlation functions are written with the joint probabilities $P\left(Q_i, Q_j\right)$ as

$$\begin{aligned} C_{ij} &= P\left(Q_i = 1, Q_j = 1\right) + P\left(Q_i = -1, Q_j = -1\right) \\ &\quad - P\left(Q_i = -1, Q_j = 1\right) - P\left(Q_i = 1, Q_j = -1\right) \\ &= 2\left\{P\left(Q_i = 1, Q_j = 1\right) + P\left(Q_i = -1, Q_j = -1\right)\right\} - 1. \end{aligned}$$

We set $K = C_{12} + C_{23} - C_{13}$. This can be represented in the following form:

$$K = 1 - 4\{P(Q_1 = 1, Q_2 = -1, Q_3 = 1) + P(Q_1 = -1, Q_2 = 1, Q_3 = -1)\}. \tag{5}$$

This representation implies the LG inequality:

$$K \leq 1. \tag{6}$$

As we know, for example, Leggett and Garg (1985) and Goggin et al. (2011), for the quantum correlation functions C_{ij}, the above inequality can be violated (theoretically and experimentally).

6.2 Contextual LG Inequality: The Generalization of LG's Assumptions in the Sense of Contextuality

Here we express LG's assumptions in terms of context-dependent probabilities (Khrennikov 2009, 2010b). Note that these probabilities cannot be represented in common Kolmogorov probability space. Therefore one can consider such contextual probabilistic models as non-Kolmogorovian probabilistic models.

Readers may compare our contextual generalization with the original LG inequality, shown in Sect. 6.1. We present the original LG derivation by considering time as a context parameter.

(A1) There exists a joint probability $P_{\mathcal{C}}(Q_1, Q_2, Q_3)$ under a certain condition of experiments (context) $\mathcal{C}$. And the Kolmogorovness of $P_{\mathcal{C}}(Q_1, Q_2, Q_3)$ is ensured within the context $\mathcal{C}$:

$$P_{\mathcal{C}}(Q_1, Q_2) = \sum_{Q_3 = \pm 1} P_{\mathcal{C}}(Q_1, Q_2, Q_3),$$

$$P_{\mathcal{C}}(Q_2, Q_3) = \sum_{Q_1 = \pm 1} P_{\mathcal{C}}(Q_1, Q_2, Q_3),$$

$$P_{\mathcal{C}}(Q_1, Q_3) = \sum_{Q_2 = \pm 1} P_{\mathcal{C}}(Q_1, Q_2, Q_3),$$

and

$$
\begin{aligned}
P_{\mathcal{C}}(Q_1) &= \sum_{Q_2=\pm 1} P_{\mathcal{C}}(Q_1,Q_2) = \sum_{Q_3=\pm 1} P_{\mathcal{C}}(Q_1,Q_3) \\
&= \sum_{Q_2=\pm 1}\sum_{Q_3=\pm 1} P_{\mathcal{C}}(Q_1,Q_2,Q_3), \\
P_{\mathcal{C}}(Q_2) &= \sum_{Q_1=\pm 1} P_{\mathcal{C}}(Q_1,Q_2) = \sum_{Q_3=\pm 1} P_{\mathcal{C}}(Q_2,Q_3) \\
&= \sum_{Q_1=\pm 1}\sum_{Q_3=\pm 1} P_{\mathcal{C}}(Q_1,Q_2,Q_3), \\
P_{\mathcal{C}}(Q_3) &= \sum_{Q_2=\pm 1} P_{\mathcal{C}}(Q_2,Q_3) = \sum_{Q_1=\pm 1} P_{\mathcal{C}}(Q_1,Q_3) \\
&= \sum_{Q_1=\pm 1}\sum_{Q_2=\pm 1} P_{\mathcal{C}}(Q_1,Q_2,Q_3).
\end{aligned}
$$

(A2) Consider three different contexts $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$; then there exists a context $\mathcal{C}$ unifying these contexts such that

$$
\begin{aligned}
P_{\mathcal{C}_A}(Q_1,Q_2) &= \sum_{Q_3=\pm 1} P_{\mathcal{C}}(Q_1,Q_2,Q_3), \\
P_{\mathcal{C}_B}(Q_2,Q_3) &= \sum_{Q_1=\pm 1} P_{\mathcal{C}}(Q_1,Q_2,Q_3), \\
P_{\mathcal{C}_C}(Q_1,Q_3) &= \sum_{Q_2=\pm 1} P_{\mathcal{C}}(Q_1,Q_2,Q_3).
\end{aligned}
$$

From these assumptions, one can obtain the inequality (5) for K given by

$$
K = 1 - 4(P_{\mathcal{C}}(Q_1 = 1, Q_2 = -1, Q_3 = 1) + P_{\mathcal{C}}(Q_1 = -1, Q_2 = 1, Q_3 = -1)). \tag{7}
$$

6.3 Violation of Inequality in Optical Illusions

As we explained in Sect. 2, we have three kinds of experimental data based on the angle of the Schröder stair: (A) it changes randomly, (B) it ranges from 0 to 90, (C) it ranges from 90 to 0. These contexts of experiments are denoted by $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$. Let X_θ be a random variable which takes ± 1. The event

Table 1 **(Left)** $(\theta_1, \theta_2, \theta_3) = (0, 10, 20)$; **(Right)** $(\theta_1, \theta_2, \theta_3) = (40, 45, 50)$

	C_{12}	C_{23}	C_{13}
$\mathcal{C}_A$	1.000	0.964	0.964
$\mathcal{C}_B$	0.917	0.833	0.750
$\mathcal{C}_C$	0.917	1.000	1.000

	C_{12}	C_{23}	C_{13}
$\mathcal{C}_A$	0.091	0.091	0.127
$\mathcal{C}_B$	0.375	0.625	0.083
$\mathcal{C}_C$	0.625	0.375	0.167

Table 2 The triple of angles $(0, 10, 20)$

$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	$\mathcal{C}_A, \mathcal{C}_A$	$\mathcal{C}_A, \mathcal{C}_B$	$\mathcal{C}_A, \mathcal{C}_C$	$\mathcal{C}_B, \mathcal{C}_A$	$\mathcal{C}_B, \mathcal{C}_B$	$\mathcal{C}_B, \mathcal{C}_C$	$\mathcal{C}_C, \mathcal{C}_A$	$\mathcal{C}_C, \mathcal{C}_B$	$\mathcal{C}_C, \mathcal{C}_C$
$\mathcal{C}_A$	1.000	1.214	1.047	0.870	1.083	0.917	1.036	1.250	1.083
$\mathcal{C}_B$	0.917	1.130	0.964	0.786	1.000	0.833	0.953	1.167	1.000
$\mathcal{C}_C$	0.917	1.130	0.964	0.786	1.000	0.833	0.953	1.167	1.000

The values of K for various combinations of contexts. For the contexts $(\mathcal{C}_A, \mathcal{C}_C, \mathcal{C}_B)$, K approaches its maximal value

in which a subject says "left side is front" corresponds to the result $X_\theta = +1$. Then, from the repeated trials for each experimental context, we have the experimentally obtained values of the joint probabilities:

$$P_{\mathcal{C}_A}(X_0, X_{10}, \cdots, X_{90}), \quad P_{\mathcal{C}_B}(X_0, X_{10}, \cdots, X_{90}), \quad P_{\mathcal{C}_C}(X_0, X_{10}, \cdots, X_{90}).$$

The correlation functions are given by

$$C_{12} = 2\{P_{\mathcal{X}}(X_{\theta_1} = 1, X_{\theta_2} = 1) + P_{\mathcal{X}}(X_{\theta_1} = -1, X_{\theta_2} = -1)\} - 1,$$
$$C_{23} = 2\{P_{\mathcal{Y}}(X_{\theta_2} = 1, X_{\theta_3} = 1) + P_{\mathcal{Y}}(X_{\theta_2} = -1, X_{\theta_3} = -1)\} - 1,$$
$$C_{13} = 2\{P_{\mathcal{Z}}(X_{\theta_1} = 1, X_{\theta_3} = 1) + P_{\mathcal{Z}}(X_{\theta_1} = -1, X_{\theta_3} = -1)\} - 1.$$

Here, the triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is given by a combination of the contexts $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$. We show the values of C_{12}, C_{23} and C_{13} in Table 1.

We estimate the LHS of the inequality:

$$K(\theta_1, \theta_2, \theta_3) = C_{12} + C_{23} - C_{13}.$$

Tables 2 and 3 show the value of K with respect to $(\theta_1, \theta_2, \theta_3) = (0, 10, 20)$ and $(40, 45, 50)$. The value of K exceeding one is seen in several cases.

Table 3 The triple of angles $(40, 45, 50)$

$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	$\mathcal{C}_A, \mathcal{C}_A$	$\mathcal{C}_A, \mathcal{C}_B$	$\mathcal{C}_A, \mathcal{C}_C$	$\mathcal{C}_B, \mathcal{C}_A$	$\mathcal{C}_B, \mathcal{C}_B$	$\mathcal{C}_B, \mathcal{C}_C$	$\mathcal{C}_C, \mathcal{C}_A$	$\mathcal{C}_C, \mathcal{C}_B$	$\mathcal{C}_C, \mathcal{C}_C$
$\mathcal{C}_A$	0.055	0.099	0.015	0.589	0.633	0.549	0.339	0.383	0.299
$\mathcal{C}_B$	0.339	0.383	0.299	0.873	0.917	0.833	0.623	0.667	0.583
$\mathcal{C}_C$	0.589	0.633	0.549	1.123	1.167	1.083	0.873	0.917	0.833

The values of K for various combinations of contexts. For the contexts $(\mathcal{C}_C, \mathcal{C}_B, \mathcal{C}_B)$, K approaches its maximal value

6.4 Statistical Analysis

We start from the random variable:

$$K = Q_1 Q_2 + Q_2 Q_3 - Q_1 Q_3.$$

Here, K takes $-4, -2, 0$ or $+3$ since Q_i takes $+1$ or -1. The probability distribution of K is not known, but it has a mean value μ and variance σ^2, and their statistical estimates can be found. To find the confidence interval in such a situation, we apply the simplest method of nonparametric statistics, namely, the method based on the *Chebyshev inequality*.[1] (However, note that this method gives us only a rough estimate for the confidence interval.)

We can apply the Chebyshev inequality to sample the mean of K

$$P\left(|m - \mu| > c\right) \leq \frac{\sigma^2}{nc^2}$$

with positive constant c. Here, m is a sample mean of independent random variables $K_1, \ldots, K_n$:

$$m = \frac{K_1 + K_2 + \cdots + K_n}{n}.$$

Although we do not know the value of σ^2, we can estimate σ^2 with unbiased sample variance. Then μ is estimated by m with confidence interval $[m - c, m + c]$.

We take the 80 % confidence level. In the case that the order of the contexts is $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = (\mathcal{C}_A, \mathcal{C}_C, \mathcal{C}_B)$, and the angle $\theta = 0, 10, 20$ (this case has a maximum value of K) , we estimate the value of K as follows.

[1] This method was recently used (Khrennikov et al. 2014) for analysis of statistical data from the Vienna test for the Bell-type inequality, the Eberhard inequality, which finally closed the fair sampling loophole. In this test, because of the presence of slight drift depending on experimental setting, one cannot assume Gaussianity of data and it seems that usage of the Chebyshev inequality is the simplest way to resolve this problem.

$$K = 1.250 \pm 0.213.$$

Statistical analysis shows that the violation of the LG inequality is statistically significant. This nonparametric method is a very conservative analysis since the confidence interval is much wilder than the interval calculated by the usual method of assuming Gaussian distribution for the random variables.

A violation of the contextual LG inequality by statistical data collected for observations of the Schröder stair rotated for different angles supports the *contextual cognition paradigm* presented in the series of works (Khrennikov 2010b; Asano et al. 2013a; Khrennikov 2010a; Asano et al. 2011b,a, 2012a,b, 2013b). Our experimental statistical data is fundamentally contextual. The brain does not have a priori prepared "answers" to the question about the right/left structure of the Schröder stair for the fixed angle θ. Answers are generated depending on the context. There are practically no (at least not so many), so to say, "absolute mental quantities"; "answers" to the same question vary essentially depending on context. This conclusion is not surprising in the framework of cognitive science and psychology, where various framing effects are well known. Thus we demonstrate the applicability of a statistical test of contextuality borrowed from quantum physics. We can also consider this study as a step towards the creation of a unified mathematical picture of the world: physical and mental phenomena can be described by the same equations, cf. Khrennikov (2010b).

7 Non-Kolmogorov Model of Bistable Perception

In this section, we model the bistable perceptions of Schröder stairs with non-Kolmogorov probability. In psychology, the idea of unconscious inference is well-known in order to describe the process of visual perception (von Helmholtz 1866). In the scheme of unconscious inference, it is assumed that there exists a state of sensation, that is, the stimulus of an image, and that perception is the result of decision-making with sensation and memory bias, which comes from personal experience and so on. Recently, the mathematics of unconscious inference has become sophisticated; compare with Bayesian inference (Asano et al. 2012).

Here we interpret the sensation as the pre-recognized perception which is not biased. Then the state of sensation is expressed as the quantum superposition of two alternative perceptions, (L) or (R). (We will express this in Eq. (8) in the next section.) The reason why we use quantum superposition is to describe the mental fluctuation that the subject feels as ambiguity toward the figure.

From the mathematical point of view, the aim of this study is to define joint probability which describes the interaction between sensation and perception. To define joint probability accompanied with context dependency, the concept of adaptive dynamics is available. We note that the mathematical formulation of joint probability based on adaptive dynamics is to be seen in the paper of Asano et al. (2013a). In the next section, we construct the mathematical model of bistable perception in the manner of adaptive dynamics. In Eqs. (10) and (11), we express the probability of perception given as the marginal of the joint probability.

7.1 Quantum Adaptive Approach to Definition of Non-Kolmogorovian (Joint) Probabilities for Unconscious Inference

Let $\mathcal{H}_{\mathrm{S}}, \mathcal{H}_{\mathrm{P}}$ be Hilbert spaces, and let $\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{P}}$ be a tensor product of $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{P}}$. Suppose that $\mathcal{H}_{\mathrm{S}} = \mathcal{H}_{\mathrm{P}} = \mathbb{C}^2$. A state of sensation (resp., a state of perception) is expressed as a density operator on $\mathcal{H}_{\mathrm{S}}$ (resp., $\mathcal{H}_{\mathrm{P}}$). Let $\{|e_L\rangle, |e_R\rangle\}$ be a fixed orthonormal basis in two-dimensional space $\mathbb{C}^2$. After an input signal comes to the receiver, the state of sensation is given by the density operator

$$\rho_{\mathrm{S}} = |s\rangle\langle s|$$

corresponding to the state vector

$$|s_\theta\rangle = \sqrt{x_\theta}\,|e_L\rangle + \exp(\mathrm{i}\phi)\sqrt{1-x_\theta}\,|e_R\rangle, \quad x_\theta \in [0,1],\ \phi \in \left[0, \frac{\pi}{2}\right]. \tag{8}$$

Here x_θ is a parameter of the probability distribution of sensation:

$$P(\lambda_\theta = L) = |\langle e_L, s_\theta\rangle|^2 = x_\theta, \quad P(\lambda_\theta = R) = |\langle e_R, s_\theta\rangle|^2 = 1 - x_\theta. \tag{9}$$

On the other hand, before the information of sensation is transmitted to the system of perception, the state of perception is not parameterized. The initial state of sensation is given by a pure state ρ_0 such as

$$\rho_0 = |e_+\rangle\langle e_+|; \quad |e_+\rangle = \frac{1}{\sqrt{2}}|e_L\rangle + \frac{1}{\sqrt{2}}|e_R\rangle.$$

Note that the state ρ_0 is interpreted as a neutral state of two perceptions $X_\theta = L, R$.

The state on the entire system $\mathcal{H}_S \otimes \mathcal{H}_P$ is written as

$$\rho_S \otimes \rho_0 = |s_\theta\rangle\langle s_\theta| \otimes |e_+\rangle\langle e_+|.$$

Further, the initial state of perception ρ_0 on $\mathcal{H}_P$ interacts with the sensation state ρ_S. The interaction is given by a unitary operator on $\mathcal{H}_S \otimes \mathcal{H}_P$ such as

$$U = |e_L\rangle\langle e_L| \otimes (|e_L\rangle\langle e_+| + |e_R\rangle\langle e_-|) + |e_R\rangle\langle e_R| \otimes (|e_R\rangle\langle e_+| + |e_L\rangle\langle e_-|),$$

where $|e_\pm\rangle$ denotes $(|e_L\rangle \pm |e_R\rangle)/\sqrt{2}$.

We define the state after the interaction by lifting:

$$\mathcal{E}^* \rho_S = U(\rho_S \otimes \rho_0)U^*.$$

The state $\mathcal{E}^*\rho_S$ is an entangled state which is expressed as

$$\begin{aligned} |\psi\rangle &= U(|s_\theta\rangle \otimes |e_+\rangle) \\ &= \sqrt{x_\theta}\,|e_L\rangle \otimes |e_L\rangle + \exp(\mathrm{i}\phi)\sqrt{1-x_\theta}\,|e_R\rangle \otimes |e_R\rangle \quad \in \mathcal{H}_S \otimes \mathcal{H}_P. \end{aligned}$$

We can calculate the joint probability with the aid of the following formula (the basic formula of the formalism of quantum adaptive dynamics (Asano et al. 2013a)):

$$\begin{aligned} P(\lambda_\theta = i, X_\theta = j) &= \mathrm{tr}\left\{\left(E_i \otimes E_j\right)\mathcal{E}^*\rho_S\right\} \\ &= \left|\left\langle e_i \otimes e_j, \psi\right\rangle\right|^2, \end{aligned}$$

where E_i is the projection of $|e_i\rangle\langle e_i|$ $(i = L, R)$. Then we have

$$P(\lambda_\theta = L, X_\theta = R) = P(\lambda_\theta = R, X_\theta = L) = 0$$

for any $x_\theta \in [0, 1]$. Therefore the interaction describes the unbiased inference, that is, $P(\lambda_\theta = i) = P(X_\theta = i)$.

In order to describe the bias effect, let us introduce the two Hermitian operators A and B which act on the Hilbert space $\mathcal{H}_S$ and $\mathcal{H}_P$, respectively. In general, A and B are given by the following matrix representation with respect to the basis $\{e_L, e_R\}$:

$$A = \begin{pmatrix} 1-p & \xi \\ \xi & q \end{pmatrix} \quad (p \in [0,1],\ \xi > 0),$$

$$B = \begin{pmatrix} p & \eta \\ \eta & 1-q \end{pmatrix} \quad (q \in [0,1],\ \eta > 0).$$

Here the parameters p and q correspond to conditional probabilities $P(X_\theta = R|\lambda_\theta = L)$ and $P(X_\theta = L|\lambda_\theta = R)$, respectively, and non-negative real numbers ξ and η characterize the intensity of the bias effect. (We discuss the details later.)

Note that we can represent A and B through their spectral decompositions as

$$A = \sum_{i=0}^{1} a_i S_i, \quad B = \sum_{j=0}^{1} b_j T_j.$$

Here a_i (resp. b_j) is an eigenvalue of A (resp. B), and S_i (resp. T_j) is a projection onto its eigenspace. Put

$$c_0 = a_0,\ \ c_1 = a_1,\ \ c_2 = b_1,\ \ c_3 = b_2$$

and

$$C_0 = S_0 \otimes E_L, \quad C_1 = S_1 \otimes E_L,$$

$$C_2 = T_0 \otimes E_R, \quad C_3 = T_1 \otimes E_R.$$

Then the joint probability $P(\lambda_\theta, X_\theta)$ is given by the following formula; see again (Asano et al. 2013a):

$$P(\lambda_\theta = i, X_\theta = j) = \frac{\mathrm{tr}\left\{f\left(E_i \otimes E_j\right)\mathcal{E}^*\rho_{\mathrm{S}}\right\}}{\mathrm{tr}\left\{f\left(I \otimes I\right)\mathcal{E}^*\rho_{\mathrm{S}}\right\}},$$

where f is a map (superoperator) on the set of bounded operators on $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{P}}$, and f is given by

$$f(\cdot) = \sum_{k=0}^{3} c_k C_k (\cdot) C_k^*.$$

Note that some eigenvalues (c_k) are negative, so this superoperator does not preserve the positivity of operators. This is one of the formal mathematical distinguishing features of quantum adaptive dynamics. As was pointed out in the introduction, this simplifies the construction of dynamical maps. Note that, for a moment, quantum adaptive dynamics is a phenomenological theory. In particular, the form of the f-superoperator was found to adjust theory to the experimental data on recognition of the ambiguous figure. Note also that, in spite of the violation of positivity of the f-superoperator, the joint probabilities $P(\lambda_\theta = i, X_\theta = j)$ are positive.

One can represent the probabilities $P(\lambda_\theta, X_\theta)$ as follows:

$$P(\lambda_\theta = L, X_\theta = L) = \frac{\sum_{i=0}^{1} a_i \langle s, P_i s\rangle \langle e_L, P_i e_L\rangle}{\langle s, As\rangle + \langle s, Bs\rangle},$$

$$P(\lambda_\theta = R, X_\theta = L) = \frac{\sum_{i=0}^{1} a_i \langle s, P_i s\rangle \langle e_R, P_i e_R\rangle}{\langle s, As\rangle + \langle s, Bs\rangle},$$

$$P(\lambda_\theta = L, X_\theta = R) = \frac{\sum_{j=0}^{1} b_j \langle s, Q_j s\rangle \langle e_L, Q_j e_L\rangle}{\langle s, As\rangle + \langle s, Bs\rangle},$$

$$P(\lambda_\theta = R, X_\theta = R) = \frac{\sum_{j=0}^{1} b_j \langle s, Q_j s\rangle \langle e_R, Q_j e_R\rangle}{\langle s, As\rangle + \langle s, Bs\rangle}.$$

We can calculate the value of $P(X_\theta = L)$ with the aid of the following formula:

$$\begin{aligned} P(X_\theta = L) &= P(\lambda_\theta = L, X_\theta = L) + P(\lambda_\theta = R, X_\theta = L) \\ &= \frac{\langle s, As\rangle}{\langle s, As\rangle + \langle s, Bs\rangle} \\ &= \frac{x_\theta (1-p) + (1-x_\theta) q + 2\xi \cos\phi \sqrt{x_\theta (1-x_\theta)}}{1 + 2(\xi + \eta)\cos\phi \sqrt{x_\theta (1-x_\theta)}}. \end{aligned} \tag{10}$$

Similarly, we have

$$\begin{aligned} P(X_\theta = R) &= P(\lambda_\theta = L, X_\theta = R) + P(\lambda_\theta = R, X_\theta = R) \\ &= \frac{\langle s, Bs\rangle}{\langle s, As\rangle + \langle s, Bs\rangle} \\ &= \frac{x_\theta p + (1-x_\theta)(1-q) + 2\eta \cos\phi \sqrt{x_\theta (1-x_\theta)}}{1 + 2(\xi + \eta)\cos\phi \sqrt{x_\theta (1-x_\theta)}}. \end{aligned} \tag{11}$$

As was discussed in the previous section, the combination of a few contexts is essential to describe the bias effect in unconscious inference. In Eqs. (10) and (11), ϕ works as a parameter-encoding context. If one takes $\phi = \pi/2$ and

$$p = P\left(X_\theta = R | \lambda_\theta = L\right), \quad q = P\left(X_\theta = L | \lambda_\theta = R\right),$$

then the probability of perception is written as the conventional form of classical probability theory:

$$\begin{aligned} P\left(X_\theta = L\right) &= x_\theta\left(1 - p\right) + \left(1 - x_\theta\right) q \\ &= P\left(\lambda_\theta = L\right) P\left(X_\theta = L | \lambda_\theta = L\right) + P\left(\lambda_\theta = R\right)\left(X_\theta = L | \lambda_\theta = R\right) \end{aligned} \tag{12}$$

for the fixed context of experiment—since we have $x_\theta = P\left(\lambda_\theta = L\right)$ in Eq. (9).

On the other hand, if ξ and η are essentially larger than one, then the probability $P\left(X_\theta = R\right)$ goes to the ratio of ξ to η:

$$P\left(X_\theta = R\right) \simeq \frac{\xi}{\xi + \eta}.$$

Hence, $P\left(X_\theta\right)$ with respect to large ξ and η is independent of the probability of sensation $P\left(\lambda_\theta\right)$. That is, ξ and η characterize the bias effect.

7.2 Method for Data Analysis

Here we show the method for determining the parameters $\{\phi, \xi, \eta, p, q, x_\theta\}$ in Eqs. (10) and (11); the result of our tests are shown in Sect. 5.

As written in Sect. 2, we use three different contexts. Generally, 11! permutations are permitted as the order of showing figures. We consider a permutation σ as a context of our test.

First, we arrange the test so that we randomly select the permutations, for example, our test for group (A). In this case, it is reasonable that the perception is not affected by bias. Therefore we can say this experimental context is an "*unbiased context*." (Although, more precisely, it means the biases are neutralized.) Then, we can calculate the value of x_θ from data of group (A):

$$x_\theta = P\left(\lambda_\theta = L\right) = P_A\left(X_\theta = L\right).$$

Next, we search for an angle θ which satisfies $P\left(X_\theta = L\right) = 1$ or $P\left(X_\theta = L\right) = 0$ for any contexts (permutations) σ. From the result shown in Fig. 2, we can find the probability distributions $P\left(X_0\right)$ and $P\left(X_{90}\right)$ which are estimated as follows.

$$x_0 = P_A(X_0 = L) = P_B(X_0 = L) = P_C(X_0 = L) = 1,$$
$$x_{90} = P_A(X_{90} = L) = P_B(X_{90} = L) = P_C(X_{90} = L) = 0.$$

Then we can calculate the value of p and q as follows.

For angle $\theta = 0$, we obtain $x_0 = 1$ by the experimental facts. Assign this to x_θ of Eq. (10), then we obtain $p = 0$ since the experimental facts show the probability $P(X_0 = L) = 1$ for any contexts (permutations) σ. Similarly, for angle $\theta = 90$, we obtain $x_{90} = 0$. Assign this to x_θ of Eq. (10), then we obtain $q = 0$ since $P(X_{90} = L) = 0$ for any contexts (permutations) σ. Finally we can rewrite Eqs. (10) and (11) as

$$P_\sigma(X_\theta = L) = \frac{P_A(X_\theta = L) + 2\xi \cos\phi \sqrt{P_A(X_\theta = L) P_A(X_\theta = R)}}{1 + 2(\xi + \eta)\cos\phi \sqrt{P_A(X_\theta = L) P_A(X_\theta = R)}}, \tag{13}$$

$$P_\sigma(X_\theta = R) = \frac{P_A(X_\theta = R) + 2\eta \cos\phi \sqrt{P_A(X_\theta = L) P_A(X_\theta = R)}}{1 + 2(\xi + \eta)\cos\phi \sqrt{P_A(X_\theta = L) P_A(X_\theta = R)}}. \tag{14}$$

Here $P_\sigma(X_\theta)$ means the probability of perception in the test with respect to the permutation σ. Note that the parameters ξ and ϕ depend on the context of σ.

Third, we search *the pure context which provides the strongest bias*, that is, the context which induces the most different probability distribution from the probability distribution for unbiased context. In our tests, we can see that two contexts corresponding to the clockwise rotation $\sigma_{\upharpoonright}$ and counterclockwise rotation $\sigma_{\downharpoonright}$ are pure contexts. Then we fix the parameter of the context at $\phi = 0$ since the value of $\cos\phi$ of Eqs. (13) and (14) is maximum. This means that the strongest bias affects the perception process. Thus, the probability $P_\sigma(X_\theta)$ for the pure context σ can be characterized by parameters ξ,η, and $P_A(X_\theta)$. Therefore, if we have the data of the test for a pure context, we can estimate the value of ξ and η for the pure contexts with some statistical methods, for example, the non-linear least square method. Hereafter, we put the values (ξ, η) is $(0, 1.5)$ (resp. $(1.5, 0)$), which correspond to the probabilities for a pure context $\sigma_{\upharpoonright}$ (resp. pure context $\sigma_{\downharpoonright}$).

Let us consider the context which is neither $\sigma_{\upharpoonright}$ or $\sigma_{\downharpoonright}$. For example, "clockwise rotation first, counterclockwise rotation second," that is, the sequence of angles are given by

$$0, 10, 20, 30, 40, 90, 80, 70, 60, 50, 45.$$

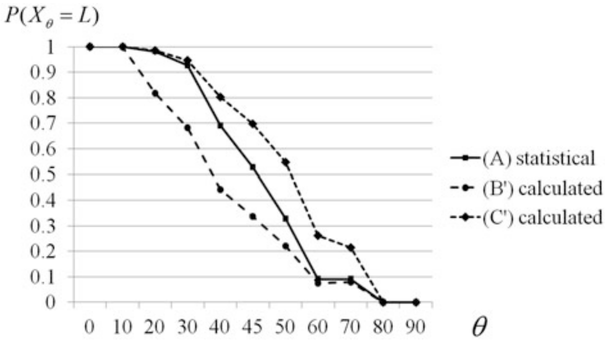

Fig. 3 We calculate the probability for clockwise rotation (B′) and counterclockwise rotation (C′) with our model. We also show the experimental result of random rotations (A) as a reference

We can naturally predict that the probability of X_θ for the first four figures is the same as that for the clockwise rotation. However, at the angle $\theta = 90$ or 80, the subject does not understand the direction of rotation. Therefore the bias effect at $\theta = 90$ or 80 is not strong; and it is similar with the effect for random rotation. We can parameterize such intermediate level of bias by the phase ϕ.

In our test, we can divide all the contexts (permutations) into two groups: (1) the contexts which induce a bias effect in the direction toward the strongest bias in the pure context $\sigma_{\upharpoonright}$ (2) the contexts which induce a bias effect in the direction toward the other strongest bias in the pure context (C).

For a context of group (1), we can calculate $P\,(X_\theta = L)$ with the data of (A) as follows.

$$P\,(X_\theta = L) = \frac{P_A\,(X_\theta = L) + 3\cos\phi\sqrt{P_A\,(X_\theta = L)\,P_A\,(X_\theta = R)}}{1 + 3\cos\phi\sqrt{P_A\,(X_\theta = L)\,P_A\,(X_\theta = R)}}. \tag{15}$$

Here we put $(\xi, \eta) = (0, 1.5)$. On the other hand, for the context of group (2), we can calculate this with the following formulae.

$$P\,(X_\theta = L) = \frac{P_A\,(X_\theta = L)}{1 + 3\cos\phi\sqrt{P_A\,(X_\theta = L)\,P_A\,(X_\theta = R)}}. \tag{16}$$

Here we put $(\xi, \eta) = (1.5, 0)$.

With Eqs. (15) and (16), we can calculate the value of $P\,(X_\theta = L)$ if we estimate the probability $P_A\,(X_\theta = L)$ or $P_A\,(X_\theta = R)$ from the data of the test for group (A). In Fig. 3, we show (A) the statistically estimated probability of

$P_A\,(X_\theta = L)$ which is the same as that in Fig. 2, (B') the calculated probability $P\,(X_\theta = L)$ for the pure context of clockwise rotation $\sigma_{\upharpoonright}$, (C') the calculated probability $P\,(X_\theta = L)$ for the pure context of counterclockwise rotation $\sigma_{\downharpoonleft}$. Since the context in (B') or (C') is a pure context, we fix ϕ of Eqs. (15) or (16) at zero. In the plot of (B') and (C') in Fig. 3, the probability at the middle range of angle θ is different to the unbiased probability (A) as well as the experimental facts shown in Fig. 2.

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Strategic Choice in Hilbert Space

Charles E. Smith Jr. and Christopher Zorn

1 Introduction

A small-but-growing number of social scientists have in recent years begun to explore the purchase of formalisms and a probability theory originally developed to accommodate nonclassical experimental results in particle physics. Motivated by the desire to explain empirical outcomes that do not fit comfortably with the axioms of rational choice theory—and that are thus related tenets of classical probability theory—these scholars have begun to examine and embrace the theory of probability namesaked for Max Born as a general framework for understanding and modeling choice. Born's (1926) account of probability differs from the familiar, classical context of Kolmogorov both

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in terms of its mathematical exposition/foundation and with respect to its governing axioms. The Born theory is expressly geometric as opposed to set-theoretic. Its axioms are formalized in the (usually) complex planes of Hilbert spaces, where distances are most generally conceptualized as metrics between (sometimes high-dimensional) spaces as opposed to points. In this framework, certain of Kolmogorov's set-theoretic axioms can be relaxed, and empirical results that do not agree with them can be systematically accommodated.

Perhaps the most prominent publication in this emergent literature appeared in the *Proceedings of the Royal Society* more than a half-decade ago (Pothos and Busemeyer 2009). This paper is exemplary of the genre in its focus on the poor fit between certain strictures of rational choice theory and well-established empirical results from behavioral economics.[1] Specific attention is given to Savage's (1954) foundational "sure-thing principle," the dominant axiom that requires a decision-maker with a preference order that is invariant across categories of a conditioning variable to produce the same order in the absence of such conditioning. The sure-thing principle is a bedrock of set-theoretic rational choice and intimately related to Kolmogorov's law of total probability, which in turn is a bedrock of classical statistical models of decision-making. Pothos and Busemeyer illustrate these links and demonstrate that the classical standard in probabilistic modeling of decision/state transitions (the Markov model) forecloses on the possibility of accommodating empirical violations of the sure-thing principle via its compliance by design with Kolmogorov's version of the law of total probability.

In contrast, Born's law of total probability admits an interference term that accommodates the prospect that a decision-maker is not constrained to a single preference order at a specific point in time.[2] This is a generalization of a more basic measure-theoretic aspect of decision models formalized in Hilbert spaces. In Hilbert's geometry, individual preferences are not presumed to be fixed in advance of measurement. Indeed, if they are, the geometry simplifies to familiar models such as the Euclidean and weighted Euclidean frameworks of spatial voting theory, and Kolmogorov's set-theoretic axioms apply.

Pothos and Busemeyer's (2009) paper, as well as like-minded work from these and other authors (e.g., Khrennikov 2009), has not gone without

[1]The primary example results in the Pothos–Busemeyer paper are organized conceptually around prisoner's dilemma experiments and the oft-replicated findings of Tversky and Shafir (1992) that illustrate sure-thing-principle violations in certain forms of the game.

[2]In both set-theoretic and Euclidean-geometric formalizations of decision theory, acts of measurement reveal orders or points distinguished distance-wise along or by straight lines. The Hilbert space is more general. Euclidean distance in Hilbert space is a special case. Likewise, a geometric equivalent of Kolmogorov's set-theoretic law of total probability is a special case of Born's theory.

criticism. In their recent book, Busemeyer and Bruza (2012) ably answered many critiques from what we imagine as a not-really-hypothetical sequence of challenges from journal and book referees that the authors "might have" encountered over the course of their research programs. Criticism, however, continues. Subsequently, Pothos and Busemeyer (2013) headlined a commentary section in an issue of the journal *Behavioral and Brain Science* with an article embracing the use of Born's probability as "providing a new direction for cognitive modeling (p. 255)." Respondents on the one hand took issue with the agnosticism of the research program regarding the appearance of nonclassical results in the absence of a physical theory/recognition of nonclassical operations in the human brain, and on the other hand regarding, more or less, the very existence of the research program given less complicated alternatives.

It is difficult to sympathize with these critiques. As advanced as it is, the science of brain activity has only just now begun to contemplate experiments that can reveal nonclassical physical operations. Agnosticism by Pothos–Busemeyer (and others) regarding specific structures in neuro-circuitry in advance of empirical evidence firmly establishing the reality and particulars of said structures seems quite proper. At the same time, knee-jerk criticism of complexity is reminiscent of the marginalized perspectives on nearly all of formal theory, as well as much of quantitative empirical research. Political science is complicated. A very general formalism—one that with restrictions produces familiar models and results but also can be extended such that what have previously been considered "errors" are predictable in the positive theory—deserves careful attention.

The aim of this note is to convey some of the broad contours of this existing work, to take some initial steps toward the formalization of a general model of K-valued choice using nonclassical probabilities that can be represented in Hilbert spaces, and to offer some preliminary suggestions for how such tools might be useful in the study of political phenomena. Two reactions to the existing literature motivate this chapter. One is the current lack of attention to this emergent research agenda in political science. The other is the heavily inductive nature of much of the work produced thus far. The modest agenda here is to ponder some familiar political science problems within the confines of familiar frameworks, then ponder them again in the more general framework, with an eye toward motivating new, more-deductive research. We begin with a discussion of models of survey responses and opinions.

2 A Single Survey Response

For nearly a century in the field of political science, understanding how opinions fit together and modeling survey responses have been almost parallel efforts. The pathbreaking work of the 1930s, 1940s, and 1950s all rested on the assumption that the way citizens think and behave closely mimicked the way they answered survey questions about their thoughts and actions.[3] Converse's landmark (1964) paper upended the discipline's notions of simple correspondences between measured and actual preferences. Soon after, random utility models for discrete choice emerged with McFadden's (1974) work in economics and, in short order, in political science: Achen's important (1975) paper had political scientists fascinated with instrumentation and measurement error. Around the same time, social psychologists began to interrogate the Gricean assumptions and to develop theories of the survey response that addressed emotional, social, and other influences (Hippler et al. 1987; Schwarz et al. 1991; for a summary, see Strack and Schwarz 2006). Developments along each of these lines have continued and become increasingly sophisticated. Today, the best academic surveys in political science do not just acknowledge and model random measurement error but increasingly seek to seize and control nonrandom error inside the process of survey design (e.g., King et al. 2004; King and Wand 2007; Street et al. 2015).

From the earliest work through to today, however, one assumption has remained so ubiquitous as not to warrant ordinarily a formal statement: distance relations between response options and respondent preferences are presumed to be Euclidean, with locations of respondents on latent variables situated on lines or curves. Indeed, when researchers invoke set theory and dominance relations to represent a utility-maximizing choice by a respondent (R) between, say, options A and B, as in:

$$\begin{aligned}\Pr(A) &= \Pr(U_A \geq U_B) \\ &= \Pr(||R - A|| \leq ||R - B||)\end{aligned}$$

it is certainly not customary to pause and remind readers that the space underneath these relations is a Euclidean plane. $\mathfrak{R}^1$ is ordinarily implicit, and fixed locations of A, B, and R in classical theoretical models are deterministic

[3]The linguistic roots of these assumptions are summarized in Grice's (1975) "cooperative principle" as applied to the surveyer/respondent conversation.

up to some degree of stochastic error (also formalized in Euclidean space). Inconsistencies such as $\Pr(A) > \Pr(B) \mid ||R - A|| > ||R - B||$ are foreclosed prior to measurement as well as after it.

As noted in the introduction, choices in a Hilbert space-based decision theory are not modeled via sets but instead via subspaces. As a result, instead of representing the states of a choice set (or evolutions thereof) in terms of classical probability functions, states in the Hilbert space are represented by unit-length vectors. Consider an example wherein l represents a respondent's partisan preference for majority control of the US Congress, and assume the binary response options ($r = 1$ and $d = 0$) are exhaustive. In Hilbert space, these options are represented as a weighted pair of vectors $a\,|d\rangle$ and $a\,|r\rangle$ in a composite state $|\psi\rangle$ equal to their simple sum

$$\psi = a\,|d\rangle + b\,|r\rangle\,. \tag{1}$$

A respondent's preference would transition from the composite form when measured, in advance of which the probability of observing d is the squared length of the weight vector, $\Pr(d) = |a|^2$. Likewise $\Pr(r) = |b|^2$ and $|a|^2 + |b|^2 = 1$. The new state $|\psi'\rangle$ is simply the observed outcome, or perhaps a state "on the way" toward an observed outcome as the respondent ponders his or her options.

The Hilbert state mechanics that model the evolution of the transition from the composite state $|\psi\rangle$ toward an observed state can be visualized as in Fig. 1. Here, the initial state $|\psi\rangle$ has transitioned from perfect indecision

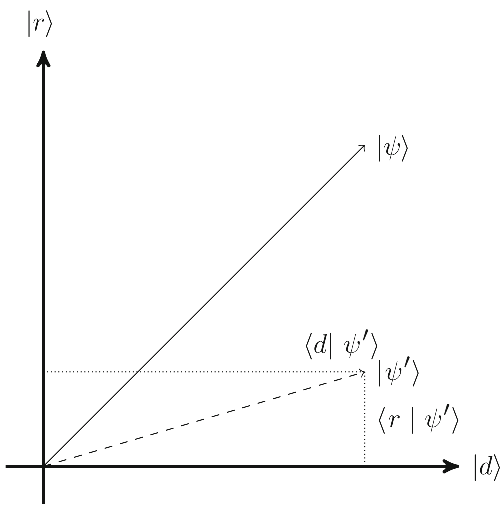

Fig. 1 Transitions in Hilbert space

over the options r and d to a new state (the dashed line vector) $|\psi'\rangle$ in clear favor of d. The projections (the bracket vectors) from the new state vector to the ket vectors representing the options $|r\rangle$ and $|d\rangle$ are the new probability amplitudes. Represented in the figure as densely dotted lines, the longer of the two—$\langle d \mid \psi'\rangle$—is interpreted as the probability amplitude associated with transitioning from the new state $|\psi'\rangle$ to the Democratic choice option. Similarly, the shorter projection $\langle r \mid \psi'\rangle$ maps to the probability of choosing the Republican option.

Almost to the same extent that this geometry seems exotic, its conceptualization of response measurement is familiar. In their justly famous "Simple Theory of the Survey Response," Zaller and Feldman (1992) begin with three axioms, paraphrased here:

1. *Ambivalence*: Most people have opposing considerations on any given issue.
2. *Accessibility*: The accessibility of those considerations at any given time depends on a "stochastic sampling process" where more recently thought-about considerations are likely to be oversampled/have higher probabilities of being accessed.
3. *Response*: Answers to survey questions are averages across considerations that can be/are accessed at the time of questioning.

Even after more than two decades, these axioms and their conceptual footings are still central to the discipline's understanding of survey responses and decision-making more generally. It is notable that the Hilbert formalization makes plain a decision-maker's reasoning over considerations and options (plural) in advance of making a decision or announcing a choice on a survey. Indeed, the ambivalence axiom seems to readily imply that, in advance of an actual decision and the recording of a response, people are "in effect" best regarded as being associated with both one option and the other. This is a fundamental characteristic of the Hilbert space-based models. The composite form described above that moves a simple, binary choice from one to two dimensions seems indeed a very natural route to capture the full complexity of ambivalence.

The response axiom, likewise, fits well in the Hilbert formalism. In Hilbert space, the response process is formulated from beginning to end as a transition of probability emerging, essentially, from the recording and summarization of trial outcomes in the mind of the respondent regarding competing considerations. This process is not unlike repeated flips of a coin to determine how the coin is weighted. The Hilbert space response then emerges stochastically, and is in part a function of the time spent by the decision-maker pondering the considerations. The familiar classical model of transition probabilities arises from a memoryless, rapid, and near-deterministic Markov process. Pothos and Busemeyer (2009, p. 4) show that the probability of choosing (say) the Democratic option in such a binary choice is equal to:

$$\Pr(\text{Democratic}) = \left(\frac{\mu}{1+\mu}\right) \cdot \{1 - \exp[-(\mu+1)\cdot t]\} + \frac{\exp[-(\mu+1)\cdot t]}{2}. \quad (2)$$

In contrast, in the Hilbert space, the model of state transition is owed to Erwin Schrödinger (1935). Pothos and Busemeyer demonstrate that this can be written as:

$$\Pr(\text{Democratic}) = \left[\frac{1}{2} + \frac{\mu}{1+\mu^2} \cdot \sin\left(\frac{\pi}{2} \cdot t\right)^2\right]. \quad (3)$$

Figure 2 illustrates the fundamental differences in the two perspectives. Consider the origin in the figure as the instant just after the survey interviewer puts the preference question to the respondent favoring (or beginning to favor) *d* and time elapsing in seconds (or fractions thereof) on the *X*-axis as the respondent ponders in advance of actually answering. In both models, the respondent's answer remains probabilistic until the response is decided, though these two curves present strikingly different viewpoints—and potentially, theoretically motivated and competitive perspectives—on the decision-making process as it unfolds.

Zaller and Feldman do not pose their ambivalence and response axioms as formal propositions that respondents are actually doing math "in their minds" with potentially competing considerations, but instead present them as "as-if" models of the process. It is notable, however, that Shrödinger's equation sketches a process that seems remarkably faithful to these "as-if" proposals, whereas the quick collapse to near-certainty of the Markov transition seems more like an alternative hypothesis for understanding survey responses, to be judged against the Zaller–Feldman tradition. Perhaps a carefully designed experiment that manipulates time-to-response, using subjects somehow matched beforehand so as to be similarly conflicted on a question, could reveal whether one account has more empirical purchase than the other.

In discussing their accessibility axiom, Zaller and Feldman (1992) point to what was then already a large body of empirical evidence in support of recency effects in cognitive psychology (see their footnote 2). They also reference supporting evidence for several "deductions" that are consistent with

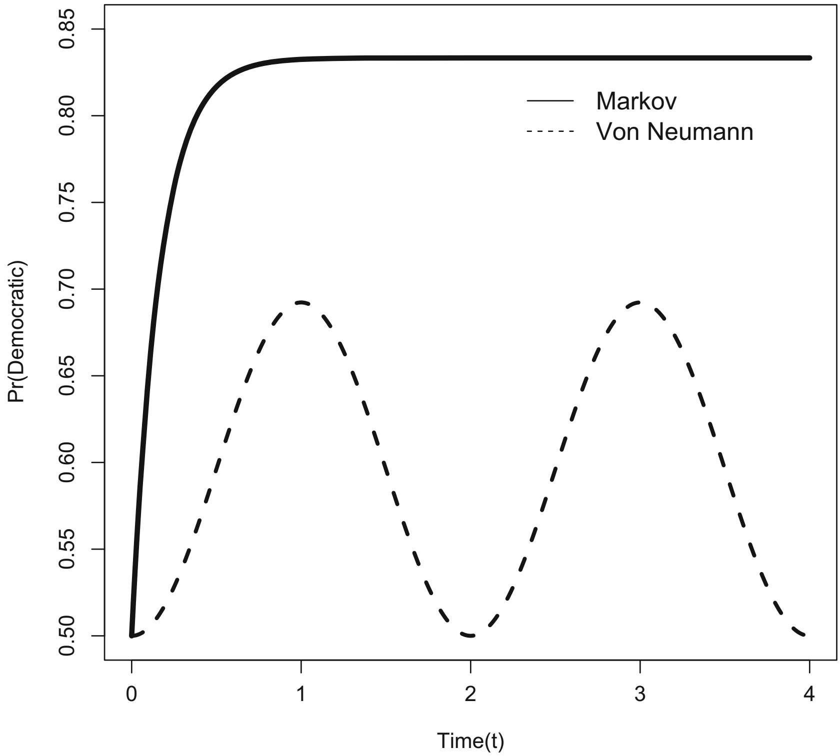

Fig. 2 Markovian and Hilbert transition probabilities ($\mu = 5$)

the accessibility axiom. Among these are the presence of "response effects" in surveys, which include as special cases order effects, endorsement effects, race (and other demographic) effects of the interviewer, priming effects, reference group effects, and framing effects. They write:

> The mechanism responsible for each of these effects appears to be a tendency for people to answer questions at least partly on the basis of ideas that have been made momentarily salient to them. (Zaller and Feldman 1992, p. 602)

A growing body of work makes use of the Hilbert space formalisms to model question order effects (e.g., Trueblood and Busemeyer 2011; Wang and Busemeyer 2013), with some focusing exclusively on order effects. This work exists largely in isolation from the theory of the survey response set out by Zaller and

Feldman.[4] Yet many (if not most) of the sort of response effects that Zaller and Feldman discuss—including framing effects and conditionality—can in principle be understood in a fashion analogous to Wang and Busemeyer's work on order effects. Perhaps the most important point made by Wang and Busemeyer is an intuitive non-Euclidean probability explanation of what it means when some individuals demonstrate order effects whereas others do not (2013, pp. 28–29).

Current efforts to study order effects tend to consider the unions of the (usually binary) responses. That is, for two response variables $Y_1 \in \{0, 1\}$ and Y_2 similar, they look to see if $\delta \neq 0$ in a model of the form:

$$\Pr(Y_{1i} = Y_{2i}) = f[\mathbf{X}_i\theta + \delta(\text{question order}_i)]. \tag{4}$$

Some such models also include an interaction between question order and one or more of the components of $\mathbf{X}$. In contrast, Wang and Busemeyer's implication is that the key parameter θ can vary across individuals, so that estimating a model of the form

$$\theta_i = g(\mathbf{X}_i\gamma) \tag{5}$$

would constitute a direct way of assessing order (and other response) effects. Here, $g(\cdot)$ is an appropriate link function that maps the linear/additive component $\mathbf{X}_i\gamma$ onto the range of θ.

Wang and Busemeyer define $\theta = R\cos(\phi)$, which means $\theta \in [-1, 1]$. A natural linear link for a function of that form is a normalized arctangent link $g(z) = \frac{\arctan(z)}{\pi/2}$, which has a domain on $\mathfrak{R}^1$ and a range of [-1,1]. They note that everything that is a component of θ is observable (see their Eq. (2)). Thus, assuming we can effectively measure θ_i for a sample of survey respondents, estimating (5) is straightforward. In this view, any given individual's θ_i for a particular pair or set of "similar" items would be an increasing function of their level of political information. Interestingly, Wang and Busemeyer's analysis suggests that high-information individuals have high values of θ, and that they correspondingly should also experience high levels of interference/order effects. This prediction runs counter to much of the conventional wisdom about order effects (e.g., Moore 2002), which posits that order effects are more

[4] Wang and Busemeyer's paper in particular is based almost entirely on the paper by Moore (2002) in terms of its framing, and relies for the most part on his data as well.

prominent among relatively uninformed respondents. So here again, we see potential for deductive research projects that pit formal models as empirical competitors.

3 Multiple Choices

More comprehensive model development for order effects or others (e.g., priming effects, reference group effects, framing effects) that involve response options over more than one question requires us to generalize the formalisms described above. Lacy's (2001a,b) set-theoretic "theory of nonseparable preferences in survey responses" is the disciplinary standard for considering such effects in the classical context. In set theory, for $\mathbf{K} = \{K, \tilde{K}\}$, preferences ($x$) are *separable* on K when, given

$$(x_K, x_{\tilde{K}})\, R\, (x'_K, x_{\tilde{K}}),$$

it is true also that

$$(x_K, x'_{\tilde{K}})\, R\, (x'_K, x'_{\tilde{K}})$$

(e.g., Schwartz 1977; Stoetzer and Zittlau 2015). That is, if x_K is preferred over x'_K when each is paired with a commodity not in K and preferences are separable, then the same relation will hold when they are paired with another non-K commodity ($x'_{\tilde{K}}$). Nonseparability occurs in the set when the relation does not hold. For example, suppose there are three commodities $\{SYW\}$—saltines (S), yogurt (Y), and water (W)—all denominated in, say, ounces, consumed across a fixed time frame. Suppose further that a consumer prefers solid food (S) to a mixture (Y), and is more hungry than thirsty. The consumer thus prefers $(1, 0, 1)$ to $(0, 1, 1)$. If the consumer's preferences are separable between the $\{SY\}$ group and W, she must also prefer $(1, 0, 0)$ to $(0, 1, 0)$. Such would imply, however, that she is not at all thirsty; in fact, she might well prefer $(0, 1, 0)$ to $(1, 0, 0)$. If so, the commodity group $\{SY\}$ is not separable from W.

Political scientists have characterized preference separability in various ways. Lacy (2001a), for example, states that "(A) person has nonseparable preferences when her preference on an issue depends on the outcome of other issues." Empirically, the existence of separable preferences has come under some attack, with Lacy and Paolino (1998) asserting that "Voter preferences for presidential and congressional candidates may be nonseparable," and Smith et al. (1999) claiming that "Around half of all respondents show some tendency toward nonseparable preferences." More recent work (e.g., Healy et al. 2010) has indicated that, in the context of voting, irrelevant events can have a small but detectable effect on voting behavior.

As with the one-question example in the previous section, Hilbert space models of options over more than one question are geometric. A conceptual lever into this problem is first to consider the classical (Euclidean) geometric representation of separability owed to Enelow and Hinich (1984). Lacy (2001a) generalized the Enelow/Hinich spatial model to respondent preferences over N issues in a survey. A stylized version of that model is defined in Appendix 1. Here we will consider a corner of it and extend the example of voter preferences for partisan control of governance (G) as a combination of preferences across options: $G = [D_E D_L, D_E R_L, R_E D_L, R_E R_L]$, where D and R identify the partisan options and E and L index executive and legislative branches. Separability and nonseparability of preferences over the institutions are defined in the weighted Euclidean norm:

$$||S_E - S_L||_{\mathbf{A}} = \sqrt{\alpha_{11}(R_E - D_E)^2 + 2\alpha(R_E - D_E)(R_L - D_L) + \alpha_{22}(R_L - D_L)^2} \tag{6}$$

$$\mathbf{A} = \begin{bmatrix} \alpha_{11} & \alpha \\ \alpha & \alpha_{22} \end{bmatrix}.$$

Specifically, when A is null and distance is simple Euclidean, preferences across the institutions are separable. When the off-diagonal elements of A are nonzero, the respondent/voter will be inclined either to unify or to divide their partisan support across the institutions.

Recalling that, in the Hilbert space, options d and r receive composite consideration, consider two vectors and their product:

$$\begin{aligned} |e\rangle &= a_0 |d\rangle + a_1 |r\rangle \\ |l\rangle &= b_0 |d\rangle + b_1 |r\rangle \\ |e\rangle \otimes |l\rangle &= a_0 b_0 |dd\rangle + a_0 b_1 |dr\rangle + a_1 b_0 |rd\rangle + a_1 b_1 |rr\rangle , \end{aligned} \tag{7}$$

where

$$|a_0 b_0|^2 + |a_0 b_1|^2 + |a_1 b_0|^2 + |a_1 b_1|^2 = 1.$$

Here, the institutional preferences also get composite consideration, and preference separability is defined by the existence of the vector product in

(7)—that is, a vector product with coefficients that can be reduced from a more aggregated form into this one via factorization.[5]

The weighted Euclidean version of this problem is restrictive against certain set-theoretic formulations of nonseparability. Likewise, via the requirement that the off-diagonal coefficients be equal, the Euclidean formulation forecloses on the modeling of all classes of order effects that are not both symmetric and identically signed. These latter restrictions are also structural features of set theory. Indeed, both the set-theoretic and the spatial modeling traditions are mute on the question of whether the order of introduction (or an actual consideration by the voter) of the institutional options matters across voters with nonseparable preferences. In the set-theoretic example above, $\{SY\}$ is "an" event recorded from the subset K without any regard or reference to the order in which the commodities were introduced as options to the consumer. In the spatial example, the off-diagonal elements of the transition matrix must be equal (if they are not, the space will be non-Euclidean in arbitrary ways). The classical versions thus foreclose, among other things, on the possibility that a voter may seek to unify when first given one institutional control outcome and divide when first given the other.

In stark contrast, the Hilbert-space version—which is four-dimensional—is extraordinarily general and of potential use to researchers in simultaneously pondering a wide array of the sorts of effects well-known to structure choices on surveys. Of keen interest to us is a current formulation of the two-question problem by Yukalov and Sornette (2011). These authors refine a version (there are many) of nonseparable preferences in Hilbert space that, but for one important difference, mimics the classical formulation of the problem of strategic choice. The important difference is that, once the target event over which a voter/respondent would reason in making an antecedent choice is known with certainty, interference effects in the model responsible for the structure of strategic considerations disappear, and the antecedent consideration reduces to a naïve/sincere choice. The classical, probabilistic formulation of this problem instead posits certainty regarding a future outcome as the most powerful inducement to decide the antecedent question. Here again, we see an opportunity for theoretically motivated empirical research. The party balancing phenomenon we have used as a toy example in this chapter is a candidate, as the structure of the problem is similar to strategic choice. We

[5] A very preliminary—but more general—K-dimensional model of survey responses is presented in Appendix 2.

thus can imagine manipulations on a survey designed to compare empirical outcomes across the different theoretical accounts/predictions.

Last, we note that the Hilbert formalisms may help inform empirical research outside the confines of surveys (e.g., Smith and Zorn 2011). To take but one example, numerous scholars are interested in voting behavior in small groups, and note that the dynamics described above also have potentially important implications for collective choice situations where voters proceed sequentially and decision rules are non-unanimous. Such contexts are widespread, from Congressional committees to corporate board rooms to faculty meetings. The key aspect of them that renders the non-Euclidean account especially promising is that, at times, individuals are tasked with casting votes in circumstances where the outcome—either proximate or ultimate—is already determined and known to the voter. Consider, for example, voting on certiorari in the U.S./ Supreme Court. That Court has an almost completely discretionary agenda; certiorari is the process by which the justices decide which cases will be heard and decided on the merits. Cases denied certiorari are not reviewed, and the decision of the lower court is allowed to stand. Once a petition for certiorari has been placed on the Court's "discuss list," it is discussed in conference; following that discussion, the justices vote to grant or deny certiorari in order of seniority (most senior to least), with the Chief Justice voting first. At least four votes (out of nine justices) are required to grant certiorari.

Researchers have shown that justices vote in ways broadly consistent with a "strategic" theory of certiorari (e.g., Schubert 1958; Caldeira et al. 1999). That is, in circumstances where a given outcome on the merits is likely, the justices' certiorari votes are often cast in ways that are consistent with their preferences over the (known or likely) final disposition of the case on the merits, rather than on the basis of their "sincere" preferences about whether or not they wish the Court to revisit the lower court's decision. An example would be a relatively liberal justice deciding how to vote on a case decided conservatively in the court below. If a significant majority of the Supreme Court is conservative, then sophisticated voting would take the form of the justice in question voting against certiorari, in order to avoid a likely affirmation of the lower court decision—and the corresponding extension of the (undesirable) precedent it set—rather than voting in favor of certiorari.

There is evidence, however, that from time to time the justices make plain and public "errors" vis-á-vis this model. For example, Justices Brennan and Marshall famously published dissents from denials of certiorari in death penalty cases after voting to grant those petitions, despite knowing beyond all doubt that their preferred outcomes on the merits—that capital punishment be proscribed by the Eighth and Fourteenth Amendments to the U.S. Constitution—would receive no more than two (and not the requisite five) votes on the merits. Similarly, Justice Stewart joined Justices Marshall and Brennan in dissents from denial after voting to grant a petition in a case that would revisit a precedent in First Amendment law set just three years earlier in *Miller v. California* (1973) in what again was almost certainly a case that would have gone against their preferences if the case had been heard.

These are anecdotes, to be sure. Moreover they have easy explanations outside the context of classical accounts of strategic voting precisely because we know about them via the rare decision to dissent publicly from denial of certiorari. However, the sequential voting process means that justices who are of lesser seniority very often already know the outcome of the certiorari decision when their time to cast a vote arrives, and in certain cases may already sense the purpose of the more senior justices who have voted to grant the petition.[6] Might there be instances where the outcomes *on the merits* are known with certainty to the junior justices, and might this certainty free them to cast a sincere vote when they might otherwise behave strategically? These questions merit further study.

3.1 Concluding Comments

This chapter has been a largely speculative contribution. Its aim has been to introduce some basic features of an emergent literature derivative of formalisms unfamiliar to most political scientists, and to ponder some research ideas that might bring new texture to established and important problems in the discipline. In concluding, we note that, while the discussion above focuses on decision-making, we believe that a host of contemporary paradigms in political science (and the social sciences more generally) are not only consistent with nonclassical approaches, but in fact often directly imply them, and could benefit significantly from integrating such insights.

To take but one example, consider the widely used (and ascendant) Neyman–Rubin paradigm for causal inferences (e.g., Keele 2015). That tradition has its roots in the notion of counterfactuals; that is, of asking "what would happen if. . .?" Such a view of causality seems fully emergent

[6]In the most extreme case, when the Chief Justice and the three most senior associate justices all vote in favor of certiorari, a majority of the Court knows that the case will be granted irrespective of how they vote. Note that a similar dynamic also holds when the justices vote on the merits, though in that instance it requires five votes to decide the outcome.

in Hilbert space problems, and in fact our description of the Hilbert space formulation of such questions has a certain Rubinesque, potential outcomes feel to it. In the classic representation of the so-called "fundamental problem of causal inference," the root of the problem lies in the fact that no single experimental subject is able both to receive and not receive the treatment at the same time. In one sense, Rubin and his acolytes solve the problem with time itself, in theory administering the treatment to each subject multiple times (times chosen at random). By contrast, because we view such questions through a lens where a subject can "be one thing and another at the same time," we might consider the alternative of using a cross-section of (say) the treated to model responses, followed by out-of-cross-section predictions of what "would have been" in the other condition.

Appendix 1: Lacy's Model of Survey Responses in an *N*-Dimensional Euclidean Space[7]

Euclidean Assumption Set 1. *Each respondent i has a fixed ideal point* $\Theta_i = (\theta_1, \ldots, \theta_n) \in E^n$*, where n indexes the issues to be probed in a survey and* E^n *is Euclidean space.*

Euclidean Assumption Set 2. *A respondent's preferences are representable by a weighted Euclidean norm, such that for two vectors of survey responses,* $\boldsymbol{r} = (r_1, \ldots, r_n)$ *and* $\boldsymbol{r}' = (r'_1, \ldots, r'_n)$, $r \succ r'$ *if and only if:*

$$\|\boldsymbol{\Theta}_i - r\|_A^2 < \|\boldsymbol{\Theta}_i - r'\|_A^2$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Euclidean Assumption Set 3. *Each survey question j presents a respondent with a set of responses that approximate the jth element of respondent i's ideal point,* θ_{ij}*, with some error,* ϵ_{ij}*. The unconstrained response to question j is* $p_{ij} = \theta_{ij} + \epsilon_{ij}$.

[7]The first three assumption sets are exact replicas of Lacy (2001a). The fourth is a stylized combination of his assumptions labeled 4 and 5 in the same paper (pp. 8–12).

Euclidean Assumption Set 4. *When respondent i answers question j, he or she may offer a constrained response, $r(j|(r_1^*, \ldots, r_{j-1}^*, r_{j+1}^o, \ldots, r_m^o))$, given his or her beliefs about the status quo, r^o, on other issues probed in the survey.*

Appendix 2: Toward a Model of Survey Responses in a *K*-Dimensional Hilbert Space

Hilbert Assumption Set 1. *Assume that survey responses can be represented as considerations over issues and potential actions mapped by basis vectors in a real or complex Hilbert space ($\mathcal{H} \equiv \otimes_i \mathcal{H}_i$). For one action or issue question (say, A), there can be multiple considerations (indexed by* i*) A_i. Between any pair of considerations over a pair of issues or actions, A and B, the sum $A + B$ means that either A or B will be considered, and the product AB implies that they will be considered together. A decision on* C *may or may not imply composite considerations (e.g. ABC).*

Hilbert Assumption Set 2. *Options exist over issues or action questions. For instance, over the action question "How will you vote in the congressional election?" may be the simple options "Democratic" or "Republican." Likewise, over the issue question "Do you support or oppose implementation of the ACA?" may be the simple options "Support" or "Oppose." Multiple options over one consideration are also admitted, for example: "Bush," "Walker," or "Rubio" over the question of "How will you vote in a Republican primary?" Index the options of the ith consideration $\mu = 1, 2, 3, \ldots, M$, and note that the options may include both positive (e.g., "to be for" Walker) and negative (e.g., "to be against" Walker) variants.*

Hilbert Assumption Set 3. *Option Vectors. The μth option of the ith consideration of an issue, say A, is denoted as the ket vector $|A_{i\mu}\rangle$, and the (time-invariant) space containing all possible issue options on the considerations is $\mathcal{H}_i \equiv \bar{\mathcal{L}}\{|A_{i\mu}\rangle\}$.*

Hilbert Assumption Set 4. *The prepared state of the ith issue response at time t is*

$$|\psi_i(t)\rangle = \sum_{\mu} c_{i\mu}(t) \, |A_{i\mu}\rangle ,$$

where the expansion weights ($c_{i\mu}$) may be individualized to fit the actions of a single decision-maker. This state is a member of $\mathcal{H}_i$, which, by definition, because *it is a Hilbert space, has a well-defined scalar product and norm. The associated scalar product is*

$$\langle \psi_i(t_1) \mid \psi_i(t_2) \rangle = \sum_{\mu} c^*_{i\mu}(t_1) c_{i\mu}(t_2)$$

and the norm is

$$\| \, |\psi_i(t)\rangle \, \| \equiv \sqrt{\langle \psi_i(t \mid \psi_i(t) \rangle}.$$

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Voters' Preferences in a Quantum Framework

Polina Khrennikova and Emmanuel Haven

1 Introduction

The domain of quantum-like models that aim to accommodate the irrational behavior of decision-makers in various cognitive and social contexts are by now widely applied in interdisciplinary areas of cognitive science, economics, and finance. During recent years, quantum probability and quantum dynamical models have been successfully applied to describe a variety of problems such as paradoxes and probability judgment fallacies (e.g. conjunction and disjunction effects, order effects, the Allais paradox (where humans violate von Neumann–Morgenstern expected utility axioms), the Ellsberg paradox (where humans violate Aumann–Savage subjective utility axioms)). In the work contained in references (Aerts et al. 2014, 2010; Asano et al. 2011; Bagarello 2015; Bagarello and Haven 2015; Busemeyer et al. 2014; Busemeyer and Bruza 2012; Busemeyer et al. 2011, 2006; Conte et al. 2009; Danilov and Lambert-Mogiliansky 2010; De Barros and Suppes 2009; Franco 2009; Dzhafarov and Kujala 2012; Haven and Khrennikov 2013; Khrennikov et al. 2014;

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Khrennikov and Haven 2009; Khrennikova 2014a; Khrennikova and Haven 2015; Khrennikova et al. 2014; Khrennikova 2014b; Plotnitsky 2009; Pothos and Busemeyer 2009; Trueblood and Busemeyer 2011; Wang and Busemeyer 2013; Zorn and Smith 2011), a quantum representation of data that exhibits 'nonclassicality' is considered.

Recently, the field has been enriched by applications to politics, in particular the contexts of voting in the process of elections have been actively considered. The pioneering work by Zorn and Smith (2011) was the first proposal for the accommodation of observables and voters decision-making states in a Hilbert space, instead of the classical Euclidean space that was actively pursued in models of the classical politics literature, starting with the well known spatial theory by Enelow and Melvin (1984), followed by Fiorina (1992), Smith et al. (1999), Finke and Fleig (2013), and Lacy (2001). The limitations of the static representation of the choice outcomes in a one-dimensional Euclidean space, such as a lack of dynamical representation of the decision-maker's state evolution as well as the non-influence of contextual factors (such as the mental state of the voter, memory, uncertainty, and the impact of new information), were first pointed out by Zorn and Smith (2011).

Politics is regarded as a vital area of social science and strongly relies on the assumptions of voters rationality, implying a stability of preferences. People would naturally follow the same principles, that is, the axioms of rationality, in their political decisions as in other situations, like investment decisions. Zorn and Smith highlight the fact that voters, who strive to maximize their returns in terms of for example support for their ideological position on some policy, would retain a consistency of preferences. If the outcomes of the elections are uncertain the voters are supposed to analyze the information following the axioms of classical probability theory (Kolmogorov 1950).

This is, however, not the case in reality. The most illuminating example is the situation in the US political arena. The phenomenon of 'divided government' that emerged during many election periods over the last decennia is not consistent with the notion of 'stable preferences of voters' that follows the axioms of expected utility (Von Neumann and Morgenstern 1953) and which has a well defined ranking of their preferences on their utility function of "political choices." Moreover, Alvarez and Schousen (1993) discusses that the ticket splitting behavior is often perceived in the traditional literature on politics as something uncommon and even undesirable.[1]

At the same time, Fiorina (1992) has explored this phenomenon from a strategic perspective. She attributed such preference reversal to a so-called 'non-

[1]We omit in this review a discussion on the social implications of the emergence of divided government.

separability' of preferences. This phenomenon has been proved empirically in various opinion polling studies and surveys, see Brams et al. (1997), Finke and Fleig (2013), Lacy (2001), Lacy and Niou (1998, 2000), Smith et al. (1999). Nonseparability is defined in the literature on politics as the conditioning of decisions on each other (Smith et al. 1999; Lacy and Niou 2000; Lacy 2001). However, the roots of this phenomenon are more complex than just a simple conditioning of the probabilities of events in a Bayesian way. This problem was first discussed in Zorn and Smith (2011). They proposed a quantum representation of voters' belief states, where the strong interrelation of their decisions for different political issues could be captured through a quantum feature called entanglement.

The ideas of Zorn and Smith (2011) were supported and developed further in Khrennikova et al. (2014), where a dynamical representation of the evolution of voters' belief states with the aid of a quantum master equation was presented. Also, a numerical simulation of such a state dynamics was performed. Building upon these first successful models, the studies performed in Khrennikova (2014a); Khrennikova and Haven (2015) sought to analyze some concrete statistics on voters' preferences, in order to motivate further applications of the quantum representation. The data obtained from the opinion poll study made by Smith et al. (1999) showed:

(a) The existence of an order effect in voters' responses.
(b) The violation of the Bayesian updating of voters' preferences that could be accommodated in a classical Kolmogorovian probability space. See the analysis in Khrennikova and Haven (2015).

To add more robustness to these empirical findings, an additional dataset on the 2008–2014 US election outcomes was collected; see Khrennikova and Haven (2015). The analysis of the voting frequencies displayed the same phenomena: (a) the nonseparability of preferences; (b) nonclassicality, that is, the violation of the core law of classical probability, the *law of total probability* (Kolmogorov 1950).

We explain such a strong interrelation of voter preferences through the notions of superposition and entanglement. Note that irrationality, biases, and nonseparability lead to a deeper uncertainty than a classical probabilistic uncertainty. In fact, the aforementioned features (a) and (b) of the statistical data on the US election outcomes reflect this deep nonclassical uncertainty that is present in voters' preferences.

The time dynamics are an additional source of such nonseparability, that is, the new information that constantly reaches the voters and resolves their uncertainty changes their beliefs. We have encoded the external environment surrounding the voters in the process of an election campaign in the quantum master equation. This equation is highly complex. Therefore, its Markovian approximation is typically applied (Ohya and Volovich 2011; Ingarden et al. 1997). This equation is known as the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation. In physics the process of the resolution of the superposition type uncertainty leading to the equilibrium state is called *decoherence*. Thus, one can say that we model decoherence of voters' beliefs about political parties. This is the political science application of a general theory of decision using decoherence as developed in a series of papers. For an extended list of references see Ohya and Tanaka (2016), Khrennikov (2016). This approach, based on the theory of open quantum systems, forms part of the general quantum theory of decision-making. For a comprehensive introduction see the monographs by Busemeyer and Bruza (2012), Haven and Khrennikov (2013).

One of the main environmental factors is the informational impact of the mass media on voters' belief states. The mass media is the main source of political information. In quantum terms, it is the main source of the decoherence of voters' beliefs. This assertion is based on numerous studies of opinion formation, see for example, Zaller (1992), Graber (1989), Arterton (1984).

2 The Nonseparability Phenomenon in the US Political System

The nonseparability phenomenon that is characterized by a special interrelation of choices has been the main cause of the emergence of the so-called divided government in the US governmental structure (Fiorina 1992). This means that some voters (in fact Smith et al. (1999) showed in an opinion poll interview on 937 respondents that more than 30 % of the US electorate exhibited this feature) do not vote consistently in the Presidential and Congressional elections. They split their ballots. As a result, the legislative and executive powers are formed from opposite parties. Since the US political system is a two-party system, the situation of a "gridlock" emerges, where for instance a Democratic President is set off by a Republican Congress. The American governmental structure is experiencing this situation currently (following the 2014 midterm Congress elections).

An example of how to distinguish between the separability (isolation of preferences) and the nonseparability (preferences are evaluated simultaneously) of preferences is illustrated in Lacy (2001). If a voter is asked to rank his or her preferences in terms of the least desirable outcomes, given a set of two issues, for example Presidential and Congressional elections[2]:

- Completely separable preferences of a partisan Democrat would be: $(DD) \succ (DR)$ or $(RD) \succ (RD)$ or $(DR) \succ (RR)$ (reverse order of preferences for a Republican partisan).
- Completely nonseparable preferences of a voter who:
  (i) prefers divided government, no matter which party wins the House/Presidential elections: $(RD) \succ (DR) \succ (DD) \succ (RR)$;
  (ii) who prefers that the same party controls both powers no matter which party it is: $(DD) \succ (RR) \succ (RD) \succ (DR)$[3]

Note that the midterm elections even further amplify the preference reversal of voters. This effect is mainly a cause of the time that passes by where new information reaches the voters and changes their belief states. As noted in the 'paradox of multiple elections' (Brams et al. 1997) the more information that is known to the decision-maker on the outcome of the other issues in question (e.g. the outcome of the Congress elections) the more ability he or she has to interrelate the choices in a strategic way. Lacy (2001) corroborates the undeniable role of the time dynamics as a key factor leading to the change of voters' preferences. In other words, as time passes, the belief state of the electorate changes as a result of the impact of new information. However, nonseparability exists even in simultaneous election contests. This effect can be vindicated by the notion of entanglement of preferences.

[2]For simplicity of illustration, Lacy considered only the House elections, as a part of the Congressional elections.

[3]Case (ii) is more difficult to detect in the context of real elections. Voters that do not exhibit a violation of the transitivity axiom would be considered as voters with stable (separable) preferences.

2.1 The Impact of New Information: Order Effect and the Mass Media

If we generalize the findings of each single survey on opinion polling, we can say that the mass media is a constant source of new information that changes the state of the decision-maker (the electorate). The probabilities for the decision change under the impact of the external environment can be called the mass-media bath. According to Zaller (1992) the following impact factors can be distinguished:

(1) the loading of the news: when the news is loaded equally with positive and negative messages about e.g. a candidate's policy, the belief state of the decision-maker remains unchanged;
(2) the level of political awareness of the voter that is contained in his or her memory plays an important role in his or her resistance to any new pieces of information.

In other words, the voters have accumulated more evidence to support their 'base rate' belief. As a consequence, the swing voters can become the easiest target for mass-media campaigns.

2.2 Violation of Classical Probabilistic Framework in Different Contexts

The law of total probability (henceforth LTP) that is derived with the aid of Bayesian conditional probabilities (Kolmogorov 1950), denotes the total probability of an outcome, given its realization through some distinct events. According to the normative rules of modern decision theory, the Bayesian conditioning of the probability of events aids the decision-maker to process new information and arrive from some prior probability (base rate) to a posterior probability (of a belief, preference, decision). The disjunctions of the conditional realizations of some events (i.e., given mutually exclusive scenarios of their realization) sum up to unity, if the outcome of the baseline event is certain. LTP is derived from the conditional probability defined by Kolmogorov (1950, p. 6) as:

$$P(B|A) = P(B \cap A)/P(A). \tag{1}$$

given that $P(A) > 0$. Here A, B are some events. This formula is also known as *Bayes formula* and defines the conditional probability of an event.

If we assume that the event B can only occur jointly with one of the mutually exclusive events $A_1, A_2, A_3 \ldots$, and so on, where A_n, $n : 1, 2, 3 \ldots$, we obtain by the addition rule of probabilities that the total probability of an event B can be expressed by the formula:

$$P(B) = \sum_n P(B \cap A_n). \tag{2}$$

The events $B \cap A_1$, $B \cap A_2$, $B \cap A_3$, $B \cap A_j$ are mutually exclusive, that is, disjoint. By using the multiplication rule from Bayes' formula for conditional probability, see (1), we can rewrite the formula as:

$$P(B) = \sum_n P(B|A_n)P(A_n). \tag{3}$$

In the case of the US Presidential and Congressional elections, we can represent them mathematically with the aid of dichotomous random variables. Due to the feature of the two-party system, the Congressional elections are characterized by two outcomes: C_d and $C_r : P(C_d) + P(C_r) = 1$ and the Presidential elections by P_d and P_r, where $P(P_d) + P(P_r) = 1$. Democrats are denoted by an index d and Republicans by r. As such, the total probability for the outcome of voters' preferences can be written as:

$$P(C_d) = P(P_d)P(C_d|P_d) + P(P_r)P(C_d|P_r). \tag{4}$$

In a similar way the total probability for $P(C_r)$ can be derived.

If LTP does not hold, we experience a violation of 'classicality', as a result of super/sub-additivity of the disjunctions or the violation of Bayes' formula for the conditional probability. We repeat that the derivation of (2) is model dependent. It is based on the following features of the classical Kolmogorovian probability model: additivity of the probability measure and Bayes' formula of conditional probability. Consequently, the violation of Bayes' formula is one of the sources of violation of LTP. For psychological data exhibiting a violation of the Bayes' updating scheme, see Khrennikova (2014b). In psychology and behavioral economics the super and sub-additivity of agents' beliefs are coined the 'disjunction effect.' Its psychological origins have been widely explored, for example, in the fundamental work by Tversky and Kahneman (1983).

In brief, the main cause for the disjunction effect to occur is the emergence of special belief states of the decision-maker when he or she is in the condition of uncertainty. The disjunction effect is deeply rooted in the contextuality of human beliefs and decisions.

The nonclassicality of voters' beliefs deeply resonates with the quantum features of superposition and entanglement. These quantum notions can provide a better understanding of the uncertainty of beliefs that the voters hold, as well as of the complexity of their interrelation. Another type of nonseparability coined by Zorn and Smith (2011) and known as 'pseudo-classical' emerges that is much more like the way the cognitive states of the voters are related in the process of casting ballots.

2.2.1 Interference of Statistical Data

From a quantum standpoint, the violation of LTP has its origins in the interference of probabilities, when they are not measured. The social interpretation for the political context is that the cognitive state of the voter is in a superposition of different beliefs.

After observing a violation of LTP in our analysis, we accommodated the statistical data in the formula of quantum probability, which can be regarded as a generalization of the classical probabilistic scheme. For a detailed discussion see Haven and Khrennikov (2013). A quantum-like[4] formula of total probability that is an extension of Eq. (4) with the so-called interference term can be denoted as:

$$Pr(C_d) = Pr(P_d)Pr(C_d|P_d) + Pr(P_r)Pr(C_d|P_r) + 2\lambda_d\sqrt{Pr(P_d)Pr(C_d|P_d)Pr(P_r)Pr(C_d|P_r)}. \tag{5}$$

The λ_d can be represented either as $\cos\theta_d$, a trigonometric interference, or $\cosh\theta_d$, a hyperbolic interference. The formula with the trigonometric interference can be derived by using the complex Hilbert space formalism of quantum mechanics (QM). The quantum formula of total probability allows for a representation of random variables by noncommuting operators for the observables, Congressional and Presidential elections. The hyperbolic interference cannot be derived in this way. To obtain interference terms of such high magnitudes ($\lambda_d > 1$ is a $\cosh\theta$ interference), one has to use

[4]The term 'quantum-like' is an umbrella word to denote both the traditional quantum formalism and its generalizations, including the usage of the mathematical apparatus only. The quantum-like models do not strictly obey the formalism of quantum mechanics.

a generalization of QM, the so-called 'hyperbolic quantum mechanics,' see Haven and Khrennikov (2013). To sum up, the interference term provides an indication of the amplitude of the probabilistic interference. Below we present some interference amplitudes for the probabilities of voters' beliefs.

For the data from Smith et al. (1999) that was analyzed in Khrennikova (2014a, 2016), we found that the probabilities of voters' beliefs and preferences for the for example Democratic President and the Republican Congress experienced an interference of the following magnitudes:

- For the observable outcomes 'Democratic Congress' the interference of probabilities was negative, i.e. destructive:

$$\cos\theta = -0.257.$$

- For the observable outcomes 'Republican Congress' the interference of probabilities was destructive as well:

$$\cos\theta = -0.216.$$

  These interference effects could be represented as the cosinus of some angle (phase) of the mental state wave function of the respondents.
- We also analyzed the interference for the observable outcome 'Don't know' which we believe can be considered as a firm preference. In fact the voters with such preferences were found to be mostly influenced by the new information on the outcomes of the presidential elections in the (Smith et al. 1999) study:

$$\cosh\theta = 3.84.$$

This interference of a high magnitude was due to a very strong contextuality effect. We accommodated such an interference of states in a hyperbolic Hilbert space. Furthermore, we were able to perform a state reconstruction with a generalized Born Rule that enabled us to motivate the possibility of modeling the voters' belief state ψ and its transition to an eigenvector with the mathematics and concepts from QM.

In order to support the claim that voters' belief states can exhibit quantum features we also found an interference in the data of 2008–2014 US elections (the total statistics on Presidential and Congressional elections). For details see, Khrennikova and Haven (2015). We performed the calculation for the observable outcome $C = D$ (Democratic Congress) and of course a similar procedure can be performed for $C = R$. For the data of 2008–2010 (i.e., the voters' 2010 Congress preferences conditioned on the 2008 Presidential election outcomes) the $\cos\theta = 0.16$ and for the 2012–2014 data the $\cos\theta = 0.121$. This is a quantum type of constructive interference that can be accommodated in a Hilbert space and a state reconstruction with the Born rule can be performed in a similar way as in Khrennikova and Haven (2015).

In the next section we will present a dynamical model based on the theory of open quantum systems and survey the methods, notations, and terminology of quantum theory. We will consider notions such as entanglement of states, the Hilbert state space, a pure state, a mixed state (density operator), and the Schrödinger equation. The reader can find a more detailed elaboration of these concepts in Chap. 1 of this handbook (Haven and Khrennikov 2016).

3 A Quantum Model of the Decision-Making of Voters

We introduce briefly the modeling of the voters' belief states in the process of decision evolution in the complex Hilbert state space. Denote by H the state space of voters. In the simplest model this can be represented as the two qubit space, that is, as the four-dimensional complex Hilbert space. For instance, in the context of US elections, one qubit corresponds to the Congressional elections and another to the Presidential elections. Each qubit represents the superposition of two states $|D\rangle$, democrats, and $|R\rangle$, republicans, where we have used kets. These qubits are entangled; see Khrennikova et al. (2014) for details. Hence, H is isomorphic to $\mathbb{C}^4$. Thus, in reality we work just with four complex variables: in particular, pure states are complex vectors with four coordinates and mixed states are 4 by 4 matrices with complex elements.

In the quantum formalism, the dynamics of the state of an isolated quantum system is described by Schrödinger's equation and the dynamics of the state of a quantum system interacting with an environment is described by the quantum master equation (Ohya and Volovich 2011). In general the latter equation is too complicated mathematically to be treated directly, that is, without using some approximations. Therefore, various approximations are used. The most popular is the quantum Markovian approximation leading to the GKSL equation (Ingarden et al. 1997; Ohya and Volovich 2011).

By applying the mathematical formalism of QM to modeling the behavior of voters, one has to specify the notion of 'environment.' Note that our model is concerned with the information dynamics. For this specific purpose, it is natural to identify the environment, 'bath,' as a 'bath' of information related to all the components of the elections: the political parties, their candidates, their programs, the private life of the candidates, their economic situation, and so on.... As we have mentioned in Sect. 2.1, the information that reaches the voters' cognitive states primarily floods in from the mass media. Thus we coin it the 'mass-media bath'.

'Isolated dynamics' correspond to the ignorance of the impact of such a bath and considers voters as ignoring the aforementioned information flow from outside. It is clear that even in physics the notion of an isolated quantum system is merely an abstraction. In reality there are no completely isolated systems, neither in physics nor in social science. However, in some contexts one can ignore the impact of the environment with some degree of approximation. This provides for the possibility of splitting the generator of dynamics into two parts: the first one corresponding to the ignorance of the environment (that generates Schrödinger's dynamics); and the second one representing the impact of the 'election environment' (generated through the mass-media bath). Note that, although the notion of an isolated system matches well a variety of physical contexts, it is not so useful in social science in general and, in particular, in politics.

One of the main distinguishing features of the solutions of the Markovian quantum master equation, the GKSL equation, is that for a wide class of equations a nonstationary solution $\rho(t)$ stabilizes to a stationary solution ρ_{decision} representing the prognosis about the distribution of the final beliefs of the electorate.

As has already been emphasized, in comparison to the Schrödinger equation, the quantum master equation can transform pure states into mixed states. Such a process is called *decoherence.* This is a dynamical equation in the space of density operators. Therefore the limiting prognosis state ρ_{decision} can be a mixed state, even if the initial belief state was a pure state. Such a steady state is (under natural conditions) diagonal in the density matrix basis. This represents the resolution of quantum-like uncertainties of superposition and entanglement types, which are typically present in the initial state ρ_0, to the classical state-firm beliefs and decisions. In general ρ_{decision} determines only the probabilities of voters' decisions. In this regard, one proceeds by assigning the voters a decision that is associated with the highest probability.

We now write the Markovian approximation of the quantum master equation, the GKSL equation (Ohya and Volovich 2011):

$$\frac{d\rho}{dt}(t) = -\frac{i}{\gamma}[\mathcal{H}, \rho(t)] + L(\rho(t)), \tag{6}$$

where $\mathcal{H}$ is a Hermitian operator acting in H and L is a linear operator acting in the space of linear operators $B(H)$ in H (such maps are often called *super-operators*). Typically the operator $\mathcal{H}$ represents the state dynamics in the absence of the environment. However, in general $\mathcal{H}$ can also contain contributions from the impact of the environment. The superoperator L has to map density operators onto density operators, that is, it has to preserve for instance Hermiticity and positive definiteness. These conditions constrain essentially the class of possible generators L. By adding some additional condition, which role we are not able to discuss in this chapter, that is, the so called *complete positive definiteness*, we describe the class of such generators precisely. They have the form:

$$L\rho = \sum_k \alpha_k [C_k \rho C_k^* - (C_k^* C_k \rho + \rho C_k^* C_k)/2] = \sum_k \alpha_k \left[C_k \rho C_k^* - \frac{1}{2} \left\{ C_k^* C_k, \rho \right\} \right]. \tag{7}$$

Operators C_k encode the special features of the social environment. To provide an overview of how such a model could work, a numerical simulation was performed in Khrennikova et al. (2014). The results show the powerful role of the election environment in shaping the final preferences of voters. This model works for voters that are potentially able to change their preferences: the preferences are not firm, and new information can affect and reveal them.

4 Discussion: What Next?

Some substantial results in exploring the origins of the phenomenon of nonseparability have been achieved. A more complex aim to represent the dynamical decision processes of the voters has been realized as well, with the aid of the mathematical and conceptual framework of QM. The research that we have discussed in this chapter pertains to empirical and theoretical contributions to the field of human decision-making. Undeniably, the field of application of quantum models to voters' political choices is still in its infancy and would benefit from further empirical studies and simulations, so an accurate match can be developed between the psychological and mathematical variables that we have introduced in a 'social analogue' of the GKSL model.

We believe that further studies on voters preferences (political issues as well as the contextual factors, including the political environment) would allow the GKSL model and the strand of related QL models to progress further. The models could become explanatory rather than merely being of a descriptive character. Hence, they could potentially provide for a viable alternative to accommodate human cognitive states and their resolution from uncertainty.

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Quantum Structure in Cognition Origins, Developments, Successes, and Expectations

Diederik Aerts and Sandro Sozzo

1 The Combination Problem in Concept Theory

That concepts exhibit aspects of 'contextuality,' 'vagueness,' and 'graded typicality' was already known in the 1970s since the investigations of Rosch (1973). These studies questioned explicitly the traditional view that 'concepts are containers of instantiations' and, additionally, although not explicitly stated, there was already the suspicion that 'the human mind combines concepts not following the algebraic rules of classical logic even if the combinations are conjunctions or disjunctions.' In particular, conceptual gradedness led scholars to introduce elements of probability theory in structuring and representing concepts. A possible way to at least preserve a set-theoretical basis was the fuzzy set approach (Zadeh 1989). According to this proposal, concepts would combine in such a way that the conjunction of two concepts satisfies the 'minimum rule of fuzzy set conjunction' and the disjunction of two concepts satisfies the 'maximum rule of fuzzy set

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disjunction.' In this way, one could still maintain that 'concepts can be represented as (fuzzy) sets and combine according to set-theoretic rules.' However, a whole set of experimental findings in the last 30 years has revealed that the latter does not hold, thus raising the so-called 'combination problem.'

(i) 'Guppy effect.' Osherson and Smith measured the 'typicality' of specific exemplars with respect to the concepts *Pet* and *Fish* and their conjunction *Pet-Fish* (Osherson and Smith 1981), and they found that people rate an exemplar such as *Guppy* as a very typical example of *Pet-Fish*, without rating *Guppy* as a very typical example either of *Pet* or of *Fish* (the 'Pet-Fish problem').[1] Interestingly enough, this guppy effect violates the minimum rule of fuzzy set conjunction.

(ii) 'Overextension and underextension effects.' Hampton measured the 'membership weight' of several exemplars with respect to specific pairs of concepts and their conjunction (Hampton 1988a) and disjunction (Hampton 1988b), finding systematic deviations from fuzzy set modeling. Adopting his terminology, if the membership weight of an exemplar x with respect to the conjunction 'A and B' of two concepts A and B is higher than the membership weight of x with respect to one concept (both concepts), we say that the membership weight of x is 'overextended' ('double overextended') with respect to the conjunction (briefly, x is overextended with respect to the conjunction). If the membership weight of an exemplar x with respect to the disjunction 'A or B' of two concepts A and B is less than the membership weight of x with respect to one concept (both concepts), we say that the membership weight of x is 'underextended' ('double underextended') with respect to the disjunction (briefly, x is underextended with respect to the disjunction). We have recently performed a similar cognitive test on conceptual conjunctions of the form 'A and B' (Sozzo 2015; Aerts et al. 2015a), detecting systematic overextension and also double overextension.[2]

(iii) 'Deviations from classicality in conceptual negation.' More recently, Hampton measured the membership weights of many exemplars with respect to specific pairs of concepts and their conjunction, e.g., *Tools*

[1]In a typicality measurement, subjects are asked to choose the exemplar they consider as the most typical example of a given concept, hence they have to pick the best example in a list of items.

[2]In a membership measurement, subjects are asked to decide whether a given exemplar x is a member of a given concept A. When many subjects are involved in the measurement, a membership weight $\mu(A)$ can be defined for x as a large number limit of the relative frequency of positive answers.

which Are also Weapons, and also conjunction when the second concept is negated, e.g., *Tools which Are not Weapons* (Hampton 1997). He detected overextension in both types of conjunctions, as well as deviations from classical logical behaviour in conceptual negation. We have recently performed a more general cognitive test (Sozzo 2015; Aerts et al. 2015a), detecting systematic overextension, double overextension, and violation of classical logic negation rules in conceptual conjunctions of the form '*A* and not *B*,' 'not *A* and *B*,' and 'not *A* and not *B*.'

(iv) 'Borderline contradictions.' Alxatib and Pelletier asked human subjects to estimate the truth value of a sentence such as "x is tall and not tall" for a given person x who was shown to subjects (Alxatib and Pelletier 2011). They found that a significant number of subjects estimated such a sentence as true, in particular, for borderline cases.[3]

Difficulties (i)–(iv) entail, in particular, that the formation and combination rules of human concepts do not generally follow the laws of classical (fuzzy set) logic (Osherson and Smith 1981; Hampton 1988a,b, 1997). Moreover, the corresponding experimental data cannot generally be modeled in a single classical probability space satisfying the axioms of Kolmogorov, which we have proved in various articles (Sozzo 2015; Aerts 2009a; Aerts et al. 2013a; Sozzo 2014).

Our investigation of the above 'deviations from classicality'[4] in conceptual combinations can be traced back to our studies on the axiomatic and operational foundations of quantum physics and the origins of quantum probability (see, e.g., Aerts 1986). We recognized that any decision process, for example, a typicality measurement, or a membership estimation, involves a 'transition from potential to actual,' in which an outcome is actualized from a set of possible outcomes as a consequence of a contextual interaction (of a cognitive nature) of the subject with the conceptual situation that is the object of the decision. Hence, human decision processes exhibit deep analogies with what occurs in a quantum measurement process, where the measurement context (of a physical nature) influences the measured quantum particle in a non-deterministic way. Quantum probability—which is able to formalize this 'contextually driven actualization of potential,' and not classical probability,

[3]A borderline contradiction can be formalized as a sentence of the form $P(x) \wedge \neg P(x)$, for a vague predicate P and a borderline case x, e.g., the sentence "Mark is rich and Mark is not rich."

[4]By the locution 'deviation from classicality' we actually mean that classical logical and probabilistic structures, i.e. the most traditional models of cognition, cannot account for the experimentally observed patterns.

which only formalizes a lack of knowledge about actuality—can conceptually and mathematically cope with this situation that underlies both the quantum and conceptual realms (Aerts and Aerts 1995).

The second step of our research was the elaboration of a 'state context property' (SCoP) formalism to represent abstractly any type of entity, for example, a conceptual entity, in terms of its states, contexts, and properties. In a SCoP, a concept is represented as an 'entity being in a specific state and changing under the influence of a cognitive context,' rather than as a 'container of instantiations,' and we were able to provide a quantum-theoretic model in Hilbert space that successfully describes the guppy effect (Aerts and Gabora 2005a,b) (Sect. 2).

The successive development of our research was the employment of the mathematical formalism of quantum theory in Fock space to model the overextension and underextension of membership weights measured in Hampton (1988a,b). These effects can be described in terms of genuine quantum aspects, like 'interference,' 'superposition,' and 'emergence' (Aerts 2009a,b; Aerts et al. 2013a,b). This quantum-mechanical model was successfully applied to describe borderline contradictions (Sozzo 2014). More recently, we extended the model to incorporate conceptual negation, thus faithfully representing the above mentioned experiments by ourselves on concept conjunctions and negations (Sozzo 2015; Aerts et al. 2015a) (Sect. 3).

Our results allowed us to put forward a unifying explanatory hypothesis for this whole set of experimental findings in human cognition, namely, that human thought is guided by two simultaneous processes—'quantum conceptual thought,' whose nature is 'emergence,' and 'quantum logical thought,' whose nature is 'logic' (Aerts et al. 2015b). Our investigations indicate that the former generally prevails over the latter, and that the effects, paradoxes, contradictions, and fallacies that are experimentally detected in human cognition can be considered as expressions of this dominance, rather than 'biases' of the human mind. More recently, we received further crucial confirmation of this two-layered structure in human thought, namely the stability of the deviation from classical probabilistic rules that we detected in Aerts et al. (2015a) (Sect. 4).

Our quantum-theoretic perspective also accounts for two recent experimental results we obtained, namely, the identification of 'quantum entanglement' in the conceptual combination *The Animal Acts* (Aerts and Sozzo 2011, 2014a) (Sect. 5) and the detection of 'quantum indistinguishability of the Bose–Einstein type' in specific combinations of identical concepts, such as *Eleven Animals* (Aerts et al. 2015c) (Sect. 6). These discoveries are also important, in our opinion, from the point of view of the foundations of quantum physics, since they can shed new light on two mysterious aspects of the microscopic world—entanglement and indistinguishability.

In this review chapter, we present the above results by basically following a historical reconstruction, though justified and restructured in a unitary and more general rational framework. We conclude the chapter with some epistemological remarks on the role and interpretation of our quantum-theoretic perspective within the domain of cognitive modeling, and with some hints for future developments (Sect. 7).

2 The First Steps: Potentiality and Contextuality in Decision Processes

The first move towards the development of a quantum-theoretic perspective in cognition came from our former research on the mathematical and conceptual foundations of quantum physics. In particular, we were guided by our studies on:

1. the identification of quantum structures outside the microscopic world, e.g., in the cognitive situation of the liar paradox (Aerts et al. 1999, 2000);
2. the recognition of the existence of deep analogies between quantum particles and conceptual entities with respect to 'potentiality' and 'contextuality';
3. the role played by quantum probability in formalizing experimental situations where these aspects of potentiality and contextuality occur.

It is well known from quantum physics that, in a quantum measurement process, the measurement context influences the quantum entity that is measured in a nondeterministic way, actualizing one outcome in a set of possible measurement outcomes, as a consequence of the interaction between the quantum entity and the measurement context. Suppose now that a statistics of measurement outcomes is collected after a sequence of many repeated measurement processes on an arbitrary entity, and such that (1) the measurement actualizes properties of the entity that were not actual before the measurement started, (2) different outcomes and actualizations are obtained probabilistically. What type of probability can formalize such an experimental situation? It cannot be classical probability, because this formalizes a lack of knowledge about actual properties of the entity that were already actual before the measurement started. We proved many years ago that a situation where context actualizes potential properties can instead be represented in a suitable quantum probabilistic framework (Aerts 1986).

What about a human decision process? Well, we realized that a decision process is generally made in a state of genuine potentiality, which is not of the type of a lack of knowledge of an actuality. The following example may help to illustrate this point. In Aerts and Aerts (1995), we considered a survey including the question "Are you a smoker or not?" Suppose that 21 participants over a whole sample of 100 answered 'yes' to this question. We can then consider 0.21 as the probability of finding a smoker in this sample. However, this probability is obviously of the type of a 'lack of knowledge about an actuality,' because each participant 'is' a smoker or 'is not' a smoker before the property has been tested, hence before the experiment to test it—the survey—starts. Suppose that we now consider the question "Are you for or against the use of nuclear energy?" and that 31 participants answer they are in favor. In this case, the resulting probability 0.31 is 'not' of the type of 'lack of knowledge about an actuality.' Indeed, it is very plausible for this type of question that some of the participants had no opinion about it before the survey, and hence for them the outcome was influenced by the context at the time the question was asked, including the specific conceptual structure of how the question was formulated. This is how context plays an essential role whenever the human mind is concerned with outcomes of experiments of a cognitive nature. We have shown that the first type of probability, for example, the type that models a 'lack of knowledge about an actuality,' is classical, and that the second type is nonclassical and, possibly, quantum (Aerts 1986).

The effect due to the role that context plays on a conceptual entity is equally fundamental to the effect due to the actualizing of potentialities during a decision process. Exactly as in a quantum measurement the measurement context changes the state of the quantum entity that is measured, so in a decision process the cognitive context changes the state of the concept (Aerts and Gabora 2005a,b). For example, in our modeling of the concept *Pet*, we considered the context e expressed by *Did you see the type of pet he has? This explains that he is a weird person*, and found that when participants in an experiment were asked to rate different exemplars of *Pet*, the scores for *Snake* and *Spider* were very high in this context. From our perspective, this is explained by the existence of different states for the concept *Set*, where we use the notion of 'state' in the same way as it is used in quantum theory, but also as it is used in ordinary language, for example, 'the state of affairs,' meaning 'how the affairs will react on different measurement contexts.' We call 'the state of *Pet* when no specific context is present' its ground state $\hat{p}$. The context e then changes the ground state $\hat{p}$ into a new state $p_{weird\ person\ pet}$. Typicality, from our perspective, is an observable semantic quantity, which means that it takes different values in different states. Hence, from our perspective the typicality variations as encountered in the guppy effect are due to changes of state of the concept *Pet* under the influence of a context. More specifically, the conjunction *Pet-Fish* is *Pet* under the context *Fish*, in which case the ground state p of *Pet* is changed into a new state p_{Fish}. The typicality of *Guppy*, being an observable semantic quantity, will be different depending on the state, and this explains the high typicality of *Guppy* in the state p_{Fish} of *Pet*, and its normal typicality in the ground state p of *Pet* (Aerts and Gabora 2005a).

We have developed this approach in a formal way, and called the underlying mathematical structure a SCoP system (Aerts and Gabora 2005a). To build a SCoP for an arbitrary concept S we introduce three sets, namely: the set Σ of states, denoting states by $p, q, \ldots$; the set $\mathcal{M}$ of contexts, denoting contexts by $e, f, \ldots$; and the set $\mathcal{L}$ of properties, denoting properties by $a, b, \ldots$. The 'ground state' $\hat{p}$ of the concept S is the state where S is not under the influence of any particular context. Whenever S is under the influence of a specific context e, a change of the state of S occurs. In case S was in its ground state $\hat{p}$, the ground state changes to a state p. The difference between states $\hat{p}$ and p is manifested, for example, by the typicality values of different exemplars of the concept and the applicability values of different properties being different in the two states $\hat{p}$ and p. Hence, to complete the mathematical construction of a SCoP, two functions μ and ν are also introduced. The function $\mu : \Sigma \times \mathcal{M} \times \Sigma \longrightarrow [0, 1]$ is defined such that $\mu(q, e, p)$ is the probability that state p of concept S under the influence of context e changes to state q of concept S. The function $\nu : \Sigma \times \mathcal{L} \longrightarrow [0, 1]$ is defined such that $\nu(p, a)$ is the weight, or normalization of applicability, of property a in state p of concept S. With these mathematical structures and tools the SCoP formalism copes with both 'contextual typicality' and 'contextual applicability.'

We likewise built an explicit quantum-mechanical representation in a complex Hilbert space of the data of the experiment on *Pet* and *Fish* and different states of *Pet* and *Fish* in different contexts explored in Aerts and Gabora (2005a), as well as of the concept *Pet-Fish* (Aerts and Gabora 2005b). In this way, we were able to cope with the pet-fish problem illustrated in Sect. 1, (i).

The analysis above already contained the seeds of our quantum modeling perspective for concept combinations—in particular, the notion of the state of a concept marked the departure from the traditional idea of a concept as a set, eventually fuzzy, that contains instantiations. However, this analysis was still preliminary, and a general quantum-mechanical modeling required further experimental and theoretic steps, as will become clear in the following section.

3 Modeling Concept Combinations in Fock Space

We present here our quantum modeling perspective in Fock space for the combination of two concepts. This is successful in describing the classically problematical results illustrated in Sect. 1, (ii) (concept conjunction and disjunction), (iii) (concept negation), and (iv) (borderline contradictions).

Let us firstly consider the membership weights of exemplars of concepts and their conjunctions/disjunctions measured by Hampton (1988a,b). He identified systematic deviations from classical (fuzzy) set conjunctions/disjunctions, an effect known as 'overextension' or 'underextension' (see Sect. 1). We showed in Aerts (2009a) that a large part of Hampton's data cannot be modeled in a classical probability space satisfying the axioms of Kolmogorov (1950) and Pitowsky (1989). For example, the exemplar *Mint* scored in Hampton (1988a) the membership weight $\mu(A) = 0.87$ with respect to the concept *Food*, $\mu(B) = 0.81$ with respect to the concept *Plant*, and $\mu(A \text{ and } B) = 0.9$ with respect to their conjunction *Food And Plant*. Thus, the exemplar *Mint* exhibits overextension with respect to the conjunction *Food And Plant* of the concepts *Food* and *Plant*, and no classical probability representation exists for these data. More generally, the membership weights $\mu(A)$, $\mu(B)$, and $\mu(A \text{ and } B)$ of the exemplar x with respect to concepts A, B, and their conjunction 'A and B,' respectively, can be represented in a classical Kolmogorovian probability model if and only if they satisfy the following inequalities (Sozzo 2015; Aerts 2009a)

$$\mu(A \text{ and } B) - \min(\mu(A), \mu(B)) \leq 0 \tag{1}$$

$$\mu(A) + \mu(B) - \mu(A \text{ and } B) \leq 1 \tag{2}$$

A violation of (1) entails, in particular, that the minimum rule of fuzzy set conjunction does not hold, as in the case of *Mint*. A similar situation occurs in the case of disjunctions. We showed in Aerts (2009a) that a large part of Hampton's data cannot be modeled in a classical Kolmogorovian probability space. For example, the exemplar *Sunglasses* scored in Hampton (1988b) the membership weight $\mu(A) = 0.4$ with respect to the concept *Sportswear*, $\mu(B) = 0.2$ with respect to the concept *Sports Equipment*, and $\mu(A \text{ or } B) = 0.1$ with respect to their disjunction *Sportswear Or Sports Equipment*. Thus, the exemplar *Sunglasses* exhibits underextension with respect to the disjunction *Sportswear Or Sports Equipment* of the concepts *Sportswear* and *Sports Equipment*, and no classical probability representation exists for these data. More generally, the membership weights $\mu(A)$, $\mu(B)$, and $\mu(A \text{ or } B)$ of the exemplar x with respect to concepts A, B, and their disjunction 'A or B,' respectively, can be represented in a classical Kolmogorovian probability model if and only if they satisfy the following inequalities (Aerts 2009a)

$$\max(\mu(A), \mu(B)) - \mu(A \text{ or } B) \leq 0 \tag{3}$$

$$0 \leq \mu(A) + \mu(B) - \mu(A \text{ or } B) \tag{4}$$

A violation of (3) entails, in particular, that the maximum rule of fuzzy set disjunction does not hold, as in the case of *Sunglasses*.

In a first attempt to elaborate a quantum mathematics model for the data in Hampton (1988a,b) we were inspired by the quantum two-slit experiment.[5] Consider, for example, the disjunction of two concepts. This led us to suggest the following Hilbert space model. One could represent the concepts A and B by the unit vectors $|A\rangle$ and $|B\rangle$, respectively, of a Hilbert space $\mathcal{H}$, and describe the decision measurement of a subject estimating whether the exemplar x is a member of A by means of a dichotomic observable represented by the orthogonal projection operator M. The probabilities $\mu(A)$ and $\mu(B)$ that x is chosen as a member of A and B, that is, its membership weights, are given by the scalar products $\mu(A) = \langle A|M|A\rangle$ and $\mu(B) = \langle B|M|B\rangle$, respectively. The concept 'A or B' is instead represented by the normalized superposition $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$ in $\mathcal{H}$. If $|A\rangle$ and $|B\rangle$ are chosen to be orthogonal, that is, $\langle A|B\rangle = 0$, the membership weights $\mu(A), \mu(B)$, and $\mu(A \text{ or } B)$ of an exemplar x for the concepts A, B, and 'A or B' are given by

$$\mu(A) = \langle A|M|A\rangle \tag{5}$$

$$\mu(B) = \langle B|M|B\rangle \tag{6}$$

$$\mu(A \text{ or } B) = \frac{1}{2}(\mu(A) + \mu(B)) + \Re\langle A|M|B\rangle \tag{7}$$

respectively, where $\Re\langle A|M|B\rangle$ is the real part of the complex number $\langle A|M|B\rangle$. The term $\Re\langle A|M|B\rangle$ is called the 'interference term' in the quantum jargon, since it produces a deviation from the average $\frac{1}{2}(\mu(A) + \mu(B))$ which would have been observed in the quantum two-slit experiment in the absence of interference. In this way, the deviation from classicality in Hampton (1988a,b) would be due to quantum interference, superposition, and emergence, exactly

[5] In the present chapter we use for our modeling purposes the standard quantum-mechanical formalism that is presented in modern manuals of quantum physics (see, e.g., Dirac 1958). A basic summary of this formalism is contained in the volume including this article (Haven and Khrennikov 2015).

as quantum interference, superposition, and emergence are responsible for the deviation from the classically expected pattern in the two-slit experiment.

This 'emergence-based' model in Hilbert space succeeded in describing many nonclassical situations in Hampton (1988a,b). However, it did not work for some cases, and these were exactly the cases where logic seemed to play a role in the mechanism of conceptual combination. This led us to work out a more general model in Fock space. We present the model in the following. We omit proofs and technical details here, for the sake of brevity, inviting the interested reader to refer to the bibliography quoted in this section. We only remind readers of some basic mathematical definitions.

A Fock space is a specific type of Hilbert space, originally introduced in quantum field theory. For most states of a quantum field the number of identical quantum entities is not conserved but is a variable quantity. The Fock space copes with this situation by allowing its vectors to be superpositions of vectors pertaining to different sectors for fixed numbers of identical quantum entities. More explicitly, the k-th sector of a Fock space describes a fixed number of k identical quantum entities and is of the form $\mathcal{H} \otimes \ldots \otimes \mathcal{H}$ of the tensor product of k isomorphic versions of a Hilbert space $\mathcal{H}$. The Fock space $\mathcal{F}$ itself is the direct sum of all these sectors, hence

$$\mathcal{F} = \oplus_{k=1}^{j} \otimes_{l=1}^{k} \mathcal{H} \tag{8}$$

(where j can be ∞). In our modeling we only used Fock space for the 'two' and 'one quantum entity' case, hence $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$. This is due to considering only combinations of two concepts. The sector $\mathcal{H}$ is called 'sector 1,' while the sector $\mathcal{H} \otimes \mathcal{H}$ is called 'sector 2.' A unit vector $|\Psi\rangle \in \mathcal{F}$ is then written as $|\Psi\rangle = ne^{i\nu}|\psi\rangle + me^{i\lambda}|\Phi\rangle$, where $|\psi\rangle \in \mathcal{H}$, $|\Phi\rangle \in \mathcal{H} \otimes \mathcal{H}$, and $n^2 + m^2 = 1$. For combinations of j concepts, the general form of Fock space expressed in (8) will have to be used.

In the case of two combining entities, a Fock space $\mathcal{F}$ consists of two sectors: 'sector 1' is a Hilbert space $\mathcal{H}$, while 'sector 2' is a tensor product $\mathcal{H} \otimes \mathcal{H}$ of two isomorphic versions of $\mathcal{H}$.

It can be proved that a quantum probability model in Fock space exists for Hampton's data on conjunction and disjunction (Aerts 2009a; Aerts et al. 2013a).

Let us start with the conjunction of two concepts. Let x be an exemplar and let $\mu(A)$, $\mu(B)$, and $\mu(A \text{ and } B)$ be the membership weights of x with respect to the concepts A, B, and 'A and B', respectively. Let $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$ be the Fock space where we represent the conceptual entities. The states of the concepts A, B, and 'A and B' are represented by the unit vectors $|A\rangle, |B\rangle \in \mathcal{H}$ and $|A \text{ and } B\rangle \in \mathcal{F}$, respectively, where

$$|A \text{ and } B\rangle = me^{i\lambda}|A\rangle \otimes |B\rangle + ne^{i\nu}\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle) \tag{9}$$

The superposition vector $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$ describes 'A and B' as a new emergent concept, while the product vector $|A\rangle \otimes |B\rangle$ describes 'A and B' in terms of concepts A and B. The positive numbers m and n are such that $m^2 + n^2 = 1$. The decision measurement of a subject who estimates the membership of the exemplar x with respect to the concept 'A and B' is represented by the orthogonal projection operator $M \oplus (M \otimes M)$ on $\mathcal{F}$, where M is an orthogonal projection operator on $\mathcal{H}$. Hence, the membership weight of x with respect to 'A and B' is given by

$$\begin{aligned}\mu(A \text{ and } B) &= \langle A \text{ and } B|M \oplus (M \otimes M)|A \text{ and } B\rangle \\ &= m^2\mu(A)\mu(B) + n^2\left(\frac{\mu(A) + \mu(B)}{2} + \Re\langle A|M|B\rangle\right)\end{aligned} \tag{10}$$

where $\mu(A) = \langle A|M|A\rangle$ and $\mu(B) = \langle B|M|B\rangle$, as above. The term $\Re\langle A|M|B\rangle$ is again the interference term of quantum theory. A solution of (10) exists in the Fock space $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ where this interference term is given by

$$\Re\langle A|M|B\rangle = \begin{cases} \sqrt{1-\mu(A)}\sqrt{1-\mu(B)}\cos\theta & \text{if } \mu(A) + \mu(B) > 1 \\ \sqrt{\mu(A)}\sqrt{\mu(B)}\cos\theta & \text{if } \mu(A) + \mu(B) \leq 1 \end{cases} \tag{11}$$

(θ is the 'interference angle.') Coming to the example above, namely, the exemplar *Mint* with respect to *Food*, *Plant*, and *Food And Plant*, (10) is satisfied with $m^2 = 0.3$, $n^2 = 0.7$, and $\theta = 50.21°$.

The previous mathematical representation admits the following interpretation. Whenever a subject is asked to estimate whether a given exemplar x belongs to the concepts A, B, 'A and B', two mechanisms act simultaneously and in superposition in the subject's thought. A 'quantum logical thought', i.e. a probabilistic version of classical logical reasoning, acts where the subject considers two copies of exemplar x and estimates whether the first copy belongs to A and the second copy of x belongs to B, and further the subject applies the probabilistic version of the conjunction to both estimates. But a 'quantum conceptual thought' also acts, where the subject estimates whether the exemplar x belongs to the newly emergent concept 'A and B'. The place where these superposed processes can be suitably structured is in Fock space.

Sector 1 hosts the latter process, while sector 2 hosts the former, while the weights m^2 and n^2 measure the 'degree of participation' of sectors 2 and 1, respectively, in the case of conjunction. In the case of *Mint*, subjects consider *Mint* to be more strongly a member of the concept *Food And Plant* than they consider it to be a member of *Food* or of *Plant*. This is an effect due to the strong presence of quantum conceptual thought, the newly formed concept *Food And Plant* being found to be a better fitting category for *Mint* than the original concepts *Food* or *Plant*. And indeed, in the case of *Mint*, considering the values of n^2 and m^2, the combination process mainly occurs in sector 1 of Fock space, which means that emergence prevails over logic.

Let us now come to the disjunction of two concepts. Let x be an exemplar and let $\mu(A)$, $\mu(B)$, and $\mu(A \text{ or } B)$ be the membership weights of x with respect to the concepts A, B, and 'A or B', respectively. Let $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$ be the Fock space where we represent the conceptual entities. The concepts A, B, and 'A or B' are represented by the unit vectors $|A\rangle, |B\rangle \in \mathcal{H}$ and $|A \text{ or } B\rangle \in \mathcal{F}$, respectively, where

$$|A \text{ or } B\rangle = me^{i\lambda}|A\rangle \otimes |B\rangle + ne^{i\nu}\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle) \qquad (12)$$

The superposition vector $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$ describes 'A or B' as a new emergent concept, while the product vector $|A\rangle \otimes |B\rangle$ describes 'A or B' in terms of concepts A and B. The positive numbers m and n are such that $m^2 + n^2 = 1$, and they estimate the 'degree of participation' of sectors 2 and 1, respectively, in the disjunction case. The decision measurement of a subject who estimates the membership of the exemplar x with respect to the concept 'A or B' is represented by the orthogonal projection operator $M \oplus (M \otimes \mathbb{1} + \mathbb{1} \otimes M - M \otimes M)$ on $\mathcal{F}$, where M has been introduced above. We notice that

$$M \otimes \mathbb{1} + \mathbb{1} \otimes M - M \otimes M = \mathbb{1} - (\mathbb{1} - M) \otimes (\mathbb{1} - M) \qquad (13)$$

that is, we have applied de Morgan's laws of logic in sector 2 of Fock space in the transition from conjunction to disjunction. The membership weight of x with respect to 'A or B' is given by

$$\mu(A \text{ or } B) = \langle A \text{ or } B|M \oplus (M \otimes \mathbb{1} + \mathbb{1} \otimes M - M \otimes M)|A \text{ or } B\rangle$$

$$m^2\,(\mu(A) + \mu(B) - \mu(A)\mu(B)) + n^2\left(\frac{\mu(A) + \mu(B)}{2} + \Re\langle A|M|B\rangle\right) \qquad (14)$$

where $\mu(A) = \langle A|M|A\rangle$ and $\mu(B) = \langle B|M|B\rangle$. The term $\Re\langle A|M|B\rangle$ is the interference term. A solution of (14) exists in $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ where the interference term is given by

$$\Re\langle A|M|B\rangle = \begin{cases} \sqrt{1-\mu(A)}\sqrt{1-\mu(B)}\cos\theta & \text{if } \mu(A)+\mu(B) > 1 \\ \sqrt{\mu(A)}\sqrt{\mu(B)}\cos\theta & \text{if } \mu(A)+\mu(B) \le 1 \end{cases} \tag{15}$$

Coming to the example above, namely, the exemplar *Sunglasses* with respect to *Sportswear*, *Sports Equipment*, and *Sportswear Or Sports Equipment*, (14) is satisfied by $m^2 = 0.03$, $n^2 = 0.97$, and $\theta = 155.00°$.

The previous mathematical representation admits the following interpretation. Whenever a subject is asked to estimate whether a given exemplar x belongs to the concepts A, B, 'A or B', two mechanisms act simultaneously and in superposition in the subject's thought. A 'quantum logical thought', i.e. a probabilistic version of classical logical reasoning, acts where the subject considers two copies of exemplar x and estimates whether the first copy belongs to A or the second copy of x belongs to B, and further the subject applies the probabilistic version of the disjunction to both estimates. But a 'quantum conceptual thought' also acts, where the subject estimates whether the exemplar x belongs to the newly emergent concept 'A or B'. The place where these superposed processes are structured is again in Fock space. Sector 1 hosts the latter process, while sector 2 hosts the former, while the weights m^2 and n^2 measure the 'degree of participation' of sectors 2 and 1, respectively, in the case of disjunction. In the case of *Sunglasses*, subjects consider *Sunglasses* to be less strongly a member of the concept *Sportswear Or Sports Equipment* than they consider it to be a member of *Sportswear* or of *Sports Equipment*. This is an effect due to the strong presence of quantum conceptual thought, the newly formed concept *Sportswear Or Sports Equipment* being found to be a less well fitting category for *Sunglasses* than the original concepts *Sportswear* or *Sports Equipment*. And indeed, in the case of *Sunglasses*, considering the values of n^2 and m^2, the combination process mainly occurs in sector 1 of Fock space, which means that emergence aspects prevail over logical aspects in the reasoning process.

Let us then analyze the experiment of Alxatib and Pelletier on borderline contradictions (Alxatib and Pelletier 2011). We proved in Sozzo (2014) that our quantum-theoretic model for the conjunction correctly represents the collected data. Suppose that a large sample of human subjects is asked to estimate the truth values of the sentences "John is tall," "John is not tall," and "John is tall and not tall," for a given subject John shown to the subjects. And suppose that the fractions of positive answers are 0.01, 0.95, and

0.15, respectively (Alxatib and Pelletier 2011). This 'borderline case' is clearly problematical from a classical logical perspective and can be modeled in terms of overextension. Indeed, let us denote by $\mu(A)$, $\mu(A')$, and $\mu(A \text{ and } A')$ the probabilities that the sentences "John is tall," "John is not tall," and "John is tall and not tall" are true, and interpret them as membership weights of the exemplar *John* with respect to the concepts *Tall*, *Not Tall*, and *Tall And Not Tall*, respectively. Then (10) is solved for $m^2 = 0.77$, $n^2 = 0.23$, and $\theta = 0°$ (Sozzo 2014). The explanation for this behavior is that the reasoning process of the subject mainly occurs in sector 2 of Fock space, hence logical reasoning is dominant, although emergent reasoning is also present, which evokes the name 'contradiction' for this situation.

Let us finally come to the experiments on conceptual negation. The first studies on the negation of natural concepts were also performed by Hampton (1997). He tested membership weights on conceptual conjunctions of the form *Tools Which Are Not Weapons*, finding overextension and deviations from Boolean behavior in the negation. We recently performed a more general cognitive test inquiring into the membership weights of exemplars with respect to conjunctions of the form *Fruits And Vegetables*, *Fruits And Not Vegetables*, *Not Fruits And Vegetables*, and *Not Fruits And Not Vegetables* (Sozzo 2015; Aerts et al. 2015a). Our data confirmed significant deviations from classicality and evidenced a very stable pattern of such deviations from the classicality conditions. The data could very faithfully be represented in two-sector Fock space, thus providing support to our quantum-theoretic modeling. Moreover, they allowed us to attain new fundamental results in concept research and to sustain and corroborate our explanatory hypothesis in Sect. 4. Hence, it is worth briefly reviewing our recent results starting from the conditions for classicality of conceptual datasets, that is, the representability of empirical membership weights in a Kolmogorovian probability space.

Let $\mu(A)$, $\mu(B)$, $\mu(A')$, $\mu(B')$, $\mu(A \text{ and } B)$, $\mu(A \text{ and } B')$, $\mu(A' \text{ and } B)$, and $\mu(A' \text{ and } B')$ be the membership weights of an exemplar x with respect to the concepts A, B, their negations 'not A,' 'not B,' and the conjunctions 'A and B,' 'A and not B,' 'not A and B,' and 'not A and not B,' respectively, and suppose that all these membership weights are contained in the interval [0, 1] (which they will be in case they are experimentally determined as limits of relative frequencies of respective memberships). They are then 'classical conjunction data' if and only if they satisfy the following conditions.

$$\mu(A) = \mu(A \text{ and } B) + \mu(A \text{ and } B') \tag{16}$$

$$\mu(B) = \mu(A \text{ and } B) + \mu(A' \text{ and } B) \tag{17}$$

$$\mu(A') = \mu(A' \text{ and } B') + \mu(A' \text{ and } B) \tag{18}$$

$$\mu(B') = \mu(A' \text{ and } B') + \mu(A \text{ and } B') \tag{19}$$

$$\mu(A \text{ and } B) + \mu(A \text{ and } B') + \mu(A' \text{ and } B) + \mu(A' \text{ and } B') = 1 \tag{20}$$

(see Aerts et al. 2015a for the proof).

A large amount of data collected in Aerts et al. (2015a) violates very strongly and also very systematically (16)–(20), hence these data cannot be generally reproduced in a classical Kolmogorovian probability framework. It can instead be shown that almost all these data can be represented by using our quantum-theoretic modeling in two-sector Fock space, as above. For the sake of simplicity, let us work out separate representations for the two sectors.

Let us start with sector 1 of Fock space, which models genuine emergence. We represent the concepts A, B and their negations 'not A,' 'not B' by the mutually orthogonal unit vectors $|A\rangle$, $|B\rangle$ and $|A'\rangle$, $|B'\rangle$, respectively, in the individual Hilbert space $\mathcal{H}$. The corresponding membership weights for a given exemplar x are then given by the quantum probabilistic Born rule

$$\mu(A) = \langle A|M|A\rangle \qquad \mu(B) = \langle B|M|B\rangle \tag{21}$$

$$\mu(A') = \langle A'|M|A'\rangle \qquad \mu(B') = \langle B'|M|B'\rangle \tag{22}$$

in sector 1. The conjunctions 'A and B,' 'A and not B,' 'not A and B,' and 'not A and not B' are represented by the superposition vectors $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$, $\frac{1}{\sqrt{2}}(|A\rangle + |B'\rangle)$, $\frac{1}{\sqrt{2}}(|A'\rangle + |B\rangle)$, and $\frac{1}{\sqrt{2}}(|A'\rangle + |B'\rangle)$, respectively, in $\mathcal{H}$, that is, sector 1 of Fock space, which expresses the fact 'A and B,' 'A and not B,' 'not A and B,' and 'not A and not B' are considered as newly emergent concepts in sector 1.

Let us come to sector 2 of Fock space, which models logical reasoning. Here we introduce a new element, an insight which we had not expressed in our earlier application of Fock space (Sozzo 2015; Aerts 2009a; Aerts et al. 2013a; Sozzo 2014), and which we explain in detail in Aerts et al. (2015a). In short it comes to 'taking into account that possibly A and B are meaning-connected and hence their probability weights are mutually dependent.' If this is the case, we cannot represent, for example, the conjunction 'A and B' by the tensor product vector $|A\rangle \otimes |B\rangle$ of $\mathcal{H} \otimes \mathcal{H}$. This would indeed entail that the membership weight for the conjunction is $\mu(A \text{ and } B) = \mu(A)\mu(B)$ in sector 2, that is, would imply probabilistic independence between the membership estimations of A and B. We instead, following this new insight, represent the conjunction 'A and B' by an arbitrary vector $|C\rangle \in \mathcal{H} \otimes \mathcal{H}$, in sector 2, which in general will be entangled if A and B are meaning-dependent. If we represent the decision measurements of a subject estimating the membership of the exemplar x with respect to the concepts A and B by the orthogonal projection operators $M \otimes \mathbb{1}$ and $\mathbb{1} \otimes M$, respectively, we have

$$\mu(A) = \langle C|M \otimes \mathbb{1}|C\rangle \quad \mu(B) = \langle C|\mathbb{1} \otimes M|C\rangle \tag{23}$$

in sector 2. We have now to formalize the fact that this sector 2 has to express logical relationships between the concepts. More explicitly, the decision measurements of a subject estimating the membership of the exemplar x with respect to the negations 'not A' and 'not B' should be represented by the orthogonal projection operators $(\mathbb{1} - M) \otimes \mathbb{1}$ and $\mathbb{1} \otimes (\mathbb{1} - M)$, respectively, in sector 2, in such a way that

$$\mu(A') = 1-\mu(A) = \langle C|(\mathbb{1}-M)\otimes\mathbb{1}|C\rangle \quad \mu(B') = 1-\mu(B) = \langle C|\mathbb{1}\otimes(\mathbb{1}-M|C\rangle) \tag{24}$$

in this sector.

Interestingly enough, there is a striking connection between logic and classical probability when conjunction and negation of concepts are at stake, namely, the logical probabilistic structure of sector 2 of Fock space sets the limits of classical probabilistic models, and vice versa. In other words, if the experimentally collected membership weights $\mu(A)$, $\mu(B)$, $\mu(A')$, $\mu(B')$, $\mu(A \text{ and } B)$, $\mu(A \text{ and } B')$, $\mu(A' \text{ and } B)$, and $\mu(A' \text{ and } B')$ can be represented in sector 2 of Fock space for a given choice of the state vector $|C\rangle$ and the decision measurement projection operator M, then the membership weights satisfy (16)–(20), hence they are classical data. Conversely, if $\mu(A)$, $\mu(B)$, $\mu(A')$, $\mu(B')$, $\mu(A \text{ and } B)$, $\mu(A \text{ and } B')$, $\mu(A' \text{ and } B)$, and $\mu(A' \text{ and } B')$ satisfy (16)–(20), hence are classical data, then an entangled state vector $|C\rangle$ and a decision measurement projection operator M can always be found such that $\mu(A)$, $\mu(B)$, $\mu(A')$, $\mu(B')$, $\mu(A \text{ and } B)$, $\mu(A \text{ and } B')$, $\mu(A' \text{ and } B)$, and $\mu(A' \text{ and } B')$ can be represented in sector 2 of Fock space (see Aerts et al. 2015a for the proof).

Let us finally come to general representation in two-sector Fock space. We can now introduce the general form of the vector representing the state of the conjunction of the concepts A, B and their respective negations.

$$|\Psi_{AB}\rangle = m_{AB}e^{i\lambda_{AB}}|C\rangle + \frac{n_{AB}e^{i\nu_{AB}}}{\sqrt{2}}(|A\rangle + |B\rangle) \tag{25}$$

$$|\Psi_{AB'}\rangle = m_{AB'}e^{i\lambda_{AB'}}|C\rangle + \frac{n_{AB'}e^{i\nu_{AB'}}}{\sqrt{2}}(|A\rangle + |B'\rangle) \tag{26}$$

$$|\Psi_{A'B}\rangle = m_{A'B}e^{i\lambda_{A'B}}|C\rangle + \frac{n_{A'B}e^{i\nu_{A'B}}}{\sqrt{2}}(|A'\rangle + |B\rangle) \tag{27}$$

$$|\Psi_{A'B'}\rangle = m_{A'B'}e^{i\lambda_{A'B'}}|C\rangle + \frac{n_{A'B'}e^{i\nu_{A'B'}}}{\sqrt{2}}(|A'\rangle + |B'\rangle) \tag{28}$$

where $m_{XY}^2 + n_{XY}^2 = 1, X = A, A', Y = B, B'$. The corresponding membership weights $\mu(A \text{ and } B)$, $\mu(A \text{ and } B')$, $\mu(A' \text{ and } B)$, and $\mu(A' \text{ and } B')$ can be written as in (10). We proved in Aerts et al. (2015a) that they can be expressed in the Fock space $\mathbb{C}^8 \oplus (\mathbb{C}^8 \otimes \mathbb{C}^8)$ as

$$\mu(A \text{ and } B) = m_{AB}^2\alpha_{AB} + n_{AB}^2(\frac{\mu(A) + \mu(B)}{2} + \beta_{AB}\cos\phi_{AB}) \tag{29}$$

$$\mu(A \text{ and } B') = m_{AB'}^2\alpha_{AB'} + n_{AB'}^2(\frac{\mu(A) + \mu(B')}{2} + \beta_{AB'}\cos\phi_{AB'}) \tag{30}$$

$$\mu(A' \text{ and } B) = m_{A'B}^2\alpha_{A'B} + n_{A'B}^2(\frac{\mu(A') + \mu(B)}{2} + \beta_{A'B}\cos\phi_{A'B}) \tag{31}$$

$$\mu(A' \text{ and } B') = m_{A'B'}^2\alpha_{A'B'} + n_{A'B'}^2(\frac{\mu(A') + \mu(B')}{2} + \beta_{A'B'}\cos\phi_{A'B'}) \tag{32}$$

where $0 \le \alpha_{XY} \le 1, -1 \le \beta_{XY} \le 1, X = A, A', Y = B, B'$.

Let us consider a relevant example, *Goldfish*, with respect to the concepts *Pets*, *Farmyard Animals* and their combinations. In this case, big overextension was observed in all experiments, and also double overextension was identified with respect to the combination *Not Pets And Farmyard Animals*. *Goldfish* scored $\mu(A) = 0.93$ with respect to *Pets*, $\mu(B) = 0.17$ with respect to *Farmyard Animals*, $\mu(A') = 0.12$ with respect to *Not Pets*, $\mu(B') = 0.81$ with respect to *Not Farmyard Animals*, $\mu(A \text{ and } B) = 0.43$ with respect to *Pets And Farmyard Animals*, $\mu(A \text{ and } B') = 0.91$ with respect to *Pets And Not Farmyard Animals*, $\mu(A' \text{ and } B) = 0.18$ with respect to *Not Pets And Farmyard Animals*, and $\mu(A' \text{ and } B') = 0.43$ with respect to *Not Pets And Not Farmyard Animals*. A complete modeling in the Fock space satisfying Eqs. (29), (30), (31) and (32) is characterized by the coefficients:

1. interference angles $\phi_{AB} = 78.9°$, $\phi_{AB'} = 43.15°$, $\phi_{A'B} = 54.74°$, and $\phi_{A'B'} = 77.94°$;
2. coefficients $\alpha_{AB} = 0.12$, $\alpha_{AB'} = 0.8$, $\alpha_{A'B} = 0.05$, and $\alpha_{A'B'} = 0.03$;
3. coefficients $\beta_{AB} = -0.24$, $\beta_{AB'} = 0.10$, $\beta_{A'B} = 0.12$, and $\beta_{A'B'} = 0.30$;

4. convex weights $m_{AB} = 0.45$, $n_{AB} = 0.89$, $m_{AB'} = 0.45$, $n_{AB'} = 0.9$, $m_{A'B} = 0.48$, $n_{A'B} = 0.88$, $m_{A'B'} = 0.45$, and $n_{A'B'} = 0.89$.

Following our interpretation in the case of conjunction and disjunction, we can say that, whenever a subject is asked to estimate whether a given exemplar x belongs to the concepts A, B, 'A and not B', both quantum logical and quantum conceptual thought simultaneously act in the subject's thought. According to the former, the subject considers two copies of x and estimates whether the first copy belongs to A and the second copy of x does not belong to B. According to the latter, the subject estimates whether the exemplar x belongs to the newly emergent concept 'A and not B'. Fock space naturally captures this two-layered structure.

4 A Unifying Explanatory Hypothesis

The Fock space modeling presented in the previous section suggested to us to formulate a general hypothesis which justifies and explains a whole set of empirical results on cognitive psychology under a unifying theoretic scheme (Aerts et al. 2015b). According to our explanatory hypothesis, human reasoning is a specifically structured superposition of two processes, a 'logical reasoning' and an 'emergent reasoning'. Logical reasoning combines cognitive entities—concepts, combinations of concepts, or propositions—by applying the rules of logic, though generally in a probabilistic way. Emergent reasoning enables instead the formation of combined cognitive entities as newly emerging entities—in the case of concepts, new concepts, in the case of propositions, new propositions—carrying new meaning, connected with the meaning of the component cognitive entities, but with a connection not defined by the algebra of logic. These two mechanisms act simultaneously in human thought during a reasoning process, the first one is guided by an algebra of 'logic', the second one follows a mechanism of 'emergence.'

Human reasoning can be mathematically formalized in the two-sector Fock space presented in Sect. 3. The states of conceptual entities are represented by unit vectors of this Fock space as we have seen in the specific case of concept combinations. More specifically, 'sector 1 of Fock space' models 'conceptual emergence,' hence the combination of two concepts is represented by a superposition vector of the vectors representing the component concepts in this Hilbert space, allowing 'quantum interference' between conceptual entities to play a role in the process of emergence. 'Sector 2 of Fock space' models a conceptual combination from the combining concepts by requiring the rules of logic for the logical connective used for the combining, that is, conjunction or disjunction, to be satisfied in a probabilistic setting. This quantum-theoretic modeling suggested to us to call 'quantum conceptual thought,' the process occurring in sector 1 of Fock space, 'quantum logical thought,' the process occurring in sector 2. The relative importance of emergence or logic in a specific cognitive process is measured by the 'degree of participation' of sectors 1 and 2.

The abundance of evidence of deviations from classical logical reasoning in concrete human decisions (paradoxes, fallacies, effects, contradictions), together with our results in these two sections, led us to draw the conclusion that emergence constitutes the dominant dynamics of human reasoning, while logic is only a secondary structure. Therefore, we put forward the view that the aforementioned deviations from classicality are a consequence of the dominant dynamics whose nature is emergence, while classical logical reasoning is not a default to deviate from but, rather, a consequence of a secondary structure whose nature is logic.

There is further empirical evidence revealing that what primarily guides human subjects in concrete human decisions is emergent reasoning, though logical aspects are likewise present.

We identified a first element of evidence by comparing the behavior of the experimental data of different experiments. Consider, for example, the exemplar *Olive* and its membership weights with respect to the concepts *Fruits*, *Vegetables* and their conjunction *Fruits And Vegetables*, measured by ourselves (Aerts et al. 2015a; Sozzo 2015); and consider also its membership weights with respect to the concepts *Fruits*, *Vegetables* and their disjunction *Fruits Or Vegetables*, measured by Hampton (1988b). *Olive* scored $\mu(A) = 0.56$ with respect to *Fruits*, $\mu(B) = 0.63$ with respect to *Vegetables*, and $\mu(A \text{ and } B) = 0.65$ with respect to *Fruits And Vegetables*, that is, *Olive* was double overextended with respect to the conjunction. However, *Olive* was also double overextended with respect to the disjunction, since it scored $\mu(A) = 0.5$ with respect to *Fruits*, $\mu(B) = 0.1$ with respect to *Vegetables*, and $\mu(A \text{ or } B) = 0.8$ with respect to *Fruits Or Vegetables*. Our interpretation of these *Olive* cases is the following. People see *Olive* as an exemplar which could be considered to be a fruit, but also a vegetable. Hence it could also, and even more so, be considered to be both, a fruit 'and' a vegetable. This explains the double overextension of *Olive* with respect to the conjunction. This way of looking at *Olive* gives indeed the necessary weight to the conjunction to produce a double overextension. Equally so, people see *Olive* as an exemplar which induces doubt about whether it is a vegetable or whether it is a fruit. Hence it could also, and even more so, be considered a 'fruit or a vegetable.' This explains the double overextension of *Olive* with respect to the disjunction. This way of looking at *Olive* gives indeed the necessary weight to the disjunction to produce a very big double overextension. Note that a double overextension with respect to the disjunction does not necessarily violate the classicality conditions; on the contrary, for a classical probability, the disjunction should be double overextended. For *Olive* the overextension is so big that another one of the classicality conditions, namely the one linked to the Kolmogorovian factor, is violated. For a classical probability model we have $\mu(A)+\mu(B)-\mu(A \text{ or } B) = \mu(A \text{ and } B)$, which means $0 \leq \mu(A)+\mu(B)-\mu(A \text{ or } B)$. However, for *Olive* we have $\mu(A) + \mu(B) - \mu(A \text{ or } B) = 0.5 + 0.1 - 0.8 = -0.2 < 0$, which shows that the double overextension for the disjunction in the case of *Olive* is of a non-classical nature. How is it that *Olive* can give such weight to both conjunction and disjunction, even though conjunction and disjunction are considered in classical probability to be distinctly different? It is because the meaning of *Olive* is dominant in sector 1, where quantum conceptual structures exist, and both connectives, 'and' and 'or,' resemble each other well in this realm of conceptuality.

The second set of empirical evidence became manifest when we calculated the deviations from (16)–(20) across all exemplars in our experiment in Sozzo (2015) and Aerts et al. (2015a), where we noticed that these deviations had approximately constant numerical values. Indeed, let us introduce the following quantities.

$$I_{ABA'B'} = 1 - \mu(A \text{ and } B) - \mu(A \text{ and } B') - \mu(A' \text{ and } B) - \mu(A' \text{ and } B') \quad (33)$$

$$I_A = \mu(A) - \mu(A \text{ and } B) - \mu(A \text{ and } B') \quad (34)$$

$$I_B = \mu(B) - \mu(A \text{ and } B) - \mu(A' \text{ and } B) \quad (35)$$

$$I_{A'} = \mu(A') - \mu(A' \text{ and } B') - \mu(A' \text{ and } B) \quad (36)$$

$$I_{B'} = \mu(B') - \mu(A' \text{ and } B') - \mu(A \text{ and } B') \quad (37)$$

We were very excited ourselves to find that for every $X = A, A'$, $Y = B, B'$, I_X, I_Y, and $I_{ABA'B'}$ are constant functions across all exemplars, because this constitutes very strong experimental evidence for the nonclassical nature of what happens during concept combinations. More concretely, the last four equations give rise to values between 0, which would be the classical value, and -0.5, though substantially closer to -0.5 than to 0; and the fifth equation gives rise to a value between 0, which again would be the classical value, and -1, though closer to -1 than to 0. This is very strong evidence for the presence of nonclassicality, indeed, if the classicality conditions are violated in such a strong and systematic way: the underlying structure cannot in any way be classical. To test the regularity of this violation we firstly performed a 'linear regression analysis' of the data to check whether these quantities can be represented by a line of the form $y = mx + q$, with $m = 0$. This was the case. For I_A, we obtained $m = 3.0 \cdot 10^{-3}$ with $R^2 = 0.94$; for I_B, we obtained $m = 2.9 \cdot 10^{-3}$ with $R^2 = 0.93$; for $I_{A'}$, we obtained $m = 2.6 \cdot 10^{-3}$ with $R^2 = 0.96$; for $I_{B'}$, we obtained $m = 3.1 \cdot 10^{-3}$ with $R^2 = 0.98$; for $I_{ABA'B'}$, we obtained $m = 4 \cdot 10^{-3}$ with $R^2 = 0.92$. Secondly, we computed the 95 % confidence interval for these parameters and obtained interval $(-0.51, -0.33)$ for I_A, interval $(-0.42, -0.28)$ for $I_{A'}$, interval $(-0.52, -0.34)$ for I_B, interval $(-0.40, -0.26)$ for $I_{B'}$, and interval $(-0.97, -0.64)$ for $I_{ABA'B'}$. This means that the measured parameters systematically fall within a narrow band centered at very similar values. Next to the very strong experimental evidence for the nonclassical nature of the underlying structure, the finding of this very stable pattern of violation also constitutes strong evidence for the validity of our Fock space model, and for the dominance of emergent reasoning with respect to logical reasoning when concepts are combined. Indeed, suppose for a moment that we substitute, in place of the experimental values in our equations to test classicality, the values that would be obtained theoretically in case we apply the first sector of the Fock space equation of our Fock space model. Since interference in this equation can be negative as well as positive, and there is a priori no reason to suppose that there would be more of the one than the other, we can neglect the interference parts of the equation, since it is reasonable to suppose that they will cancel out when summing all the terms of the equations of our classicality conditions. This means that we get, for every $X = A, A', Y = B, B'$, $\mu(X \text{ and } Y) = \frac{1}{2}(\mu(X) + \mu(Y))$ (see (29)–(32)). A simple calculation shows that, for every $X = A, A', Y = B, B'$, $I_X = I_Y = -0.5$ and $I_{ABA'B'} = -1$, in this case. These are exactly the values to which our experimental violations are close, which means that our Fock space model captures the underlying structures in a systematic and deep way. The experimental values are in between these values and 0, which is the classical value, which means that logical reasoning is also present, but the emergent reasoning is dominant.

We think that these two results confirm, on one side, the general validity of our quantum-theoretic perspective in cognition and, on the other side, they constitute a very strong experimental support to the explanatory hypothesis presented in this section.

The next two sections complete our overview of the identification of quantum structures in concept combination, which also sheds new light on the mysteries that surround quantum entanglement and indistinguishability at a microscopic level.

5 Identification of Entanglement

The presence of entanglement is typically revealed in quantum physics by a violation of Bell-type inequalities (Bell 1964; Clauser et al. 1969), indicating that the corresponding coincidence measurements exhibit correlations that cannot be modeled in a classical Kolmogorovian probability framework (Aerts 1986; Pitowsky 1989).

We recently measured in a cognitive test statistical correlations in the conceptual combination *The Animal Acts*. We experimentally found that this combination violates Bell's inequalities (Aerts et al. 2013a; Aerts and Sozzo 2011) and elaborated a model that faithfully represents the collected data in complex Hilbert space (Aerts and Sozzo 2014a). *The Animal Acts* unexpectedly revealed the presence of a 'conceptual entanglement' which is only partly due to the component concepts, or 'state entanglement,' because it is also caused by 'entangled measurements' and 'entangled dynamical evolutions between measurements' (Aerts and Sozzo 2014a). Our analysis shed new light on the mathematical and conceptual foundations of quantum entanglement, revealing that situations are possible where only states are entangled and measurements are products ('customary state entanglement'), but also situations where entanglement appears on the level of the measurements, in the form of the presence of both entangled measurements and entangled evolutions ('nonlocal box situation', 'nonlocal non-marginal box situation'), due to the violation of the marginal distribution law, as in *The Animal Acts*. More specifically, *The Animal Acts* is a paradigmatic example of a 'nonlocal nonmarginal box situation', that is, an experimental situation where (1) joint probabilities do not factorize, (2) Bell's inequalities are violated, and (3) the marginal distribution law does not hold. Whenever these conditions are simultaneously satisfied, a form of entanglement appears which is stronger than the 'customarily identified quantum entanglement in the states of microscopic entities.' In these cases, it is not possible to work out a quantum-mechanical representation in a fixed $\mathbb{C}^2 \otimes \mathbb{C}^2$ space which satisfies empirical data and where only the initial state is entangled while the measurements are products. It follows that entanglement is a more complex property than usually thought. Briefly, if a single measurement is at play, one can distribute the entanglement between state and measurement but, if more measurements are considered, the marginal distribution law imposes limits on the ways to model the presence of the entanglement.

Let us now come to our coincidence measurements e_{AB}, $e_{AB'}$, $e_{A'B}$, and $e_{A'B'}$ for the conceptual combination *The Animal Acts*. In all these measurements, subjects were asked to pick the combination that they judged to be 'a good example of' the concept *The Animal Acts*. In measurement e_{AB}, participants choose among four possibilities: (1) *The Horse Growls*; (2) *The Bear Whinnies*—and if one of these is chosen, the outcome is $+1$; (3) *The Horse Whinnies*; (4) *The Bear Growls*—and if one of these is chosen, the outcome is -1. In measurement $e_{AB'}$, they choose among: (1) *The Horse Snorts*; (2) *The Bear Meows*—and in case one of these is chosen, the outcome is $+1$; (3) *The Horse Meows*; (4) *The Bear Snorts*—and in case one of these is chosen, the outcome is -1. In measurement $e_{A'B}$, they choose among: (1) *The Tiger Growls*; (2) *The Cat Whinnies*—and in case one of these is chosen, the outcome is $+1$; (3) *The Tiger Whinnies*; (4) *The Cat Growls*—and in case one of these is chosen, the outcome is -1. Finally, in measurement $e_{A'B'}$, participants choose among: (1) *The Tiger Snorts*; (2) *The Cat Meows*—and in case one of these is chosen, the outcome is $+1$; (3) *The Tiger Meows*; (4) *The Cat Snorts*—and in case one of these is chosen, the outcome is -1. We evaluate now the expectation values $E(A,B)$, $E(A,B')$, $E(A',B)$, and $E(A',B')$ associated with the measurements e_{AB}, $e_{AB'}$, $e_{A'B}$, and $e_{A'B'}$ respectively, and insert the values into the Clauser-Horne-Shimony–Holt (CHSH) version of Bell's inequality (Clauser et al. 1969)

$$-2 \le E(A',B') + E(A',B) + E(A,B') - E(A,B) \le 2 \tag{38}$$

We performed a test on 81 participants who were presented with a questionnaire to be filled out in which they were asked to choose among the above alternatives in e_{AB}, $e_{AB'}$, $e_{A'B}$, and $e_{A'B'}$. Table 1 contains the results of our experiment (Aerts and Sozzo 2011).

If we denote by $P(H,G)$, $P(B,W)$, $P(H,W)$, and $P(B,G)$ the probability that *The Horse Growls*, *The Bear Whinnies*, *The Horse Whinnies*, and *The Bear Growls*, respectively, is chosen in e_{AB}, and so for the other measurements, the expectation values are, in the large number limits,

$$\begin{aligned}
E(A,B) &= p(H,G) + p(B,W) - p(B,G) - p(H,W) = -0.7778 \\
E(A',B) &= p(T,G) + p(C,W) - p(C,G) - p(T,W) = 0.6543 \\
E(A,B') &= p(H,S) + p(B,M) - p(B,S) - p(H,M) = 0.3580 \\
E(A',B') &= p(T,S) + p(C,M) - p(C,S) - p(T,M) = 0.6296
\end{aligned}$$

Table 1 The data collected in coincidence measurements on entanglement in concepts (Aerts and Sozzo 2011)

Horse growls	*Horse whinnies*	*Bear growls*	*Bear whinnies*
$p(H, G) = 0.049$	$p(H, W) = 0.630$	$p(B, G) = 0.259$	$p(B, W) = 0.062$
Horse snorts	*Horse meows*	*Bear snorts*	*Bear meows*
$p(H, S) = 0.593$	$p(H, M) = 0.025$	$p(B, S) = 0.296$	$p(B, M) = 0.086$
Tiger growls	*Tiger whinnies*	*Cat growls*	*Cat whinnies*
$p(T, G) = 0.778$	$p(T, W) = 0.086$	$p(C, G) = 0.086$	$p(C, W) = 0.049$
Tiger snorts	*Tiger meows*	*Cat snorts*	*Cat meows*
$p(T, S) = 0.148$	$p(T, M) = 0.086$	$p(C, S) = 0.099$	$p(C, M) = 0.667$

Hence, (38) gives

$$E(A', B') + E(A', B) + E(A, B') - E(A, B) = 2.4197 \tag{39}$$

which is significantly greater than 2. This implies that (1) it violates Bell's inequalities, and (2) the violation is close to the maximal possible violation in quantum theory, that is, $2 \cdot \sqrt{2} \approx 2.8284$.

Let us now construct a quantum representation in complex Hilbert space for the collected data by starting from an operational description of the conceptual entity *The Animal Acts*. The entity *The Animal Acts* is abstractly described by an initial state p. Measurement e_{AB} has four outcomes λ_{HG}, λ_{HW}, λ_{BG}, and λ_{BW}, and four final states p_{HG}, p_{HW}, p_{BG}, and p_{BW}. Measurement AB' has four outcomes λ_{HS}, λ_{HM}, λ_{BS}, and λ_{BM}, and four final states p_{HS}, p_{HM}, p_{BS}, and p_{BM}. Measurement $A'B$ has four outcomes λ_{TG}, λ_{CG}, λ_{TW}, and λ_{CW}, and four final states p_{TG}, p_{TW}, p_{CG}, and p_{CW}. Measurement $A'B'$ has four outcomes λ_{TS}, λ_{CS}, λ_{TM}, and λ_{CM}, and four final states p_{TS}, p_{TM}, p_{CS}, and p_{CM}. Then, we consider the Hilbert space $\mathbb{C}^4$ as the state space of *The Animal Acts* and represent the state p by the unit vector $|p\rangle \in \mathbb{C}^4$. We assume that $\{|p_{HG}\rangle, |p_{HW}\rangle, |p_{BG}\rangle, |p_{BW}\rangle\}$, $\{|p_{HS}\rangle, |p_{HM}\rangle, |p_{BS}\rangle, |p_{BM}\rangle\}$, $\{|p_{TG}\rangle, |p_{TW}\rangle, |p_{CG}\rangle, |p_{CW}\rangle\}$, $\{|p_{TS}\rangle, |p_{TM}\rangle, |p_{CS}\rangle, |p_{CM}\rangle\}$ are orthonormal (ON) bases of $\mathbb{C}^4$. Therefore, $|\langle p_{HG}|\psi\rangle|^2 = p(H, G)$, $|\langle p_{HW}|\psi\rangle|^2 = p(H, W)$, $|\langle p_{BG}|\psi\rangle|^2 = p(B, G)$, $|\langle p_{BW}|\psi\rangle|^2 = p(B, W)$, in the measurement e_{AB}. We proceed analogously for the other probabilities. Hence, the self-adjoint operators

$$\mathcal{E}_{AB} = \sum_{i=H,B} \sum_{j=G,W} \lambda_{ij} |p_{ij}\rangle\langle p_{ij}| \tag{40}$$

$$\mathcal{E}_{AB'} = \sum_{i=H,B} \sum_{j=S,M} \lambda_{ij} |p_{ij}\rangle\langle p_{ij}| \tag{41}$$

$$\mathcal{E}_{A'B} = \sum_{i=T,C} \sum_{j=G,W} \lambda_{ij} |p_{ij}\rangle\langle p_{ij}| \tag{42}$$

$$\mathcal{E}_{A'B'} = \sum_{i=T,C} \sum_{j=S,M} \lambda_{ij} |p_{ij}\rangle\langle p_{ij}| \tag{43}$$

represent the measurements e_{AB}, $e_{AB'}$, $e_{A'B}$, and $e_{A'B'}$ in $\mathbb{C}^4$, respectively.

Let now the state p of *The Animal Acts* be the entangled state represented by the unit vector $|p\rangle = |0.23e^{i13.93^\circ}, 0.62e^{i16.72^\circ}, 0.75e^{i9.69^\circ}, 0e^{i194.15^\circ}\rangle$ in the canonical basis of $\mathbb{C}^4$. This choice is not arbitrary, but deliberately 'as close as possible to a situation of only product measurements,' as we explained in Aerts and Sozzo (2014a). Moreover, we choose the outcomes $\lambda_{HG} = \lambda_{BW} = +1$, $\lambda_{HW} = \lambda_{BG} = -1$, and so on, as in our concrete experiment. We proved that

$$\mathcal{E}_{AB}=\begin{pmatrix} 0.952 & -0.207-0.030i & 0.224+0.007i & 0.003-0.006i \\ -0.207+0.030i & -0.930 & 0.028-0.001i & -0.163+0.251i \\ 0.224-0.007i & 0.028+0.001i & -0.916 & -0.193+0.266i \\ 0.003+0.006i & -0.163-0.251i & -0.193-0.266i & 0.895 \end{pmatrix} \tag{44}$$

$$\mathcal{E}_{AB'}=\begin{pmatrix} -0.001 & 0.587+0.397i & 0.555+0.434i & 0.035+0.0259i \\ 0.587-0.397i & -0.489 & 0.497+0.0341i & -0.106-0.005i \\ 0.555-0.434i & 0.497-0.0341i & -0.503 & 0.045-0.001i \\ 0.035-0.0259i & -0.106+0.005i & 0.045+0.001i & 0.992 \end{pmatrix} \tag{45}$$

$$\mathcal{E}_{A'B}=\begin{pmatrix} -0.587 & 0.568+0.353i & 0.274+0.365i & 0.002+0.004i \\ 0.568-0.353i & 0.090 & 0.681+0.263i & -0,110-0.007i \\ 0.274-0.365i & 0.681-0.263i & -0.484 & 0.150-0.050i \\ 0,002-0.004i & -0,110+0.007i & 0.150+0.050i & 0.981 \end{pmatrix} \tag{46}$$

$$\mathcal{E}_{A'B'}=\begin{pmatrix} 0.854 & 0.385+0.243i & -0.035-0.164i & -0.115-0.146i \\ 0.385-0.243i & -0.700 & 0.483+0.132i & -0.086+0.212i \\ -0.035+0.164i & 0.483-0.132i & 0.542 & 0.093+0.647i \\ -0.115+0.146i & -0.086-0.212i & 0.093-0.647i & -0.697 \end{pmatrix} \tag{47}$$

in this case (Aerts and Sozzo 2014a).

This completes the quantum-theoretic modeling in $\mathbb{C}^4$ for our cognitive test. One can then resort to the definitions of entangled states and entangled measurements and to the canonical isomorphisms, $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ and $L(\mathbb{C}^4) \cong L(\mathbb{C}^2) \otimes L(\mathbb{C}^2)$ ($L(\mathcal{H})$ denotes the vector space of linear operators on the Hilbert space $\mathcal{H}$), and one can prove that all measurements e_{AB}, $e_{AB'}$, $e_{A'B}$, and $e_{A'B'}$ are entangled with this choice of the entangled state p (Aerts and Sozzo 2014a). Moreover, the marginal distribution law is violated by all measurements, for example, $p(H, G) + p(H, W) \neq p(H, S) + p(H, M)$. Since we are below Tsirelson's bound (Tsirelson 1980), this modeling is an example of a 'nonlocal non-marginal box modeling 1,' following the classification we proposed in Aerts and Sozzo (2014b).

To conclude the section we remind readers that we have used the term 'entanglement' by explicitly referring to the structure within the theory of quantum physics that a modeling of experimental data requires: (1) these data are represented, following carefully the rules of standard quantum theory, in a complex Hilbert space, and hence states, measurements, and evolutions are presented respectively by vectors (or density operators), self-adjoint operators, and unitary operators in this Hilbert space; (2) a situation of coincidence joint measurement on a compound entity is considered, and the subentities are identified following the tensor product rule of 'compound entity description in quantum theory'; (3) within this tensor product description of the compound entity entanglement is identified as 'not being a product,' whether it is for states (nonproduct vectors), measurements (nonproduct self-adjoint operators), or evolutions (nonproduct unitary transformations).

6 The Quantum Nature of Conceptual Identity

One of the most mysterious and less understood aspects of quantum entities is the way they behave with respect to 'identity,' and more specifically their statistical behavior due to indistinguishability. Indeed, the statistical behavior of quantum entities is very different from the statistical behavior of classical objects, which are instead, in principle, not identical, hence distinguishable, whenever there is more than one. The latter is governed by the Maxwell–Boltzmann (MB) distribution, while the former is described by the Bose–Einstein (BE) distribution for quantum particles with integer spin, and by the Fermi–Dirac distribution for quantum particles with semi-integer spin (we omit considering fractional statistical particles here, for the sake of brevity) (Dieks and Versteegh 2008; French and Krause 2006).

What about concepts? Consider, for example, the linguistic expression "eleven animals." This expression, when both "eleven" and "animals" are looked upon with respect to their conceptual structure, represents the combination of concepts *Eleven* and *Animals* into *Eleven Animals*, which is again a concept. Each of the *Eleven Animals* is then completely identical on this conceptual level, and hence indistinguishable. The same linguistic expression can however also elicit the thought about 11 objects, present in space and time, each of them being an instantiation of *Animal*, and thus distinguishable from each other. We recently inquired into experiments on such combinations of concepts, surprisingly finding that BE statistics appears at an empirical level for specific types of concepts, hence finding strong evidence for the hypothesis that indeed there is a profound connection between the behavior of concepts with respect to identity and indistinguishability and the behavior of quantum entities with respect to these notions (Aerts et al. 2015c). What is interesting in this respect is that we can intuitively understand the behavior of concepts with respect to identity and indistinguishability, which means that it might well be that an understanding of the behavior of quantum entities with respect to identity and indistinguishability should be searched for by making use of this analogy. In this sense, that identical concepts can be modeled exactly as identical quantum entities, that is, by using quantum theory, is not only a strong achievement for quantum cognition, but it might also incorporate a new way to reflect on this mysterious behavior of identical quantum entities.

Let us discuss these aspects both at a theoretic and an empirical level.

Let us firstly consider the SCoP structure in Sect. 2 and two states of *Animal*, namely *Cat* and *Dog*, hence the situation where *Eleven Animals* can be either *Cats* or *Dogs*. Then, the conceptual meaning of *Eleven Animals*, which can be *Cats* or *Dogs*, gives rise in a unique way to 12 possible states. Let us denote them by $p_{11,0}$, $p_{10,1}$, ..., $p_{1,10}$, and $p_{0,11}$, which stand respectively for *Eleven Cats* (and no dogs), *Ten Cats And One Dog*, ..., *One Cat And Ten Dogs* and *Eleven Dogs* (and no cats). We investigated the 'probabilities of change of the ground state $\hat{p}$ of the combined concept *Eleven Animals* into one of the 12 states $p_{11,0}$, $p_{10,1}$, ..., $p_{1,10}$, and $p_{0,11}$' in a cognitive experiment on human subjects. The subjects were presented with the 12 states and asked to choose their preferred one. The relative frequency arising from their answers was interpreted as the probability of change of the ground state $\hat{p}$ to the chosen state, that is, one of the set $\{p_{11,0}, p_{10,1}, \ldots, p_{1,10}, p_{0,11}\}$. The context e involved in this experiment is mainly determined by the 'combination procedure of the concepts *Eleven* and *Animals*' and the 'meaning contained in the new combination' for participants in the experiment (Aerts and Gabora 2005a). Hence, our psychological experiment tested whether participants follow the 'conceptual meaning' of *Eleven Animals* treating *Dogs* (*Cats*) as identical, or participants follow the 'instantiations into objects meaning' of *Eleven Animals* treating *Dogs* (*Cats*) as distinguishable.

We mathematically represent the conceptual entity *Eleven Animals* by the SCoP model $(\Sigma, \mathcal{M}, \mu)$, where $\Sigma = \{\hat{p}, p_{11,0}, p_{10,1}, \ldots, p_{1,10}, p_{0,11}\}$, $\mathcal{M} = \{e\}$, and our transition probabilities are $\{\mu(q, e, \hat{p}) \mid q \in \{p_{11,0}, p_{10,1}, \ldots, p_{1,10}, p_{0,11}\}\}$. We recognize in the structure of $\mu(q, e, \hat{p})$ that the situation is analogous to the one in which one has $N = 11$ particles that can be distributed in $M = 2$ possible states. It is thus possible, by looking at the relative frequencies obtained in the experiment, to find out whether a classical MB statistics or a quantum-type, that is, BE statistics, applies to this situation. In case that MB would apply, it would mean that things happen as if there are underlying the 12 states hidden possibilities, namely $T(n, C; 11 - n, D) = 11!/n!(11 - n)!$ in number, for the specific state of n *Cats* and $11 - n$ *Dogs*, $n = 0, \ldots, 11$. Of course, this "is" true in case the cats and dogs are real cats and dogs, hence are 'objects existing in space and time,' which is why for objects in the classical world MB statistics do indeed apply. Let us calculate the probabilities involved then. For the sake of simplicity, we assume that two probability values P_{Cat} and P_{Dog} exist such that $P_{Cat} + P_{Dog} = 1$, and that the events of making actual such an underlying state for *Cat* and *Dog* are independent. Hence the probability for n exemplars of *Cat* and $11 - n$ exemplars of *Dog* is then

$$\mu_{\mathrm{MB}}^{P_{Cat},P_{Dog}}(p_{n,11-n}, e, \hat{p}) = T(n, C; 11 - n, D) P_{Cat}^{n} P_{Dog}^{11-n} = \frac{11!}{n!(11-n)!} P_{Cat}^{n} P_{Dog}^{11-n} \tag{48}$$

Note that, under the assumption of MB statistics, $\mu_{\mathrm{MB}}^{P_{Cat},P_{Dog}}(p, e, \hat{p})$ becomes the binomial probability distribution. For example, if $P_{Dog} = P_{Cat} = 0.5$, the number of possible arrangements for the state *Eleven Cats And Zero Dogs* and for the state *Zero Cats And Eleven Dogs* is 1, hence the corresponding probability for these configurations is $\mu_{\mathrm{MB}}^{P_{Cat},P_{Dog}}(p_{0,11}, e, \hat{p}) = \mu_{\mathrm{MB}}^{P_{Cat},P_{Dog}}(p_{11,0}, e, \hat{p}) = 0.0005$. Analogously, the number of possible arrangements for the state *Ten Cats And One Dog* and for the state *One Cat And Ten Dogs* is 11, hence the corresponding probability for these configurations is $\mu_{\mathrm{MB}}^{P_{Cat},P_{Dog}}(p_{10,1}, e, \hat{p}) = \mu_{\mathrm{MB}}^{P_{Cat},P_{Dog}}(p_{1,10}, e, \hat{p}) = 0.0054$, and so on. When P_{Cat} and P_{Dog} are equal, MB distribution entails a maximum value for such a probability. In this example, this corresponds to the situation of *Six Cats And Five Dogs* and *Five Cats And Six Dogs* with $\mu_{\mathrm{MB}}^{P_{Cat},P_{Dog}}(p_{6,5}, e, \hat{p}) = \mu_{\mathrm{MB}}^{P_{Cat},P_{Dog}}(p_{6,5}, e, \hat{p}) = 0.2256$.

Let us now make the calculation for BE statistics, where we keep making the exercise of only reasoning on the level of concepts, and not on the level of instantiations. This means that the 12 different states do not admit underlying hidden states, because the existence of such states would mean that we reason on more concrete forms in the direction of instantiations. As above, we suppose that *Cat* and *Dog* have an independent elicitation probability P_{Cat} and P_{Dog} such that $P_{Cat} + P_{Dog} = 1$. Hence, the probability that there are n exemplars of *Cat* and $(11 - n)$ exemplars of *Dog* is

$$\mu_{\mathrm{BE}}^{P_{Cat},P_{Dog}}(p_{n,11-n}, e, \hat{p}) = \frac{(nP_{cat} + (11 - n)P_{dog})}{(\frac{12\times 11}{2})} \tag{49}$$

Note that as $P_{Cat} = 1 - P_{Dog}$, then $\mu_{\mathrm{BE}}^{P_{Cat},P_{Dog}}(p_{n,11-n}, e, \hat{p})$ is a linear function. Moreover, when $P_{Cat} = P_{Dog} = 0.5$, we have $\mu_{\mathrm{BE}}(p_{n,11-n}, e, \hat{p}) = 1/12$ for all values of n, thus recovering BE distribution (Aerts et al. 2015c).

Starting from the above theoretic analysis, if one performs experiments on a collection of concepts like *Eleven Animals* to estimate the probability of elicitation for each state, then one can establish whether a distribution of MB type $\mu_{\mathrm{MB}}^{P_{Cat},P_{Dog}}(p_{n,11-n}, e, \hat{p})$, or of BE type $\mu_{\mathrm{BE}}^{P_{Cat},P_{Dog}}(p_{n,11-n}, e, \hat{p})$, or a different one, holds. However, in case there are strong deviations from MB statistics, while a quasi-linear distribution is obtained, then this would indicate that, in context e, where only *Cat* and *Dog* are allowed to be states of the concept *Animal*, the statistical distribution of the collection of concepts *Eleven Animals* is of a BE type and that concepts present a quantum-type indistinguishability.

We performed a cognitive experiment with 88 participants. We considered a list of concepts A^i of different (physical and nonphysical) nature, $i = 1, \ldots, 14$, and two possible exemplars (states) p_1^i and p_2^i for each concept. Next we requested participants to choose one exemplar of a combination N^iA^i of concepts, where N^i is a natural number. The exemplars of these combinations of concepts A^i are the states p_{k,N^i-k}^i describing the conceptual combination 'k exemplars in state p_1^i and $(N^i - k)$ exemplars in state p_2^i', where k is an integer such that $k = 0, \ldots, N^i$. For example, the first collection of concepts we considered is N^1A^1 corresponding to the compound conceptual entity *Eleven Animals*, with p_1^i and p_2^i describing the exemplars *Cat* and *Dog* of the individual concept *Animal*, respectively, and $N^1 = 11$. The exemplars (states) we considered are thus $p_{11,0}^1$, $p_{10,1}^1$, …, $p_{1,10}^1$, and $p_{0,11}^1$, describing the combination *Eleven Cats And Zero Dogs*, *Ten Cats And One Dog*, …, *One Cat And Ten Dogs*, and *Zero Cats And Eleven Dogs*. The other collections of concepts we considered in our cognitive experiment are reported in Table 2.

Table 2 List of concepts and their respective states for the psychological concept on identity and indistinguishability

i	N^i	A^i	p_1^i	p_2^i
1	11	*Animals*	*Cat*	*Dog*
2	9	*Humans*	*Man*	*Woman*
3	8	*Expressions of emotion*	*Laugh*	*Cry*
4	7	*Expressions of affection*	*Kiss*	*Hug*
5	11	*Moods*	*Happy*	*Sad*
6	8	*Parts of face*	*Nose*	*Chin*
7	9	*Movements*	*Step*	*Run*
8	11	*Animals*	*Whale*	*Condor*
9	9	*Humans*	*Child*	*Elder*
10	8	*Expressions of emotion*	*Sigh*	*Moan*
11	7	*Expressions of affection*	*Caress*	*Present*
12	11	*Moods*	*Thoughtful*	*Bored*
13	8	*Parts of face*	*Eye*	*Cheek*
14	9	*Movements*	*Jump*	*Crawl*

We computed the parameters $P^{\mathrm{MB}}_{p_1^i}$ and $P^{\mathrm{BE}}_{p_1^i}$ that minimize the R-squared value of the fit using the distributions $\mu_{\mathrm{MB}}^{P_{p_1^i},P_{p_2^i}}$ and $\mu_{\mathrm{BE}}^{P_{p_1^i},P_{p_2^i}}$ for each $i = 1, \ldots, 14$. Hence, we fitted the distributions obtained in the psychological experiments using MB and BE statistics (note that only one parameter is needed as $P^{\mathrm{MB}}_{p_2^i} = 1 - P^{\mathrm{MB}}_{p_1^i}$ and $P^{\mathrm{BE}}_{p_2^i} = 1 - P^{\mathrm{BE}}_{p_1^i}$). Next, we used the 'Bayesian information criterion (BIC)' (Kass and Raftery 1995) to estimate which model provides the best fit and to contrast this criterion with the R-squared value. Table 3 summarizes the statistical analysis. The first column of the table identifies the concept in question (see Table 2), the second and third columns show $P^{\mathrm{MB}}_{p_1^i}$ and the R^2 value of the MB statistical fit, the fourth and fifth columns show $P^{\mathrm{BE}}_{p_1^i}$ and the R^2 value of the BE statistical fit. The sixth column shows the Δ_{BIC} criterion to discern between the $\mu_{\mathrm{MB}}^{P_{p_1^i}P_{p_2^i}}$ and $\mu_{\mathrm{BE}}^{P_{p_1^i}P_{p_2^i}}$, and the seventh column identifies the distribution which best fits the data for concept A^i, $i = 1, \ldots, 14$.

Note that, according to the BIC, negative Δ_{BIC} values imply that the category is best fitted by an MB distribution, whereas positive Δ_{BIC} values on row i imply the concept A^i is best fitted with a BE distribution. Moreover, when $|\Delta_{\mathrm{BIC}}| < 2$ there is no clear difference between the models, when $2 < |\Delta_{\mathrm{BIC}}| < 6$ we can establish a positive but not strong difference towards the model with the smallest value, whereas when $6 < |\Delta_{\mathrm{BIC}}|$ we are in the presence of strong evidence that one of the models provides a better fit than

Table 3 Results of statistical fit for the psychological experiment

i	$P^{\mathrm{MB}}_{p_1^i}$	R^2_{MB}	$P^{\mathrm{BE}}_{p_1^i}$	R^2_{BE}	Δ_{BIC}	Best model
1	0.55	−0.05	0.16	0.78	19.31	BE strong
2	0.57	**0.78**	0.42	0.44	−9.54	MB strong
3	0.82	0.29	**0.96**	0.79	10.81	BE strong
4	0.71	0.81	0.53	0.77	−1.69	MB weak
5	0.25	**0.79**	0.39	0.93	14.27	BE strong
6	0.62	0.59	0.61	0.57	−0.37	MB weak
7	0.72	0.41	0.64	**0.83**	12.66	BE strong
8	0.63	0.58	0.47	0.73	5.53	BE positive
9	0.45	**0.87**	0.26	0.67	−9.69	MB strong
10	0.59	0.50	0.63	0.77	7.17	BE positive
11	0.86	0.46	1.00	**0.87**	11.4	BE strong
12	0.21	0.77	0.00	0.87	6.68	BE positive
13	0.62	0.54	0.71	0.67	2.97	BE weak
14	0.81	0.20	0.91	**0.90**	20.68	BE strong

Each column refers to the 14 collections of concepts introduced in Table 2

the other model (Kass and Raftery 1995). We see that categories 2 and 9 show a strong Δ_{BIC} value towards the MB type of statistics, and that categories $1, 3, 5, 7, 11, 12$ and 14 show a strong Δ_{BIC} value towards the BE type of statistics. Complementary to the BIC, the R^2 fit indicator helps us to see whether or not the indications of Δ_{BIC} can be confirmed with a good fit of the data. Interestingly, the concepts we have identified with strong indication towards one type of statistics have R^2 values larger than 0.78 (such R^2 values are marked in bold text), which indicates a fairly good approximation for the data. Moreover, note that in all the cases with a strong tendency towards one type of statistics, the R^2 of the other type is poor. This confirms the fact that we can discern between the two types of statistics depending on the concept in question.

The interpretation of our results is thus clear. Conceptual combinations exist, like *Nine Humans*, whose distribution follows MB statistics. However, conceptual combinations, like *Eleven Animals*, *Eight Expressions of Emotion*, or *Eleven Moods*, whose distribution follows BE statistics, also exist. The conclusion is that the nature of identity in these concept combinations is of a quantum type and in these combinations the human mind treats the two states we consider as identical and indistinguishable. Also the hypothesis that 'the more easy the human mind imagines spontaneously instantiations, e.g., *Nine Humans*, the more MB, and the less easy such instantiations are activated in imagination, e.g., *Eight Expressions of Emotion*, the more BE statistics appears' is confirmed by our experiment.

We have an intuitive explanation for this empirical difference. Whenever the human mind 'imagines' two different combinations of *Eleven Animals*, say two cats and nine dogs, and five cats and six dogs, the human mind does not really take into account that this situation can be about real cats and dogs, in which case there are many more ways to put five cats and six dogs into a cage, than to put two cats and nine dogs. This is the reason why BE, not MB, appears in this case. Suppose instead that the human mind considers two different combinations of *Nine Humans*, say two elders and seven children, and four elders and five children. Then the human mind is likely to be influenced by real known families with nine sons and, in the real world, there are much more situations of families with four elders and five children, than two elders and seven children.

This pattern was confirmed by a second experiment we performed on the World Wide Web about the nature of conceptual indistinguishability (Aerts et al. 2015c).

7 Concluding Remarks and Perspectives

In the previous sections we have provided an overview of our quantum-theoretic perspective on concepts and their combinations. We have expounded the reasons that led us to develop this perspective, namely our former research on operational and axiomatic approaches to quantum physics, the origins of quantum probability, and various experimental results in cognition pointing to a deviation of human reasoning from the structures of classical (fuzzy set) and classical probability theory. We have proved that these deviations can be described in terms of genuine quantum effects, such as contextuality, emergence, interference, and superposition. We have identified further quantum aspects in the mechanisms of conceptual combination, namely entanglement and quantum-type indistinguishability. And we have proposed an explanation that allows the unification of these different empirical results under a common underlying principle on the structure of human thought.

Our quantum-theoretic perspective fits the global research domain that applies the mathematical formalisms of quantum theory in cognitive science and which has been called 'quantum cognition' (we quote here some books on this flourishing domain Khrennikov 2010; Busemeyer and Bruza 2012; Haven and Khrennikov 2013). Further, we believe that our findings in cognition may also have, as a feedback, a deep impact on the foundations of microscopic quantum physics. Indeed, let us consider entanglement. We have identified in concepts an entanglement situation where Bell's inequalities are violated within Tsirelson's bound (Tsirelson 1980), the marginal distribution law is violated, and there is 'no signaling', which implies that entangled measurements, in addition to entangled states, are needed to model this experimental situation. And this completely occurs within a Hilbert space quantum framework, which is at variance with general opinion. This theoretic scheme with entangled measurements could explain some 'anomalies,' that is, deviations from the marginal distribution law, that were recently observed in the typical Bell-type nonlocality tests with entangled photons (Adenier and Khrennikov 2006, 2007). So let us consider the quantum nature of conceptual indistinguishability. From our perspective, this is due to the human mind being able to consider specific conceptual entities without the need to imagine also their instantiations as objects existing in space and time. Hence, it could well be that quantum indistinguishability at a microscopic level is provoked by the fact that quantum entities are not localized as objects in space and time, and that nonlocality would mean nonspatiality, a view that has been put forward by one of us in earlier work for different reasons (Aerts 1990). These insights could for example have implications on quantum statistics and the so-called 'spin-statistics theorem' (French and Krause 2006; Dieks and Versteegh 2008).

We conclude this chapter with an epistemological consideration. We think that our quantum-theoretic perspective in concept theory constitutes a step towards the construction of a general theory for the modeling of conceptual entities. In this sense, we distinguish it from what is typically considered as an ad hoc cognitive model. To understand what we mean by this distinction let us consider an example taken from everyday life. As an example of a theory, we could introduce the theory of 'how to make good clothes.' A tailor needs to learn how to make good clothes for different types of people, for example, women, children, and old people. Each item of clothing is a model in itself. Then, one can also consider intermediate situations where one has models of a series of clothes. A specific body will not fit in any item of clothing: you need to adjust the parameters (length, size, etc.) to reach the desired fit. We think that a theory should be able to reproduce different experimental results by suitably adjusting the involved parameters, exactly as in a theory of clothing. This is different from a set of ad hoc models, even if the set can cope with a wide range of experimental data. There is a tendency in psychology to be critical of a theory that can cope with all possible situations it applies to. One then often believes that the theory contains too many parameters, and that it is only by allowing all these parameters to attain different values that all the data can be modeled. In case we have to do with an ad hoc model, that is a model specially made for the circumstance of the situation it models, this suspicion is grounded. Adding parameters to such an ad hoc model, or stretching the already contained parameters to other values, does not give rise to what we call a theory. A theory needs to be well defined—its rules; the permitted procedures; its theoretical, mathematical, and internal logical structure—'independent' of the structure of the models describing specific situations that can be coped with by it. Hence also the theory needs to contain a well defined description of 'how to produce models for specific situations.' Think again of the theory of clothing. If a tailor knows the theory of clothing, obviously he or she can make a garment for every human body, because the theory of clothing, although its structure is defined independently of a specific garment, contains a prescription of how to apply it to any possible specific garment. Other subtle aspects are involved with the differences between ad hoc models and models finding their origin in a theory (Aerts and Rohrlich 1998). We have raised this issue because we think it does lead to misunderstandings if attention is not paid to the difference between an ad hoc model and a model which is derived from a theory. Intuitive thoughts about the nature of a model differ, often depending on whether the model is inspired by a psychology approach—where it will then rather automatically be looked upon as an ad hoc model, in which all the data appear to be suspicious—or whether it is inspired by a physics approach—where it will rather be looked upon as resulting from a theory, and that all data that can be modeled by it have a positive aspect, validating the theory.

What is the status of the Fock space model for concept combinations? Hilbert space, hence also Fock space when appropriate, for the description of quantum entities provides models that definitely come from a theory, namely quantum theory, and hence are not ad hoc models. Is quantum theory also a theory for concepts and their combinations, and hence, if so, can we consider our models, for example, the Fock space model, as models coming from this theory? Or is quantum cognition rather still a discipline where ad hoc models are built, making use, also in a rather ad hoc way, of mathematics arising from quantum theory? An answer to this question cannot yet be given definitely, though some hypothesis can be formulated with plausibility in respect of it. We believe that, notwithstanding their deep analogies, concept entities are less crystallized and symmetric structures than quantum entities. As a matter of fact some data in Hampton (1988a,b), Sozzo (2015), and Aerts et al. (2015a) cannot be modeled in Fock space, and further experimental findings could in the future confirm such impossibility. Notwithstanding this, we believe that emergence is also the actual driver for these data that cannot be modeled in Fock space. However, this type of emergence cannot be represented in a linear Hilbert (Fock) space, and more general structures are needed. The search for more general mathematical structures capturing conceptual emergence will, by the way, constitute an important aspect of our future investigation in concept theory. On the other hand, we do believe that we have arrived at the stage of building models that come from a theory and which are not ad hoc. Indeed, although we believe that this theory will turn out to be a generalization of the actual quantum theory, its basic principles—except linearity most probably—will be present in the generalized quantum theory too. We have recently worked out an analysis where the view of the status of actual quantum cognition, as describing a quantum-like domain of reality, less crystallized than the microworld, but containing deep analogies in its foundations, is put forward (Aerts and Sozzo 2014c).

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Quantum Cognition, Neural Oscillators, and Negative Probabilities

J. Acacio de Barros and Gary Oas

1 Introduction

One common view is that humans are rational decision-makers. What constitutes *rational* is in itself a matter of debate, but perhaps a common idea of rationality is the notion that, when making decisions, humans follow the prescriptions of classical logic. Where logical true or false values are replaced with uncertainty, we have to deal with beliefs, and not with certainty. One can argue that the rules of inference over beliefs, for a rational being, should be replaced by measures consistent with an underlying Boolean algebra of propositions. If this is the case, and under some reasonable assumptions, the rules of probability theory are derived (Cox 1961; Jeffrey 1992). In other words, if one wishes to assign measures of belief in such a way that a decision-maker, when faced with new evidence, acts in a way consistent with the rules of logic, one needs to use classical probability (CP).

In the 1980s, Tversky and Kahneman examined the heuristics of decision-making with cleverly designed experiments where inferences required by

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CP were tested against actual human beliefs. In a series of results, they showed that in many situations humans did not follow CP, and later on they developed a theory to described human decisions, prospect theory, which fitted the experimental data better than the standard expected utility theory in economics (Kahneman and Tversky 1979).[1] Such was the importance of those results that Kahneman was awarded the Nobel Memorial Prize in Economics in 2002 (Tversky passed away in 1996).

In 2007, in a special session during the Association for the Advancement of Artificial Intelligence's Spring Symposium at Stanford University, a group of researchers, among them Andrei Khrennikov, Emmanuel Haven, Jerome Busemeyer, Peter Bruza, and Patrick Suppes, met to discuss applications of the quantum mechanical formalism to the social sciences. The main idea put forth was that, in relation to the social sciences, quantum mechanics could go beyond an analogy of how to deal with complementary variables (in the sense of Bohr): the quantum mechanical formalism itself could better represent situations in which CP was violated (such as Tversky and Kahneman's examples). The area of research spawned from this became known as Quantum Interactions, and the application of the quantum-mathematical formalism to psychology as Quantum Cognition (QC).

In this chapter we put forth the following three theses. First, QC is about contextuality. By this we mean that we can think of decisions as experimental outcomes, and such outcomes depend on the experimental conditions (contexts). Second, that contextuality in QC may come from the inconsistency of (perhaps learned, for cognition) conditions. To support this, we provide two examples: a neural oscillator model that shows contextuality when incompatible events are activated, and a decision-making scenario where information based on subjective beliefs are inconsistent. Our third and final thesis is that such contextual effects may be better modeled by allowing non-observable probabilities to take negative values, and not by quantum probabilities that violate CP theory. We support this by first showing that there are certain neural oscillator set-ups that result in responses that cannot be modeled in a natural way by the Hilbert space formalism of quantum mechanics (QM). Then, in another example, we not only show that the QC approach is inferior to negative probabilities (NPs),[2] as we call our generalization of CP, but also argue that, because of its inherent advantage

[1] Expected utility theory relies heavily on CP (Savage 1972; Anand et al. 2009).

[2] Some readers may object to the use of negative probabilities, since probabilities come from the ratio of two non-negative numbers. We ask them to hold their concerns until Sect. 5, where we discuss NPs in detail. However, at this point we emphasize that in our approach *no experimentally observable* event has NPs.

with respect to Bayesian approaches, NPs may be the mechanism of choice for actual biological systems dealing with contextual information.

In Sect. 2 we introduce the idea of QC, and discuss the importance of interference for most of the discussions of violations of CP. In Sect. 3 we discuss QM and the reason why contextuality is the characteristic that makes its formalism the most relevant to QC. Keeping this in mind, we describe in Sect. 4 a neurophysiologically inspired neural oscillator model that presents the same contextuality observed in experiments used to support QC. We then show that neural oscillators can model certain decisions that are not compatible with the quantum formalism. Inspired by such a model, we then present in Sect. 5 a theory of extended probabilities that describes the cases found in QC and which provides some insight into contextual decision-making. Our model seems to be computationally better than the quantum one and seems to offer better advice than the Bayesian approach. We end the chapter with some remarks and suggestions for future research.

2 Elements of Quantum Cognition

In this section we describe some of the main characteristics of QC. Here we focus on QC models that rely on state interference.[3] We believe that the main features exhibited by these models are sufficient to make our main point. However, we should remark that our arguments and accounts do not immediately generalize to the use of quantum dynamics, but only to the description of the relationship between states and observables.

To understand how QM violates CP, and how this can be applied to cognition, let us look at an example. First, following Kolmogorov (1956), we have the following definition.

Definition 1. Let Ω be a finite set, $\mathcal{F}$ an algebra over Ω, and p a real-valued function, $p : \mathcal{F} \to \mathbb{R}$. Then $(\Omega, \mathcal{F}, p)$ is a probability space, and p a probability measure, iff:

$$\text{K1.}\ \ 0 \le p\left(\{\omega_i\}\right), \quad \forall \omega_i \in \Omega$$

$$\text{K2.}\ \ p\left(\Omega\right) = 1,$$

[3] It is not our intent to give an exhaustive account of the field of QC. Readers interested in it should consult the excellent books available (e.g. Khrennikov 2010; Busemeyer and Bruza 2012; Haven and Khrennikov 2013).

K3. $p\left(\{\omega_i, \omega_j\}\right) = p\left(\{\omega_i\}\right) + p\left(\{\omega_j\}\right), \quad i \neq j.$

The elements ω_i of Ω are called *elementary probability events* or simply *elementary events*.[4]

Definition 1 implies that from elementary events and $\mathcal{F}$ we can create complex events. In a subjective interpretation the function p could be thought as a measure of rational belief (Cox 1961; Jaynes 2003). For example, a consequence of axioms K1–K2 is that, for two sets containing elementary events, A and B, if $A \subset B$, then $p(A) \leq p(B)$ follows. This property of CP is called *monotonicity*, and it is possible to show that if we relax the requirement of $\mathcal{F}$ being an algebra of events and instead allow it to be a quantum lattice, monotonicity is violated (Holik et al. 2014). In other words, QC violates CP.

An important concept is that of a random variable, defined below.

Definition 2. Let $(\Omega, \mathcal{F}, p)$ be a probability space, and let Θ be a finite set, with $\mathcal{T}$ an algebra over this set. A *random variable* $\mathbf{X}$ is a measurable function $\mathbf{X} : \Omega \to \Theta$, that is, for every $T \in \mathcal{T}$ we have $\mathbf{X}^{-1}(T) \in \mathcal{F}$.[5]

Intuitively, we can understand a random variable in the following way. To each value of Ω we assign a value in Θ, such that the function $\mathbf{X}$ determines a partition of the space Ω in different regions (consistent with $\mathcal{F}$, since $\mathbf{X}^{-1}(T) \in \mathcal{F}$). Such a partition attributes to each region of Ω a value in Θ. So, random variables can be seen as a way to represent possible outcomes of measurements that depend functionally on a probability event.

A simple example to illustrate random variables is the following. Let $\Omega = \{hh, ht, th, tt\}$ be the space of outcomes of tossing a coin twice in a row (h representing heads). If the coin is not biased, we have $p(hh) = p(ht) = p(th) = p(tt) = 1/4$. Let us say we now want to represent an experiment where whenever we get two values in a row (either heads or tails) the result is 1 (we could think of a one dollar payoff in a game), and -1 otherwise (one dollar lost). The ± 1-valued random variable $\mathbf{X}$ is the function $\mathbf{X} : \Omega \to \{-1, 1\}$ with outcomes $\mathbf{X}(hh) = \mathbf{X}(tt) = 1$ and $\mathbf{X}(ht) = \mathbf{X}(th) = -1$. From those functions we have the expected value of $\mathbf{X}$, given by

[4] It follows that any probability of an element of $\mathcal{F}$ is a real number in [0, 1].

[5] Usually there are extra constraints for defining a random variable, but we avoid such technicalities by working with discrete Ω and Θ. The above definition is sufficient for our purposes.

$$E(\mathbf{X}) = \sum_{\theta \in \Theta} \theta p(\mathbf{X} = \theta),$$

which in our example is

$$E(\mathbf{X}) = (+1) \cdot p(\mathbf{X} = +1) + (-1) \cdot p(\mathbf{X} = -1)$$
$$= (+1) \cdot \left(\frac{1}{2}\right) + (-1) \cdot \left(\frac{1}{2}\right) = 0.$$

The second moment is[6]

$$E(\mathbf{XY}) = \sum_{\theta, \phi \in \Theta} \theta \phi p(\mathbf{X} = \theta \& \mathbf{Y} = \phi).$$

A useful notation for ± 1-valued random variables is the following. Instead of writing $p(\mathbf{X} = +1)$, we write $p(x)$, and, instead of $p(\mathbf{X} = -1)$, we write $p(\overline{x})$.

A typical example of nonmonotonicity in QM is the two-slit experiment, whose main features can be seen in the Mach–Zehnder interferometer (MZI) of Fig. 1.[7] For this interferometer, imagine two different situations: situation 1,

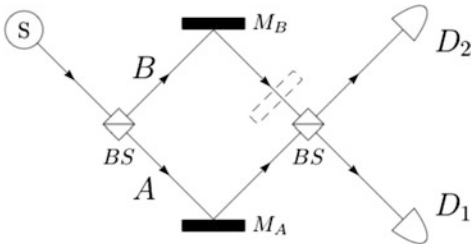

Fig. 1 Mach–Zehnder interferometer. A single photon state is emitted from a source (S) and impinges on the first beam splitter (BS), where it has equal probability of being in arm A or B. Upon reflection at mirrors M_A, M_B, the two paths are recombined at the second beam splitter. The probabilities for detection at detectors D_1 and D_2 are dependent upon the phase relation between the two alternatives at the second beam splitter. In the ideal case, the probability for detection at D_1 is unity. The *dashed box* in path B represents the choice of inserting a barrier, thereby changing the phase relationship and thus the detection probabilities

[6]For ± 1-value random variables with zero expectation, it is easy to show that the moment $E(\mathbf{XY})$ has the same value as the correlation $\rho = E(\mathbf{XY}) / (\sigma_{\mathbf{X}} \sigma_{\mathbf{Y}})$.

[7]A more detailed discussion of the MZI in the context presented here can be found in de Barros and Oas (2014), de Barros et al. (2015).

in which the arms of the interferometer are unobstructed and a detection is made on D_1 or D_2, and situation 2, in which a barrier is placed in arm B (represented by the dashed box in the figure). Following Feynman et al. (2011), a particle able to go through either arm of the interferometer has probability of detection 1 in D_1 (for an appropriate choice of lengths for the interferometer arms). However, if the particle is constrained to go through only one of the arms (because of a barrier in the other interferometer arm), the probability of detection in D_1 is $1/2$. If we consider $\mathbf{D}_1 = 1$ as the value of $\mathbf{D}_1$ when there is a detection and -1 when not (similarly for $\mathbf{D}_2$), and if we have the ± 1-valued random variable $\mathbf{A} = 1$ (or $\mathbf{B} = 1$) as corresponding to going through A (or B), and -1 otherwise, we have

$$p(d_1) = p(d_1|a)\,p(a) + p(d_1|b)\,p(b), \tag{1}$$

where

$$p(x|y) \equiv p(x,y)/p(y) \tag{2}$$

is the *conditional probability* of x given y (for $p(y) \neq 0$). The expression (2) is known as Bayes' formula, and gives the definition of conditional probability in Kolmogorov's axiomatic framework.[8] From

$$p(d_1|a) = p(d_1|b) = p(a) = p(b) = \frac{1}{2} \tag{3}$$

the observed value of $p(d_1) = 1$ of situation 1 is incompatible with Eq. (1) of situation 2. Notice that Eq. (1) requires the existence of a joint distribution $p(x,y)$, and the derivation of (3) depends not only on such a definition, but also on the additivity of probabilities from Kolmogorov's axioms.

The incompatibility between the observed probabilities for situation 1 and 2 comes from the assumption that the random variable $\mathbf{D}_1$ is the same for both situations, as this is a requirement of a joint probability distribution. However, the experimental conditions are different, and this assumption is somewhat silly: we have no reason to believe they should be the same, and indeed the data does not support this view. We call this impossibility to reconcile the probability distributions of a random variable under different experimental

[8] In Kolmogorov's theory of probability, joint probabilities are primitives, whereas conditional probabilities are defined from the joints (Khrennikov 2009). But other interpretations of probability, notably some subjective interpretations, consider conditional probabilities as more fundamental, and joint probabilities are derived from them. For some of these interpretations, probabilities are always conditional, and it may not even make sense to talk about joint probabilities (Galavotti 2005).

conditions *contextuality*, since each experiment provides an alternative context for the observation.[9]

We now return to QC. As mentioned, experiments show that human decision-making may not follow CP. For instance, in his classic work, Savage introduced a rational decision-making concept called the sure-thing principle (STP) (Savage 1972). The idea of the STP is simple:

> A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness of the purchase. So, to clarify the matter for himself, he asks whether he should buy if he knew that the Republican candidate were going to win, and decides that he would do so. Similarly, he considers whether he would buy if he knew that the Democratic candidate were going to win, and again finds that he would do so. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say. It is all too seldom that a decision can be arrived at on the basis of the principle used by this businessman, but, except possibly for the assumption of simple ordering, I know of no other extralogical principle governing decisions that finds such ready acceptance. (Savage 1972, p. 21)

Formally, let us imagine that the ± 1-valued random variable $\mathbf{X}$ corresponds to buy if $\mathbf{X} = 1$ (not buy if $\mathbf{X} = -1$), and let another ± 1-valued variable $\mathbf{A}$ be such that $\mathbf{A} = 1$ is a Democrat win and $\mathbf{A} = -1$ is a Republican win. If $\mathbf{X} = 1$ is preferred over $\mathbf{X} = -1$ when $\mathbf{A} = 1$ and also when $\mathbf{A} = -1$, then $\mathbf{X} = 1$ is always preferred, since $\mathbf{A} = 1$ and $\mathbf{A} = -1$ exhaust all possibilities for $\mathbf{A}$.

If we deal with propositions that are not certain, what a "rational" being should base his or her decisions on for preferences of propositions (say, the proposition "buy a certain piece of property" or "$\mathbf{X} = 1$") is represented in terms of probabilities. For example, given a set of propositions, say $\{P_1, P_2\}$, we can form more complex propositions by compounding them via the usual operators in propositional calculus, for example, "$P_1 \& P_2$," "P_1 or P_2," or "not P_1". If we require that the rules of inference are such that the measures of belief assigned to propositions are consistent with this composition of propositions (e.g., if you assign a high belief for P_1, then "not P_1" should be assigned a low value), then the measures of belief follow the axioms of probability above (Jaynes 2003; Galavotti 2005). Such axioms imply Savage's STP.

[9]Physicists usually refer to contextuality as a particular concept related to hidden-variables in a Kochen–Specker-like situation, and would not call the MZI contextual. Here we take a comprehensive approach to contextuality, which we define mathematically below.

To prove STP from CP, assume that

$$p(\mathbf{X} = 1|\mathbf{A} = 1) > p(\mathbf{X} = -1|\mathbf{A} = 1).$$

This is interpreted as "$\mathbf{X} = 1$ is preferred over $\mathbf{X} = -1$" if $\mathbf{A} = 1$. If we also assume

$$p(\mathbf{X} = 1|\mathbf{A} = -1) > p(\mathbf{X} = -1|\mathbf{A} = -1),$$

then, multiplying each inequality by $p\,(\mathbf{A} = 1)$ and $p\,(\mathbf{A} = -1)$, and using the above notation,

$$p(x|a)p\,(a) + p(x|\overline{a})p\,(\overline{a}) > p(\overline{x}|a)p\,(a) + p(\overline{a}|\overline{a})p\,(\overline{a})\,.$$

From $p(\mathbf{A} = 1\&\mathbf{A} = -1) = p\,(a\&\overline{a}) = 1$ and the definition of conditional probabilities we have

$$p(\mathbf{X} = 1) > p(\mathbf{X} = -1).$$

This, is clearly Savage's STP.

Savage's view of probabilities is *normative*, and not *descriptive*. A descriptive theory of decision under uncertainty tells us how actual human beings make decisions, whereas a normative theory tells us how they ought to make decisions (see Briggs 2015). In the words of Boole, "probability I conceive as to be not so much expectation, as a rational ground for expectation" (Boole 1854). Therefore, we should think of probability theory not as a description of what humans actually think (or do), but instead as what humans *should* do when faced with uncertain information.

According to such views, the STP should hold if agents are making rational decisions. However, as mentioned, actual human decision-makers do not follow the STP (Tversky and Shafir 1992; Shafir and Tversky 1992). For example, in Tversky and Shafir (1992) students were told about a game of chance, to be played in two steps. In the first step, not voluntary, players had a 50 % probability of winning \$200 and 50 % of losing \$100. The second step allowed a choice of whether or not to gamble a second time, with the same odds and payoffs. When told that they had won the first bet, 69 % of subjects accepted the second gamble, and when told they lost, 59 % also accepted. If we think of the random variables $\mathbf{A}$ and $\mathbf{X}$ as

$$\mathbf{A} = +1 \leftrightarrow \text{"Won first bet,"}$$

$$\mathbf{A} = -1 \leftrightarrow \text{"Lost first bet,"}$$

$$\mathbf{X} = +1 \leftrightarrow \text{"Accept second gamble,"}$$
$$\mathbf{X} = -1 \leftrightarrow \text{"Reject second gamble,"}$$

and

$$P(x|a) = 0.69 > P(\overline{x}|a) = 0.31,$$
$$P(x|\overline{a}) = 0.59 > P(\overline{x}|\overline{a}) = 0.41,$$

then "Accept second gamble" is preferred over "Reject second gamble" regardless of $\mathbf{A}$. However, the decision $\mathbf{X}$ was asked later on in the semester, but participants were not told whether they had won in the first step (they did not know $\mathbf{A}$). Under the unknown condition, 64 % of students rejected the second gamble, and

$$P(x) = 1 - 0.64 = 0.36 < P(\overline{x}) = 0.64,$$

a clear violation of the STP.

Violations of CP by human decision-makers are one of the main driving forces behind QC. For instance, the nonmonotonicity of probabilities in the MZI yield results that are very similar to violations of the STP. In the MZI, let us say that the statement "detector D_1 is preferred over D_2" corresponds to a higher probability of detecting a particle in D_1 instead of D_2, and let us represent such a statement in terms of the random variable $\mathbf{X}$, where

$$p(x) > p(\overline{x}) \tag{4}$$

corresponds to the previous statement. We can also represent the which-path information by a ± 1-valued random variable $\mathbf{A}$, where $\mathbf{A} = 1$ corresponds to the particle going through A and $\mathbf{A} = -1$ to the particle going through B. Clearly, in this case, we have[10]

$$p(x|a) = p(x|\overline{a}) = p(\overline{x}|a) = p(\overline{x}|\overline{a}) = \frac{1}{2}. \tag{5}$$

[10] Equal probabilities in (5) are not necessary, as biases in the interferometer could modify the conditional probabilities.

Similarly to violations of the STP, when a or $\overline{a}$, we have no reason to prefer x over $\overline{x}$ or vice versa, but CP implies, from (5), that $p(x) = p(\overline{x})$, in dissonance with (4).

Thus, nonmonotonic violations of CP, such as the STP, can be reproduced by quantum interference, as in the MZI. Thus, it should not come as a surprise that the typical QC model relies on interference. For example, Busemeyer et al. (2006) used quantum interference to model the disjunction effect observed by Tversky and Shafir (1992) (Shafir and Tversky 1992). In their model, they showed that a quantum process with interference yielded a better fit to experimental observations than a classical Markov model. There are several other uses of the quantum formalism to cognitive sciences, such as modeling the conjunction effect (Pothos and Busemeyer 2009; Busemeyer et al. 2011), order effects (Trueblood and Busemeyer 2011; Atmanspacher and Römer 2012; Wang and Busemeyer 2013; Busemeyer et al. 2014) (see also Khrennikov et al. 2014), and the "guppy" effect (Aerts 2009; Aerts et al. 2012; Aerts and Sozzo 2014). We refer the interested reader to the excellent available reviews, such as Khrennikov (2010), Busemeyer and Bruza (2012), Haven and Khrennikov (2013), and Ashtiani and Azgomi (2015).

To summarize, CP fails to describe properly actual human cognition, and we need to generalize it to account for such cases. A generalization of CP was revealed with the creation of a mathematical formalism to deal with order and context effects in physics: the quantum formalism. QC tries to use the mathematics of QM to describe systems that may have similar contextual effects to the physical ones.

3 Contextuality in Classical and Quantum Systems

We saw that contextual effects, such as the ones present in the MZI, may give rise to outcomes of experiments that are not consistent with CP. In this section, we explore the idea of contextuality as the connection to violations of CP in QC.

Let us start with the role of contextuality in QC.[11] QM was laid out about 100 years ago. However, it comes as a surprise to many that there is no consensus as to what this theory actually represents. Of course, researchers agree with the theory's predictions, but there is substantial disagreement as to

[11] For a quick review of quantum mechanics, see reference Haven and Khrennikov (2015) in this issue.

what the theory has to say about the physical systems it models. For example, is the theory about what the system actually is (ontological) or about what we can say about the system (epistemological)?

The sources of such disagreements are in the many consequences of QM that are irreconcilable with the classical views of nature. Following de Barros and Suppes (2009), we stress three main characteristics of QM that are not part of classical mechanics: nondeterminism, contextuality, and nonlocality. To single out contextuality as the relevant aspect to QC, let us analyze each separately.

Determinism, in its simplest form, comes from the idea that a past state of a system determines its future state. This is certainly true in classical mechanics, where given the state of a particle at time t_0 and the forces acting on it, the state of such particle at times $t > t_0$ are determined by

$$\frac{d^2\mathbf{r}(t)}{dt^2} = \frac{\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})}{m},$$

where $\mathbf{r}(t)$ is the position of the particle at time t, m its mass, and $\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$ the force acting on the force. From this, it follows that the state of the system at $t \geq t_0$ is completely determined by the particle's position $\mathbf{r}(t_0)$ and velocity $\dot{\mathbf{r}}(t_0)$. Determinism is also true for classical electromagnetic theory, and seems even to be consistent with thermodynamics, via the kinetic theory of gases. However, already in the late 1800s, with the discovery of radioactive decay, some physicists started to realize that some nuclear processes seemed inconsistent with the idea of determinism (Pais 1986). As knowledge about microscopic systems increased, and QM developed, physicists realized that quantum systems seemed to be different from classical ones, as they did not always allow for predictable outcomes from the state of the system: not only was the state based on less information (as position and velocity could not be simultaneously measured), but it also had an intrinsically probabilistic connection to measurement outcomes (Born's rule). Thus, QM seems to violate determinism.

However, we can argue that quantum nondeterminism is not necessary to QC for two reasons. First, as we have argued extensively elsewhere (Suppes and de Barros 1996), the discrimination between determinism and predictability is difficult, and all we can say is that certain cognitive processes may not be predictable. So, positing a nondeterministic underlying process is unnecessary. Second, cognitive models already make extensive use of stochastic processes without resorting to QM (Busemeyer and Diederich 2010). So, the use of QM in cognitive modeling seems unnecessary.

The second feature of QM we analyze is nonlocality. In 1932, Einstein, Podoslky, and Rosen (EPR) published a seminal paper (Einstein et al. 1935) where they examined the effects of a measurement on an entangled quantum system, for example, a system comprised of two particles, 1 and 2. In this paper, EPR argued that if an interaction happens with particle 1, such an interaction cannot in any way instantaneously affect particle 2, if the particles have a large spatial separation. In Bohm's version for spin one-half particles, the EPR state is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle_1 \otimes |-\rangle_2 - |-\rangle_1 \otimes |+\rangle_2 \right), \tag{6}$$

where $|+\rangle_i$ ($|-\rangle_i$) corresponds to an eigenvector of spin in direction $\hat{\mathbf{z}}$ with eigenvalue $+1$ (-1) for particle i (we use here units where $\hbar/2 = 1$). As we see from (6), if we measure the spin in the direction $\hat{\mathbf{z}}$ for particle 1 and obtain $+1$ (or -1), then we "know" for sure the result of a spin-z measurement for particle 2. Thus, according to EPR, since we cannot have any instantaneous influence of 1 in 2, a measurement in 1 yields information about 2 without disturbing it. EPR then went on and argued that such a result would imply that the description of nature given by QM was incomplete, as clearly we could know something about 2 without directly measuring it. In a surprising result, John Bell (1964, 1966) showed that EPR's view that a measurement in 1 did not disturb 2 was inconsistent with the experimental predictions of QM. Therefore, QM seems to allow for some superluminal influence.[12] This characteristic of QM is known as nonlocality.

Aspect and collaborators provided evidence for quantum nonlocality in the 1980s (Aspect et al. 1981, 1982), when they showed that a set of inequalities (known as Clauser-Horne-Shimony-Holt (CHSH) inequalities, after reference Clauser et al. (1969)) were violated. For nonsignaling systems,[13] the CHSH inequalities are necessary and sufficient conditions for the existence of a joint probability distribution (Fine 1982), which are also equivalent to the existence of a local (realistic) theory, meaning that their violation implies nonlocality.

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[12] Bell's results and the actual claims about superluminal influences are conceptually very subtle, and it would go beyond the scope of this chapter to explain them carefully. We refer the interested reader to Bell's excellent papers in Bell (2004).

[13] In physics, the nonsignaling condition is the statement that no matter, energy, or information (i.e. signal) can be sent between two spacelike separated events. It is a restriction imposed by relativity theory. In practice, this condition simply states that marginal probabilities for one observer cannot change when a second, far away, observer changes the choice of measurement, such that the choices and measurements are spacelike separated. Other terms for this property of marginal probabilities have been used (e.g. "parameter independence," "marginal selectivity").

We point out that to show nonlocality, Aspect's experiment had to show correlations between spacelike separated measurements.

With Aspect's experiment in mind, we ask ourselves whether nonlocality is relevant to cognition. Given the brain's radius is of the order of 10^{-1} m, any events within the brain would have to be correlated within a time window of 10^{-10} s for them to be separated by a spacelike interval. Since there are no cognitive processes that can be measured within such intervals of time, no empirical evidence of nonlocality in the brain should be expected. Furthermore, since cognitive processes are several orders of magnitude slower than 10^{-10} s, one could never reject classical mechanisms that explain such influences. Therefore nonlocality should not be pertinent for QC.

We are then left with the idea of contextuality. Contextuality in QM was discussed explicitly by Kochen and Specker (1967), but its roots appeared early on in the realization that experiments did not actually reveal the outcomes of a pre-existing quantity, but instead create them. In Peres's example (Peres 1995), three measurement directions, $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$, for spin $1/2$ present problems if assumed that measurements reveal the component of the spin in a given direction, that is, if we imagine that the particle has some unknown spin $\boldsymbol{\mu}$ and that measuring it in direction $\hat{\mathbf{e}}_1$ reveals the component of $\boldsymbol{\mu}$ in this direction (i.e., $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\mu}$). Since each spin measurement only yields either $+1$ or -1, if we choose our directions such that $\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 = 0$, we have

$$\boldsymbol{\mu} \cdot (\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3) = \boldsymbol{\mu} \cdot \hat{\mathbf{e}}_1 + \boldsymbol{\mu} \cdot \hat{\mathbf{e}}_2 + \boldsymbol{\mu} \cdot \hat{\mathbf{e}}_3,$$

which would yield a contradiction, since the left-hand side is zero (by our choice of directions) and the right-hand side is either ± 3, or ± 1, but never zero. This problem is resolved when we realize that an experiment to measure $\hat{\mathbf{e}}_1$ is incompatible with an experiment to measure $\hat{\mathbf{e}}_2$ or $\hat{\mathbf{e}}_3$ (spin operators do not commute), and that the contradiction comes from assuming that the values of the spin components do not change when we change the experiment.

We say a set of experimental outcomes are contextual if their values change under different conditions (see Dzhafarov and Kujala 2015, 2014,b; de Barros et al. 2015). To illustrate this, imagine three ± 1-valued random variables $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ recorded under such conditions that we never observe all three simultaneously, but only in pairs (e.g. $\mathbf{X}$ and $\mathbf{Y}$ but not $\mathbf{Z}$, or $\mathbf{Y}$ and $\mathbf{Z}$ but not $\mathbf{X}$). For simplicity assume that their expectations are all zero, $E(\mathbf{X}) = E(\mathbf{Y}) = E(\mathbf{Z}) = 0$, and that they are perfectly anti-correlated, $E(\mathbf{XY}) = E(\mathbf{XZ}) = E(\mathbf{YZ}) = -1$. Now, the assumption that a variable is the same under different experimental conditions leads to a contradiction. To see this, start with a hypothetical $\mathbf{X} = 1$ and $\mathbf{Y} = -1$ on a trial. The second correlation gives $\mathbf{Z} = -1$, but the third correlation gives $\mathbf{Z} = 1$. Clearly, $\mathbf{Z}$ when measured with $\mathbf{X}$ is different from $\mathbf{Z}$ measured with $\mathbf{Y}$, and this system is contextual.

We formalize contextuality following Dzhafarav and Kujala. Let us assume that variables are a priori contextual, and instead of calling them $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$, we include a label to describe the context. For the three correlation experimental conditions, we have the following six variables: $\mathbf{X_Y}$, $\mathbf{Y_X}$, $\mathbf{X_Z}$, $\mathbf{Z_X}$, $\mathbf{Y_Z}$, and $\mathbf{Z_Y}$. For these variables, the observed correlations are $E(\mathbf{X_Y Y_X}) = E(\mathbf{X_Z Z_X}) = E(\mathbf{Y_Z Z_Y}) = -1$, and it is straightforward to confirm that no contradiction arises from this expanded set of random variables. So, we now have a clear definition of contextuality: our system of three random variables is noncontextual if and only if it is possible to find a probability distribution consistent with the observed correlations and expectations such that $P(\mathbf{X_Y} = \mathbf{X_Z}) = P(\mathbf{Y_X} = \mathbf{Y_Z}) = P(\mathbf{Z_Y} = \mathbf{Z_X}) = 1$. In other words, a system is noncontextual if the values of the random variables do not depend on the experimental contexts, and contextual otherwise. The notion of noncontextuality (and contextuality) can be easily extended to more variables, and we refer the reader to reference Dzhafarov et al. (2015).

The most famous case of a contextual quantum system was presented in Kochen and Specker's seminal paper (Kochen and Specker 1967), where they provided a set of yes–no questions that, if answered in accord with quantum mechanical predictions, lead to inconsistencies, similar to our example above (though with 118 questions, instead of only three). As mentioned above, the CHSH inequalities are equivalent to the existence of a joint probability distribution. Its violation by QM means that Bell-type quantum systems are contextual. However, they present a special type of contextuality, where the contexts for the variables are set by the choice the experimenters make in a spacelike separated interval (thus the nonlocal character of QM). Furthermore, because they are probabilistic, unlike Kochen–Specker, they are not often discussed as examples of contextuality,[14] though they clearly are, if we think of contextuality as above. As such, QM provides other types of contextuality which do not require nonlocality, such as is the case with the MZI or with order effects.

Can contextuality be a feature of cognitive systems? Absolutely. As we saw in the examples from QC above, the cases where CP fails to describe all situations where different contexts were used to probe an answer (say, a known versus

[14]There are exceptions, such as the works of Cabello (Cabello 2013, 2011, 2014; Gühne et al. 2014).

unknown context, in the violation of the STP). Furthermore, as we show in the next section, such contextual outcomes can be modeled in a very classical way.

To summarize, in this section we discussed reasons for using QM in cognitive models. Among those reasons, we argued that only stochasticity and contextuality are relevant. To support this, in the next section we provide a neural model that fits the same outcomes as quantum cognitive models, but also provides cases where outcomes are contextual but yet not describable by QM.[15]

4 A Neural Model of Quantum Cognition

In the previous sections, we discussed the features of QM relevant to QC. We argued that contextuality is the most probable feature relevant to social systems. In this section we present, in a hopefully intuitive way, a *classic* neural oscillator model that replicates some of the characteristics of QC (Suppes et al. 2012). Our goal is that such a model might shed some light on the limitations of using QM to model cognition.

Our model relies on neurophysiological evidence that suggests cognitive processes as an activity involving large collections of synchronizing neurons. This is corroborated by EEG experiments showing the EEG data as a good representation of language or visual imagery. In this section we follow de Barros and Oas (2014), and readers interested in more technical details are referred to Suppes et al. (2012).

In our model, the mathematical behavioral stimulus–response theory (SR theory) is described by synchronized neural oscillators.[16] SR theory is one of the most successful behavioral theories, mainly because it can be mathematically formalized as a simple set of axioms. In terms of random variables $\mathbf{Z}$, $\mathbf{S}$, $\mathbf{R}$, and $\mathbf{E}$, with $\mathbf{Z} : \Omega \to E^{|S|}$, $\mathbf{S} : \Omega \to S$, $\mathbf{R} : \Omega \to R$, and $\mathbf{E} : \Omega \to E$, where S is the set of stimuli, R the set of responses, and E the set of reinforcements, a trial in SR theory has the following structure:

$$\mathbf{Z}_n \to \mathbf{S}_n \to \mathbf{R}_n \to \mathbf{E}_n \to \mathbf{Z}_{n+1}. \tag{7}$$

[15] Here we mean not describable in the sense discussed in Kochen and Specker (1975); see also de Barros (2012, 2015).

[16] It is beyond the scope of this chapter to give a fully fledged account of SR theory, and here we only attempt to describe it in an intuitive way. Readers interested in a mathematical treatment of this theory are referred to Suppes and Atkinson (1960) and Suppes (2002).

Intuitively, a trial n starts with the subject having a given state of conditioning $\mathbf{Z}_n$. Then, a stimulus $s \in S$ is sampled ($\mathbf{S}_n$), and a response $\mathbf{R}_n$ is given according to the sate of conditioning (or randomly, if no conditioning is associated with s). After a response, a reinforcement $\mathbf{E}_n$ event occurs, informing the subject of the correct answer, and this may result (with probability c) to a change in conditioning to this reinforced event, thus leading to a new state of conditioning $\mathbf{Z}_{n+1}$. In other words, learning happens with repeated reinforcement in a probabilistic way by changes in the state of conditioning.

To obtain SR theory in terms of neurons, a distal stimulus is represented in the brain by a set of synchronized neurons, and similarly for responses. Collections of neurons synchronize in phase because of their excitatory connections, and synchronize out of phase because of inhibitory connections (de Barros and Oas 2014). Because we are talking about ensembles of neurons (perhaps thousands), each set stimulus/responses can be described in a first approximation by a periodic function, which for simplicity we assume to be a cosine function. Thus, the basic unit in our model is an oscillator

$$O(t) = A(t) \cos \omega t, \tag{8}$$

where $\omega = \omega(t)$ is its time-dependent frequency. Since ω is a function of time, $O(t)$ is determined by the argument of the cosine, that is, by $\varphi(t) = \omega(t) \cdot t$. Thus, we rewrite this simple oscillator as $O(t) = A(t) \cos \varphi(t)$, and call $\varphi(t)$ the phase of $O(t)$. Firing neurons spike with the same amplitude but varying frequencies. Therefore, a collection of firing neurons can be approximately described by $A(t) = A_0$ and $\varphi(t)$, and in our model we assume interactions that affect only the phase.

So, let $O_s(t)$ be a stimulus oscillator given by

$$O_s(t) = A \cos(\omega_0 t) = A \cos(\varphi_s(t)), \tag{9}$$

and let

$$O_{r_1}(t) = A \cos(\omega_0 t + \delta\phi_1) = A \cos(\varphi_{r_1}(t)), \tag{10}$$

$$O_{r_2}(t) = A \cos(\omega_0 t + \delta\phi_2) = A \cos(\varphi_{r_2}(t)), \tag{11}$$

be the two response oscillators (Fig. 2).

To describe synchronization, we start with $O_s(t)$ and $O_{r_1}(t)$. When uncoupled their natural frequencies ω_s and ω_{r_1} are constant. From Eq. (8) their uncoupled dynamics satisfy

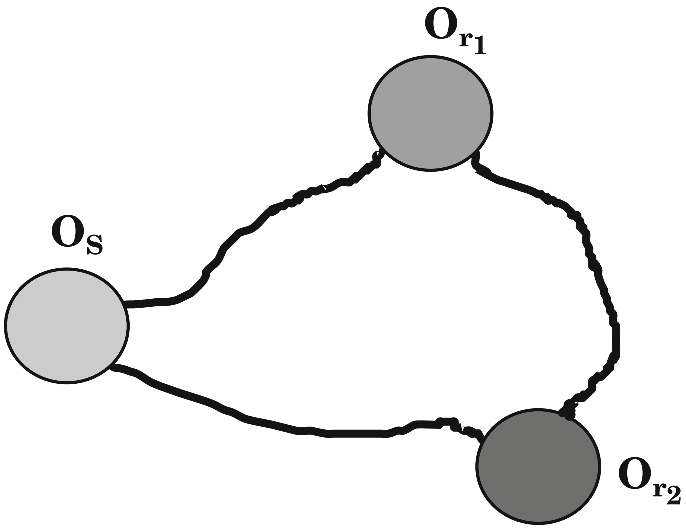

Fig. 2 Schematic representation of the SR oscillator model for two possible responses, 1 or 2, represented by the synchronization of the stimulus oscillator O_s with the response oscillators O_{r_1} or O_{r_2}. Each *circle* corresponds to groups of neurons synchronized among themselves, and the *lines* to connections between each group of neurons

$$\frac{d\varphi_s}{dt} = \omega_s, \tag{12}$$

$$\frac{d\varphi_{r_1}}{dt} = \omega_{r_1}. \tag{13}$$

If weakly coupled, their interaction does not affect the sinusoidal character of $O_s(t)$ and $O_{r_1}(t)$, but affects their relative phases, and (12) and (13) need to include an interaction term. Such a term reflects the tendency of phases either to move closer to each other for excitatory synapses or to move apart from each other for inhibitory synapses. Then, in a first approximation, we have

$$\frac{d\varphi_s}{dt} = \omega_s - k_{s,r_1} \sin\left(\varphi_s - \varphi_{r_1}\right), \tag{14}$$

$$\frac{d\varphi_{r_1}}{dt} = \omega_{r_1} - k_{r_1,s} \sin\left(\varphi_{r_1} - \varphi_s\right), \tag{15}$$

where k_{ij} are the couplings. To understand where synchronization comes from, let us define

$$\varphi_s' = \varphi_s - \omega_s t,$$
$$\varphi_{r_1}' = \varphi_{r_1} - \omega_{r_1} t.$$

Substituting in (14) and (15), we have

$$\frac{d\varphi_s'}{dt} = -k_{s,r_1} \sin\left(\left(\varphi_s' - \varphi_{r_1}'\right) + \left(\omega_s - \omega_{r_1}\right) t\right), \tag{16}$$
$$\frac{d\varphi_{r_1}'}{dt} = -k_{r_1,s} \sin\left(\left(\varphi_{r_1}' - \varphi_s'\right) - \left(\omega_s - \omega_{r_1}\right) t\right). \tag{17}$$

Equations (16) and (17) have fixed points[17] when

$$\varphi_{r_1}' - \varphi_s' = \delta\omega\, t,$$

or,

$$\varphi_{r_1} = \varphi_s.$$

In other words, (14) and (15) are stationary when synchronized.

The system (14) and (15) can be extended to N oscillators, and become

$$\frac{d\varphi_i}{dt} = \omega_i - \sum_{j \neq i} k_{ij} \sin\left(\varphi_i - \varphi_j\right). \tag{18}$$

Equations (18) are known as Kuramoto equations (Kuramoto 1984), and they are often used to describe synchronizing systems. Their advantage come from two main points. They can be exactly solved under symmetry assumptions in the limit of large N, providing insight into the nature of emerging synchronization. Second, sets of weakly coupled oscillating systems can be roughly described by Kuramoto-like equations (Izhikevich 2007). In our model, we assume Kuramoto's equations are a good approximation for the dynamics of coupled sets of neural oscillators.

From the oscillators' mathematical description, we can describe how SR theory is modeled by them. The main idea is straightforward. Once a distal stimulus is presented, an associated ensemble of neurons is activated in the brain. Neurons in this ensemble synchronize, and we describe this highly

[17] A fixed point is a point where all derivatives are zero. They are important points because they represent stationary solutions for the dynamical system. Fixed points can have stationary solutions that are either stable or unstable.

complex system by its average phase. We think of this synchronization as an activation of the stimulus representation in the brain.

Once the stimulus is activated, it may elicit a response by activating synaptically coupled oscillators (in a mechanism that may lead to spreading activation (Collins and Loftus 1975)). Similarly to stimuli, responses are represented by ensembles of synchronized neurons. Among the active responses, the selection of a particular one is done by the relative phase synchronization between the stimulus oscillator and the selected response. This phase synchronization is determined by the couplings between the stimulus and response oscillators, and the couplings are related to the state of conditioning in SR theory.

The simplest model utilizes three oscillators as introduced above. Once activated, the rate of firings within each response oscillator is due to their own dynamics and also the firings of O_s. Thus, it is reasonable to assume that they interfere, with interference meaning higher coherence when in phase and lower coherence when out of phase. Mathematically, we have, for equal amplitude oscillators, Eqs. (9)–(11). As with physical oscillators, the mean intensity is a measure of the excitation carried by the oscillations, and at response 1 it is

$$\begin{aligned} I_1 &= \left\langle (O_s(t) + O_{r_1}(t))^2 \right\rangle_t \\ &= \left\langle O_s(t)^2 \right\rangle_t + \left\langle O_{r_1}(t)^2 \right\rangle_t + \left\langle 2O_s(t)O_{r_1}(t) \right\rangle_t , \end{aligned}$$

where $\langle f(t) \rangle_{t_0}$ is the time average of $f(t)$ defined by $\langle f(t) \rangle_{t_0} = \frac{1}{\Delta T} \int_{t_0}^{t_0+\Delta T} f(t)\, dt$ $(\Delta T \gg 1/\omega_0)$. We have at once

$$I_1 = A^2 (1 + \cos(\delta\phi_1)) ,$$

and similarly

$$I_2 = A^2 (1 + \cos(\delta\phi_2)) .$$

Therefore, the intensity for r_1 or r_2 depends on the phase difference between the SR oscillators.

Since I_1 and I_2 are competing responses, the maximum contrast between them happens when one of their relative phases (with respect to the stimulus oscillator) is zero while the other is π. It is standard to normalize the difference $I_1 - I_2$ by the total intensity,

$$b = \frac{I_1 - I_2}{I_1 + I_2}. \tag{19}$$

taking values between -1 and 1. The quantity b is called the contrast.

The contrast provides a way to think about a continuum of responses between r_1 and r_2. Assume

$$\delta\phi_1 = \delta\phi_2 + \pi \equiv \delta\phi, \tag{20}$$

which yields

$$I_1 = A^2 \left(1 + \cos\left(\delta\phi\right)\right), \tag{21}$$

and

$$I_2 = A^2 \left(1 - \cos\left(\delta\phi\right)\right). \tag{22}$$

Then, to determine b all we need is $\delta\phi$, as

$$b = \cos\left(\delta\phi\right), \tag{23}$$

$0 \leq \delta\varphi \leq \pi$. So, $\delta\phi$ codes a continuum of responses between -1 and 1 or any arbitrary interval (ζ_1, ζ_2) upon rescaling.

The above discussion presents only some aspects of our oscillator model, which was designed to reproduce SR theory. It goes beyond the scope of this chapter to describe fully this model, particularly because learning, one of the central features of SR theory, is not relevant to our current purposes of showing quantum-like characteristics in neural oscillators. However, we examine in more detail two of the mathematical components of the oscillator SR model that are relevant to us here: sampling and response.

When a stimulus s_n is sampled, a collection of neurons fire synchronously, corresponding to the activation of a neural oscillator, O_{s_n}. In consonance with SR theory, we assume the activation of s_n in a way that is consistent with the random variable $\mathbf{S}_n$. In other words, from a set of s_n oscillators, we activate only one oscillator with equal probability. This is a stochastic characteristic of the theory that is not part of the dynamics, but is a classical type of stochasticity.

Once s_n is sampled, the active oscillators evolve for the time interval Δt_r, which is selected as a parameter representing the time of response computation. This evolution satisfies Kuramoto's differential equations

$$\frac{d\varphi_i}{dt} = \omega_i - \sum_{i \neq j} k_{ij} \sin\left(\varphi_i - \varphi_j + \delta_{ij}\right), \tag{24}$$

where k_{ij} is the coupling constant between oscillators i and j, and δ_{ij} is an anti-symmetric matrix representing phase differences, and i and j can be either O_{s_n}, O_{r_1}, or O_{r_2}. Equation (24) can be rewritten as

$$\frac{d\varphi_i}{dt} = \omega_i - \sum_j \left[k_{ij}^E \sin\left(\varphi_i - \varphi_j\right) + k_{ij}^I \cos\left(\varphi_i - \varphi_j\right)\right], \tag{25}$$

where $k_{ij}^E = k_{ij}\cos\left(\delta_{ij}\right)$ and $k_{ij}^I = k_{ij}\sin\left(\delta_{ij}\right)$, and this has an important physical interpretation: k_{ij}^E corresponds to excitatory couplings, and k_{ij}^I to inhibitory ones. In terms of those couplings, the evolution equation is

$$\frac{d\varphi_i}{dt} = \omega_i - \sum_{i\neq j} \left[k_{i,j}^E \sin\left(\varphi_i - \varphi_j\right) - k_{i,j}^I \cos\left(\varphi_i - \varphi_j\right)\right], \tag{26}$$

where ω_i is the oscillator's natural frequency. The solutions to (26) and the initial conditions randomly distributed at activation give us the phases at time $t_{r,n} = t_{s,n} + \Delta t_r$. The coupling strengths between oscillators determine their relative phase locking, which in turn corresponds to the computation of a given response, according to Eq. (19). The couplings are determined by reinforcement, but here we assume the values are given for each experimental condition (see Suppes et al. 2012 for details).

At this point the attentive reader may have guessed where quantum-like contextuality come from: the interference of two neural oscillators in (23). To see how interference renders quantum-like results, let us consider the following case discussed in detail in de Barros (2012). Imagine that instead of a single stimulus, O_s, we have two stimuli, O_{s_1} and O_{s_2} which can be activated separately or simultaneously. The activation of stimulus O_{s_1} leads to a response (contrast) b_1 when

$$k^E_{s_1,r_1} = k^E_{r_1,s_1} = \alpha b_1 = -k^E_{s_1,r_2} = -k^E_{r_2,s_1}, \tag{27}$$

$$k^E_{r_1,r_2} = k^E_{r_2,r_1} = -\alpha, \tag{28}$$

and

$$k^I_{s_1,r_1} = k^I_{r_2,s_1} = \alpha\sqrt{1 - b_1^2} = -k^I_{s_1,r_2} = -k^I_{r_1,s_1}, \tag{29}$$

$$k^I_{r_1,r_2} = k^I_{r_2,r_1} = 0, \tag{30}$$

where α is a convergence to a synchronization parameter (the larger the α, the faster it converges). A similar set of couplings can be obtained for the other stimulus oscillator O_{s_2} if we require it to answer b_2.

Now, from (26) and couplings (27)–(30), the system is deterministic. However, the initial conditions are not the same at every trial, and if we assume a Gaussian distribution of initial phases at each trial, the responses given to the stimulus O_{s_1} will vary around the value b_1. To code a discrete response, such as a ± 1-valued random variable $\mathbf{A}$ we say the outcome of a random variable $\mathbf{A}$ is $+1$ if the response b_1 is greater or equal to 0.5, and -1 if the response is lesser than 0.5, and we interpret the value $+1$ as an action being preferred over no action. Then, if we carefully choose the parameters in (27)–(30) such that b_1 is slightly greater than 0.5, then $\mathbf{A}$ would be $+1$ with a higher probability than -1. Thus we could say that an action is preferred, given stimulus O_{s_1}. We could perform the same type of set-up for stimulus O_{s_2}, such that whenever this stimulus is presented, an action is also preferred.

The oscillator case above is equivalent to the example presented in Sect. 2 if we think of the two distinct stimuli O_{s_1} and O_{s_2} as corresponding to "won first bet" and "lost first bet," respectively, and $\mathbf{A} = 1$ as "accept second gamble" and $\mathbf{A} = -1$ as "reject second gamble." If O_{s_1} and O_{s_2} are inconsistent stimuli, the violation of the STP comes from the probabilities of response for such an oscillator model when *both* oscillators are activated (in case of lack of knowledge), and the interference effects of the oscillations lead to the nonmonotonicity of probabilistic outcomes (de Barros 2012). In other words, because of interference, neural oscillator models may exhibit contextual quantum-like features.

A natural question now arises from our oscillator model. Since QM brings so many features in addition to stochasticity and contextuality,[18] it is worth investigating whether there are violations of CP from our neural model that cannot be described by QM. A natural starting point is the three random variable example, $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$, given in Sect. 3. It is straightforward to prove that a Hilbert space description of three observables, represented by the Hermitian operators X, Y, and Z, where we can observe them in a pairwise fashion, implies that we can observe all three simultaneously. In other words, if $[X,Y] = [X,Z] = [Y,Z] = 0$, then there exists a basis where X, Y, and Z are simultaneously diagonal. This means that if we can concoct an experiment to measure X and Y together, another to measure Y and Z, and yet another to measure X and Z, QM predicts it to be possible to create an experiment where all three observables, X, Y, and Z, are measured simultaneously. Since a simultaneous measure of three observables is a guarantee of the existence of a joint probability distribution (by simply counting how many times each

[18]Nonlocality, as we discussed in Sect. 3, is one prominent case, but there are many nontrivial results in QM that bear no clear connection to social systems, such as the no-cloning theorem (Dieks 1982), or the monogamy of entanglement (Yang 2006), to mention a few.

elementary event shows up), the three random variable example provided in Sect. 3 cannot be described by QM.[19]

However, as we showed in de Barros (2012, 2015), in more complicated three stimulus and six response oscillators, there are couplings between oscillators that give a higher probability of anti-correlation between pairwise activations of stimuli. For strong enough anti-correlations, there are no joint probability distributions, as Suppes and Zanotti (1981) proved that a joint probability exists iff

$$\begin{aligned} -1 &\leq E(\mathbf{XY}) + E(\mathbf{XZ}) + E(\mathbf{YZ}) \\ &\leq 1 + 2\min\{E(\mathbf{XY}), E(\mathbf{XZ}), E(\mathbf{YZ})\}. \end{aligned} \tag{31}$$

Thus, there are neural oscillator models that exhibit a type of contextuality that cannot be modeled by QM.

In this section we have presented a neural oscillator model that reproduces not only SR theory, but also displays the nonmonotonicity associated with contextual quantum-like behavior. We also showed that such a model poses difficulties for quantum descriptions, as it implies the theoretical existence of systems that would not be describable by QM. In the next section we introduce an alternative stochastic model that we believe could be a natural replacement for the quantum formalism in such cases where QM is not applicable.

5 Negative Probabilities

In this section we introduce the idea of negative probabilities (NPs) as a way to describe certain contextual stochastic processes. Historically, NPs were first encountered in QM, when Wigner attempted to produce a joint probability distribution for momentum and position that would give the same outcomes as quantum statistical mechanics (for a somewhat old review, see Mückenheim 1986). Wigner dismissed NPs as meaningless, and called them quasi-probability distributions.[20] Later, Dirac used NPs to approach

[19]This is a point mentioned by Kochen in Kochen and Specker (1975), but in de Barros (2014) we showed that by increasing the Hilbert space and adding a fourth variable corresponding to context, we can artificially reproduce the correlations that violate a joint probability distribution.

[20]Perhaps very much in the same way that mathematicians had problems with negative numbers. For instance, as late as the 1800s, the famous mathematician Augustus De Morgan stated the following (De Morgan 1910, p. 72). "Above all, he [the student] must reject the definition still sometimes given of the quantity $-a$, that it is less than nothing. It is astonishing that the human intellect should ever have tolerated such an absurdity as the idea of a quantity less than nothing; above all, that the notion should

problems in quantum electrodynamics (Dirac 1942), and Feynman used them to describe the two-slit and spin of particles (Feynman 1987). Dirac and Feynman's views were similar to Wigner's, but they thought of NPs as a nice accounting tool that could perhaps be as useful as negative numbers in mathematics. However, not all quantum mechanical set-ups allow for NPs (e.g., the two-slit experiment can be shown to allow for NPs only under certain counterfactual reasoning (de Barros and Oas 2014; de Barros et al. 2015)). We emphasize that here NPs always mean that a joint probability distribution takes negative values for non-observable events (such as joint values of position and momentum), but is always non-negative for observable events.

Before we delve further into our discussion of NPs, we formally define it (from now on we follow de Barros et al. (2015)). We start with a preliminary definition related to marginal expectations that are observable.

Definition 3. Let Ω be a finite set, $\mathcal{F}$ an algebra over Ω, and let $(\Omega_i, \mathcal{F}_i, p_i)$, $i = 1, \ldots, n$, a set of n probability spaces, $\mathcal{F}_i \subseteq \mathcal{F}$ and $\Omega_i \subseteq \Omega$. Then $(\Omega, \mathcal{F}, p)$,where p is a real-valued function, $p : \mathcal{F} \to [0, 1]$, $p(\Omega) = 1$, is *compatible* with the probabilities p_i's iff

$$\forall \left(x \in \mathcal{F}_i\right) \left(p_i\left(x\right) = p\left(x\right)\right).$$

Furthermore, the marginals p_i are *viable* iff p is a probability measure.

Intuitively, we can think of the p_i's as observable marginal probabilities on subspaces of a larger sample space Ω. Then such marginals are *viable*[21] if it is possible to "sew" them together to produce a larger probability function over the whole Ω (Dzhafarov and Kujala 2013, 2014a; de Barros et al. 2015).

As mentioned, in QM the marginals are not always viable, but are compatible with a real-valued function p that has the characteristic of being somewhere negative. This motivates the following definition.

Definition 4. Let Ω be a finite set, $\mathcal{F}$ an algebra over Ω, P and P' real-valued functions, $P : \mathcal{F} \to \mathbb{R}$, $P' : \mathcal{F} \to \mathbb{R}$, and let $(\Omega_i, \mathcal{F}_i, p_i)$, $i = 1, \ldots, n$, a set of n probability spaces, $\mathcal{F}_i \subset \mathcal{F}$ and $\Omega_i \subseteq \Omega$. Then $(\Omega, \mathcal{F}, P)$ is an NP space, and P an NP, if and only if $(\Omega, \mathcal{F}, P)$ is compatible with the probabilities p_i's and

have outlived the belief in judicial astrology and the existence of witches, either of which is ten thousand times more possible."

[21] A term coined by Halliwell and Yearsley (2013).

N1. $\forall\left(P'\right)\left(\sum_{\omega_i\in\Omega}\left|P\left(\{\omega_i\}\right)\right|\leq\sum_{\omega_i\in\Omega}\left|P'\left(\{\omega_i\}\right)\right|\right)$

N2. $\sum_{\omega_i\in\Omega}P\left(\{\omega_i\}\right)=1$

N3. $P\left(\left\{\omega_i,\omega_j\right\}\right)=P\left(\{\omega_i\}\right)+P\left(\left\{\omega_j\right\}\right),\ \ i\neq j.$

In this definition we replaced Kolmogorov's non-negativity axiom with a minimization of the L1 norm of P. There is an intuitive reason to do this: we seek a quasi-probability distribution that is as close to a proper distribution as possible. This departure from a proper norm is the motivation for the following definition.

Definition 5. Let $(\Omega,\mathcal{F},P)$ be an NP space. Then, the *minimum L1 probability norm*, denoted M^*, or simply *minimum probability norm*, is given by $M^*=\sum_{\omega_i\in\Omega}\left|P\left(\{\omega_i\}\right)\right|$.

In de Barros et al. (2015) we proved that P is a probability (and therefore $(\Omega,\mathcal{F},P)$ is a probability space) if and only if $M^*=1$. Since M^* can be greater than one for systems with NP, and since NPs come from the impossibility of defining a proper probability distribution that can put together the different marginals, we interpret M^* as a measure of contextuality. In other words, not only does the existence of NPs lead to contextuality, but the more they depart from a proper distribution the more contextuality there is (de Barros et al. 2015).

An important result for NPs relies on the following definition (de Barros et al. 2015).

Definition 6. Let Ω be a finite set, $\mathcal{F}$ an algebra over Ω, and let $(\Omega_i,\mathcal{F}_i,p_i)$, $i=1,\ldots,n$, a collection of n probability spaces, $\mathcal{F}_i\subseteq\mathcal{F}$ and $\Omega_i\subseteq\Omega$. Then the probabilities p_i are *contextually biased*[22] if there exists an a in $\mathcal{F}_i$ and in $\mathcal{F}_j$, $i\neq j$, $b\neq a\neq b'$, $\sum_{\forall b\in\mathcal{F}_j}p\left(a\cap b\right)\neq\sum_{\forall b'\in\mathcal{F}_i}p\left(a\cap b'\right)$.

In references Abramsky and Brandenburger (2011), Al-Safi and Short (2013), Oas et al. (2014), and Loubenets (2015) it was independently proven that NPs (in the sense we use above) exist if and only if the marginals p_i are not

[22] Here we adopt and adapt the terminology of Dzhafarov and Kujala (2014a).

contextually biased. Thus, it follows that for many systems where proper joint probability distributions cannot be defined, we can still define NPs if such systems are not contextually biased. Another way is to say that a collection of probabilities p_i are compatible if and only if they are not contextually biased.

We now present an example of a nontrivial application of NPs to decision-making (de Barros 2014). In this example, Deana is a decision-maker who wants to bet on the stock market (well, some "simple" version of it). She wants to invest in three companies, creatively named here X, Y, and Z. Since she knows nothing about X, Y, and Z, she contacts three "experts," Alice, Bob, and Carlos, who provide her with expected outcomes of X, Y, and Z. However, each expert is specialized only in two of the companies, but not all. Furthermore, perhaps because of a bias, experts may give information that is inconsistent. For example, say we create the following ± 1-valued random variables, $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$, corresponding to their beliefs of a stock value going up if $+1$ and down if -1. Our experts all agree that the probabilities of stocks of X, Y, and Z going up are the same as going down, and therefore we can say that

$$E\left(\mathbf{X}\right) = E\left(\mathbf{Y}\right) = E\left(\mathbf{Z}\right) = 0. \tag{32}$$

But since Alice only knows about X and Y, she can only tell us that her belief is

$$E_A\left(\mathbf{XY}\right) = -1, \tag{33}$$

where we put a subscript on the expectation to emphasize that it is Alice's subjective belief.[23] Equation (33) has the simple interpretation: Alice believes that if X's stocks go up/down then Y's will go down/up with certainty. Bob's and Carlos's beliefs are that

$$E_B\left(\mathbf{XZ}\right) = -\frac{1}{2}, \tag{34}$$

and

$$E_C\left(\mathbf{YZ}\right) = 0. \tag{35}$$

It is easy to see from (31) that (33)–(35) are not viable, but (32) imply the probabilities that lead to Alice, Bob, and Carlos's expectations are compatible. Therefore, there exists an NP distribution consistent with (32)–(35).

[23] An example of inconsistent information like the one we present here is not easily translatable into objective interpretations of probabilities.

What is Deana to do with the inconsistent information she got from Alice, Bob, and Carlos? A standard approach is to start with a prior distribution and use their information to update the posterior using Bayes' theorem. However, as demonstrated in de Barros (2014), such an approach has a shortcoming: it does not tell us anything new about the triple moment $E\left(\mathbf{XYZ}\right)$. The Bayesian approach does not update the triple moment, and its value comes purely from Deana's prior knowledge.

The lack of update for the triple moment presents a difficulty. To show it, take the deterministic (and consistent with a proper joint probability) case where Deana was given $E_A\left(\mathbf{XY}\right) = E_B\left(\mathbf{XZ}\right) = E_C\left(\mathbf{YZ}\right) = 1$. It is immediately clear from these correlations that $E\left(\mathbf{XYZ}\right) = 1$. So, since the experts are *not* disagreeing, and since their judgment leads to a specific value for the triple moment, why should Deana not take this into account? Why should her bet on the triple moment be related simply to her prior knowledge of it? This seems to be a failure of the Bayesian approach. The situation is different for NPs. Because we assume that a joint quasi-probability distribution exists (albeit negative) and that the best joints (as they are not unique) minimize the L1 norm, as they are the closest to a "rational" and consistent joint, then we are constrained to only the best joints. In the deterministic case of 1 correlations, this leads to the correct prediction that the triple moment is 1.

The minimization of the L1 norm also has a consequence for the inconsistent pairwise expectations (33)–(35). It restricts the possible values for the triple moments to the range (de Barros 2014)

$$-\frac{1}{4} \leq E\left(\mathbf{XYZ}\right) \leq \frac{1}{2}.$$

Given that the Bayesian approach provides no information to Deana, it should be possible to devise a Dutch Book between NPs and Bayesian approaches for certain situations.[24] NP provides normative information that goes beyond the Bayesian approach.

The situation is a little better between NPs and QC. As we mentioned, the $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ example is only describable via QM with supplementary assumptions, as done in de Barros (2014), where an extra dimension to the Hilbert space was added corresponding to the internal states of belief of Alice, Bob, and Carlos. However, the triple moment correlation needs to be explicitly given in the state vector, and there are no arguments to limit its values. So, QC

[24]A Dutch Book is the name given to a strategy that would allow one of the gamblers to win for sure over the other gamblers in a game (Anand et al. 2009).

is in better shape than the Bayesian approach because even though it does not provide an advantage over the other approaches, it at least makes it explicit that the triple moment is included ad hoc.

We end this section with some comments about the meaning of NPs. In this chapter we take Feynman and Dirac's views: NPs are a useful accounting tool. However, there are ways to interpret them. For example, Khrennikov showed that in the frequentist interpretation of von Mises, NPs appear when we have sequences in the usual Archimedian metric that violate the principle of stabilization, and therefore do not converge to a specific probability value.[25] In those cases, a p-adic metric makes such sequences convergent, and NPs appear as the p-adic limiting case (Khrennikov 1993a,b,c, 1994a,b, 2009). Abramsky and Brandenburger (2011, 2014) interpret NPs in the context of sheaf theory. For them NP comes from two independent types of events belonging to different types. One type of event erases recordings of the other type, and this allows for the observed correlations. Finally, closely related to Abramsky and Brandenburger's, is Szekely's interpretation, which thinks of NPs P as related to a proper probability p via a convolution equation $P * f = p$, which always exists (Ruzsa and Székely 1983; Székely 2005). This convolution means that for a random variable $\mathbf{X}$ whose probability distribution is P, there exists two other random variables, $\mathbf{X}_{+}$ and $\mathbf{X}_{-}$ with proper probability distributions (p and f, respectively) and such that $\mathbf{X} = \mathbf{X}_{+} - \mathbf{X}_{-}$. Our interpretation, though, can be subjective: NPs are an accounting tool, but provide us with the best subjective information about systems which do not have an objective probability distribution, as it is the closest distribution to a proper one (via normalization of the L1 norm).

6 Final Remarks

In this chapter we have discussed how some of the well-known examples used in QC are connected to the contextuality of the two-slit experiment. A neural oscillator model was introduced based on reasonable neurophysiological assumptions that reproduce behavioral SR theory and the nonmonotonic character of QC. Such a neural model produces outcomes for certain situations that are not naturally modeled by QM, as in the case with six-response oscillators. However, such examples could be modeled by NP. More importantly, NP not only provided a way to describe such systems, but was also normative.

[25] Pseudo-random sequences may have this property.

QC comes from the idea that human decision-making is better described by the mathematics of QM, with its probability associated with density operators in a Hilbert space. However, there are possible situations where QC is not appropriate, such as the **X**, **Y**, and **Z** example. Furthermore, we saw that the **X**, **Y**, and **Z** example shows up in neural oscillator models that reproduce standard SR theory, but also in decision-making situations. Therefore, as an extended probability theory, QC is too restrictive, leaving out perhaps important situations. Furthermore, QC is mostly descriptive, not offering, as far as we are aware, any normative power. We contrast this with NP, which describes many of the QC systems (those with compatible probabilities), but also those created by inconsistent oscillators or inconsistent information. Given how NP offers normative information via the minimization of the L1 norm, which is computationally simple for biological (as well as computer) systems, perhaps it is not unreasonable to hypothesize that such processes actually happen in our brain. This, we believe is an exciting perspective, and we hope to investigate it further in the future.

Bayesians have problems with not updating their triple moment, even when faced with indirect information about them. This suggests the existence of a Dutch Book. An interesting question is how such a Dutch Book could be constructed. For example, if Alice, Bob, and Carlos are subject to confirmation biases, could we model it (similar to Fine's prism model (Fine 1982, 2009)) and show that NP outperforms Bayesianism? Furthermore, if our L1 norm hypothesis for the brain is correct, wouldn't human decision-makers unconsciously follow a strategy that would win bets with "rational" Bayesians? These questions also present a research program that we believe will be fruitful.

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Quantum-Like Type Indeterminacy: A Constructive Approach to Preferences à la Kahneman and Tversky

A. Lambert-Mogiliansky

1 Introduction

To many people it may appear unmotivated and artificial to turn to quantum mechanics (QM) when investigating human behavioral phenomena. However, the founders of QM, including Bohr and Heisenberg, recognized early on the similarities between the two fields. In particular Bohr was influenced by the psychology and philosophy of knowledge of Harald Höffding. The similarity stems from the fact that in both fields the object of investigation cannot (always) be separated from the process of investigation.[1] QM and in particular its mathematical formalism was developed to respond to that epistemological challenge. It is therefore legitimate to explore the value of the mathematical formalism of quantum mechanics in the study of human behavioral phenomena. The type-indeterminacy (TI) model proposes to use elements of that formalism to model uncertain preferences. The basic idea is that the Hilbert space model of QM can be thought of as a general contextual

[1]In the words of Bohr (1971) "the impossibility of a sharp separation between the behavior of an atomic object and the interaction with the measuring instrument which serves to define the condition under which the phenomenon appears". In psychology investigating a person's emotional state affects the state of the person. In social sciences "revealing" one's preferences in a choice can affect those preferences.

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predictive tool particularly well suited to describing experiments in psychology or in "revealing" preferences. This chapter provides an introduction to the TI model and some of its developments. For a complete exposition see (Lambert-Mogiliansky et al. 2009).

2 Our Approach

The well-established Bayesian approach suggested by Harsanyi to model incomplete information consists of a chance move that selects the type (i.e., their preference and private information) of player and informs each one of his or her own type. For the purposes of this chapter, I emphasize the following essential implication of this approach: all uncertainty about a player's type exclusively reflects the other players' *incomplete knowledge* of it. This follows from the fact that a Harsanyi type is fully determined. It is a complete well-defined characteristic of a player that is known to him or her. Consequently, from the point of view of the other players, uncertainty as to the type can only be due to lack of information. Each player has a probability distribution over the type of the other players, but his or her own type is fully determined and known to him or her.

This brings me to the first important point at which we depart from the classical approach. We propose that, in addition to informational reasons, the uncertainty about preferences is due to indeterminacy: prior to the moment a player acts, his or her (behavior) type is indeterminate. The state representing the player is a superposition of potential types.[2] It is only at the moment when the player selects an action that a specific type is actualized. It is not merely revealed but rather determined in the sense that, prior to the choice, there is an irreducible multiplicity of potential types. Thus we suggest that, in modeling a decision situation, we do not assume that the preference characteristics (we do not address private information) can always be fully known with certainty (neither to the decision-maker nor even to the analyst). Instead, what can be known is the state of the agent, as a vector in a Hilbert space which encapsulates all existing information to predict how the agent is expected to behave in different decision situations.

This idea, daringly imported from QM to the context of decision and game theory, is very much in line with Tversky and Simonson (Kahneman and Tversky 2000) according to whom "there is a growing body of evidence that supports an alternative conception according to which preferences are

[2]A superposition is a linear combination such that the squares of the coefficients sum to one.

often constructed—not merely revealed—in the elicitation process. These constructions are contingent on the framing of the problem, the method of elicitation, and the context of the choice." This view is also consistent with that of cognitive psychology, which teaches one to distinguish between objective reality and the proximal stimulus to which the observer is exposed, and further to distinguish between those and the mental representation of the situation that the observer eventually constructs. More generally, this view fits in with the observation that players (even highly rational ones) may act differently in game theoretically equivalent situations that differ only in seemingly irrelevant aspects (framing, prior unrelated events, etc.). Our theory as to why agents act differently in game theoretically equivalent situations is that they are not in the same state; that is, they are not the same agents: (revealed) preferences are contextual because of (intrinsic) indeterminacy.

The basic analogy with physics, which makes it appealing to adopt the mathematical formalism of QM to the social sciences, is the following. We view an observed play, decisions, and choices as something similar to the result of a measurement (of the player's type). A decision situation is then similar to an experimental set-up to measure the player's type. This is modeled as an operator (called observable) and the resulting behavior as an eigenvalue of that operator. The analogy to the noncommutativity of observables (a very central feature of QM) is, in many empirical phenomena, like the following well-known experiment conducted by Leon Festinger (the father of the theory of cognitive dissonance). In that experiment people were asked to perform a very boring task. They would sort a batch of spools into lots of 12 and turn a square peg a quarter turn to the left. They were then told that one subject was missing and asked to convince a potential female subject in the waiting room to participate. In one group they were offered \$1, and in the other group \$20, for expressing enthusiasm for the task. Some refused, but others accepted. Those who accepted for \$20 later admitted that they thought the task was dull. Those who accepted for \$1 maintained that the task was enjoyable. The experiment aimed at showing that attitudes change as a response to cognitive dissonance. The dissonance faced by those who were paid \$1 was between the cognition of being a "good guy" and of being ready to lie for a dollar. Changing one's attitude to the task resolves the dissonance. Similar phenomena have been documented in hazardous industries, with employees showing very little caution in the face of danger. Here too, experimental and empirical studies (e.g., Ben-Horin 1979) exhibited attitude changes among employees following their decision to work in a hazardous industry. More generally, suppose that an agent is subject to the same decision situation in two different contexts (the contexts may vary with respect to the decision situations that precede the investigated one, or with respect to the framing in the presentation of the decision, cf. Selten 1998). If we do not observe the same decision in the two contexts, then the classical approach is to say that the two decision situations are not the same: they should be modeled so as to incorporate the context. In many cases, however, such an assumption, that is, that the situations are different, is difficult to justify. And so, the standard theory leaves a host of behavioral phenomena unexplained: the so-called behavioral anomalies (cf. McFadden 1999).

In contrast, I propose that the observed decisions are not taken by an agent in the same state. The context, for example, a past decision situation, is viewed as an operator that does not commute with the operator associated with the investigated decision situation; its operation on the agent has changed his or her state. As in QM, the phenomenon of noncommutativity of decision situations (measurements) leads us to conjecture that an agent's preferences are represented by a state that is indeterminate but which gets determined (with respect to any particular type characteristic) in the course of interaction with the environment. Our approach allows us to go beyond the cognitive dissonance argument, as I show below in an example of application of the TI model.

The objective of this chapter is to provide the basic elements of a theoretical framework that extends the Bayes–Harsanyi model to accommodate various forms of the so-called behavioral anomalies. I attempt to provide a model for the Kahneman–Tversky (KT) man as opposed to what McFadden calls the "Chicago man" (McFadden 1999). Our work is related to Random Utility Models (RUMs) as well as to behavioral economics. RUM models have proven very useful tools for explaining and predicting deviations from standard utility models. However, RUMs cannot accommodate the kind of drastic and systematic deviations characteristic of the KT man. These models are based on a hypothesis of "the primacy of desirability over availability" and they assume stable taste templates. In RUMs, preferences are noncontextual by construction, while in the proposed TI model, (actualized) preferences are contextual. Behavioral economics has contributed a wide variety of theories (see Camerer et al. 2011). Often the proposed explanations address a very specific deviation (e.g., "trade off contrast" or "extremeness aversion", Kahneman and Tversky 2000). Important insights have been obtained by systematically investigating the consequences on utility maximization of "fairness concerns" (Rabin 1993), "cost of self-control" (Gul and Pesendorfer 2001), or "concerns for self-image" (Benabou and Tirole 2002). Yet, other explanations appeal to bounded rationality, for example, "superficial reasoning" or "choice of beliefs" (Selten 1998; Akerlof and Dickens 1982). In contrast, the TI model is a framework model that addresses structural properties of preferences, that is, their intrinsic indeterminacy.

3 Basic Ingredients of the TI Model

The object of my investigation is individual choice behavior, which I interpret as the revelation of an agent's preferences in what I call a Decision Situation (DS). In the basic model I focus on nonrepeated, nonstrategic decision situations. Examples of such DSs include the choice between buying a Toshiba or a Compaq laptop, the choice between investing in a project or not, the choice between a sure gain of \$100 or a bet with probability 0.5 to win \$250 and 0.5 to win \$0. When considering games, I view them as decision situations from the perspective of a single player.[3]

3.1 State, Measurement, and Observable

An agent is represented by a *state* which captures the agent's expected behavior in the decision situation under consideration. Mathematically, a state $|\varphi\rangle, |\varphi\rangle$ is a vector in a Hilbert space H of finite or countably infinite dimensions over the field of the real numbers R.[4] A key ingredient in the formalism of indeterminacy is the principle of superposition. This principle states that the linear combination of any two states is itself a possible state.[5] The principle of superposition implies that, unlike the Harsanyi type space, the state space is non-Boolean.[6]

The notion of "measurement" is a central one in my framework. A measurement is an operation (or an experiment) performed on a system. It yields a result, the outcome of the measurement. A defining feature of a measurement

[3]All information (beliefs) and strategic considerations are embedded in the definition of the choices. Thus the agent's play of C is a play of C given his information (knowledge) about the opponent.

[4]In quantum mechanics the number field is that of complex numbers. However, for our purposes the field of real numbers provides the structure and the properties needed (see e.g. Beltrametti and Cassinelli 1981; Holland 1995).

[5]I use the term "state" to refer to "pure state." Some people use the term to refer to a mixture of pure states, which combines indeterminacy with elements of incomplete information. They are represented by so-called density operators.

[6]The distributivity condition defining a Boolean space is dropped for a weaker condition called orthomodularity. The basic structure of the state space is that of a logic, i.e., an orthomodular lattice. For a good presentation of quantum logic, a concept introduced by Birkhoff and Von Neuman (1936), and further developed by Mackey (2004, 1963), see Cohen (1989).

is the so-called first-kindness property. This refers to the fact that if one performs a measurement on a system and obtains a result, then one will get the same result if one again performs the same measurement on the same system immediately afterwards. Thus, the outcome of a first-kind measurement is reproducible, but only in a next subsequent measurement. First kindness does not entail that the first outcome is obtained when repeating a measurement if other measurements were performed on the system in between.

3.2 One Decision Situation

A DS A can be thought of as an experimental set-up where the agent is invited to choose a particular action among all the possible actions allowed by this DS. To every DS A, we associate an observable, namely, a specific symmetric operator on H which, for notational simplicity, we also denote by A. The actual implementation of the experiment, that is, the act of choice, is represented by a measurement of the associated observable A. The outcome, that is, the choice made, is information about the agent's preferences.

If A is the only decision situation we consider, we can assume that its eigenvectors, which we denote by $|1_A\rangle, |2_A\rangle, \ldots, |n_A\rangle$, all correspond to different eigenvalues, denoted by $1_A, 2_A, \ldots, m_A$ respectively. As A is symmetric, there is a unique orthonormal basis of the relevant Hilbert space H formed with its eigenvectors. It is thus possible to represent the agent's state as a superposition of the vectors of this basis:

$$|\varphi\rangle = \sum_k \lambda_k |k_A\rangle,$$

where $\lambda_k \in R$, $\forall\ k \in \{1, \ldots, m\}$ and $\sum_k (\lambda_k)^2 = 1$.

According to the so-called reduction principle, the result of a measurement of A can only be one of its eigenvalues. If the result is m_A, that is, the player selects action m_A, the superposition $\sum_i \lambda_i |\, i_A\rangle$ "collapses" onto the eigenvector associated with the eigenvalue m_A. The probability that the measurement yields the result m_A is equal to $m_A|\varphi^2\rangle = {\lambda_m}^2$.[7] The coefficients λ_m, called "amplitudes of probability," play a key role when studying sequences of measurements.

[7] For simplicity I assume that all eigenvalues are "non-degenerated."

In our theory an agent is represented by a state. We shall also use the term "type" to denote a state degenerated to one eigenvector, say $|m_A\rangle$. An agent in this state is said to be of type m_A. An agent in a general state $|\varphi\rangle$ is hence a superposition of all types relevant to the DS under consideration. Our notion of type is closely related to the notion used by Harsanyi. A type captures all the agent's characteristics of relevance (taste, subjective beliefs) for uniquely predicting his or her choice in a given situation. In contrast to Harsanyi we shall not assume that there exists an exhaustive description of the agent that enables us to determine the agent's choice uniquely in all possible decision situations *simultaneously*. Instead, our types are characterized by an irreducible uncertainty that is revealed when the agent is confronted with a sequence of choices.

Remark Clearly, when only one DS is considered, the above description is equivalent to the traditional probabilistic representation of an agent by a probability vector $(\alpha_1, \ldots, \alpha_n)$ in which α_k is the probability that the agent will choose action k_A and $\alpha_k = \lambda_k{}^2$ for $k = 1, \ldots, n$. The advantage of the proposed formalism consists in enabling us to study several decision situations and the interaction between them.

3.3 More than One Decision Situation

When studying more than one DS, say A and B, it turns out that a key question is whether the corresponding observables are commuting operators in H, that is, whether $AB = BA$. The question of whether two DSs can be represented by two commuting operators is an empirical one.

When dealing with commuting observables, it is meaningful to speak of measuring them simultaneously. Whether we measure first A and then B or first B and then A, the probability distribution on the joint outcome $p\,(i_A \wedge j_B) = \sum_k \lambda_k{}^2$. So (i_A, j_B) is a well-defined event. Formally, this implies that the two DSs can be merged into a single DS. When we measure it, we obtain a vector as the outcome, that is, a value in A and a value in B. In particular, in accordance with the calculus of probability, we see that the conditional probability formula holds:

$$p_{AB}\,(i_A \wedge j_B) = p_A\,(i_A)\,p_B\,(j_B|i_A)\,.$$

For example, consider the following two decision problems. Let *A* be the DS of choosing between a week's vacation in Tunisia and a week's vacation in Italy. And let *B* be the choice between buying €1000 of shares in Bouygues Telecom or in Deutsche Telecom. It is quite plausible that *A* and *B* commute, but whether or not this is in fact the case is of course an empirical question. If *A* and *B* commute we expect a decision on portfolio (*B*) not to affect the decision-making regarding the location for vacation (*A*). And thus the order in which the decisions are made does not matter.

Remark The type space associated with type characteristics represented by commuting observables is equivalent to the Harsanyi type space. When all DSs commute, a type-indeterminate (TI) agent cannot be distinguished from a classical agent. In particular, if the DSs *A* and *B* together provide a full characterization of the agent, then all types $i_A j_B$ are mutually exclusive: knowing that the agent is of type $1_A 2_B$ it is certain that he or she is not of type $i_A j_B$ for $i \neq 1$ and/or $j \neq 2$. All uncertainty about the agent's choice behavior is due to our incomplete knowledge about his or her type, and it can be fully resolved by making a series of suitable measurements.

3.4 Noncommuting Decision Situations

It is when we consider DSs associated with observables that do not commute that the predictions of the TI model differ from those of the probabilistic model. In such a context, the quantum probability calculus, that is, Born's rule: $p\left(\langle i_A|\varphi\rangle\right) = \langle i_A|\varphi\rangle^2$, generates cross-terms, also called interference terms. These cross-terms are the signature of indeterminacy. In the next section we demonstrate how this feature of the TI model captures the phenomenon of cognitive dissonance (in our 2009 article we also provide a model of the framing).

For simplicity, assume that the two DSs *A* and *B* have the same number *n* of possible choices, which means that the observables *A* and *B* have (nondegenerated) eigenvalues $1_A, 2_A, \ldots, n_A$ and $1_B, 2_B, \ldots, n_B$ respectively and each one of the sets of eigenvectors $\{|1_A\rangle, |2_A\rangle, \ldots, |n_A\rangle\}$ and $\{|1_B\rangle, |2_B\rangle, \ldots, |n_B\rangle\}$ is an orthonormal basis of the relevant Hilbert space. Let $|\,\varphi\rangle$ be the initial state of the agent

$$|\varphi\rangle = \sum_i \lambda_i \,|i_A\rangle = \sum_j \gamma_j \,|j_B\rangle.$$

We note that since each set of eigenvectors of the respective observables forms a basis of the state space there exists a unitary operator S such that S is a basis transformation $n \times n$ matrix with elements $\langle j_B | i_A \rangle$. This matrix plays an important role in practical applications of the theory. For ease of presentation we write $\mu_{ij} = \langle j_B | i_A \rangle$ and we can write $|j_B\rangle$ as

$$|j_B\rangle = \sum \mu_{ij} \, |i_A\rangle \,,$$

implying

$$|\varphi\rangle = \sum_i \sum_j \gamma_j \mu_{ij} \, |i_A\rangle$$

If the agent plays A directly, he chooses i_A with probability $p_A\left(i_A\right) = \left(\sum_i \gamma_j \mu_{ij}\right)^2$. If he first plays B, he selects action j_B with probability ${\gamma_j}^2$ and his state is projected onto $|j_B\rangle$. The agent then selects action i_A in DS A with probability ${\mu_{ij}}^2$. So the (*ex ante*) probability for i_A is $p_{AB}\left(i_A\right) = \sum \gamma_j^2 {\mu_{ij}}^2$, which is in general different from $\left(\sum \mu_{ij}\right)^2$. Playing B first changes the way A is played. The difference stems from the so-called interference term which is the sum of cross-terms generated when taking the square of the sum.

This is illustrated in Fig. 1 for two DSs: the Dictator Game (DG)[8] with two options G for "Generous" and E for "Egoist," and the Ultimatum Game (UG)[9] with two options, as well, A for "Accept" and R for "Reject." We are interested in the probability for G in the DG. The dotted line corresponds to the probability for G in the direct play of the DG and the dash-dotted lines for the probability of G when the agent first plays the UG which changes his state into A or R and thereafter when he plays the DG and G obtains.

Some intuition about interference effects may be provided using the concept of "propensity" due to Popper (1992). Imagine an agent's mind as a system of propensities to act (associated with the different possible actions). As long as the agent is not required to choose an action in a given DS, the

[8]The Dictator Game is one in which one player chooses how to divide a pie and the other has no choice but to accept.

[9]The Ultimatum Game is one where one player chooses how to divide a pie and the other may either accept, in which case each player gets the share proposed by the first player, or the other player may refuse, in which case both get zero.

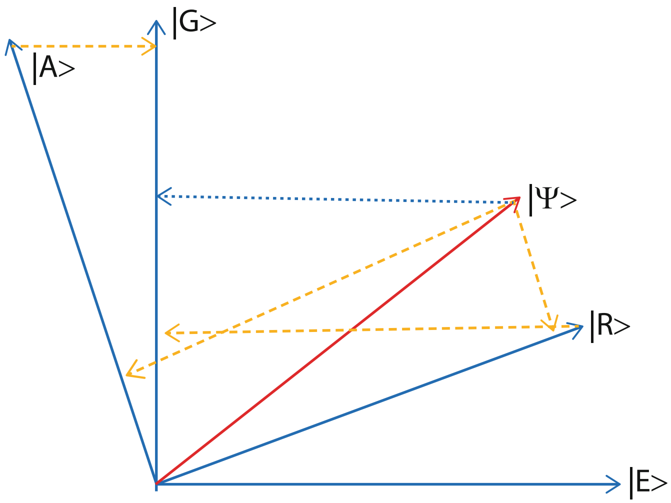

Fig. 1 Two non-compatible measurements

corresponding propensities coexist in his mind; the agent has not "made up his mind." A decision situation operates on this state of "hesitation" to trigger the emergence of a single type (of behavior). But as long as alternative propensities are present in the agent's mind, they affect choice behavior by increasing or decreasing the probability of the choices under investigation.

An illustration of this kind of situation may be supplied by the experiment reported in Knetz and Camerer (2000). The two DSs studied are the Weak Link (WL) game[10] and the prisoner's dilemma (PD) game. They compare the distribution of choices in the PD game when it is preceded by a WL game and when only the PD game is being played. Their results show that playing the WL game affects the play of individuals in the PD game. The authors appeal to an informational argument, which they call the "precedent effect."[11] However, they cannot explain the high rate of cooperation (37.5 %) in the last round of the PD game (Table 5, p. 206). Instead, I propose that the WL game and the PD game are two DSs that do not commute. In such a case we expect a

[10]The Weak Link Game is a type of coordination game where each player picks an action from a set of integers. The payoffs are defined in such a manner that each player wants to select the minimum of the other players but everyone wants that minimum to be as high as possible.

[11]The precedent effect hypothesis is as follows: "The shared experience of playing the efficient equilibrium in the WL game creates a precedent of efficient play strong enough to . . . lead to cooperation in a finitely repeated PD game" (Knetz and Camerer 2000, p. 206).

difference in the distributions of choices in the (last round of the) PD game depending on whether or not it was preceded by a play of the WL or another PD game.

Remark When A and B do not commute, they cannot have simultaneously determinate values: the state of the agent is characterized by irreducible uncertainty. Equivalently, and in contrast with the commuting case, two noncommuting observables cannot be merged into one observable. There is no probability distribution on the event "to have the value i_A for A and the value j_B for B." The conditional probability formula does not hold.

As G. W. Mackey expresses it: "When A and B do not commute there are limitations to the degree to which the probability distribution of the corresponding observables may be simultaneously concentrated near to single points" (Mackey 2004, p. 78). In our context these limitations can be interpreted as reflecting cognitive limitations. When the agent knows what he or she prefers in one situation, he or she cannot but feel hesitant in some other (noncommuting) DS.

4 Type Indeterminacy in Social Sciences: An Example of Application

A central feature of TI is that choices can alter the state of the agent, which may imply noncommutativity of choice behavior. Of course, not all instances of noncommutativity in decision theory call for Hilbert space modeling. For instance, in standard consumer theory, choices do have implications for future behavior, that is, when goods are substitutes or complements. The Hilbert space model of preferences is useful when we expect choice behavior to be consistent with the standard probabilistic model, because nothing justifies a modification of preferences. Yet, actual behavior contradicts those expectations. An example of application of the TI model that we next develop is the modeling of cognitive dissonance (CD).

The kind of phenomena I have in mind can be illustrated as follows. Numerous studies show that employees in risky industries (like nuclear plants) often neglect safety regulations. The puzzle is that before moving into the risky industry those employees were typically able to estimate the risks correctly. They were reasonably averse to risk and otherwise behaved in an ordinary rational manner. A TI model of CD defines the choice options as follows. Let A be a decision about jobs; a_1: take a job with a hazardous task (adventurous type), a_2: stay in a safe task (habit-prone type). Let B be a decision about behavior at the risky workplace; the choices are b_1: use safety equipment (risk-averse type), b_1: don't use safety equipment (risk-loving type). We assume A and B are noncommuting so their eigenvectors span the same space. We shall compare two scenarios.

First scenario. The hazardous task is introduced in an existing context. It is imposed on the workers. They are given only the choice to use or not to use safety equipment (B). We write the initial state of the worker as

$$|\varphi\rangle = \lambda_1 |a_1\rangle + \lambda_1 |a_2\rangle, \lambda_1{}^2 + \lambda_2{}^2 = 1.$$

We write the eigenvectors of A in terms of the eigenvectors of B:

$$\begin{aligned} |a_1\rangle &= \langle b_1 |a_1\rangle |b_1\rangle + \langle b_2 |a_1\rangle |b_2\rangle \\ |a_2\rangle &= \langle b_1 |a_1\rangle |b_1\rangle + \langle b_2 |a_2\rangle |b_2\rangle \end{aligned}$$

Substituting for $|a_1\rangle$ and $|a_2\rangle$, the probability that a worker chooses to use safety equipment is

$$\begin{aligned} p_B(b_1) &= \langle b_1 |\varphi\rangle^2 = [\lambda_1 \langle b_1 |a_1\rangle + \lambda_2 \langle b_1 |a_2\rangle]^2 \\ &= \lambda_1^2 \langle b_1 |a_1\rangle^2 + \lambda_2^2 \langle b_1 |a_2\rangle^2 + 2\lambda_1\lambda_2 \langle b_1 |a_1\rangle \langle b_1 |a_2\rangle . \end{aligned}$$

Second scenario. First A then B. The workers choose between taking a new job with a hazardous task or staying with the current safe routine. Those who choose the new job then face the choice between adopting safety measures or not. Those who turn down the new job offer are asked to answer a questionnaire about their choice in the hypothetical case where they are confronted with a risky task. The *ex ante* probability for observing b_1 is

$$\begin{aligned} p_{BA}(b_1) &= p_A(a_1) p_B(b_1|a_1) + p_A(a_2) p(b_1|a_2) \\ &= \lambda_1^2 \langle b_1 |a_1\rangle^2 + \lambda_2^2 \langle b_1 |a_2\rangle^2 \end{aligned}$$

The empirically documented phenomenon of "cognitive dissonance" can now be formulated as

$$P_{BA}(b_1) < p_B(b_1),$$

which occurs in our model when $2\lambda_1\lambda_2 \langle b_1|a_1\rangle \langle b_1|a_2\rangle$ is strictly positive.

The contribution of the indeterminacy approach is twofold. First, the TI model explains the appearance of "cognitive dissonance." Indeed, if coherence is such a basic need, as proposed by L. Festinger and his followers, why does dissonance occur in the first place? In the TI model "dissonance" arises when resolving indeterminacy in the first DS because of the cognitive limitations on possible preference types. Second, the TI model features a dynamic process such that the propensity to use safety measures is actually altered (reduced) as a consequence of the act of choice. This dynamic effect of coherence (which arises when choosing in the second DS) is reminiscent of the psychologists' "drive-like property of coherence" leading to a change in attitude.

5 Discussion 1

Our approach to decision-making yields the result that the type of agents, rather than being exogenously given, emerges as the outcome of the interaction between the agent and the DSs. This is modeled by letting a DS be represented by an operator (observable). Decision-making is modeled as a measurement process. When the observables commute, the corresponding type space has the properties of the Harsanyi type space. From a formal point of view, this reflects the fact that all (pure) types are then mutually exclusive. When the observables do not commute, the associated pure types are not all mutually exclusive. Instead, an agent who is in a pure state after the measurement of an observable will be in a different pure state after the measurement of another observable that is incompatible with the first one. As a consequence, the type space cannot be associated with a classical probability space and we obtain an irreducible uncertainty in behavior.

In the TI model, any type (state) corresponds to a probability measure on the type space which allows one to make predictions about the agent's behavior. It is in this sense that the TI model generalizes Harsanyi's approach to uncertainty. The more controversial feature of the TI model as a framework for describing human behavior is related to the modeling of the impact of measurement on the state, that is, how the type of agent changes with decision-making. The rules of change are captured in the geometry of the type space and in the projection postulate. It is more than justified to question whether this seemingly very specific process should have any relevance to the social sciences. It has been shown that the crucial property that gives all its structure to the process of change can be stated as a "minimal perturbation principle." The substantial content of that principle is that we require that when a coarse DS resolves some uncertainty about the type of agent, the remaining uncertainty is left unaffected. In behavioral terms, this can be expressed as follows. When confronted with the necessity to make a choice, the agent only "makes the effort" to select his or her preferred item, while leaving the order relationship between the other items uncertain, as at first. It may be argued that the minimal perturbation principle is quite demanding. But we do not expect the TI model to be a fully realistic description of human behavior. Rather, we propose it as an idealized model of agents characterized by the fact that their type changes with decision-making.

The next section briefly presents an extension of the TI model to dynamic optimization that reveals how conceptually fruitful the model can be.

6 Dynamic Optimization: A Theory of Self-Management

The idea that an individual's choice of action (behavior) determines his or her inner characteristics (preference, attitudes, and beliefs) rather than (exclusively) the other way around has been present in people's minds throughout history and has been addressed in philosophy, psychology, and more recently in economics. Nevertheless, the dominant view, particularly in economics, is based on a postulate: individuals are endowed with an identity (preferences, attitudes, and beliefs) that explain their behavior. This postulate is hard to reconcile with a host of experimental evidence. In an extension of the basic TI model Lambert-Mogiliansky and Busemeyer (2012) consider dynamic individual optimization and show that it provides a model where identity arises from decision-making and can explain patterns of self-control.

In a dynamic context, the TI model induces a game among potential incarnations of the individual. In each period, these potential incarnations represent conflicting desires or propensities to act. We formulate the decision problem in terms of a game between a multiplicity of (one-period lived) players, the selves. They are linked to each other through two channels: (1) the selves share a common interest in the utility of the future incarnations of the individual and (2) they are connected to each other in a process of state transition (which captures indeterminacy). In each period, the current selves form intentions to act. One action is played by the individual but the whole profile of (intended) actions matters to tomorrow's identity, because of the state transition process. This creates a strategic concern among contemporaneous selves. In particular when the selves pool, the individual's preferences are unchanged, while if they choose different actions, preferences are modified. We define a Markov perfect equilibrium among the selves where the state variable is the individual's identity. In our model, behavior affects future preferences (identity) and in particular a concern for identity (self-image) arises endogenously, because identity determines future expected utility. Choice behavior exhibits deviations from standard utility maximization. This is characterized by some degree of self-control: some selves may refrain from short-run gains (and pool with others) to secure a desirable identity. It can also feature dynamic inconsistency because, as preferences are modified, the choices made by the individual through time are not consistent with a stable preference order.

6.1 Basic Ingredients of the Model

In each period t, the individual faces a DS A^t corresponding to the finite set of available actions in period t. We restrict the one-period players' strategy set to pure actions. The possible preferences (or eigentype[12]) over the profiles of actions are denoted by $e_{M,i}$ where M defines the complete measurement corresponding to A^t. The eigentypes $e_{M,i}$ of M are associated with the eigenvectors $|e_{M,i}$ of the operator, which form a basis of the state space. The state vector representing the individual can therefore be expressed as a superposition:

$$| s^t\rangle = \sum_i \lambda_i |e_{M,i}\rangle$$

where $e_{M,i}$ are the (potential) selves relevant to DS A^t. This formulation means that the individual cannot generally be identified with a single true self. He or she does not have a single true preference. Instead he or she is intrinsically "conflicted," which is expressed by the multiplicity of potential selves.

Decision-making is modeled as the measurement of the preferences, and it is associated with a transition process from the initial state and (intended) actions to a new state. The rules that govern the state transition process reflect the intrinsic indeterminacy of the individual's type or preferences.

Formally, a transition process is a function from the initial state and (intended) actions to a new state. It can be decomposed into an outcome mapping and a transition mapping. The first mapping defines the probability for the possible choices of action when an individual in state s is confronted

[12] In this section the term "eigen type" refers to the specific preferences, while the term "type" is used as a synonym for a general state.

with DS A. The second mapping indicates where the state transits as we confront the individual with DS A and obtain outcome a.

Let the initial state be $|s^t\rangle = \sum_i \lambda_i^t |e_{M,i}\rangle$. The standard Hilbert space formulation yields that if we, for instance, observe action a_1, the state transits onto:

$$\left|s^{t+1}\right\rangle = \sum_{j=1} \lambda'_j \left|e_{M,j}\right\rangle \tag{1}$$

where $\lambda'_j = \frac{\lambda_j^t}{\sqrt{\sum_{k^t} (\lambda_k^t)^2 (s_k^* = a_j)}}$ and $\sum_{k^t} (\lambda_k^t)^2 (s_k^* = a_j)$ is the sum over the probabilities for the selves who pool in choosing a_j.

As in the basic TI model the distinction with the classical model appears when the individual faces a series of at least two consecutive noncommuting DSs, A^t and A^{t+1}. The probability for the second play depends on the selves' play with respect to pooling respectively separation in DS A^t. When no player chooses the same action, the choice of a^t separates out a single player (some $e_{M,i}^t$), while when several players pool in choosing the same action, the calculus of the probability involves the square of a sum. This expresses the fact that the player is a nonseparable system with respect to A^t and A^{t+1}.

When dealing with multiple selves, the issue as to how to relate the utility of the selves (here the players) to that of the individual is not self-given. The problem is reminiscent of the aggregation of individual preferences into a social value. We adopt the following definition of the utility of self (or player) $e_{M,i}^t$ of playing $a_i{}^t$ and all the other $-i$ t-period players playing $a_{-\mathrm{i}}{}^t$

$$U_{e_{M,i}}\left(a_i^t; s^t\right) + \delta_{e_{M,i}} \sum_{k=t}^{T} EU\left(s^{k+1}\right)\left[a_i^t, a_{-i}^t; s^t\left(a^t = a_i^t\right)\right]$$

The utility for $e_{M,i}^t$ of playing a_i^t is made of two terms. The first term is the utility in the current period evaluated by self $e_{M,i}^t$. This term only depends on the action chosen by $e_{M,i}^t$. And the second term is the expected utility of the individual evaluated by the future selves. Self $e_{M,i}^t$ puts some weight $\delta_{e_{M,i}}$ on that expected utility. The second term depends indirectly on the whole profile of action in the current period through the state transition process $s^{t+1}(a_i{}^t; s^t)$.[13]

[13]The utility function may recall a Bernoulli function in the following sense. With some probability, the self survives (his preferred action is played by the individual); and with the complementary probability, he is "out of the game." The formulation in Eq. (4) means that he maximizes utility conditional on

In each period, the current selves move simultaneously. They know the current state resulting from the previous (actual and intended) play. We have common knowledge among the selves about the payoff functions of all current and future selves and common knowledge of rationality. The selves' payoffs are functions of the current actions and the current state as defined in the previous section. Together, this means that we are dealing with a separable dynamic game of complete information and that it seems most appropriate to restrict ourselves to Markov strategies: a strategy for a self is a function from the current state to the set of actions available at period t. We accordingly focus on Markov perfect equilibria.

Definition A Markov perfect equilibrium of the game is characterized by $a_i^{t^*}$

$$a_i^{t*} = \text{argmax}_{a_{i \in A^t}} U_{e_{M,i}} \left(a_i^t; s^t\right) = \delta_{e_{M,i}} \sum_{\tau=t+1}^{T} EU^* \left[s^\tau \left(a^t; s^t\right)\right]$$

In all periods $t = 1, \ldots, T$ and for all $e_{M,i}, M, \ i = 1, \ldots, n$.

If all DSs commute, the state variable evolves through Bayesian updating. The individual eventually learns who he or she is and behaves as a classical decision-maker who maximizes discounted expected utility. In the TI model, the concern for identity arises exclusively as a consequence of the noncommutativity of successive DSs.

When the action set is sufficiently rich so as to sort out fully the preferences, all selves will have "conflicting" preferences with respect to the short-run choice. If the MPE (Markovian Perfect Equilibrium) is characterized by pooling, some selves must be exercising self-restrain: they refrain from immediate reward for the sake of the individual's future utility—this is an instance of self-control. Generally, i.e., in a standard DS, the set of actions is limited relative to the possible preferences (because a DS is generally a coarse measurement). In that case we will talk about self control when selves with conflicting interest (*begin*) with respect to the current DS (*end*) pool.

In Busemeyer and Lambert-Mogiliansky 2012 we show how the MPE can be used to define and characterize generic classes of behavior corresponding

surviving. The probability for survival depends on the initial coefficients of superposition and his own and other selves' choices. But the selves do not take that into account. The approach is justified on the following grounds: being "out of the game" cannot be valued. The self ceases to "exist," which is neither good nor bad. In other words, there is no reason to assume that selves have a "survival instinct"; they are simply mental constructs. A self is defined as rational when he maximizes his conditional utility, which is well-defined for any sequence of DSs.

to a balanced and a conflicted individual respectively. I also derive some comparative statics on the likelihood of the conflicting behavior with respect to among others the size and sign of the interference effect.

7 Discussion 2

Self-perception theory is based on two postulates: (1) "individuals come to 'know' their own attitude and other internal states partially by inferring them from observations of their own behavior and/or the circumstances in which behavior occurs; (2) thus the individual is functionally in the same position as an outside observer, an observer who must necessarily rely upon those same external cues to infer the individual inner state (Bem 1972, p. 2)." Self-perception theory's own postulates are fully consistent with the hypothesis of (nonclassical) indeterminacy which overturns the classical postulate of pre-existing identity, attitudes, and preferences. With indeterminacy of the inner state, behavior (the action chosen in a decision situation) shapes the state of preferences/attitudes through a state transition process (see next section). Indeterminacy means intrinsic uncertainty about individual identity so the individual may not know his or her own attitudes, preferences, and beliefs. And as in self-perception theory, it is by observing his own action that he infers (learns) his state (of beliefs and preferences). In self-perception theory, inner states are not accessible without some instruments to measure them. Such a "descriptor" includes "cues" that can be manipulated to obtain widely different perceptions (in an experiment a feeling could be identified as anger or euphoria depending on the question asked). This is consistent with the most basic feature of indeterminacy, namely that the property of a system does not pre-exist observation. Therefore different measuring instruments may give various incompatible but equally true accounts of the same state.

I also argue that the three basic departures from the classical model in Benabou and Tirole, that is, imperfect knowledge, recall, and willpower, are in many respects equivalent to giving up the classical dogma of a pre-existing (deterministic) individual identity and replacing it by indeterminacy. Indeterminacy implies imperfect knowledge because of intrinsic uncertainty: there is no set of "true preferences" (to be learned). Instead, an individual is represented by a superposition of potential types. Indeterminacy implies imperfect recall because no type is the true type forever. The (preference) state keeps changing with the action taken, so yesterday's correctly inferred information about oneself may simply not be valid tomorrow. Indeterminacy implies "imperfect willpower" because it involves selves that are multiple both simultaneously (multiplicity of potentials) and dynamically (by force of the noncommutativity of DSs). Therefore, there are necessarily conflicting desires and issues of self-control and self-monitoring. Moreover, in a world of indeterminate agents, actions aimed at shaping one's identity are fully justified from an instrumental point of view (it determines future expected utility). In particular there is no need to add any additional concerns for self-image (as in Benabou and Tirole) or diagnostic utility (Bodner and Prelec, 2003). The TI model provides a simple and rigorous setting that relies on one single departure from the standard setting. Some of our comparative statics results are similar to those in Benabou and Tirole and consistent with a host of empirical data. My contribution proposes an alternative explanation in terms of a fundamental characteristic of the mind: its intrinsic indeterminacy.

In our approach we formalize internal conflicts and explain features of self-management without using time preferences, which were the almost exclusive focus of earlier work on dynamic inconsistency. Moreover, we can connect to another branch of research related to identity and self-image, extensively investigated in psychology (especially in self-perception theory) and more recently in economics, for example, Benabou and Tirole (2002), whose predictions are delivered by the model, some of which are novel. I characterize generic classes of personality/behavior: a balanced, weakly decisive but behaviorally stable character, and a highly conflicted, strongly decisive, and behaviorally unstable character.

8 Concluding Remarks and Challenges

The TI model introduces quantum-like indeterminacy in decision theory. The basic idea is to view DSs as experimental set-ups that measure preferences. Two measurements can be incompatible in the sense that they measure Bohr complementary type characteristics: the agent cannot have a determinate value in both simultaneously. A major implication is that preferences are modified along with the choices made.

This approach has shown itself to be fruitful in formalizing the constructive approach to preferences suggested by Kahnemann and Tversky. In the TI model preferences are not merely elicited in choices, they are not read off some master list. Instead they are created in the process of elicitation (decision-making). Different ways of eliciting those preferences leads to different (actualized) preferences. The TI model can explain a variety of behavioral anomalies including among others cognitive dissonance, framing effects, or preferences reversal. A significant advantage is that it offers a unified framework for explaining phenomena that are currently explained by a variety of theories. Our approach appeals to a fundamental characteristic of preferences: indeterminacy. By so doing it captures and connects two central themes in behavioral economics, namely contextuality and bounded rationality, in the following sense. TI agents are not capable of comparing all choice options simultaneously: different choice sets correspond to different DSs which may not commute. As a consequence the "revealed" preferences are contextual to the choice set.

The extension of the basic TI model to dynamic decision-making shows its potential in addressing issues like concerns for identity, time inconsistency, and self-control. It also provides a model for internal conflict between selves which complements the traditional multiple self models.

The multiple-self model arising from optimal dynamic decision-making with a TI agent shows the way forward for the development of a theory of games with TI players (see Martinez- Martinez and Lambert-Mogiliansky 2014). The importance of this development is profound and very much in line with the spirit of both game theory and modern physics. The point is to make interactions the central building stone of the theory while the agents arise endogenously. To quote Mermin (1998) in his article titled "What is Quantum Mechanics Trying to Tell Us": "correlations have physical reality, that to which they correlate does not—the rest is commentary." Similarly, I would like to contend that the TI model proposes that "interactions have social reality, those who interact do not." The challenges linked to developing a theory of games with TI players are numerous; some are conceptual, others are technical. The TI model invites a revolution in the way of thinking about behavior and social interaction that is expected to bring many surprising insights.

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Quantum Models of Human Causal Reasoning

Jennifer S. Trueblood and Percy K. Mistry

How do people reason about causes and effects? If you wake up in the morning with a stomach ache, how do you infer that it was the seafood you had for dinner rather than stress that caused your stomach to hurt? Human causal reasoning has intrigued scholars as far back as Hume and Kant and currently involves researchers from a variety of fields including cognitive science, developmental psychology, and philosophy. Many researchers approach this topic by developing models that can explain the processes by which people reason about causes and effects. In this chapter, we review these modeling approaches and comment on their strengths and weaknesses. We then introduce a new approach based on quantum probability theory.

1 Classical Probability Models of Causal Reasoning

Some of the first models of causal reasoning were centered around the idea that people use the covariation between causes and effects as a basis for causal judgments (Jenkins and Ward 1965; Kelley 1973). These approaches trace

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their roots back to Hume (1987) and are based on the idea that causation is inferred from the constant conjunction of events as perceived by our sensory system. While models based on covariation can account for many situations, they all face the same ultimate problem—covariation does not necessarily imply causation. As such, these models cannot account for situations where covariational relations are not perceived as causal. For example, we would never think that ice cream consumption causes shark attacks even though shark attacks increase at the same time as ice cream sales (because both increase during summer). To overcome this issue, Cheng (1997) and Novick and Cheng (2004) combined covariational information with domain specific prior knowledge to create the *power PC theory*. According to this theory, reasoners infer causal relations in order to understand observable regularities between events. The model can explain why covariation sometimes reveals causation but other times does not. While power PC theory has been able to account for many behavioral findings, some studies have shown that people's causal judgments deviate significantly from the predictions made by the model (Lober and Shanks 2000; White 2005).

Another approach to modeling causal reasoning uses causal graphical models (CGMs), which represent causal relationships using Bayes' calculus (Kim and Pearl 1983; Pearl 1988). CGMs are quite successful at explaining and predicting causal judgments, and their predictions are generally accepted as normative. CGMs can account for causal inferences driven by intervention-based, observational, and counterfactual processes (Hagmayer et al. 2007), which are often difficult to discriminate in traditional probabilistic models. They have also been used to explain causal learning, where people learn the relationship between variables through observation or intervention (Griffiths and Tenenbaum 2005, 2009). Some researchers have even combined different probabilistic approaches by integrating power PC theory with CGMs (Griffiths and Tenenbaum 2005; Lu et al. 2008). Beyond causal reasoning and learning, CGMs have been applied to decision-making (Hagmayer and Sloman 2009), classification (Rehder and Kim 2009, 2010), and structured knowledge (Kemp and Tenenbaum 2009).

While CGMs have been quite successful in accounting for human causal reasoning, several recent empirical studies have reported violations of the predictions of these models. All CGMs must obey a condition called the local Markov property, which states that if we know about all the possible causes of some event Z, then the descendants (i.e., effects) of Z may give us information about Z, but the nondescendants (i.e., noneffects) cannot give us any more information about Z. Recently, several studies have provided evidence that people's causal inferences often violate the local Markov condition (Rottman and Hastie 2014; Park and Sloman 2013; Rehder 2014; Fernbach and Sloman

2009; Waldmann et al. 2008; Hagmayer and Waldmann 2002). Relatedly, other studies have shown people often ignore relevant variables. For example, Fernbach et al. (2010) found that people ignore alternative causes in predictive causal reasoning (i.e., reasoning about an effect given information about causes), but not in diagnostic causal reasoning (i.e., reasoning about causes given information about the effect).

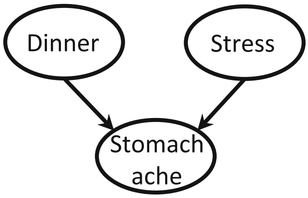
To overcome the issues mentioned above, CGMs are often elaborated through the inclusion of hidden variables (i.e., latent variables that are not explicitly part of the causal system being studied, but are added to the mental reconstruction of the causal system by individuals as part of their reasoning process). While these elaborated CGMs often provide good accounts of data (Rehder 2014), they are difficult to conclusively test. Further, the inclusion of hidden variables is typically post hoc, added when a basic CGM fails to capture data. As an alternative approach, we suggest expanding the set of probabilistic rules of basic CGMs by using quantum probability theory (Trueblood and Pothos 2014). Our approach can be considered as a generalization of Bayesian causal networks. The essential idea is that any CGM can be generalized to a quantum Bayes net by replacing the probabilities in the classic model with probability amplitudes in the quantum model (Tucci 1995; Busemeyer and Bruza 2012). In the next sections, we review CGMs in more detail and introduce quantum Bayes nets as generalizations of these models.

2 Causal Graphical Models

CGMs describe causal relationships as directed acyclic graphs (DAGs) representing a set of random variables and their conditional dependencies. For example, suppose that either an unusual dinner or the presence of stress can cause your stomach to hurt. In this example, the three variables—stomach ache, dinner, and stress—are represented as nodes in the DAG (Fig. 1). Edges between the nodes represent conditional dependencies. In Fig. 1, edges connect dinner and stomach ache as well as stress and stomach ache. Nodes that are not connected by an edge are conditionally independent. In our example, dinner and stress are conditionally independent and thus there is no edge connecting them.

The probability of a node taking a particular value is determined by a probability function that takes as input the values of any parent nodes. These probabilities are specified in conditional probability tables. Consider the stomach scenario where all three variables have two possible values: $A =$ stomach ache is present (true/false), $D =$ dinner is unusual (true/false), $S =$ stress is present (true/false). The CGM can answer questions such as

Dinner	
t	f
.1	.9



Stress	
t	f
.6	.4

		Stomach ache	
Dinner	Stress	t	f
t	t	.8	.2
t	f	.7	.3
f	t	.2	.8
f	f	.1	.9

Fig. 1 A CGM of the stomach ache scenario. There are two possible causes of a stomach ache—an unusual dinner or stress. The three variables are represented as a DAG with conditional probability tables

"What is the probability that dinner was unusual, given that your stomach aches?" by using the formula for conditional probability:

$$p(D = t|A = t) = \frac{p(A = t, D = t)}{p(A = t)} = \frac{\sum_{j\in\{t,f\}} p(A = t, D = t, S = j)}{\sum_{i,j\in\{t,f\}} p(A = t, D = i, S = j)} \tag{1}$$

where the joint probability function $p(A = t, D = i, S = j) = p(A = t|D = i, S = j)p(D = i)p(S = j)$ because D and S are conditionally independent. We can now calculate the desired probability $p(D = t|A = t)$ using the conditional probability tables in Fig. 1:

$$\begin{aligned} &p(D = t|A = t) \\ &= \frac{(0.8 \times 0.1 \times 0.6) + (0.7 \times 0.1 \times 0.4)}{(0.8 \times 0.1 \times 0.6) + (0.7 \times 0.1 \times 0.4) + (0.2 \times 0.9 \times 0.6) + (0.1 \times 0.9 \times 0.4)} \\ &\approx 0.346 \end{aligned} \tag{2}$$

The probabilities of other combinations of variables (e.g., $p(D = f|A = f)$) follow similar calculations.

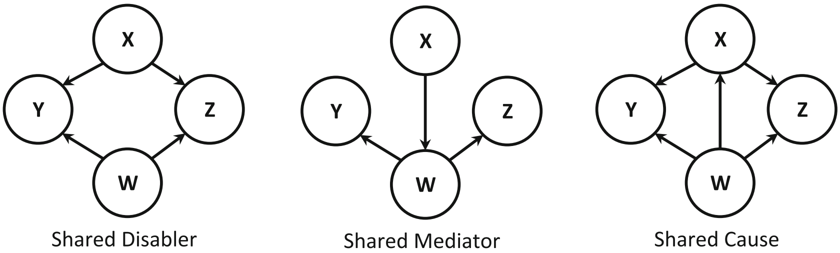


Fig. 2 Three different ways to elaborate a common cause structure with an additional variable W

All CGMs obey the local Markov property, which states that any node in a Bayesian network is conditionally independent of its nondescendants (i.e., noneffects) given its parents (i.e., direct causes). Consider a situation where a variable X causes Y and Z (represented by the DAG: $Y \leftarrow X \rightarrow Z$). The local Markov property implies that if you know X, then Y provides no additional information about the value of Z. Mathematically, we have $p(Z|X) = p(Z|X, Y)$. There is empirical evidence that people's causal judgments do not always obey the local Markov property. For example, Rehder (2014) presented participants with causal scenarios involving three variables (e.g., an economic scenario with variables describing interest rates, trade deficits, and retirement savings) and asked them to infer the value of an unknown target variable given information about one or two of the remaining variables. Rehder found that information about nondescendants influenced judgments even when the values of the parent nodes (i.e., direct causes) were known, showing a direct violation of the local Markov property. (We discuss this experiment in more detail in a later section.)

In order to account for the observed violations of the local Markov property, Rehder (2014) augmented CGMs by including an additional variable that severed as either a shared disabler, shared mediator, or shared cause. For example, in a common cause structure where X causes Y and Z (i.e., $Y \leftarrow X \rightarrow Z$), the structure can be elaborated in several different ways by the inclusion of a fourth variable W as shown in Fig. 2. While such an approach can provide a good account of the data, it is difficult to conclusively test because participants are never questioned about the hidden variable W. As an alternative approach, we propose generalizing CGMs to quantum Bayes nets.

3 Quantum Bayes Nets

In our quantum Bayes nets, we replace the classical probabilities in the conditional probability tables of a CGM with quantum probability amplitudes as proposed by Tucci (2012, 1995) and Moreira and Wichert (2014). Consider the situation where there are two causally related variables X and Y such that $X \rightarrow Y$. Further assume that these two variables are binary (true/false). In quantum probability theory, the observables X and Y are represented by Hermitian operators:

$$X = x_t P_{x_t} + x_f P_{x_f} \tag{3a}$$

$$Y = y_t Q_{y_t} + x_f Q_{y_f} \tag{3b}$$

where x_i and y_i are eigenvalues and P_{x_i} and Q_{y_i} are projectors onto corresponding eigen-subspaces. The probability of a concrete value, such as x_t (we use the notation x_t as shorthand for $X = t$), is given by Born's rule:

$$p_\rho(x_t) = \langle P_{x_t}\psi|\psi\rangle = ||P_{x_t}\psi||^2 \tag{4}$$

where ψ is a pure state and $\rho = |\psi\rangle\langle\psi|$ is the corresponding density operator. Suppose we want to answer the question "What is the probability Y is false, given that X is true?" In this situation, we first calculate the output state ρ_{x_t} as defined by the projection postulate (see the chapter *A Brief Introduction to Quantum Formalism*) and then apply Born's rule:

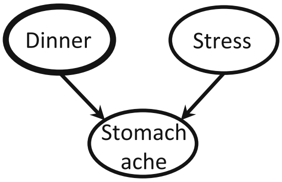
$$p_\rho(Y = f|X = t) = p_{\rho_{x_t}}(y_f) = \langle Q_{y_f}\psi_{x_t}|\psi_{x_t}\rangle = ||Q_{y_f}\psi_{x_t}||^2. \tag{5}$$

We then use these conditional probabilities for our network rather than the classical ones used in a CGM.

Consider the stomach ache scenario again. In the classical model, in order to answer the question "What is the probability that dinner was unusual, given that your stomach aches?," we needed to calculate joint probabilities such as $p(A = t, D = i, S = j)$. We can determine these probabilities from the conditional probability tables of the CGM by writing $p(A = t, D = i, S = j) = p(A = t|D = i, S = j)p(D = i)p(S = j)$. We take a similar approach in our quantum Bayes net. First, let the three observables, stomach ache, dinner, and stress, be represented by Hermitian operators A, D, and S with the respective projectors P, Q, and R. Now, we define joint probabilities by Born's rule:

$$p_\rho(A = t, D = i, S = j) = p_{\rho_{d_i,s_j}}(a_t)p_\rho(d_i)p_\rho(s_j) = ||P_{a_t}\psi_{d_i,s_j}||^2||Q_{d_i}\psi||^2||R_{s_j}\psi||^2 \tag{6}$$

Dinner	
t	f
$.3162e^{i\theta dt}$	$.9487e^{i\theta df}$



Stress	
t	f
$.7746e^{i\theta st}$	$.6325e^{i\theta sf}$

		Stomach ache	
Dinner	Stress	t	f
t	t	$.8944e^{i\theta at\mid dt,st}$	$.4472e^{i\theta af\mid dt,st}$
t	f	$.8367e^{i\theta at\mid dt,sf}$	$.5477e^{i\theta af\mid dt,sf}$
f	t	$.4472e^{i\theta at\mid df,st}$	$.8944e^{i\theta af\mid df,st}$
f	f	$.3162e^{i\theta at\mid df,sf}$	$.9487e^{i\theta af\mid df,sf}$

Fig. 3 A quantum Bayes net generalization of the stomach ache scenario. The dinner node in the DAG has a thick border to indicate that it is considered before stress. The tables contain probability amplitudes rather than probabilities. These amplitudes were determined from the CGM shown in Fig. 1

where the output state is given by

$$\psi_{d_i,s_j} = \frac{R_{s_j} Q_{d_i} \psi}{||R_{s_j} Q_{d_i} \psi||}. \tag{7}$$

If the observables D and S do not commute, then the output state will depend on the order in which these two variables are considered so that $\psi_{d_i,s_j} \neq \psi_{s_j,d_i}$. As a consequence, $p(A = t|D = i, S = j) \neq p(A = t|S = j, D = i)$.

Figure 3 shows a quantum Bayes net generalization of the stomach ache CGM shown in Fig. 1. For this example, the probabilities in the CGM have been replaced by probability amplitudes in the quantum Bayes net. These amplitudes are related to classical probabilities by taking the squared magnitude of the amplitudes. For example, the probability that dinner was unusual is given by

$$\begin{aligned} p(D = t) &= ||0.3162e^{i\theta dt}||^2 = (0.3162e^{i\theta dt})\overline{(0.3162e^{i\theta dt})} \\ &= (0.3162e^{i\theta dt})(0.3162e^{-i\theta dt}) \\ &= (0.3162)^2 e^{i(\theta dt - \theta dt)} = 0.1 \end{aligned} \tag{8}$$

which is the same as the classical probability in the CGM. Note that the term $e^{i\theta dt}$ is simply the phase of the amplitude.

When determining the conditional probabilities of the stomach ache given information about dinner and stress, the order in which dinner and stress are considered matters. In the quantum Bayes net in Fig. 3, we assume that information about dinner is always processed before information about stress. Psychologically, we would say that an individual thinks about dinner and stress separately, always starting with dinner. In the figure, we used a thick border on the dinner node to indicate that this variable is processed first. If we wish to switch the order and have stress processed before dinner, then we would need to define a different set of conditional probabilities. In other words, we have two different conditional probability tables describing the probability of the stomach ache given information about dinner and stress—one table describing the situation where dinner is considered before stress (as shown in Fig. 3) and another table describing the situation where stress is considered before dinner (not shown in the figure). Even though there are two different conditional probability tables for the quantum version of the stomach ache scenario, these tables are related to one another. In quantum probability theory, noncommutative observables (such as dinner and stress) are related by a unitary transformation, which preserves lengths (the state vector must have length equal to one) and inner products.

For the stomach ache scenario, we started with a CGM and generalized this to a quantum Bayes net by designating a processing order (dinner before stress) and changing the classical probabilities into probability amplitudes. Note that our decision that dinner should be processed before stress was arbitrary. We could have easily specified the reverse order (stress processed before dinner). Thus, there are at least two different ways to generalize the CGM in this example. In general, there will often be multiple ways to generalize a CGM to a quantum Bayes net. As a consequence, if we start with a quantum Bayes net, it is not necessarily the case that we can derive a well-defined CGM. The conditional probability tables of a quantum Bayes net will always have classical probability analogs, which are derived by squaring the probability amplitudes in the quantum tables. However, when a quantum Bayes net involves noncommutative observables, the corresponding CGM is ill-defined. This is because noncommutative observables result in different conditional probabilities tables for the same causal situation. This is not allowed in a traditional CGM. Thus, the behavior of a quantum Bayes net will often be fundamentally different than the behavior of a CGM.

4 Implications

Noncommutative quantum Bayes nets make several interesting predictions about human behavior. In the next sections, we discuss these predictions and supporting empirical evidence.

4.1 Order Effects

Quantum Bayes nets with noncommutating observables naturally predict order effects. Consider a causal scenario where X and Y cause Z (represented by the DAG: $X \rightarrow Z \leftarrow Y$). In an experiment, participants might be asked to judge $p(Z|X, Y)$ where information about X precedes information about Y. An order effect occurs when final judgments depend on the sequence of information so that $p(Z|X, Y) \neq p(Z|Y, X)$. Classical probability models such as CGMs have difficulty accounting for order effects due to the commutative property because $p(X, Y|Z) = p(Y, X|Z)$ implies $p(Z|X, Y) = p(Z|Y, X)$ by Bayes' rule. To account for order effects, classical probability models need to introduce extra events such as O_1, so that X is presented before Y, and such as O_2, so that Y is presented before Z. Then, it is possible that $p(Z|X, Y, O_1) \neq p(Z|X, Y, O_2)$. However, without a theory about O_1 and O_2, this approach simply redescribes the empirical result. Further, in many empirical studies of order effects, the order of presentation is randomly determined so that order information such as O_1 and O_2 is irrelevant.

A large number of empirical studies have shown that order of information plays a crucial role in human judgments (Hogarth and Einhorn 1992). Order effects arise in a number of different situations, ranging from judging the guilt of a defendant in a mock trial (Furnham 1986; Walker et al. 1972) to judging the likelihood of selecting balls from urns (Shanteau 1970). Recently, Trueblood and Busemeyer (2011) found evidence for order effects in causal reasoning. In this experiment, participants made causal judgments about ten different scenarios where there was a single effect and two binary (present/absent) causes. For example, in one scenario, participants were asked about the likelihood of a fictitious person, Mary, losing weight over the next month (the effect) given that she did not make any changes to her diet (absent cause) and began an exercise program (present cause).

The participants (N = 113) provided likelihood judgments of the effect (e.g., Mary losing weight) on a 0–100 scale at three different times: (1) before reading either cause, (2) after reading one of the causes, and (3) after reading the remaining cause. Participants judged the present cause before the absent cause for a random half of the scenarios. The order of the causes was reversed (i.e., absent followed by present) for the other half of the scenarios. The results of the experiment showed a large, significant order effect ($p < 0.001$) across the ten scenarios. The presence of order effects in causal judgments provides support for quantum Bayes nets with noncommutating observables.

4.2 Violations of the Local Markov Condition

The local Markov condition of CGMs stipulates that any node in a DAG is conditionally independent of its nondescendants when its direct causes are known. For example, in the common effect structure $X \rightarrow Z \leftarrow Y$, this property implies that the two causes X and Y are conditionally independent. In other words, if X and Y are binary, then $p(Y = i|X = t) = p(Y = i|X = f)$ for $i \in \{t, f\}$ and similarly when X and Y are swapped. In a quantum Bayes net where X and Y do not commute, there is a natural dependency between these two variables. That is, knowing the value of X influences our beliefs about Y. This dependency can result in violations of the local Markov condition so that $p(Y = i|X = t) \neq p(Y = i|X = f)$. By the definition of conditional probability, $p(Y = i|X = j) = p(Y = i, X = j)/p(X = j)$. In a CGM, X and Y are independent so that the joint probability $p(Y = i, X = j) = p(Y = i)p(X = j)$. Thus, $p(Y = i|X = j) = p(Y = i)$ for all i, j. In a quantum Bayes net, $p(Y = i, X = j) = ||Q_{y_i} P_{x_j} \psi||^2$. If X and Y do not commute, then it is clearly the case that $||Q_{y_i} P_{x_j} \psi||^2 \neq ||Q_{y_i} \psi||^2 ||P_{x_j} \psi||^2$, leading to violations of the local Markov condition.

Rehder (2014) empirically demonstrated that people often violate the local Markov condition in their causal judgments. In his task, participants were given two causal situations with an unknown target variable and were asked to select the situation where the target variable was more probable. For example, in the common effect structure $X \rightarrow Z \leftarrow Y$, participants had to decide whether the target variable Y was more likely be true in a situation where $X = t$ or in a situation where $X = f$. According to CGMs, we expect participants' choice proportions to be equal on average because $p(Y = t|X = t) = p(Y = t|X = f)$. However, Rehder (2014) found that on average people selected the causal situation where $X = t$ more often than the one where $X = f$, suggesting people judged $p(Y = t|X = t) > p(Y = t|X = f)$. He also showed similar violations with other causal structures such as chain structures ($X \rightarrow Y \rightarrow Z$) and common cause structures ($Y \leftarrow X \rightarrow Z$).

4.3 Anti-discounting Behavior

Noncommutating observables can also account for anti-discounting behavior in causal reasoning. The term discounting refers to the situation where one cause casts doubts on another cause. In the common effect structure $X \rightarrow Z \leftarrow Y$, knowing the value of X could cast doubt on the value of Y such that $p(Y|Z, X) < p(Y|Z)$. In many causal scenarios, discounting is considered normatively correct (Morris and Larrick 1995). For example, it is normatively correct to judge $p(Y = t|Z = t) > p(Y = t|Z = t, X = t)$ because knowing $X = t$ sufficiently explains the value of the effect $Z = t$ and consequently renders the other cause Y redundant. When X is unknown, as in $p(Y = t|Z = t)$, there is a greater chance the effect was brought about by Y.

Rehder (2014) found that many people display anti-discounting behavior. That is, people judge an unknown target cause Y as highly likely to be based on the presence of the alternative cause $X = t$, resulting in judgments where $p(Y = t|Z = t) < p(Y = t|Z = t, X = t)$. Similar to violations of the local Markov property, quantum Bayes nets can explain anti-discounting behavior by the noncommutativity of X and Y, which produces a causal dependency between these variables.

4.4 Reciprocity

The term reciprocity describes the situation where a person judges the probability of one variable given another to be the same as the probability when the variables are swapped, $p(X|Y) = p(Y|X)$. This phenomenon is similar to the inverse fallacy (Koehler 1996; Villejoubert and Mandel 2002) where people equate posterior and likelihood probabilities. If H represents a hypothesis and D represents data, the inverse fallacy occurs when $p(H|D) = p(D|H)$, where the first term is the posterior and the second term is the likelihood. The inverse fallacy has been observed in a number of different medical judgment problems, where clinicians are asked to judge the likelihood of a disease based on a set of symptoms (Meehl and Rosen 1955; Hammerton 1973; Liu 1975; Eddy 1982). The fallacy has also been demonstrated in the famous *taxicab problem* (Kahneman and Tversky 1972), where individuals are asked to judge the likelihood that a cab was in a wreck given its color (blue or green). Results of this experiment showed that most people judged $p(H|D)$ as $p(D|H)$.

The *law of reciprocity* (Peres 1998) in quantum probability theory simulates that if two events X and Y are represented by single dimensional

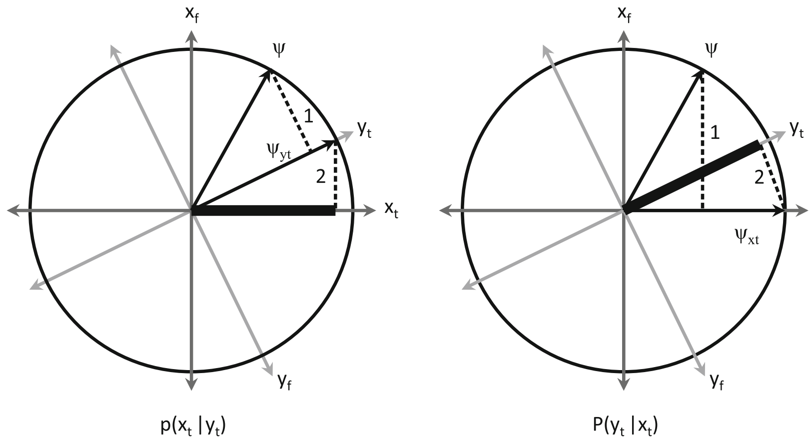


Fig. 4 The *law of reciprocity* in quantum probability theory. In the *left panel*, $p(x_t|y_t)$ is calculated by a series of two projections. First, ψ is projected onto the y_t subspace (labelled projection 1) and then normalized to yield the output state ψ_{y_t}. Next, the output state is projected onto the x_t subspace (labelled projection 2), and the probability is calculated by squaring the length of the projection (represented by the thick black line). In the *right panel*, the series of projections is reversed to calculate $p(y_t|x_t)$

subspaces, then $p(X|Y)$ is equivalent to $p(Y|X)$. We illustrate this result in Fig. 4 for the probabilities $p(x_t|y_t)$ and $p(y_t|x_t)$. In the left panel, we calculate $p(x_t|y_t)$ by first projecting the state ψ onto the y_t subspace and then normalizing to produce the output state ψ_{y_t}. This new state is then projected onto the x_t subspace and the conditional probability is the length of this projection squared as defined by Born's rule (represented by the thick black bar in the figure). In the right panel, we calculate $p(y_t|x_t)$ following a similar procedure. First, we project the state onto the x_t subspace and then normalize to produce the output state ψ_{x_t}. This revised state is then projected onto the y_t subspace and the conditional probability is the length of the projection squared. As shown in the figure, the two conditional probabilities are the same (i.e., the thick black bars are the same length). Note that not all quantum models can account for reciprocity and the inverse fallacy. Only quantum models that make the specific assumption that different outcomes are represented by single dimensional subspaces can explain these findings.

5 Conclusions

One could argue that CGMs have been one of the most successful approaches in modeling human causal reasoning. These models can account for casual deductive and inductive reasoning in a large number of situations. Besides causal reasoning, CGMs have been applied to a variety of other domains including classification (Rehder 2003; Rehder and Kim 2009, 2010) and decision-making (Hagmayer and Sloman 2009).

However, there has been recent evidence that people's judgments often deviate from the rules of CGMs. There are at least two possible ways to modify CGMs in order to account for these findings. One method involves elaborating CGMs through the inclusion of additional nodes and edges in the network. These hidden variables provide flexibility to the models and help them accommodate a wider range of human behavior. However, the addition of hidden variables to a CGM is often ad hoc and these additional variables are difficult to test conclusively.

In this chapter, we have suggested an alternative approach using quantum probability theory. Instead of elaborating a CGM with extra nodes and edges, we suggest changing the probabilistic rules used to perform inference. In our approach, we replace the classical probabilities of a CGM with quantum ones to yield quantum Bayes nets. By using quantum probabilities, we allow for variables to be noncommutative. We show that quantum Bayes nets with noncommutative observables can account for a variety of different behavioral phenomena including order effects, violations of the local Markov condition, anti-discounting behavior, and reciprocity.

Quantum probability theory has successfully explained numerous findings in cognition and decision-making including violations of the sure-thing principle (Pothos and Busemeyer 2009), interference effects in perception (Conte et al. 2009), conjunction and disjunction fallacies (Busemeyer et al. 2011), violations of dynamic consistency (Busemeyer et al. 2012), interference of categorization on decision-making (Busemeyer et al. 2009), and order effects in survey questions (Wang and Busemeyer 2013). We feel that quantum probability theory also has great potential to explain human causal reasoning. The results we have considered here make us optimistic about this approach in the future.

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A Quantum Probability Model for the Constructive Influence of Affective Evaluation

Lee C. White, Emmanuel M. Pothos, and Jerome R. Busemeyer

1 Introduction

People experience simple affective evaluations every day. Commonplace events, such as listening to breakfast radio, deliberating over what to have for lunch, or unwinding in front of the television in the evening, all have the potential to generate a positive or negative affective impression, depending on how we feel about the music on the radio, the menu choices, or the television program we are watching. Affective evaluation is a fundamental and basic activity of the human cognitive system and is central to most theories of cognition and emotion (Musch and Klauer 2003). Research on affective priming suggests that affective evaluations can be formed automatically,

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independently of other cognitive processes, without fully processing the features of the stimulus and can be generated in response to novel stimuli (e.g., Bargh et al. 1992; Damasio 1994; Duckworth et al. 2002; Fazio et al. 1986; Greenwald et al. 1989; LeDoux 1996; Zajonc 1980). What is perhaps more surprising is that the process of articulating an affective evaluation might also be *constructive*. That is, the simple affective impression that we might form in response to music, food, or a television program, can alter relevant representations depending on whether or not we are required to state our affective impression. This was the main premise in recent research by White et al. (2013, 2014), which was inspired by cognitive applications of quantum probability (QP) theory.

The idea that cognitive processes can be constructive is not new. For a cognitive process to be constructive, the information on which it acts needs to be altered in some way, as a consequence of being subjected to that process. Many researchers have argued that, for example, making a choice between alternatives can be constructive, so that the act of choosing actually influences the subsequent preferences for the relevant alternatives (e.g., Ariely and Norton 2008; Kahneman and Snell 1992; Payne et al. 1993; Sharot et al. 2010; Sherman 1980; Slovic 1995). However, as will be seen in this review, White et al.'s (2013, 2014) research is new in that it extends the conditions in which cognitive processes can be constructive and that QP is used as a basis for modelling this effect.

We will begin this chapter with a brief summary of the research on constructive processes in judgment and decision-making and the rationale for using QP to model such processes. We then describe a QP model for the constructive role of articulating an affective impression and the empirical research that has been undertaken to support the model. We end by considering some of the limitations of this model and discuss directions for future research.

2 Constructive Processes in Judgment and Decision-Making

The constructive role of a judgment was first discussed and empirically explored by Brehm (1956) who employed a free-choice paradigm. Female shoppers were presented with eight appliances and asked to rate how desirable they were. They were then offered two appliances they had rated equally desirable and told they could choose one as a gift to compensate them for their trouble in taking part in the study. After a short time they were asked to re-evaluate the two items and it was observed that the chosen item was rated more highly than the rejected item, compared with the initial ratings. These results have been interpreted as providing support for Festinger's (1957) well known theory of cognitive dissonance, which states that when people detect a discrepancy between their behaviour and their preferences, they experience psychological tension which they seek to reduce by changing their preferences to match their behaviour (see also Bem's (1967) self-perception theory for an alternative explanation). So, in Brehm's (1956) study, presumably participants experienced psychological tension as they considered the desirable and undesirable aspects of the chosen and rejected appliances. This in turn motivated them to alter their preferences to reduce the tension, so that the chosen appliance was seen as more desirable and the rejected item less desirable.

The central observation in the free choice paradigm, that participants will like the chosen item more and the rejected item less, has been replicated a number of times (e.g., Egan et al. 2007; Lieberman et al. 2001; for reviews see Ariely and Norton 2008; Slovic 1995). However, the free choice paradigm has not been without criticism. It is argued by some (e.g., Chen 2008; Chen and Risen 2009) that it does not take into account the idea that choice reveals preference. Using the free-choice paradigm, it is not possible to say whether preference shifts result from the constructive influence of making a choice or are simply predicted by existing preferences which the rating scale cannot precisely measure.

Research by Sharot et al. (2010) addressed this criticism. In one experiment, participants were asked to choose between two holiday destinations. After they initially indicated how happy they would be with various holiday destinations, they then made a blind choice between destinations (participants had been led to believe that the study was about subliminal decision-making). They were then informed which destination they had chosen, before they were asked again to rate the destinations. The results showed a change in preferences that reflected their choices. The chosen holiday destinations were rated more positively, even though the participants' actual choices had been made with the holiday destinations masked. Additionally, the choice-induced preference change was not observed when participants were given a choice that had been made by a computer. The experiments conducted by Sharot et al. (2010) and other work by Egan et al. (2010) with children and nonhuman primates appear to have settled the debate over whether choice can potentially alter preferences.

But there is, arguably, a rationale for why our choice for a particular option might increase our preference for that option. It helps people to reduce uncertainty, psychological tension, or post-choice doubts. That the articulation of a simple affective impression can also be constructive would seem more surprising, and it is difficult to motivate such a possibility within existing theory. However, the application of QP theory in this area will lead us to an interesting prediction regarding the scope of constructive influences in cognition. Specifically, we will see that QP theory predicts a limitation in how the cognitive system can represent uncertain information, which means that a judgement (i.e., a choice or affective evaluation) must (sometimes) be constructive, simply because of how potentialities regarding the different options translate into certainty for a specific option.

3 Why Use QP to Model Cognitive Processes?

QP is a formal framework for assigning probabilities to observable events (Hughes 1989; Isham 1989). It originated in attempts to explain some of the paradoxical findings in physics, such as the double-slit experiment, which had defied classical interpretations. QP and classical probability (CP) are derived from different axioms. CP is derived from the Kolmogorov axioms, which assign probabilities to events defined as sets, and QP is derived from the von Neumann axioms, which assign probabilities to events defined as subspaces of a vector space. The use of CP principles in cognitive modeling has been popular and it is generally assumed that probabilistic computations in judgment and decision-making conform to the principles of CP theory, at least in some cases (e.g., Oaksford and Chater 2009; Tenenbaum et al. 2011). Furthermore, an argument can be made, especially through the Dutch Book theorem (e.g., Howson and Urbach 1993), that CP theory provides a rational basis for making decisions.

But as this volume attests, more recently, increasing interest has been shown for using QP in decision-making theory, as an alternative to CP, with a number of different QP based modeling approaches being used (e.g., Aerts and Aerts 1995; Basieva and Khrennikov 2014; Bordley and Kadane 1999; Bordley 1998; Busemeyer et al. 2011; Khrennikov and Haven 2009; Lambert-Mogiliansky et al. 2009; Pothos and Busemeyer 2009; Trueblood and Busemeyer 2011; Wang and Busemeyer 2013; Yukalov and Sornette 2008, 2009). Because CP and QP are based on different axioms, QP incorporates certain unique features, which do not exist in CP, such as incompatibility, superposition, and entanglement. These features have been used to help physicists understand, for example, interference effects and how measuring the state of a system can actually sometimes create a property of the system. Advocates for the use of QP theory in cognition argue that phenomena analogous to those observed in physics are also present in human decision-making. For example, interference effects between possibilities/questions in the cognitive system can be seen in research on order effects in decision-making (e.g., Hogarth and Einhorn 1992) and the constructive effects of measurement on a system can be seen in the research we described previously, on the constructive role of choice. So QP appears to offer principles, which fairly naturally incorporate effects (such as interference and constructive influences), for which there is empirical evidence in psychology, but which have been difficult to model within CP frameworks (for overviews see Busemeyer and Bruza 2011; Pothos and Busemeyer 2013).

One concept from QP theory, in particular, is important for modeling constructive effects of articulating an affective evaluation—and this is *superposition*. Most decision-making models based on CP theory assume that, whilst the cognitive system can change from moment to moment, at any one point in time the system is considered to be in a definite state, with respect to the decision or judgment to be made. So, in the example of providing an affective evaluation for a visual stimulus, within a CP framework, we would assign probabilities to the person experiencing positive or negative affect. Perhaps, if the affective content of the image is more ambiguous, there may be more fluctuation and change in the observer's affective state over time, until the final state is reached. However, importantly, at any one moment in time, the person is assumed to be in a definite positive or negative affective state.

Modeling the same situation using QP theory is different. Much of the difference between CP and QP theory derives from the fact that the former is a set-theoretic representation of probabilities, whereas the latter is a geometric representation of probabilities. In QP theory, events are represented as subspaces in a multidimensional Hilbert space (see Chapter "A Brief Introduction to Quantum Formalism"). Thus the person's affective state, with respect to the visual stimulus, would be represented by amplitudes across all relevant possibilities. So, as long as there is a nonzero weight for different possibilities, the person is said to be in a superposition. This means that at any one moment in time his or her affective state is consistent with neither possibility. Instead, there is potential for all possibilities, and which affective state will be eventually selected cannot be ascertained until the system is measured (i.e., a judgment or affective evaluation is made).

Exactly what happens when the system is measured leads us to another important difference between CP and QP. In CP, the measurement of a system at a given point in time is assumed to represent the state of the system at the point in time just before the measurement was made. Alternatively, in QP, taking a measurement of a system creates rather than records a property of the system (Peres 1998). This means that the state of the cognitive system following a measurement is constructed from the interaction between the measurement and the superposition state (Bohr 1958). Applying these ideas to cognition, the act of judgment would alter the cognitive state, so that, in the case of our visual stimuli, whereas the state was a superposition one prior to the measurement, following a measurement, the state becomes consistent with a specific affect.

These two fundamental principles of QP theory, superposition and the requirement that measurement creates rather than records a property of the system, offer a natural and straightforward way in which to model constructive processes in judgment and decision-making. In what follows, we describe the research of White et al. (2013, 2014), who provided a demonstration of these ideas, using a simple experimental paradigm, designed to examine how measurement of the cognitive system might influence affective evaluations of visual stimuli.

4 A Design for Investigating the Constructive Influence of Articulating Affective Evaluations

In White et al.'s (2013, 2014) research, fictitious adverts were created which had positive or negative affective content. For example, a positive advert for insurance showed a group of happy students and a negative advert for insurance showed a burnt out kitchen (see Fig. 1).

In a within-subjects experiment, participants were asked to consider pairs of images presented sequentially in either a positive and then negative order or vice versa. When they viewed images in the double rating condition they were required to provide a simple affective rating of the first image and were then again asked for a rating for the second image. In the single rating condition, they viewed the first image but provided no rating, instead moving on to view and rate the second image (see Fig. 1).

The reactions that people might have to the individual images can be easily anticipated as the valence of the images was predetermined to be either positive or negative. The question posed by this research was this: if the same participant views the same sequence of positive and negative images twice, with the only difference being whether or not they rated the first image, would we expect to see a difference in ratings for the second image? In other words, does articulating an impression for the first image change how someone sees the second image? The results of the research showed that there was a difference between the ratings of the two identical second images. Specifically, when participants viewed a positive image followed by a negative image, they found that ratings of the second image in the single rating condition were

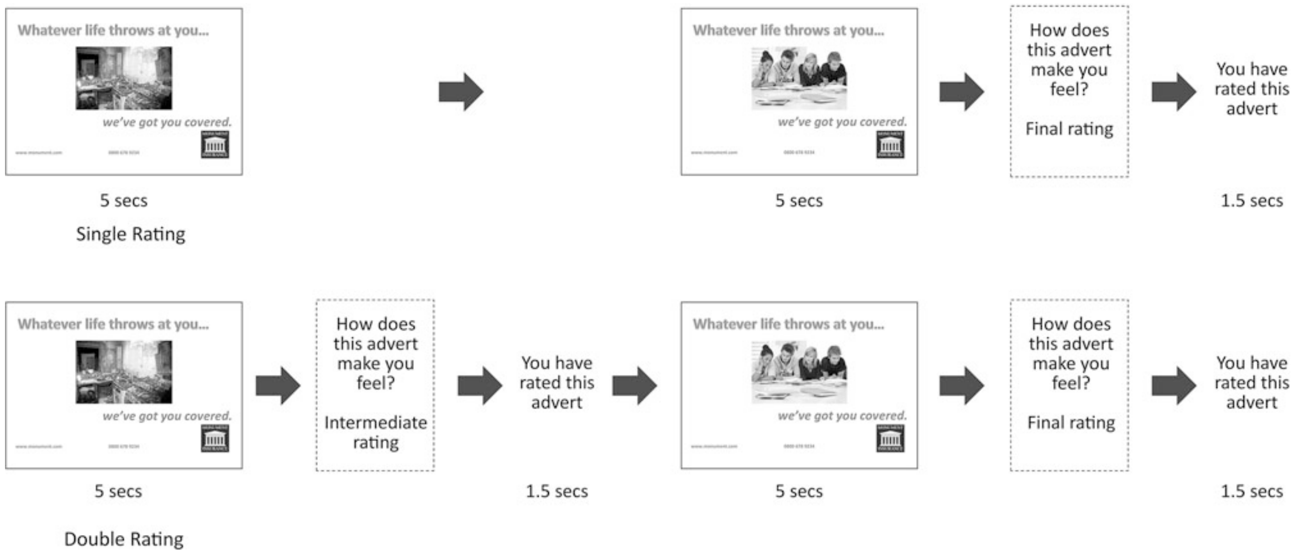

Fig. 1 Sample adverts and procedure used in White et al. (2014) Experiment 2. *Source*: White et al. (2014)

significantly more positive than ratings of the same negative image in the double rating condition. Similarly, when participants viewed a negative image followed by a positive image, they found that ratings of the second image in the single rating condition were significantly less positive than ratings of the same positive image in the double rating condition.

Clearly one potential influence on the rating of the second image, suggested by previous research, could have been the order in which the images were viewed (e.g., Hogarth and Einhorn 1992; Moore 2002; Ross and Simonson 1991). For example, Ross and Simonson (1991) found that when asked to evaluate an event which was composed of positive and negative components, participants found more satisfactory the event where the positive component came last. Moore's (2002) research, using data from a Gallup poll, found that, if participants were first asked to rate the honesty of American Vice President Al Gore, before rating the honesty of President Bill Clinton, Clinton's honesty was rated lower than if participants were asked to rate Clinton's honesty first. In both examples, the response to the preceding question about the first component can be seen to change the way the second question is evaluated. These types of order effect have been extensively considered in the literature in terms of recency, primary, and contrast effects (e.g., Anderson 1981; Hogarth and Einhorn 1992; Miller and Campbell 1959; Payne et al. 1993; Tversky and Griffin 1990; Wang and Busemeyer 2013; Wang et al. 2014). Arguably, these effects can also be considered to be constructive, in that the response to the first component in the sequence changes the information that is relevant to the judgment of the second component. For example, how participants regard Gore's honesty changes information that is used to evaluate Clinton's honesty.

However, White et al.'s (2013, 2014) design controls for the potential influence of order effects, by having participants view an identical sequence of positive and then negative images twice. The only difference was whether or not they provided an intermediate rating of the first image. Thus any difference between the ratings of the second images could not be explained as an order effect. Instead, White et al. proposed a cognitive model, using QP principles, which could predict the results of the experiment. It is worth noting that, possibly, a CP model augmented with some mechanism for how decisions/measurements could alter the relevant representations could well account for the results of White et al. The advantage of a QP approach is that a constructive influence is unavoidable, whenever the underlying state is a superposition one.

5 A QP Model for the Constructive Influence of Affective Evaluation

In order to describe White et al.'s QP model it is necessary to review briefly some principles of QP theory, as they apply to cognitive modeling. In QP theory, a cognitive state is represented as a state vector (of length 1) in a Hilbert space, which is a multidimensional vector space, with some additional properties. In such a space, different possibilities are represented as subspaces. For example, consider a fictitious person, Bill, to whom we have presented an image. Figure 2 shows a three-dimensional space with Bill's cognitive state represented by the cognitive state vector. We will assume two possible responses that Bill might have, following the presentation of the image. Bill might be happy, which is represented by a one-dimensional subspace (a ray) or Bill might be unhappy, represented by a two-dimensional subspace (a plane; for now we do not need to concern ourselves with how the dimensionality of subspaces for different possibilities is determined).

Now if we want to determine the probability that Bill will be unhappy in response to the image, we project the cognitive state vector, representing Bill's current cognitive state, onto the relevant subspace (the projection is shown by the dotted line in Fig. 2). The laying down or *projection* of a vector onto a particular subspace is a critical operation in QP. One of the fundamental theorems in QP is that the squared length of a projection, along a subspace, determines the probability that the associated possibility is true of the system

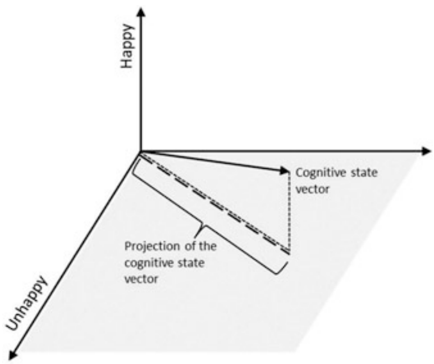

Fig. 2 Projection: the cognitive state vector is projected onto the two-dimensional plane (indicated by the *shaded area*), corresponding to the "unhappy" possibility. The projection is denoted by the *dashed line* and its length *squared* is the probability that the hypothetical person, Bill, will decide he is unhappy. *Source*: White et al. (2014)

represented by the state vector. Thus, the squared length of the projection in the unhappy subspace is the probability that Bill will say he is unhappy. And if Bill does say that he is unhappy, the cognitive state vector will change to become a vector of length 1 along the line of projection shown by the dashed line in Fig. 2.

An important principle in determining the positioning of subspaces in relation to each other is the degree of correspondence between the possibilities that they represent. Thus the possibility that someone can be happy is assumed to be mutually exclusive to the possibility that the person is unhappy, hence the subspaces representing being happy and being unhappy are orthogonal. If the cognitive state vector is in the happy subspace, the projection to the unhappy subspace must be zero. In other words, if $Prob(unhappy) = 1$ then the $Prob(happy) = 0$. If there is a greater degree of correspondence between possibilities then the angle between the subspaces will be smaller, meaning that it is easier to project from one subspace to another. The degree of overlap of the cognitive state vector onto the relevant subspace is what determines the probability, with a greater degree of overlap, suggesting a higher probability.

Finally, we also need to consider how QP theory represents dynamic processes, such as how the introduction of new information might influence the cognitive system. For example, suppose Bill is told that he has won the lottery. We assume that this makes him happy and represent this in the model by rotating the cognitive state vector towards the happy subspace. This in turn increases the projection on the happy ray and therefore the probability that Bill would say that he is happy if we asked. Note that such transformations of the state vector, to model dynamic processes, are termed "unitary" and are the simplest kind of dynamic transformation employed in QP theory. Strictly speaking, to model changes towards a specific subspace we need open-system dynamics; compare with Asano et al. (2011a, b). Unitary dynamics are applicable to situations where we can assume a limited or no interaction between the system of interest and its environment. This assumption is relaxed in open systems dynamics which are technically more complex. For some QP cognitive models, unitary dynamics can be thought of as an approximation to the more realistic open system dynamics.

We can now take these principles and show how they can be used to devise a model to describe the empirical results found in White et al. (2013, 2014). In this model, a two-dimensional real space is used to represent the different possibilities in the experiment. There are two rays at a 90° angle to each other to represent the mutually exclusive possibilities of someone experiencing either positive or negative affect in relation to the visual stimulus (Fig. 3a). In fact, in White et al.'s experiment, participants were required to rate their happiness

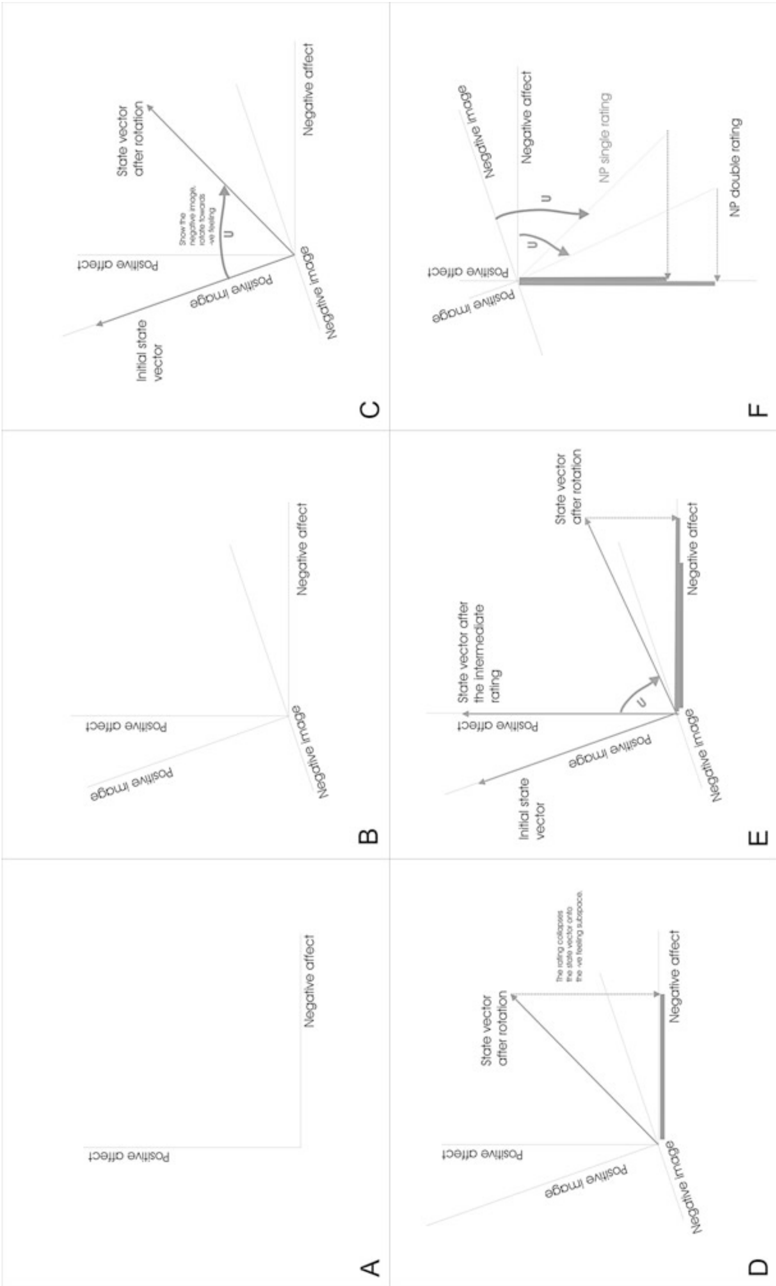

Fig. 3 Quantum Probability Model for the constructive role of measurement in the present experiments in the PN condition (**a–e**) and NP condition (**f**). *Source*: White et al. (2014)

on a nine-point scale. So strictly speaking, there are nine possible outcomes which should be represented by a nine-dimensional vector space. However, as a simplifying approximation, we use a two-dimensional space for the purposes of explanation.

In addition to the rays representing positive and negative affect, there is a set of rays to represent the combined, affective, perceptual impact of processing a positive or negative image (Fig. 3b). These are also orthogonal to each other because the two images were designed to be unrelated in terms of the theme of their content (excluding affective polarity), even though, of course, the images both corresponded to adverts. The assumption of unrelatedness, together with the obvious point that the affective content of the two images is opposite, allows the placement of the rays for each at approximately right angles to each other. In QP terms, this means that, if a naïve observer is thinking about one advert, it is extremely unlikely that he or she will spontaneously think of the other advert too. There are various subtleties here. One is that if the two stimuli were *totally* unrelated, then one would expect that change of the state vector relative to one ray to be irrelevant, regarding the relation of the state vector to the other ray. We think that, given the materials, these assumptions are reasonable. But, in any case, the assumptions are directly testable in an empirical experiment. These assumptions lead to a prediction of a very specific influence of a judgment on a subsequent judgment. If they are wrong, the prediction would not hold.

It can also be seen in Fig. 3b that there is a relatively small angle between the positive affect ray and the positive image. This is because it is assumed that, when someone sees a positive image, it is likely to generate positive affect. In other words, if the cognitive state vector was in the positive image subspace, then projection to the positive affect subspace would be greater than the projection to the negative affect subspace. This simply reflects the idea that there would be a higher probability of someone saying that they felt positive affect in relation to the positive image. Thus, the positioning of the affect and image rays, relative to each other, is fairly automatic, depending on how the possibilities interrelate with one another.

With these minimal assumptions about how the relevant possibilities are represented, we are now in a position to demonstrate how the model describes the experimental effect observed, whereby articulating an affective evaluation, with respect to the first image, influences the rating of the second image. Consider, for example, the positive to negative order of presentation (PN). The ψ symbol is used to denote the participant's cognitive state in relation to the images. In the PN condition, the participant first sees the positive image, and so ψ is initially in the positive image ray (Fig. 3c), since it is assumed that the only influence on the participant's cognitive state at that point in time is the act of processing the image. In the PN single rating condition, the participant is next shown the second negative image without rating the first positive image. The effect of this is a fixed rotation of the state vector towards the negative affect ray. This is a key psychological assumption of the model. Seeing the negative visual stimulus does not put the participant into an absolute negative state but rather it induces a change towards negative affect. For example, if someone is feeling in a good mood as they are about to watch their favorite television program, they are likely subsequently to feel worse if they are then told that they have lost, say, £1000, than if they are told that they have lost £1. Their final affective state in this situation will depend on how their initial affective state is compounded with the negativity they feel in response to the two possibilities. This assumption is intuitive and can be seen in a number of psychological theories (e.g., Hogarth and Einhorn 1992; see also relative judgment models of perceptual differences, Laming 1984; Stewart et al. 2005).

So the participant's final affective state is a function of their initial affective state and the degree of change in the negative direction. In the QP model this is represented as a fixed rotation of the cognitive state vector towards the ray for negative affect. Once the participant has seen the second image, they are asked to provide a rating. Remember that the smaller the angle between ψ and the negative affect ray, the greater the projection of ψ onto the negative affect ray. Longer projections, in QP, equate to a higher probability that the participant will provide a negative rating, which in the model suggests that they will provide a more negative rating (Fig. 3d).

We now come on to the importance of the idea of superposition in the model. Unless the cognitive state vector is wholly within the positive or negative affect subspace, the participant can be considered to be in superposition with respect to their affective evaluation of visual stimuli. This means that there is potential for either a positive or negative affective evaluation but, importantly, it also means that their cognitive state is not consistent with either of those possibilities, before they state their evaluation. Once they have made an evaluation, ψ has to move to the subspace consistent with their judgment. It is this aspect that makes QP theory constructive and one which can be distinguished from CP theory in which uncertainty relates to lack of knowledge of the real value of the state.

We can now examine how the model describes the alternative PN condition, where the participant provides a double rating (Fig. 3e). Providing a rating to the first visual stimulus forces a transition of ψ to the positive affect subspace.

This is the key difference between what happens in the single and double rated conditions (see Fig. 3c vs Fig. 3e). The additional change in the state vector, following a rating of the first stimulus, combines with the change induced after the presentation of the second stimulus. So in the PN condition, ψ transitions to the positive affect subspace, as a result of the first rating. As can be seen in Fig. 3e, this positions ψ closer to the negative affect ray. This means that when the second negative stimulus is presented, ψ is moved through the same fixed rotation towards the negative subspace, but because it is starting from a closer position to the negative affect ray, it ends up with a smaller angle between the state vector and the negative affect ray (Fig. 3e). This in turn indicates a greater probability of a negative rating, which is interpreted as meaning that the participant will provide a more negative rating than that provided in the single rating condition.

The direction in which the state vector is rotated is determined by the possible outcomes in the experiment. So in the case of the PN condition, it makes sense that the impact of presenting the negative image should be a rotation away from the positive image subspace and towards the negative affect subspace, which in this experiment has to be clockwise. A consistency consideration means that the same direction of rotation applies for the single rating condition. It should be noted that these assumptions provide sufficiently accurate approximations for this specific experimental paradigm, but would need to be suitably generalized for a more general experimental one.

In summary, it can be seen how the QP model for the PN condition for single and double rating of visual stimuli describes the empirical results. An intermediate judgment in the PN condition will result in a more negative rating of the second stimulus, compared with the rating of the same stimulus, when there was no intermediate rating of the first stimulus. Exactly the opposite prediction is made for the condition where a negative and then positive image is presented (NP; Fig. 3f).

Psychologically, it could be argued that the QP model is describing a process of differing contrasts depending on whether or not the participant makes an intermediate evaluation. Consider, for example, the PN condition. The rating for the second negative stimulus will depend on the contrast between the stimulus and the participant's cognitive state, prior to seeing the stimulus. In the case of the single rating procedure, a participant's cognitive state would correspond to the positive stimulus, which will be contrasted with the subsequent negative stimulus, when that is revealed to the participant. In the double rating procedure, where the participant has provided a rating of the first positive image, assuming he or she has rated it positively (a reasonable assumption given the stimuli were predesigned to evoke positive or negative affect), the participant's cognitive state will correspond to positive affect. So when the participant sees the following negative image the contrast is between positive affect and the negative image, which will emphasize the negativity of the image, leading to a more negative rating.

The QP model, drawing on previous work (e.g., Busemeyer et al. 2011), provides some insight into the psychological processes that underlie the constructive influence of articulating an affective impression. Initially, the cognitive state represents the first positive or negative image, and this is represented in the model by the cognitive state vector, which coincides with the positive or negative image ray. When an intermediate judgment is made, this changes the cognitive state to one of either positive or negative affect, with respect to the first image. This change is represented in the model by a collapse of the state vector onto either the positive or negative affect ray. This is like an abstraction process, whereby some of the information about the first image is forgotten and attention is focused on information related to its affective properties. It is also the critical constructive step in the model: the intermediate rating changes the mental state in a certain way. This means that having made the intermediate rating and having changed the cognitive state, when the second oppositely valenced image is presented, it is viewed from the perspective of a different cognitive state than it would have been if no intermediate rating had been given and the cognitive state was still the initial one. As the second image is opposite in valence to the first, when the cognitive state is a pure affective one, there is a greater contrast in the impression made by the second image. Without the intermediate rating, the differences between the images concern aspects of their affective quality, but also differences between the images that are not related to affect, so the affective contrast between the first and second image is less pronounced.

6 Alternative Explanations

In three experiments reported in White et al. (2014), the predictions of the QP model were confirmed. An intermediate rating of the first positive stimulus led to a more negative rating of the second image (and vice versa for the negative to positive ordering). There are however some alternative explanations for these results which are worth exploring. The "belief-adjustment model" (Hogarth and Einhorn 1992) describes how evidence can be combined to form a view about a hypothesis. The model is pertinent because it describes how sequences of information and intermediate evaluations can impact on the overall judgment. Hogarth and Einhorn (1992) compared studies which employed either an end of sequence (EoS) methodology, whereby evidence was presented sequentially and then evaluated in one go at the end of the sequence, or a step by step (SbS) methodology, whereby information was presented and evaluated sequentially with a final judgment being made at the end of the sequence. The EoS methodology is akin to White et al.'s single rating condition and the SbS approach is similar to the double rating condition. According to Hogarth and Einhorn's (1992) review of the evidence, there is "primacy in 19 of 27 EoS studies and recency in 16 of 16 SbS studies" (p. 6).

There is some similarity between this finding and the results of White et al.'s experiments. In the double rating condition, the second stimulus was evaluated more intensively, which is similar to the recency effect, in that the last item in the sequence had a greater impact. However, there are some clear differences between Hogarth and Einhorn's (1992) work and the experimental situation considered by White et al. in that the latter considered how a judgment about an earlier stimulus influences the judgment of a completely different, unrelated second stimulus. Furthermore, the belief-adjustment model was based on a review of studies in which several pieces of information (e.g., 2–17 items) were used and Hogarth and Einhorn's (1992) own experiments used an initial description followed by at least two pieces of information. The belief-adjustment model does provide a computational framework but, in their analysis, White et al. (2014) show that this framework does not predict the influence of the intermediate rating and instead predicts no difference between the single and double rated conditions.

Another possible explanation for White et al.'s empirical results is what Tversky and Kahneman (1974) called the "anchoring-and-adjustment heuristic." One approach to judging the affective value of a novel stimulus might be to base your judgment on the known affective value of a previously seen stimulus and then adjust from that base until you reach a satisfactory value for the new stimulus. In their original demonstration, Tversky and Kahneman (1974) asked participants first to make a comparative assessment (e.g., "Is the percentage of African countries in the UN higher or lower than 10?") before making an absolute judgment (e.g., "What is the exact percentage of African countries in the UN?"). They observed that the latter, absolute judgment was biased towards the comparison value provided in the first judgment (e.g., the median estimates for starting values of 10 and 65, were, respectively, 25 and 45), even when that initial value was randomly generated in the participant's presence by spinning a "wheel of fortune" numbered 1–100.

Since this early work there have been many demonstrations of the anchoring-and-adjustment heuristic, including the assessment of gambles (e.g., Chapman and Johnson 1994; Schkade and Johnson 1989), the responses given to general knowledge questions (e.g., Epley and Gilovich 2001, 2006; Jacowitz and Kahneman 1995; Strack and Mussweiler 1997), property price estimation (Northcraft and Neale 1987), and judgments of future effort and performance (Switzer and Sniezek 1991). Research has also shown that both experts and non-experts are similarly affected by anchors (e.g., Englich and Mussweiler 2001; Northcraft and Neale 1987), which people use even when they have been forewarned not to (e.g., Wilson et al. 1996). There have been demonstrations of anchors that are random or uninformative influencing the absolute judgement (Russo and Shoemaker 1989; Tversky and Kahneman 1974). For example, Russo and Shoemaker (1989) first asked people for the last three digits of their telephone number and then added that to 400 before using that result in the question "Do you think Attila the Hun was defeated in Europe before or after [insert their answer plus 400] AD?".

The empirical evidence as it relates to anchoring appears to converge to at least two mechanisms that explain the cognitive processes underpinning anchoring, depending on whether or not anchors are self-generated. The selective accessibility model is a theory in which it is argued that the anchoring effect is a result of the enhanced accessibility of anchor-consistent information (Mussweiler and Strack 1999, 2000; Strack and Mussweiler 1997). Epley and Gilovich (2001, 2006) argue that self-generated anchors act as a heuristic by simplifying the cognitive process of making the judgment with respect to an absolute value.

In relating the research on anchoring-and-adjustment to White et al.'s experimental paradigm, it should be noted that there are some clear differences. There is no explicit relationship between the first and second stimuli in their experiments. Participants are asked to view and consider each stimulus independently. There is no comparison question, as used in the standard anchoring experiment. Participants, in the double rating condition, make an absolute judgment with regard to each stimulus. It should also be noted that White et al.'s experiments do not fit with standard definitions of anchoring. For example, Chapman and Johnson (2002, p. 122), in their review, define anchoring as an outcome where "the influence of an anchor ... renders the final judgement too close to the anchor ... thus, anchoring is defined as assimilation rather than contrast." In the White et al. experiments, in the presence of an anchor (i.e., in the double rating condition), subsequent ratings show greater contrast with previous ratings, whereas in the absence of the first rating (i.e., the single rating condition), greater assimilation is observed. That is, the rating of the second image is closer to the valence of the first image than the rating of the second image in the double rating condition.

Nevertheless, demonstrations of anchoring are common and it might be that the White et al. experimental results are a special case of anchoring. What drives the observed result is possibly the availability of a rating (whatever the source) after the first image, rather than the act of measurement by the participant about his or her own feelings. White et al. (2014) addressed this question in Experiment 3 by showing participants a randomly generated rating from a hypothetical participant for the first stimulus, before they rated the second stimulus. They also re-analysed data from previous experiments, comparing ratings for the first and second rated stimuli, in the double rating condition. No evidence for anchoring was found and instead they observed a replication of the key interaction seen in Experiments 1 and 2.

7 Conclusions and Future Directions

There remain some outstanding questions regarding White et al.'s (2014) work, which also suggest future directions for research. Some of these questions are methodological. As we have discussed above, the QP model has been simplified. Strictly speaking judgments should be represented by a nine-dimensional vector space because participants use a nine-point scale to indicate their level of affect, with anchors "1: very unhappy to 9: very happy." However, the QP model uses a two-dimensional vector space, so that judgments are represented as being either very unhappy or very happy. We have argued that this does not affect the generality of the theory. However, it is important to demonstrate that the same key interaction can be observed when participants are required to consider the image and then make a simple choice between being either happy or unhappy, that is, a binary judgment. Such a demonstration would support more directly the specific model we have described.

There is yet another alternative explanation for the results described previously, which concerns the length of time that participants had to process the stimuli in the single and double rating conditions. In the single rating condition, they view the first stimulus for five seconds, before being presented with the second stimulus for rating. In the double rating condition, they view the first stimulus for five seconds and then have no time limit on giving their response, taking as long as they like to consider the image before rating it. This could mean, then, that a difference arises between the single and double rating conditions, because people process the first stimulus in the double rating condition for longer. This potentially allows for more deliberative or strategic processing. In turn, there is the possibility that the stimulus creates a more lasting impression with, perhaps, greater saliency or accessibility of the stimulus as a standard of reference for the second stimulus. In the double rating condition, the impression a stimulus makes on a participant, therefore, could be different from the impression made by the same stimulus in the single rating condition, simply because they have longer to process it.

Research on affective priming suggests that judgments about a stimulus's goodness or pleasantness are not dependent on the length of time that a participant has to process it (e.g., Bargh et al. 1992; Duckworth et al. 2002; Fazio et al. 1986; Greenwald et al. 1989). Affective content can be processed relatively quickly and, in spite of the speed in which it is processed, can still have an influence on subsequent judgments. It seems reasonable to suppose that, in either the single or double rating condition, initial exposure will lead participants to form rapidly an affective impression of the stimulus. But it is also possible that, as participants have longer than 500 milliseconds to view the first stimulus, affective priming is not relevant, as the longer time scale provides them with ample time to process the stimulus more deliberately. But, in order to rule out the length of time that they have to process the first stimulus as an explanation, it would be a simple experiment to control for the amount of time that they have to make their ratings.

There are also questions about boundary conditions for the effects that have been observed. There may be something specific to affective impressions, which have enabled us to confirm the prediction of the QP model for White et al.'s (2014) results. Perhaps the apparent impression we have that we can entertain positive and negative emotions concurrently (Brehm and Miron 2006) makes them more ambiguous and, therefore, easier to change. Or possibly the effect is specific to visual stimuli. We've all probably experienced a debate about the aesthetics of a painting and the varying, complex reactions that people can have to art. Possibly, the greater potential for ambiguous affective reactions in relation to visual stimuli makes them more susceptible to the constructive influence of measurements. But, the QP model should apply regardless of the stimuli used or the types of judgment made. Extensions that employ either different types of stimuli or different categories of judgment would help to establish whether the results only apply to the affective evaluation of visual stimuli.

The potential for QP theory to offer a new perspective on the constructive role of judgment resonates with the thinking of a significant figure in physics, Wolfgang Pauli, as highlighted by Haven and Khrennikov (2013). They reproduce a quote from one of Pauli's unpublished essays, where, in discussing complementarity in physics, he argues that "any 'observation' of unconscious contents entails fundamentally indefinable repercussions of the conscious on these very contents" (Meier 2001, p. 185). The concept of superposition is new in psychology, and the review of White et al.'s (2013, 2014) research suggests that the transition from superposition to definiteness in QP theory has potential for formalizing the constructive influence that articulating an affective evaluation can have on the cognitive system. The application of QP theory can be seen to be fairly straightforward and requires only minimal assumptions regarding the relevant psychological processes. Moreover, QP cognitive models are a bit closer to the process, since, for example, the assessment of probabilities often involves sequential operations. Also, as presently relevant, the effect of measurement in QP theory is another specific way in which representations change as a result of cognitive operations. The incorporation of process assumptions in cognitive models is considered to be an important direction for their development (Jones and Love 2011; cf. Newell 1990).

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Is there Room in Quantum Ontology for a Genuine Causal Role for Consciousness?

Paavo Pylkkänen

It may be said, indeed, that without bones and muscles and the other parts of the body I cannot execute my purposes. But to say that I do as I do because of them, and that this is the way in which the mind acts, and not from the choice of the best, is a very careless and idle mode of speaking. I wonder that they cannot distinguish the cause from the condition, which the many, feeling about in the dark, are always mistaking and misnaming. (Plato, The Phaedo)

1 Introduction

Does consciousness have causal powers? Does it make a difference to the effects of information processing whether or not the system is conscious of a given item of information? Are our actions at least sometimes determined

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by our conscious free will? Since Libet's (1985) work on the neuroscience of free will, the notion that the conscious will is not the original determinant of action has won increasing support. For example, Velmans's (1991) work suggests that consciousness "is neither necessary for any type of mental ability nor does it occur early enough to act as a cause of the acts or processes typically thought to be its effects" (Van Gulick 2014, p. 36). The radical upshot of this line of thinking is the claim that "the sorts of mental abilities that are typically thought to require consciousness can all be realized unconsciously in the absence of the supposedly required self-awareness" (ibid.). In Libet's famous studies, conscious self-awareness is present, but Van Gulick notes that many claim that it occurs too late to be the cause of the relevant actions: "self-awareness or meta-mental consciousness according to these arguments turns out to be a psychological after-effect rather than an initiating cause, more like a *post facto* printout" (ibid.). Van Gulick adds, however, that the arguments are controversial and that many theorists regard the empirical data as no real threat to the causal status of consciousness (for a recent discussion of the issue from various viewpoints, see e.g. Pockett et al. 2006).

But how are we to understand the causal status of consciousness? In philosophy of mind there has been a long debate about the problem of mental causation. Many philosophers assume that consciousness is in some sense a nonphysical property. But this immediately gives rise to the problem of understanding how something nonphysical could possibly influence something physical. A key idea to be explored in this chapter is that the ontological interpretation of quantum theory might throw new light upon this perennial issue. This interpretation suggests that a new type of active information is playing a key causal role in physical processes at the quantum level. Now, when one examines the various suggestions about the putative causal powers of consciousness, many of them refer to the role of information, in one way or another. This then suggests a strategy for the present chapter. We will first consider how the various suggestions about the causal status of consciousness involve information before asking whether such information in mental and conscious states could be connected to information at the quantum level. In this way we could begin to understand mental causation, and the causal role of conscious experiences in particular, in a new way. Of course, this is a big and difficult issue and we can only sketch the solution in a single chapter. However, even this sketch will hopefully illustrate the great potential of quantum theory when trying to meet some of the grand challenges facing the social sciences.

2 Van Gulick and Revonsuo on the Causal Efficacy of Consciousness

In his useful review of the suggestions about the causal role of consciousness Van Gulick (2014, pp. 34–42) says that consciousness is thought to provide the organism with (a) more flexible control; (b) better social coordination; (c) more integrated representation; (d) more global informational access; (e) increased freedom of will; and (f) intrinsic motivation. In this section I will briefly explicate these (as well as some of Revonsuo's 2006 related ideas) and then, in the next section, discuss how they connect with the notion of information. Note that the aim in this chapter is not to evaluate critically these suggestions. The aim is rather to indicate, for the sake of the discussions that follows, that there is at least a reasonable possibility that consciousness has a genuine causal role, and that this connects strongly with the notion of information. For a more detailed discussion the reader is advised to consult the references given below, as well as in Van Gulick (2014, pp. 35–42) and Revonsuo (2006). Let us now consider a number of suggestions about the causal role of consciousness.

It is common to claim that conscious mental processes provide a flexible and adaptive type of control, as opposed to unconscious automatic processes (Anderson 1983). Even if these latter can be quick, they are also relatively fixed and predetermined, and thus not particularly effective in unexpected situations (Penfield 1975). Also, when the challenge is to learn new skills, conscious attention is typically assumed to be important at the early stages of learning (Shiffrin and Schneider 1977).

It has been suggested that organisms that are conscious of their own and others' mental states have a better ability to interact, cooperate, and communicate. The idea is that such meta-mental or "higher-order" consciousness would enable a better capacity for social coordination, which in turn can be thought to provide adaptive advantage (Humphreys 1982; Van Gulick 2014, p. 38).

It has further been suggested that conscious experiences enable a more unified and integrated representation of reality, which allows for a more flexible response in various situations (Campbell 1994; Van Gulick 2014, pp. 38–39; Tononi and Koch 2015).

It is a well-known suggestion that information in conscious mental states is globally available to a number of different mental subsystems or "modules", and can thus be made use of in many different ways in behavior (Baars 1988). In contrast, it is argued that non-conscious information is usually available only to special mental modules and has a more limited effect upon behavior and action (Fodor 1983). (However, Rosenthal 2009 thinks it is unclear that a state's potential to have global effects coincides with its being conscious.)

When it comes to free will, it seems that conscious experience not only presents us with the options to choose from (at least sometimes), it also seems to be a prerequisite for such freedom. Mustn't one be conscious to be able to make a free choice at all (Van Gulick 2014, p. 41)? One should note that researchers such as Velmans have suggested that there can be unconscious free will; but it is not obvious that a decision made unconsciously can be considered truly free.

Finally, it has been suggested that certain conscious states, such as pleasure and pain, have an intrinsic motivating force (e.g., attraction) as an indivisible part of the experience itself. The idea is that such a force cannot be reduced to nonconscious properties (for a brief account of the various viewpoints on this issue, see Van Gulick 2014, pp. 41–2).

Revonsuo (2006) has considered the causal powers of consciousness (or the "phenomenal level" as he calls it) in the light of various studies on blindsight, implicit perception, nonconscious visually guided actions, and similar phenomena. He acknowledges that there are complex information processing mechanisms in the brain that in themselves are nonconscious or, in his terms, "realize no phenomenal level of organization." However, he emphasizes that such nonconscious "zombie systems" seem to have only limited causal powers in guiding organism–environment interaction, whereas the contribution of consciousness (or the "phenomenal level") seems to be decisive for meaningful interactions with our environment.

He further considers disorders, such as epileptic automatisms and sleep-walking, which seem to turn the whole person into a nonconscious zombie, and notes that a careful examination of such zombies reveals that nonconscious organism–environment interaction, while complex, is typically pointless. He concludes (2006, pp. xxiii–xxiv):

> other types of disorders show that the simulated phenomenal world in the brain has unique causal powers in determining the behavioral trajectories of our physical bodies. In the light of the evidence from these disorders, consciousness surfaces as a causally potent biological system with unique causal powers. Therefore, we need not worry about epiphenomenalism any longer.

We note here that Revonsuo's reference to the way in which the simulated phenomenal world in the brain determines behavioral trajectories of bodies is interestingly analogous to Bohm's notion that active information encoded in the quantum field determines the trajectories of particles at the quantum level (we will discuss this latter idea below). We also note that to avoid truly epiphenomenalism or reductionism, Revonsuo needs to show how conscious experiences *qua* conscious could possibly play a genuine causal role in guiding the physical organism without violating the laws of physics (or the causal closure of the physical domain). This is of course connected to the problem of mental causation, a solution to which we are trying to sketch in this chapter.

3 How the Causal Efficacy of Consciousness Connects with Information

Let us now see how the above suggestions make a link between consciousness and information. We can understand "more flexible control" as flexibility in the way that information can be used to guide the organism. It seems that consciousness makes possible such flexibility. Unconscious information just "acts" when it is activated, according to an automatic routine. If there are items of unconscious information that imply mutually exclusive actions, then presumably the "stronger" information wins, and this may take place without conscious experience ("stronger" here may be assumed to correspond to e.g. a higher level of neural activity). However, it seems possible that when a person is conscious of an item of information, at least some (automatic) activity of that information can be suspended. Also, it seems obvious that at least in some situations a person can review a number of different options, and choose the one that seems best in the given situation. (In this way consciousness, flexible control, and free will seem related.) Of course, which option is in the end chosen may not be the result of a completely "free" choice, but is instead determined by some further information which arises when reviewing the options, with a content like "it is reasonable to do X" (cf. Bohm 1990).

We also noted that it has been suggested that organisms that are conscious of their own and others' mental states have a better ability to interact, cooperate, and communicate. "Conscious of" can here be understood to include "having meta-level information about." This connects with higher order theories of consciousness which assume that what makes a given mental state conscious is that there exists a higher level of (typically) unconscious mental state, which has the content that one is in the first-order mental state or activity (Rosenthal 1997). Thus, consciousness is not assumed to be a neural or computational property, but rather something that arises when initially nonconscious mental states are related in a suitable way. It seems quite natural to think about such meta-mentality in terms of information. We could say that meta-mentality involves higher-order "information about information" rather than just first-order "information about the environment." In these terms, higher-order theories of consciousness suggest that consciousness essentially involves information about information. A simple possibility would be to postulate that what makes a given informational state conscious is that there exists a higher level of (typically) unconscious information, which has the content that one is in the first-order informational state. When it comes to the causal efficacy of consciousness, the question is whether having meta-level information (and consciousness) in this sense implies a better ability to interact, cooperate, and communicate. Below I will briefly note how in the Bohmian scheme active information at a given level can organize the behavior of elements at a lower level. The challenge here, too, is to find out whether being conscious of active information gives the organism some special advantages when it comes to interaction, cooperation, and communication.

We further mentioned the suggestion that conscious experiences enable a more unified and integrated representation of reality, which allows for a more flexible response in various situations. To understand this feature better, we can usefully quote van Gulick (2014, pp. 38–9):

> Conscious experience presents us with a world of objects independently existing in space and time. Those objects are typically present to us in a multi-modal fashion that involves the integration of information from various sensory channels as well as from background knowledge and memory. Conscious experience presents us not with isolated properties or features but with objects and events situated in an ongoing independent world, and it does so by embodying in its experiential organization and dynamics the dense network of relations and interconnections that collectively constitute the meaningful structure of a world of objects.

This reminds us about the fact that the information we meet in consciousness is highly integrated and structured and also meaningful in various ways. Van Gulick acknowledges that non-experiental sensory information can also have an adaptive effect on behavior (e.g., as seen in reflexes). However, he draws attention to the work of Lorenz (1977) and Gallistel (1990), which suggest that conscious experience provides a more integrated representation of reality, which in turn enables more flexible responses. If we consider this feature in informational terms, it seems that a certain kind of information only becomes available and, especially, flexibly usable to the organism in conscious experience. This connects with the previously mentioned issues of flexible control and free will, in the sense that consciousness, flexible control, free will, and unified and integrated representations are all interconnected. Unified and integrated representations, especially when consciously experienced, provide the "free will" rich information about the available options which enables flexibility in the control of the organism.

There are a number of other researchers who emphasize that consciousness involves an integrated representation in the form of a "virtual reality" or "world-simulation." Revonsuo, for example, characterizes conscious experience in dreams as a complex, organized, temporally progressing world-simulation. During waking we also experience subjectively an internal, phenomenal, simulated world, which we take to be the "real" world, when consciousness happens to be online with the external physical world (Revonsuo 2015, p. 65).

And as we have already seen, for Revonsuo the simulated phenomenal world in the brain is causally efficacious in that it determines the behavioral trajectories of our physical bodies. Here we can ask what the nature of a world-simulation is. It seems natural to think of it as some kind of structure of information that is meaningful and has phenomenal properties. And given that this world-simulation guides the organism, it is natural to think of it as a kind of active information in the Bohmian sense that will be explained later.

Let us then move on to consider the suggestion that information in conscious mental states is globally available to a number of different mental subsystems or "modules" and can thus be made use of in many different ways in behavior. This feature, together with the issues discussed previously, helps to explain the flexible control that consciousness seems to enable. We saw above that information in conscious experience is typically very rich in its content—it is unified and integrated. If consciousness further means that such information becomes globally available to many different subsystems, it clearly becomes easier to understand why consciousness enables more flexible control. To put it briefly, the idea is that consciousness both enables the sort of information that flexible control requires, and it also makes it possible for such information to reach the subsystems that are required in the execution of the control.

In recent years much attention has been given to Tononi's integrated information theory of consciousness (Tononi and Koch 2015; Oizumi et al. 2014). There are various reasons why Tononi thinks the concept of information is needed in a theory of consciousness. To account for the fact that consciousness is differentiated (i.e., that each experience has a specific set of phenomenological distinctions), a system of mechanisms must specify a differentiated conceptual structure via a process of in-forming (we will see later that Bohm's notion of active information likewise refers to a process of in-forming, though in a somewhat different sense). Tononi further says that to account for the irreducible unity of consciousness (i.e., that each experience is irreducible to non-interdependent components), there has to be integrated information, in the sense that the conceptual structure specified by the system is irreducible to that specified by non-interdependent subsystems. More technically, the presence of integration (characterized by big phi or Φ) means that a partitioning of a system of mechanisms would destroy several cause–effect repertoires and change others.

Tononi's theory tries to explain what consciousness is in terms of the notion of information. But the theory also suggests that consciousness as integrated information makes a difference to the behavior of the organism. Tononi and Koch (2015, p. 11) write: "a brain having a high capacity for information integration will better match an environment with a complex causal structure varying across multiple time scales, than a network made of many modules that are informationally encapsulated." And given the hypothesis that consciousness is integrated information, this implies that it enables a better match with the environment and consequently more adaptive behavior.

We have already briefly considered the relation of free will and consciousness above, and will return to this issue below. Van Gulick's review also drew attention to the suggestion that certain conscious states, such as pleasure and pain, have an intrinsic motivating force (e.g., attraction) as an indivisible part of the experience itself. The idea is that such force cannot be reduced to nonconscious properties. This suggests that consciousness not only enables information to be integrated and globally available, but that it also involves (perhaps gives rise to) "forces," such as attraction. Again, we will return below to consider this interesting suggestion when discussing the notion of active information.

Van Gulick's review (as well as Revonsuo's and Tononi's theories) make a reasonably strong case for the idea that consciousness has genuine causal powers. Now, presumably each particular argument for such causal efficacy is subject to potentially serious criticisms, but I think that it is fair to say that together they imply that the question is at least an open one. It at least *seems* to make a difference to the behavior of an organism whether or not it is conscious. I have also drawn attention to the way many of the suggestions about the causal efficacy of consciousness involve a link between consciousness and information. In the rest of the chapter I will try to understand this link better by discussing it in the context of a new notion of active information that is extended all the way into physics. However, before doing that I want to meet briefly another challenge. For as was already hinted at above, contemporary philosophers of mind often suggest that consciousness cannot have genuinely causal powers if we stay within the physicalist scientific world picture. We need to address this issue briefly before proceeding.

4 Philosophy of Mind: Does Consciousness Have No Causal Power?

Much of contemporary Anglo-American analytical philosophy is committed to physicalism, which means that philosophers assume that everything is physical, or everything is in an appropriate way dependent (or "supervenient") upon the physical. However, many philosophers find it difficult to simply reduce the mental to the physical, and they thus defend a doctrine known as "nonreductive physicalism." This typically holds that mental properties are nonphysical properties that, however, depend or supervene upon the physical. Note that "mental" here is not taken to be synonymous with "conscious," but includes even such possibly nonconscious properties as intentionality (in the sense of the "directedness" or "aboutness" of mental states).

The trouble with nonreductive physicalism is that it seems to leave the mental as causally inefficacious or epiphenomenal. If the mental is nonphysical, it seems impossible to understand how it could be the cause of physical effects. Even the notion of mental-physical dependence or supervenience doesn't seem to help here. Some philosophers (e.g., Stephen Yablo, David Lewis, and Jaegwon Kim) have developed some ingenious ways to make the idea of genuine mental causation plausible (see Ritchie 2008). However, it seems that even these fail to tell us how mental properties (conceived as nonphysical) could possibly influence the physical course of events. There thus seems to be no genuine causal role for mental properties in contemporary nonreductive physicalism. This is a very unsatisfactory situation. However, to go back to, say, interactive substance dualism seems equally unsatisfactory. Nagel (2005) has succinctly summarized the situation: "neither dualism nor materialism seems likely to be true, but it is not clear what the alternatives are."

Note that this apparent epiphenomenalism of the mental is particularly troublesome for our above discussion about the causal role of conscious experience. It is not at all obvious that conscious experiences are physical or material in any traditional sense (remember e.g. Chalmers's 1996 discussion of the "hard problem" of consciousness). Thus contemporary nonreductive physicalism seems forced to declare consciousness to be an epiphenomenon.

Reductive physicalism resolves the issue trivially by assuming that conscious experiences are physical states. But for those who do not understand how conscious experience could possibly be a physical state, this “resolution” is not of much value.

We have noted that nonreductive physicalism implies that consciousness is epiphenomenal, but how seriously should we take the nonreductive physicalists’ arguments? For if one examines the views of many of the leading physicalists (whether reductive or nonreductive), one is struck by the fact that hardly any attention is given to what seems to be the most fundamental of the natural sciences, namely (fundamental) physics. This seems to be in violation of the very principles the physicalists have usually set themselves, namely that they ought to base their metaphysics upon the best theories in the natural sciences. A particularly sharp criticism of such tendencies in philosophy has recently been made by Ladyman and Ross (2007, p. vii). They write, for example, that “standard analytic metaphysics (or ‘neo-scholastic’ metaphysics as we call it) contributes nothing to human knowledge and, where it has any impact at all, systematically misrepresents the relative significance of what we do know on the basis of science.” Such “neo-scholastic” metaphysics also includes analytic philosophy of mind, in so for as this gives little attention to the results of modern science, including fundamental physics. Ladyman and Ross’s view is extreme, but I think they are correct in drawing attention to certain weak points in contemporary philosophy of mind. If we want to claim that the physical world leaves no room for the causal powers of consciousness, we should justify our view on the basis of the best theories in physics. And as we will see in the next section, it is not clear that, say, quantum theory excludes in principle the causal powers of consciousness. On the contrary, a natural extension of quantum theory might well make room for mental properties and even conscious experience in our scientific world picture.

5 Information in the Ontological Interpretation of Quantum Theory

Can quantum theory throw any new light upon the nature of information, which might also help us to understand the relationship between consciousness and information, and the causal powers of consciousness? I suggest that the best place to start exploring this issue is David Bohm’s interpretation of quantum theory, in its later form developed in cooperation with Basil Hiley

(Bohm and Hiley 1987, 1993; see also Pylkkänen et al. 2016; for Bohm's early work on quantum theory and the mind, see Pylkkänen 2014).

To understand the significance of Bohm's work for the mind–matter problem it is necessary to understand the development of physics in the twentieth century. When quantum theory was emerging, physicists were trying to make sense of puzzling features such as wave–particle duality and, a little later, entanglement. In particular they were attempting to develop ontological models of quantum systems such as electrons. In the 1920s Louis de Broglie came up with the idea of an electron being a particle guided by a pilot wave, while Schrödinger was trying to describe the electron as some kind of a physical field. These models had some difficulties, though in retrospect we can see that at least de Broglie's ideas could have been developed further (Bacciagaluppi and Valentini 2009). What happened however was that the so-called "Copenhagen interpretation" won the day in the 1920s. There are actually many different versions of this interpretation, but it is typical of them that they emphasize epistemology—in the sense of our ability to predict the statistical results of measurement—rather than ontology—in the sense of a model of what quantum reality may be like, including when we are not making measurements. As a result, physicists were not able to offer a new notion of objective physical reality, which philosophers could then use when discussing ontological issues, such as the mind–matter relationship.

It is here that Bohm comes in. In the early 1950s, after discussions with Einstein in Princeton, he independently rediscovered de Broglie's theory and formulated it in a more coherent way, providing a first consistent realistic model of quantum systems (Bohm 1952). Bohm's interpretation was initially resisted, but is today more and more widely acknowledged as one of the key possible interpretations of quantum theory. Later on further ontological models were proposed, for example Everett's (1957) "many worlds" interpretation and Ghirardi et al.'s (1986) objective collapse theory, and currently the nature of quantum reality is intensively debated within the philosophy of physics community (see e.g. the anthology *The Wave Function: Essays on the Metaphysics of Quantum Mechanics*, edited by Alyssa Ney and David Albert (2013)). We do not know which ontological interpretation (if any) is correct, but each may reveal something significant about the nature of physical reality at a very fundamental level. One should note that there are by now also different versions of the Bohm theory. Much attention has in recent years been given to a minimalist version known as "Bohmian mechanics" (see e.g. Goldstein 2013; for a balanced discussion of the relation between de Broglie's and Bohm's approaches, see Holland (2011)). Bohm himself developed from the mid-1970s, with Basil Hiley, a philosophically more radical version they called the "ontological interpretation," culminating in their 1993 book *The Undivided Universe*.

How, then, might Bohm's theory be relevant to the mind–matter relationship and to the causal status of consciousness in particular? The theory postulates that an electron is a particle, always accompanied by a new type of field, which guides its behavior—thus the name "pilot wave theory" which is sometimes used. Jack Sarfatti has characterized the Bohmian electron imaginatively by saying that it consists of a "thought-like" pilot wave, guiding a "rock-like" particle. This metaphor suggests that matter at the quantum level is fundamentally different from the sort of mechanical matter of classical physics that is presupposed in philosophy of mind by typical materialists. If even the basic elements that constitute us have "thought-like" and "rock-like" aspects, then it is perhaps not so surprising that a very complex aggregate of such elements (such as a human being) has a body, accompanied by a mind that guides it.

But, one might think, this is merely a vague metaphor. Now, Bohm himself realized in the early 1980s that the pilot wave might be more literally "thought-like" in a very interesting sense. He considered the mathematical expression of the so-called quantum potential, which describes the way the pilot wave affects the particle. He realized that the quantum potential, and thus the effect of the wave upon the particle, only depends on the form or shape of the wave, not on the size or amplitude of the wave (mathematically, the quantum potential depends only on the second spatial derivative of the amplitude of the wave). He went on to suggest that the quantum wave is literally putting form into, or in-forming, the motion of the particle along its trajectory, rather than pushing and pulling it mechanically.

Note that we are here talking about information for the electron, not information for us—we are thus thinking about information as an objective commodity that exists out there in the world, independently of us, guiding and organizing physical processes. The form of the quantum wave reflects the form of the environment of the particle—for example the presence of slits in the famous two-slit experiment. In this experiment, electrons arrive one by one at the detecting screen at localized points, suggesting that they are particles. Yet as we keep on watching, the individual spots build up an interference pattern, suggesting that each individual electron *also* has wave properties. Remember that in the Bohm theory the electron is seen as a particle *and* a wave. In the two-slit experiment the particle goes through one of the slits. The wave goes through both slits, interferes and guides or in-forms the particle in such a way that an interference pattern is formed as many electrons pass through the slit system. It thus seems that with the help of the notion of active information we can have a realist interpretation of the quantum theory, without the usual puzzles, such as Schrödinger's cats, many worlds, or the consciousness of the observer producing physical reality (for details see Bohm and Hiley 1987, 1993).

What happens with the electron is somewhat analogous to a ship on autopilot, guided by radar waves that carry information about the environment of the ship. The radar waves are not pushing and pulling the ship, but rather in-forming the much greater energy of the ship. Bohm generalized this into a notion of "active information"—which applies in situations where a form with smaller energy enters and informs a larger energy. We see this not only with various artificial devices, but also in the way the form of the DNA molecule informs biological processes, and even in the way forms act in human subjective experience (for example, seeing the form of a shadow in a dark night and interpreting it as "danger" may give rise to a powerful psychosomatic reaction). Indeed, Bohm (1990) sketched out how the active information approach could be developed into a theory of mind and matter.

While the radar-wave analogy helps us to understand the Bohmian electron, it is important to realize that the quantum potential has some radically holistic properties that go beyond what is implied by such mechanical analogies. In particular, in the many-body system there can be a nonlocal connection between particles that depends on the quantum state of the whole, in a way that cannot be expressed in terms of the relationships of the particles alone. Bearing in mind that this quantum state involves active information, we can note an interesting connection to Tononi's idea of integrated information. It is likely that the many-body quantum state involves the most radically holistic (integrated) information that science has thus far detected, thus making it interesting to consider its role when trying to understand consciousness as integrated information.

6 Bohm's Sketch for a Theory of the Relation of Mind and Matter

Bohm proposed that we understand mental states as involving a hierarchy of levels of active information. We typically not merely think about objects in the external world, but we can also become aware of our thinking. He suggested that such meta-level awareness typically involves a higher level of thought. This higher level gathers information about the lower level. But because its essential nature is active information, it not merely makes a passive representation of the lower level. Rather, the higher level also acts to organize the lower level, somewhat analogously to the way the active information in the pilot wave acts to organize the movement of the particle. (In particular, the higher level of thought can organize the content in the lower level into a coherent whole. This could be seen as a kind of "integrated information" and suggests yet another connection with Tononi's integrated information theory of consciousness.) And of course, we can become aware of this higher level of thought from a yet higher level, and so on.

How then does mind, understood as a hierarchy of levels of active information, connect with matter in the Bohmian scheme? First of all, he suggested that it is natural to extend the quantum ontology. So just as there is a pilot wave that guides the particle, there can be a super-pilot wave that guides the first-order pilot wave, and so on. (He claimed that such an extension is "natural" from the mathematical point of view.) Now it seems that we have two hierarchies, one for mind and another for matter. His next step was to postulate that these are the same hierarchy, so that there is only one hierarchy. This then allows, at least in principle, for a new way of understanding how mind can affect the body. Information at a given level of active information in the mind can act downwards, all the way to the active information in the pilot waves of particles in, say, the synapses or neural microtubules, and this influence can then be amplified to signals in the motor cortex, leading to a physical movement of the body.

Bohm's proposal differs strongly from the usual theories in cognitive neuroscience. Most neuroscientists ignore quantum considerations and seek the "neural correlates of consciousness" in some macroscopic neural phenomena, which can presumably be understood in terms of classical physics. Yet Bohm is proposing that mind, understood as a hierarchy of levels of active information, is implemented in (or perhaps even identical with) a hierarchy of super-quantum fields. However, these fields are not separate from the macroscopic neural processes. On the contrary, the role of the fields is in the end to gather information about the manifest neural processes and, on the basis of what this information means, to organize and guide them.

One should acknowledge that it is a tremendous challenge to work out an empirically testable theory along the Bohmian lines. The ideas described above provide a scheme for such an endeavor, rather than a fully developed theory. Bohm and Pylkkänen (1992) were discussing ways to develop the scheme in the late 1980s and early 1990s. In a later development, Hiley and Pylkkänen

(2005) discussed the prospects of applying the Bohm scheme to Beck and Eccles's quantum model of synaptic exocytosis (for a review of Beck and Eccles's model, as well as other quantum approaches to consciousness, see Atmanspacher 2011). While this may be a small step forward, problems remain. For example, Henry Stapp (2005) has pointed out that the sort of interference of the mind upon the laws of quantum mechanics that the Bohmian scheme involves can lead to serious problems with special relativity. This is a challenge that future research along Bohmian lines needs to face. A possible way for meeting this challenge is opened up by a recent study on the nature of nonlocal quantum information transfer by Walleczek and Grössing (2016).

While the possibility of non-negligible quantum effects in the brain is often dismissed as implausible, there are interesting recent advances in quantum biology. And it is already part of mainstream neuroscience that the retina acts to amplify the effects of individual photons. Also, researchers such as Stuart Hameroff and Roger Penrose (2014) have discussed in great detail how quantum effects might play a role in neural processes via quantum coherence and collapse in neural microtubules. Connecting the Hameroff–Penrose work with the Bohm scheme is one potentially fruitful line for future research. Indeed I have begun to explore these connections together with Hameroff and the philosopher Rocco Gennaro, who is a specialist on higher-order (HO) theories of consciousness (which seem to fit together with Bohm's idea of the mind as a hierarchy of levels of information). (For an early result of this cooperation, focusing on combining HO theories with Penrose and Hameroff's orchestrated objective reduction (ORCH-OR) hypothesis, leading to "deeper order thought" (DOT), see Hameroff et al. 2014.)

Note that Bohm introduced a new category, namely information, to the debate. Is information physical or mental? He suggested that it is simultaneously both physical and mental, or has these two as its aspects. This sort of view is called a double-aspect theory in philosophy of mind. The traditional worry with double-aspect views is that the underlying thing, which has the aspects, is left as a mystery. The hypothesis that information is the fundamental, underlying feature of reality can be seen as a way to alleviate this worry.

7 Understanding Consciousness in the Active Information Scheme

A common criticism of contemporary theories in the philosophy of mind—such as identity theory and functionalism—is that they leave out conscious experience, instead of explaining it (Searle 1992). How might conscious experience fit into the active information scheme? In particular, is it possible to understand the causal status of consciousness in this scheme? While Bohm saw nature as a dynamic process where information and meaning play a key dynamic role, he assumed that "99.99 per cent" of our meanings are not conscious (Bohm in discussion with Renée Weber 1987, p. 439). Thus, for example, he thought it obvious that the particles of physics are not conscious. But how can one then address the problem of consciousness in this scheme? In other words, why is there sometimes conscious experience associated with the activity of information (as seems obvious at least with humans and higher animals)? Why doesn't all the activity of information in humans proceed "in the dark," as it seems to do in physical and biological processes in general? And does the presence of consciousness make a causal difference? Bohm himself did not say much about the hard problem of consciousness (he died a little before the hard problem was made the center of attention by David Chalmers in the 1994 Tucson consciousness conference). However, I have suggested that the most natural context to explore this issue is some version of an HO theory of consciousness (Pylkkänen 2007, p. 247). Let us here expand somewhat on this idea.

As we saw above, the basic idea of higher-order theories of consciousness, when expressed in terms of the notion of information, is to postulate that what makes a given mental state (or level of information or mental activity) conscious is that there exists a higher level of (typically) unconscious information, which has the content that one is in the first-order mental state or activity.

Note also that David Chalmers famously suggested that we tackle the hard problem of consciousness with a double-aspect theory of information. The idea is that information is a fundamental feature of the world, which always has both a phenomenal and a physical aspect. Now, we could take this idea to the Bohm scheme and postulate that active information, too, has phenomenal properties. This then raises the question about what we should think about the active information in the pilot wave of an electron. Does it, too, have phenomenal properties in some sense? Bohm went as far as to say that electrons have a "primitive mind-like quality," but by "mind" he was here referring to the "activity of form," rather than conscious phenomenal experience in any full sense.

I think that it is reasonable to combine Chalmers's hypothesis to active information, but we need to restrict the hypothesis. For example, we could say that a certain kind of active information (e.g., a holistic active information that is analogous to quantum active information) has the potentiality for phenomenal properties, but a potentiality that is actualized only in suitable circumstances (e.g., when a given level of active information is the intentional target of a higher level of active information; or if we want to follow an approach similar to that of Tononi, we could say that suitably integrated active information is conscious). Of course, this also opens up the possibility for genuine artificial consciousness. If we could implement quantum-like holistic active information in an artificial system and set up a suitable higher-order relationship of levels in the system, phenomenal properties should actualize themselves, according to this hypothesis. (Or, in a Tononian approach, if active information is suitably integrated in an artificial context, it would be conscious.)

We should acknowledge that Bohm and Hiley's proposal about active information at the quantum level is radical and somewhat controversial, for they are in effect suggesting that this type of information ought to be acknowledged as a fundamental—perhaps *the* fundamental—category of physics. Indeed, they wrote in 1984: "the notion of a particle responding actively to information in the [quantum] field is ... far more subtle and dynamic than any others that have hitherto been supposed to be fundamental in physics." This proposal is still mostly ignored within the physics community. There are some technical issues with the proposal, but in my view a major reason for its being ignored is that it goes so much against the prevalent mechanistic way of thinking in physics. However, some leading thinkers do take it seriously, for example Smith (2003). Also, an interesting adaptation of the active information scheme to neuroscience has been proposed by Filk (2012). In the field of the social sciences, Khrennikov (2004) has made imaginative use of the proposal and the Bohm theory has also been applied to financial processes by Choustova (2007) and Haven (2005). Of course, the notion of "quantum information" has been widely discussed in recent years (see e.g. Bouwmeester et al. 2000). The advantages of the concept of active information over quantum information, when discussing some quantum experiments, have been argued for by Maroney (2002); see also Maroney and Hiley (1999).

To summarize: Bohm's suggestion was that a natural extension of his ontological interpretation of the quantum theory can include mental processes and even conscious experience into a single coherent view. From the point of view of the question about the causal powers of consciousness this view is particularly promising, for it makes it—at least in principle—possible to understand how conscious experience, via its effects upon information, could make a difference to physical processes. If we can provide an intelligible theory about how conscious experience can make a difference to information, this scheme provides a view of how such informational differences can then affect manifest physical processes (see also Hiley and Pylkkänen 2005). We have hinted that this question can be approached within some of the already existing available theories of consciousness—for example, higher order theories or Tononi's integrated information theory.

8 Active Information and the Causal Powers of Consciousness

The view described above sketches out how information content might affect manifest physical processes (e.g., bodily behavior) in a way that is coherent with the principles of physics. We have already touched on the question of the causal role of consciousness in the active information scheme. Let us now consider this role in more detail. First of all, how can we understand the idea that consciousness enables more flexible control in the context of the active information view? More flexible control means, for example, that the organism is able to choose from among different options the one that best fits the situation, instead of having to follow mechanically one of the options. In Bohmian terms this means that consciousness enables the organism to suspend the activity of information. The way this works is that one is aware of information that means something like "It is reasonable to consider different options before acting." And when one finally acts, this is based on information that means "It is reasonable to do X." In other words, flexible control in the Bohmian view seems to involve higher-order, meta-level information that we are conscious of (while typically, according to higher order theories, we need not be conscious of the higher-order thought itself).

When it comes to better social coordination, Bohm's view involves a notion he calls "common pools of information" (Bohm 1990). This notion applies strikingly well at the quantum level (e.g., in the Bohmian view of superconductivity) where the behavior of a system of particles can in some situations be organized by information in the so called many-body wave function. The particles act together in an organized way (e.g., electrons may pass obstacles in a wire, which results in very low resistance). Information at the level of human cognition operates presumably according to different principles from information at the quantum level. However, when a group of people communicate with each other (e.g., in a group discussion) they begin to build up a common pool of information. This enables the group to develop common intentions and carry out common actions (see e.g. Tuomela 2013). Suppose, for example, that a group of eight people need to carry a very heavy grand piano upstairs along a narrow staircase. They need to exchange information and make sure that they each understand what they are supposed to do. Again, it is hard to imagine that such joint tasks requiring collective coordination could take place without some consciousness of the shared information. However, it is an experimental question to ask to what extent such collective action is possible without conscious awareness. Going back to our above example, it does seem difficult to act without conscious awareness at least in a situation where the group needs to carry the piano through a very narrow opening. While the mainstream literature in the field of collective or shared intentionality does not consider quantum principles, there is at the very least an interesting analogy between Bohm's notion of common pools of information at the quantum level and the notion of collective intentionality in social ontology. Some researchers have even explored whether social phenomena might involve quantum principles more literally. See, for example, Alexander Wendt's (2015) recent ground-breaking study, as well as Flender et al.'s (2009) radical approach to the shared intentionality of the mother-infant relationship, making use of quantum principles in a phenomenological context.

We have also considered the suggestion that consciousness enables more unified and integrated representation. The tricky question here is whether the information first gets unified and integrated in preconscious processes, and is then presented to consciousness; or whether consciousness plays a role in the very unification and integration of the information (Van Gulick seems to favor the latter alternative). I am inclined to think that much of the unification and integration takes place (largely) without consciousness, but that consciousness is needed for such information to be flexibly usable in the control of behavior (of course, in the Tononian approach one would say that sufficiently integrated information *constitutes* consciousness). In the Bohmian picture it is assumed that typically such information tends to act, even if it is not consciously attended to. Conscious attention may then make the response of information stronger, or lead to the suspension of action and reflection of the different options.

The idea that consciousness involves more global access can also be naturally understood in terms of the notion of active information. If information is consciously attended to, this may start what Bohm (2003) calls a "signa-somatic" flow: the significance of the information acts somatically toward a more manifest level in the brain. Global access means that the significance can affect many different modules.

When it comes to free will, Bohm used to emphasize that true freedom is typically limited by our lack of knowledge—both about the consequences of one's actions and about our true motives. He refers to Schopenhauer when he writes: “though we may perhaps be free to choose as we will, we are not free to will *the content of the will* … Is there any meaning to freedom of will when the content of this will is … determined by false knowledge of what is possible” (Bohm 1986). In a more positive vein, he writes:

> How, then, is it possible for there to be the self-awareness that is required for true freedom? … I propose that self-awareness requires that consciousness sink into its implicate (and now mainly unconscious) order. It may then be possible to be directly aware, in the present, of the actual activity of past knowledge, and especially of that knowledge which is … false … Then the mind may be free of its bondage to the active confusion that is enfolded in its past. (Ibid.)

By “the implicate order” Bohm above refers (roughly) to the more subtle levels of active information which include long-term memory and from which the part of the content of conscious experience unfolds. It is clear that for Bohm free will requires consciousness. However, it is not enough that we are conscious of the options that we typically face in a situation when we are about to make a choice. We also need to be aware of—and thus free from—falsity in the past knowledge that we typically unconsciously hold and on the basis of which we tend to react and make our choices.

Let us finally consider intrinsic motivation in the light of the Bohmian view. What is interesting here is that Bohm emphasizes that information is typically active (while passive information is a special case). One possibility is that the presence of consciousness increases the level of activity of the information. Thus, for example, consciousness of information with an attractive content may be needed to awaken desire or make that desire more intense. At the same time conscious awareness of the negative consequences of carrying out a particular desire may lead to the suspension of action. In Bohmian terms, all these phases involve active information. For example, desire informs us to carry out a certain action X, while information about the consequences of the action may result in information with the content “It is not reasonable to do X.”

9 Concluding Discussion

I have drawn on fundamental physics to support the idea that conscious experiences can, at least in principle, be causally efficacious in a physical world, contrary to what much of contemporary physicalism suggests. Yet we have admittedly only scratched the surface of this difficult topic. Basically, I have assumed that consciousness (understood as something that arises due to higher order information and/or information integration) can influence lower-level information, and information in turn can influence physical processes “signa-somatically,” as Bohm would put it.

The Bohmian view we considered suggests that nature can be understood as a two-way movement between the aspects of soma (the physical) and significance (information, meaning, the mental). Consciousness comes in here, but only at the higher, subtler levels, where, say, suitable higher-order relations (and/or a sufficient degree of information integration) prevail, depending upon which theory of consciousness we are relying upon. Thus the active information view is consistent with the idea—also supported by recent experimental work—that much of our most sophisticated brain functions work totally independently of consciousness. Yet the active information view also makes room for the genuine causal powers of consciousness, and in this way can accommodate such causal efficacy of consciousness as is suggested by Van Gulick, Revonsuo, and others. Bohm himself did not address very explicitly the causal powers of consciousness, but I think it is reasonable to assume that his scheme makes such powers in principle possible. To explain that scheme fully is, however, not possible here, and the interested reader is referred to a more detailed study (Pylkkänen 2007).

One important potential criticism of the active information approach has to do with the notion of information that is presupposed. Is it really justified to use the term “information” to describe the sorts of processes connected to the quantum field? One could examine this question in the light of the recent developments in the philosophy of information (e.g., Floridi 2015). Floridi distinguishes between environmental and semantic information; and semantic information can be further distinguished into factual and instructional information. The quantum active information is *about* something (the environment, slits, etc.), it is *for* the particle and it helps to *bring about* something (a certain movement of the particle). This suggests that it is semantic and has both factual and instructional aspects, though this issue needs to be explored more carefully in future research. Also, Maleeh and Amani (2012) have usefully considered active information in relation to Roederer’s (2005) notion of pragmatic information, suggesting that only biological systems are capable of “genuine” information processing. I think one can argue that Bohmian quantum information potential involves genuine information processing (indeed, the most fundamental kind of genuine information processing science has thus far discovered), but this will also need to be explored in future research.

I would like to end by reflecting upon the quote from Plato’s *Phaedo* (1892) provided at the start of the chapter. Plato there thinks it obvious that our physical actions depend upon "the choice of the best," while a typical materialist would say that insofar as physical actions are determined, they are determined by the physical state in a previous moment (including "bones and muscles"). Now, I think that the active information view allows for a naturalistic grounding of Plato's view. In their 1984 article Bohm and Hiley note that there are good reasons for expecting that quantum theory, and therefore the notion of a quantum information potential, would be relevant when we are studying consciousness itself, as based on the material structure of the brain and nervous system:

> it may well be that in our mental processes, the quantum information potential is significant (as is, for example, suggested by the fact that information regarded as correct is active in determining our behaviour, while as soon as it is regarded as incorrect, it ceases to be active). The quantum theory may then play a key part in understanding this domain. (1984, p. 269)

The above implies that our veridicality judgments play a key role in determining whether or not information acts. For example, if I judge a shadow in a dark night to mean "an assailant" and thus "danger," this typically gives rise to a powerful psychosomatic reaction; if I a little later notice that it was merely a shadow of a branch (i.e., that the earlier judgment was incorrect), I will typically calm down. We could expand the idea toward Plato by assuming that our ethical judgments (e.g., "the choice of the best") can typically also affect the way information is activated, and consequently our behavior. The quantum theoretical active information scheme enables such activity of information to reach all the way to the level of fundamental physics, and in this way we can begin, in a new way, to make sense of a perennial puzzle in Western philosophy, namely the place and role of minds, meanings, and morals in the physical world.

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Big Challenges Section

Why Quantum?

Andrei Khrennikov

1 Introduction

Application of the methods of quantum mechanics (QM) outside of physics is a novel and rapidly developing area of multidisciplinary research unifying quantum information and probability, open quantum systems, and the foundations of probability with molecular biology, genetics and epigenetics, cognitive psychology, decision-making, economics and finance, and social science and politics.[1] The multidisciplinary structure may induce communication problems related to the use of quantum terminology.

[1]See, for example, de Barros and Suppes (2009), de Barros (2012), Accardi et al. (2008, 2009), Aerts et al. (2014a, 2013a,b), Asano et al. (2010a,b, 2011, 2013b, 2012a,b,c, 2013a, 2014, 2015), Atmanspacher et al. (2002, 2009), Atmanspacher and Filk (2012, 2013), Basieva et al. (2010), Basieva and Khrennikov (2012, 2014), Blutner et al. (2013), Bruza et al. (2009, 2010), Busemeyer and Bruza (2012), Busemeyer and Wang (2007), Busemeyer et al. (2006a,b, 2008, 2009, 2011), Cheon and Takahashi (2010), Cheon and Tsutsui (2006), Conte et al. (2006, 2008, 2009), Dzhafarov and Kujala (2012a,b, 2013, 2014), Fichtner et al. (2008), Ezhov and Khrennikov (2005), Ezhov et al. (2008), Haven and Khrennikov (2009, 2012, 2013), Ishio and Haven (2009), Khrennikov (2003, 2004, 2006, 2008, 2009, 2010), Khrennikov and Basieva (2014a,b), Khrennikov et al. (2014), Khrennikov and Haven (2013), Khrennikova et al. (2014), Khrennikova (2012), Lambert-Mogiliansky and Busemeyer (2012), Ohya and Volovich (2011), Pothos and Busemeyer (2009, 2013), Pothos et al. (2013), Yukalov and Sornette (2011), Wang and Busemeyer (2013), Wang et al. (2013).

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First of all, I will clarify for experts in quantum foundations, quantum information, and quantum probability (QP) the possibility of the application of QM methods outside of physics in Sect. 2 (this motivating section will also be useful for nonphysicists using the mathematical apparatus of QM). I discuss the necessity of the generalization of this formalism to cover all problems generated by the probabilistic description of bio- and sociophenomena. I then introduce the basic elements of quantum physics—see Haven and Khrennikov (2016) for more detail (and, especially, its advanced parts such as the theory of open quantum systems)—to experts in cognition, psychology, economics and finance, social science, molecular biology, and genetics, which will also serve to motivate them to explore the mathematical methods of QM, in Sect. 3. I also discuss the distinguishing features of *quantum adaptive dynamics*—a generalization of the standard theory of open quantum systems which is used in this chapter.

2 Quantum Biophysics or Quantum Information Biology?

From the very beginning I must emphasize that applications of the methods of QM to the modeling of cognition have no direct relation to *physical quantum processing of information by the brain*, for example, theories of Penrose (1989, 1994) and Hameroff (1994a,b). The superposition of mental states is the key issue here. In principle, state superpositions can be treated from the purely information viewpoint. For the moment, there is no commonly accepted neurophysiological model of the creation of mental superpositions; compare with Khrennikov (2011). However, the appearance of such superpositions in the process of decision-making can be (at least indirectly) confirmed by statistical data collected in molecular biology and cognitive psychology (Hofstader 1983, 1985; Shafir and Tversky 1992; Tversky and Shafir 1992; Croson 1999; Inada et al. 1996) which leads to violation of the laws of classical probability (CP)[2] and, in particular, violation of the law of total probability

[2]See, for example, Asano et al. (2010a,b, 2011, 2013b, 2012a,b,c, 2013a, 2014, 2015); Atmanspacher et al. (2002, 2009); Atmanspacher and Filk (2012, 2013); Basieva et al. (2010); Basieva and Khrennikov (2012, 2014); Blutner et al. (2013); Bruza et al. (2009, 2010); Busemeyer and Bruza (2012); Busemeyer et al. (2006a); Busemeyer and Wang (2007); Busemeyer et al. (2006b, 2008, 2009, 2011); Cheon and Takahashi (2010); Cheon and Tsutsui (2006); Conte et al. (2006, 2008, 2009); Dzhafarov and Kujala (2012a,b, 2013, 2014); Khrennikov (2003, 2004, 2006, 2008, 2009, 2010); Khrennikov and Basieva (2014a); Khrennikov et al. (2014); Khrennikov and Basieva (2014b); Khrennikov and Haven (2013); Khrennikova et al. (2014); Khrennikova (2012); Lambert-Mogiliansky and Busemeyer (2012); Ohya and Volovich (2011); Pothos and

(LTP). As was shown by Feynman and Hibbs (1965), in quantum physics the superposition of states leads to violation of LTP. In Khrennikov (2010) it was shown that the inverse is true as well: for probabilistic data of any origin, violation of LTP allows for the representation of states by complex probability amplitudes—*the constructive wave function approach.* The entanglement of mental states is another key issue. In physics, Bell's inequality is used as a statistical test of entanglement. The same can be done for decision-making, for example, by humans, see Conte et al. (2008), Bruza et al. (2010).

As in the case of quantum physics,[3] we can proceed by treating the mathematical formalism of QM as an *operational formalism* describing measurements (Khrennikov 2010); see also Plotnitsky (2015). Neither QM nor cognitive science can explain why the systems under study produce such random outputs. Moreover, according to the Copenhagen interpretation, it is in principle impossible to provide some "explanation" of quantum behavior, for example, by using a more detailed description with the aid of so-called hidden variables. In spite of this explanatory gap, QM is one of the most successful scientific theories. One may hope that a similar operational approach would finally lead to the creation of a novel and fruitful theory of decision-making. In recent studies by D'Ariano et al. (Chiribella et al. 2010; D'Ariano 2007; Chiribella et al. 2012; D'Ariano 2011; D'Ariano and Jaeger 2009) quantum theory is asserted to have been derived from a set of purely operational postulates. Such an approach can be applied to the theory of decision-making as a special theory of (self-)measurements.

I would also emphasize that recently the *informational interpretation* of the quantum state has started to play an important role in QM and, especially, in quantum information theory.[4] In this interpretation QM formalism is merely about the information processing related to experiments. The information interpretation matches my aims perfectly (although its creators probably would not support the attempts to apply it outside of physics).

Of course, cognition is a special biophysical phenomenon. Therefore, the quantum-mental analogy has to be used with some reservation. Consider, for

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Busemeyer (2009, 2013); Pothos et al. (2013); Yukalov and Sornette (2011); Wang and Busemeyer (2013); Wang et al. (2013); Haven and Khrennikov (2016); Penrose (1989, 1994); Pothos and Busemeyer (2009, 2013); Pothos et al. (2013); Yukalov and Sornette (2011); Wang and Busemeyer (2013); Wang et al. (2013).

[3]Bohr always pointed out that quantum theory describes the results of measurements and emphasized the role of an observer (Bohr 1987); he stressed that the whole experimental arrangement has to be taken into account. Heisenberg and Pauli had similar views; see Plotnitsky (2006, 2009, 2011) for detailed analysis.

[4]See the works of A. Zeilinger and C. Brukner (Zeilinger 1999, 2010; Brukner and Zeilinger 1999a,b, 2009), C. Fuchs et al. (Caves et al. 2002; Fuchs 2002a,b, 2007; Fuchs and Schack 2011) and M. D'Ariano et al. (Chiribella et al. 2010; D'Ariano 2007; Chiribella et al. 2012; D'Ariano 2011; D'Ariano and Jaeger 2009).

example, the issue of nonlocality. The brain is a very small physical system (compared with distances covered by propagating light); see de Barros and Suppes (2009). Therefore "mental nonlocality" (restricted to information states produced by a single brain) is not as mystical as the physical nonlocality of QM. In principle, the possibility of the future "explanation" of cognition, for example, in terms of nonlocal hidden variables, is not ruled out.

I hope that previous considerations have justified the possibility of application of QM formalism to model information processing by biosystems (from proteins and cells to brains and social systems) and, in particular, to model decision-making. However, one may ask[5]:

> Why should it be precisely the QM formalism? May be its generalization would be more adequate to problems of information biology?

Pragmatically the use of QM formalism can be treated as the first trial for probing nonclassical probabilistic methods. And at the first stage of its application outside of physics, for example, in cognitive psychology, this strategy was very successful.

Later, however, more detailed studies (Khrennikov 2010; Khrennikov et al. 2014) demonstrated that some generalizations are needed indeed. As we already know, even problems of quantum physics cannot be completely covered by the standard quantum formalism based on the representation of quantum observables by Hermitian operators. A more general theory of quantum instruments (a part of the theory of open quantum systems) was developed and applied to a variety of problems, especially in quantum optics. Here the generalized quantum observables are represented by *positive-operator valued measures* (POVMs); see Haven and Khrennikov (2016).

The necessity of the application of generalized quantum observables, POVMs, to cognition and decision-making was emphasized in Khrennikov (2010) and Khrennikov et al. (2014). (This handbook contains a brief introduction to the theory of quantum instruments with a model of decision-making (Basieva and Khrennikov 2016).) Applications of POVMs to cognition and decision-making were presented in Asano et al. (2010a,b, 2011, 2015), Basieva and Khrennikov (2014), to molecular biology in Basieva et al. (2010), Asano et al. (2015) and to epigenetics in Asano et al. (2013a).

[5] This question was asked by A. Zeilinger during a lecture given by the author in Vienna, May 2014.

However, it seems that even the use of quantum instruments does not solve all problems (Khrennikov et al. 2014): cognition and the biological processing of information in general have a more complex structure than the one covered by the "standard generalized quantum observables." Generalizations of POVMs were invented and explored in decision-making (Khrennikov 2009; Basieva and Khrennikov 2012); but they did not satisfy the normalization condition $\sum_j M_j = I$, see (Haven and Khrennikov 2016, Sect. 8).

Recently a novel formalism known as *quantum adaptive dynamics* was developed (Asano et al. 2013b) and applied to a variety of problems of cognition modeling and molecular biology (Asano et al. 2015, 2013b, 2012a,b,c, 2013a). We shall discuss this briefly below.

Finally, we point to a new model, so-called *hyperbolic quantum mechanics*, which has been applied to a series of problems of cognition and decision-making (Khrennikov 2010). Here, probability amplitudes are valued not in the field of complex numbers, but in the algebra of hyperbolic numbers, numbers of the form $z = x + jy$, where x, y are real numbers and j is the generator of the algebra satisfying the equation $j^2 = +1$. Hyperbolic amplitudes describe interference which is stronger than the one given by complex amplitudes used in QM: the interference term is given by the hyperbolic cosine, opposite to the ordinary trigonometric cosine in QM.

3 Open Quantum Systems, Adaptive Dynamics

The dynamics of an isolated quantum system is described by the Schrödinger equation (Haven and Khrennikov 2016). In the standard quantum framework measurements are mathematically represented by orthogonal projectors (P_i) onto eigen-subspaces corresponding to the observed values (a_i). A quantum observable can be formally represented as a Hermitian operator $A = \sum_i a_i P_i$. The probability of obtaining the fixed value a_i as the result of measurement is given by *Born's rule*. Let a system have the physical state given by a normalized vector ψ of the complex Hilbert space H, that is, $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$, where $\langle\cdot|\cdot\rangle$ is the scalar product on H. (Such states are called *pure states*.) Then (Born's rule (Haven and Khrennikov 2016)):

$$p(a_i) = \langle P_i\psi|\psi\rangle = \|P_i\psi\|^2 \tag{1}$$

and the post-measurement state is given by

$$\psi_{a_i} = P_i\psi/\|P_i\psi\|. \tag{2}$$

In the finite dimensional case $\psi = (z_1, \ldots, z_n), z_j \in \mathbf{C}, \sum_j |z_j|^2 = 1$. By Born's rule probability is expressed through complex "probability amplitudes" given by the coordinates of ψ. Born's rule (1) is the basis of the QP calculus: the calculus of such amplitudes.

However, the situation that an isolated quantum system propagates in space–time and then suddenly meets a measurement device is too ideal. In the real world a quantum system interacts with other systems, that is, the presence of an environment cannot be ignored. In particular, measurement devices can also be treated as special environments. The corresponding part of quantum theory is known as the theory of *open quantum systems.* Here the dynamics of a state is described by the *quantum master equation,* the Markovian approximation of which is known as the *Gorini–Kossakowski–Sudarshan–Lindblad* (GKSL) equation, for example, Ohya and Volovich (2011).[6] This equation has numerous applications to problems in quantum physics, especially quantum optics. It was applied to model decision-making (in games of the prisoner's dilemma type) in a series of papers (Asano et al. 2010a,b, 2011); see also Khrennikova et al. (2014) for applications for decision-making in political science. However, the GKSL equation is an approximation and its derivation is based on a number of assumptions essentially narrowing the domain of its applications; see Khrennikova et al. (2014) for analysis of the validity of these assumptions in the modeling of cognition. One of the main assumptions is that the environment is huge compared to the system under study. We cannot apply the GKSL equation to model the state dynamics of an electron interacting with another electron considered as the environment.

In decision-making typically there are two brain functions, for example, sensation and perception (Accardi et al. 2016), which interact and produce the output of one of them, for example, the output perception. In such a situation, the GKSL equation is not applicable. In our work (Asano et al. 2013b) we developed a general theory of *quantum adaptive dynamics* generalizing the standard theory of open quantum systems. Here the state dynamics is described by a more general class of state transformations than in the standard theory.

[6]Note that the quantum master equation (as well as the Schrödinger equation) is a linear first-order (with respect to time) differential equation. Linearity is one of the fundamental features of quantum theory. It is very attractive even from the purely operational viewpoint, since it simplifies calculations dramatically. In fact, no mathematical tool more advanced than matrix calculus is required. The question whether QM can be treated as the linearization of a more complex nonlinear theory has been actively discussed in quantum foundations. For the moment, it is commonly accepted that QM is fundamentally linear, although strong reasons in favor of the linearization hypothesis have been presented. This problem is also very important for cognition but lacks even preliminary analysis, unlike the situation in physics.

In the paper (Accardi et al. 2016) of this handbook we use the quantum adaptive dynamics to describe the process of the creation of perceptions from sensations.

Roughly speaking, the approach relaxes the standard constraints on the class of state transformations. In particular, in the theory of open quantum systems all state transformations are the so-called *completely positive maps,* see Basieva and Khrennikov (2016) in this handbook. An adaptive dynamical map need not be completely positive nor even simply positive.

The tricky point is that in quantum physics, for a given state, one can, in principle, measure any observable. It seems that in problems of decision-making this (very strong) assumption has to be relaxed. I work with generalized states which permit measurements of only special (state-dependent) classes of observables. In this chapter only two observables are considered: sensation and perception. The class of adaptive dynamical maps is essentially larger than the class of (completely) positive maps. This simplifies the modeling of concrete phenomena, for example, recognition of ambiguous figures (Accardi et al. 2016). Thus it seems that complete positivity is the distinguishing feature of the operational quantum formalism for physical systems. For biological systems and, in particular, cognitive systems, this strong mathematical assumption can be relaxed. (Of course, it makes the mathematical formalism essentially more complicated). In any event creation of quantum information biology stimulates applications of state transformations which are not completely positive.

We finish this chapter with emphasizing the role of creation of quantum information biology, as the general operational formalism describing probabilistic behavior of biological systems, from proteins and cells to cognitive and ecological systems. Moreover, this was the crucial step towards unification of the mental and physical phenomena, without the reduction of mental processes to physical ones. The same mathematical formalism describes processing of information by electrons, photons, and other quantum systems as well as by a variety of biological systems.

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Quantum Principles and Mathematical Models in Physics and Beyond

Arkady Plotnitsky

1 Introduction

The history of mathematical modeling beyond physics—in biology, neuroscience, psychology, and economics—has been and still is dominated by classical mathematical models (hereafter C-models), primarily of probabilistic and statistical nature, models inspired by classical statistical physics or chaos (or complexity) theory. More recently, however, quantum mathematical models (hereafter Q-models), that is, mathematical models based in the mathematical formalism of quantum theory, have acquired a certain currency in mathematical modeling, especially in psychology and economics, my primary subjects here, beyond quantum mechanics (QM) (Pothos and Busemeyer 2013).[1] My abbreviations mirror Dirac's distinction between c-numbers and q-numbers, and more than merely mirror it: the variables used in Q-models are q-numbers. This chapter examines some of the reasons for using Q-models in these fields.

[1]By quantum *theory* I refer to the standard versions of QM or quantum field theory, rather than alternative theories of quantum phenomena, such as Bohmian theories, although the corresponding mathematical models have been used in psychology and economics. By quantum phenomena I refer to those physical phenomena in considering which Planck's constant, h, must be taken into account; by quantum objects I refer to those entities in nature that are responsible for the appearance of quantum phenomena.

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In order to do this, I shall revisit the reasons for the use of Q-models in physics itself, to which a large portion of this chapter is devoted. These models are based in the mathematics of Hilbert spaces over *complex* numbers, with Hilbert-space *operators* (q-numbers) taking the role of variables in the equations of QM, as against the *functions of real variables* that serve in this role in classical physics. In particular, in the words of Heisenberg's and Dirac's parallel titles of their famous books, I shall examine the fundamental *principles that led* to the development of quantum theory, including more recently quantum information theory, and consider a possible role for similar principles *in using* Q-models beyond quantum theory vs C-models (Heisenberg 1930; Dirac 1930). My emphasis reflects the fact that fields such as psychology and economics borrow Q-models from quantum theory, rather than derive them from their own internal principles, while the foundational research in quantum theory, especially quantum information theory, is still concerned with deriving Q-models from such fundamental principles. Nevertheless, the principle perspective may help us to understand better a possible and possibly necessary role for Q-models beyond physics as well.

2 The Principle Thinking in Physics and Beyond

I begin by explaining the concept of principle, as it is to be understood in this chapter, following Einstein's distinction between "constructive" and "principle" theories (Einstein 1919). This distinction implies two contrasting, although in practice sometimes intermixed, types of theories. According to Einstein, "constructive theories" aim "to build up a picture of the more complex phenomena out of the materials of a relatively simple formal scheme from which they start out" (Einstein 1919, p. 228). Einstein's example of a constructive theory in classical physics is the kinetic theory of gases, which "seeks to reduce mechanical, thermal, and diffusional processes to movements of molecules—i.e., to build them up out of the hypothesis of molecular motion," described by the laws of classical mechanics (Einstein 1919, p. 228).

By contrast, principle theories "employ the analytic, not the synthetic, method. The elements which form their basis and starting point are not hypothetically constructed but empirically discovered ones, general characteristics of natural processes, *principles* that give rise to mathematically formulated criteria which the separate processes or the theoretical representations of them have to satisfy" (Einstein 1919, p. 228; emphasis added). Thermodynamics, Einstein's example of a classical principle theory (parallel to the kinetic theory of gases as a constructive theory), is a principle theory because it "seeks by analytical means to deduce necessary conditions, which separate events have to satisfy, from the universally experienced fact that perpetual motion is impossible" (Einstein 1919, p. 228). Einstein's special and general relativity are principle theories as well, which fact occasioned his reflections on the subject, although both, especially general relativity, have constructive dimensions to them.

Principles, then, are "empirically discovered, general characteristics of natural processes, ... that give rise to mathematically formulated criteria which the separate processes or the theoretical representations of them have to satisfy." This definition will be adopted here, but with the following qualification. Principles are not so much empirically discovered as formulated, *constructed*, on the basis of empirically discovered or established evidence. It would be difficult to see "the impossibility of perpetual motion" as empirically given, rather than as a principle formulated on the basis of such evidence.

Constructive theories tend to be, and are often aimed to be, *realist*: they describe, usually causally, the corresponding objects in nature and their behavior by way of mathematical models, assumed to idealize how nature works at the simpler, or deeper, level thus constructed by a theory. This characterization will serve this chapter as the definition of a realist theory. Such a theory offers a description of the processes underlying and connecting the observable phenomena considered, on the model of classical mechanics, from which quantum theory departs. *By "reality" itself I shall refer to that which actually exists or is assumed to exist.* In the case of physics, it is nature that is generally assumed to exist independently of our interaction with it, and to have existed when we did not exist and to continue to exist when we will no longer exist.

All modern, post-Galilean, physical theories proceed by way of idealized mathematical models, whether realist, descriptive, or only predictive, as are some of the models used in quantum theory or its interpretations, models that are only predictive and moreover are only probabilistically or statistically predictive. Heisenberg adopts this view of QM or quantum electrodynamics, following Bohr and "the spirit of Copenhagen," as Heisenberg called it (Heisenberg 1930, p. iv).[2] This suggests a general definition of a mathematical model that I shall adopt here. *A mathematical model is a mathematical structure or set of mathematical structures that enables any type of relation, descriptive or predictive, to the (observed) phenomena or objects considered.*

Realist mathematical models are descriptive models: they are idealized mathematical descriptions of physical processes. The probabilistic or statistical

[2] I distinguish "the spirit of Copenhagen" from "the Copenhagen interpretation," because there is no single such interpretation.

character of quantum predictions must be equally maintained by realist interpretations of these theories or alternative theories (such as Bohmian theories), because it corresponds to what is observed in quantum experiments, concerning which only probabilistic or statistical predictions are possible. The repetition of identically prepared experiments, in general, leads to different outcomes, the difference that, unlike in classical physics, cannot be improved beyond a certain limit (defined by Planck's constant, h) by improving the conditions of measurement, a fact reflected in the uncertainty relations. This leads to the quantum probability principle, the QP principle, arguably the most important principle defining Q-models in physics and beyond. At the same time, predictive interpretations usually assume, as that of Bohr did, the concept of reality. If, however, *realism*, as defined here, presupposes a description or at least a conception of reality, this alternative concept of *reality* is that of "reality *without* realism" (Plotnitsky and Khrennikov 2015). Assuming this concept of reality as operative is itself a principle, "the RWR principle."

It follows that a principle theory could be realist or not, by, in the first case, unavoidably acquiring constructive dimensions. Constructive theories are, as I explained, by definition realist and are usually causal, unless one uses a given construction as a kind of heuristic device within a predictive (principle) theory. It is also true that a given theoretical construction may be revealed or argued merely to provide a predictive mechanism for a given theory.[3]

Constructive theories may and customarily do involve principles, such as the equivalence principle in general relativity, or the principle of causality, dominant throughout the history of modern physics, from Galileo on, until QM put it into question. This principle, as defined, for example, by Kant (this definition has been commonly used since), states that, if an event takes place, it has a cause of which it is an effect (Kant 1997, pp. 305, 308). I shall refer to this form of causality as *classical* causality.[4] Such causal influences are also commonly, although not always, assumed to propagate from past or present towards future. This requirement is strengthened by special relativity theory, which restricts causes to those occurring in the backward (past) light cone

[3]There are arguments to that effect concerning the status of space–times in general relativity (Butterfield and Isham 2001).

[4]I distinguish causality—which is an ontological category, describing reality—from determinism, which is an epistemological category, describing part of our knowledge of reality, specifically our ability to predict the state of a system (ideally) exactly at any moment of time once we know its state at a given moment of time. Determinism is sometimes used in the same sense as causality, and in the case of classical mechanics (which deals with single objects or a small number of objects), causality and determinism, as defined here, coincide. Once a classical system is large, one can no longer predict its behavior exactly, but only statistically.

of the event that is seen as an effect of this cause, while no event can be a cause of any event outside the forward (future) light cone of that event. These restrictions follow from the assumption that causal influences cannot travel faster than the speed of light in a vacuum, *c*. Principle theories do not require classical causality, which becomes difficult to assume in quantum physics, especially without violating locality. Relativistic "causality," as the prohibition of the possibility of physical influences towards the past, may be maintained in the absence of classical causality, although relativity itself is (locally) classically causal.

The distinction between constructive and principle theories is, thus, not unconditional, as Einstein realized. There is, however, an asymmetry between them: a constructive theory always involves principles, at least philosophical principles, while a principle theory need not involve constructive strata at the ultimate level considered by the theory and thus need not be realist.

An appeal to fundamental principles need not imply that there is some permanent, Platonist, essence to such principles. Principles change as our experimental findings and our theories change, and we cannot anticipate or control all of these changes. The principles of QM, such as the QP principle, replaced, within a new scope, some of the main principles of classical physics, which continue to remain operative within the proper scope of classical physics and some of them extend to quantum theory. There could be such changes within the same physical scope, as in the case of general relativity theory vs Newton's theory of gravity. Some principles of quantum theory have changed as well, both in view of extending the scope of theory to quantum field theory and with the scope of QM, for example, in quantum information theory. It is true that the QP principle has remained in place throughout the history of QM, although the mathematical expression of the principle has been refined a few times. But that does not mean that it may not be abandoned at some point.

3 Quantum Mechanics as a Principle Theory

QM or higher-level quantum theories are principle theories, at least if one follows the spirit of Copenhagen in interpreting them. It is, again, the first theory that, in this interpretation, is strictly principle insofar as it precludes a constructive theorization of quantum objects and processes. This aspect of QM, at least as matrix mechanics, was manifest from the outset, in contrast to Schrödinger's wave mechanics, which was constructive, even though it proceeded from certain principles as well. Schrödinger's mathematics, equivalent to that of Heisenberg, could be interpreted in the spirit of Copenhagen. If understood in this spirit, QM is a principle theory by definition, because it is not possible to configure constructively the ultimate entities, quantum objects, from which the observable quantum phenomena are built or due to which these phenomena arise, unless one sees quantum objects as *constructed* by quantum theory as fundamentally *unconstructible*. It follows that, in this interpretation, QM is nonrealist: it divorces itself from the *description* of quantum objects and processes and relates to quantum phenomena only in terms of predictions, in general probabilistic or statistical in character. Einstein hoped that nature will eventually allows us to do better as concerns the descriptive or realist capacity of quantum theory or, as he thought, an alternative theory that would eventually replace it. Bohr, on the other hand, thought that nature *might not* allow us to do better in dealing with quantum phenomena, which is not the same as that it *will not*. The question remains open and continues to be intensely debated, although most hold on to Einstein's hope, and the type of view adopted by Bohr (or following Bohr here) remains a minority view.

According to Bohr, "in quantum mechanics we are not dealing with an arbitrary renunciation of a more detailed analysis of atomic phenomena, but with a recognition that such an analysis is *in principle* excluded," beyond a certain point (Bohr 1987, vol. 2, p. 62). This statement is also an annunciation of a new principle of quantum reality, defined above as the principle of "reality without realism," the RWR principle. This principle maintains both the existence, *reality*, of quantum objects and the impossibility of representing or even conceiving of the nature of this reality, and hence the impossibility of realism, at least as things stand now. The RWR principle emerged in the 1930s in the Bohr–Einstein debate. Heisenberg, in his initial approach to QM, merely abandoned the project of describing the motion of electrons because he thought that such a description was unachievable at the time, rather than "*in principle* excluded" (Heisenberg 1925). His approach may be seen as guided by the combination of a certain "*proto*-RWR" principle and the QP principle. As a consequence of the RWR principle or even Heisenberg's proto-RWR principle, classical causality becomes impossible. As Schrödinger observed, with some disparagement, if there is no definable physical state, one cannot assume that the system's state evolves causally (Schrödinger 1935a, b, p. 154).

In both Heisenberg's initial approach to QM and Bohr's interpretation of the theory the key principles were:

(1) The principle of discreteness or the QD principle, according to which all observed quantum phenomena are individual and discrete in relation to

each other, which is not the same as the atomic discreteness of quantum objects themselves.

(2) The principle of the probabilistic or statistical nature of quantum predictions, the QP principle (in effect correlative to the QD principle), which also reflects a special, non-additive, nature of quantum probabilities and rules, such as Born's rule, for deriving them.

(3) The correspondence principle, which, as initially understood by Bohr, required that the predictions of quantum theory must coincide with those of classical mechanics at the classical limit, but was given by Heisenberg a more rigorous form, "the mathematical correspondence principle," requiring that the equations of QM convert into those of classical mechanics at the classical limit.

Bohr's interpretation of QM added a new principle:

(4) The complementarity principle, which stems from the concept of complementarity, introduced by Bohr and which requires: (a) a mutual exclusivity of certain phenomena, entities, or conceptions; and yet (b) the possibility of considering each one of them separately at any given point, and (c) the necessity of considering all of them at different moments for a comprehensive account of the totality of phenomena that one must consider in quantum physics.

The RWR principle could be inferred from the complementarity principle, given that the latter prevents us from ascertaining the composition of the "whole from parts," because the complementary parts never add to a whole in the way they do in classical physics.

Bohr's initial comment on Heisenberg's discovery shows a clear grasp of what was at stake: "in contrast to ordinary mechanics, the new quantum mechanics does not deal with a space–time description of the motion of atomic particles. It operates with manifolds of quantities which replace the harmonic oscillating components of the motion and symbolize the possibilities of transitions between stationary states in conformity with the correspondence principle. These quantities satisfy certain relations which take the place of the mechanical equations of motion and the quantization rules [of the old quantum theory]." (Bohr 1987, vol. 1, p. 48)

Heisenberg's matrix scheme essentially amounted to the Hilbert-space formalism (with Heisenberg's matrices as operators), introduced by von Neumann shortly thereafter (1932). Von Neumann's formalism provided a rigorous mathematical foundation to Heisenberg's scheme, by then developed more properly by Heisenberg himself, Bohr and Pascual Jordan, and, differently (in terms of q-numbers), by Dirac. Most crucial is the fact that Heisenberg's matrices were (re)invented by him from the physical principles coupled to a mathematical construction leading to an algebra Heisenberg had to define, beginning with the multiplication rule. This multiplication is noncommutative and is also accompanied by tensor calculus in a Hilbert space, which were, arguably, the most essential mathematical features of Heisenberg's scheme.

Bohr's complementarity principle may be seen as the physical principle behind quantum noncommutativity. Conversely, noncommutativity becomes the mathematical expression of the complementarity principle, even though noncommutativity was discovered first, from the QD and QP principles. The QP principle itself is given its mathematical expression via the complex Hilbert-space structure cum conjugation, inherent in this structure, and Bohr's rule. This structure is, however, in turn coupled to complementarity, a coupling also manifested in the uncertainty relations. Finally, insofar as the complementarity principle implies the RWR principle, the latter, too, is mathematically expressed in noncommutativity.

The new character of quantum theory introduced by Heisenberg was bound to have an impact on the very practice of theoretical physics in the quantum domain. Indeed, it may be argued that a new way of doing theoretical physics has, especially with Dirac, effectively taken quantum theory over ever since, whatever the philosophical attitudes of the practitioners themselves may be. In this new paradigm, the practice of theoretical physics is transformed into working with the mathematical apparatus of the theory to enable this apparatus to provide correct predictions, rather than trying to develop an idealized mathematical description of the physical processes considered. Dirac spoke, in describing his discovery of his famous equation for the (free) relativistic electron, of most of his work as "playing with equations," to which expression the present analysis gives a more rigorous meaning (Dirac 1962). This is how Dirac discovered his equation, still guided, however, by the physical principles of quantum theory (Plotnitsky 2015).

4 Quantum Theory and Principles of Information Processing

I would now like to consider a more recent approach to deriving quantum theory, in this case the finite-dimensional one, from fundamental principles, those of quantum information theory, which are related to but different from the principles considered thus far. There are several recent projects that exemplify this approach. I shall focus on D'Ariano and coworkers' theory because it is expressly principle in character (Chiribella et al. 2011). It has a constructive dimension as well, introduced by the idea of quantum cellular automata, though this aspect of their framework will not be considered here.

The program is inspired in part by "a need for a *deeper understanding of quantum theory* itself from fundamental principles" (which, the authors contend, has never been really achieved) and is motivated by the development of quantum information theory.[5] In part for that reason it deals with discrete variables and the finite-dimensional Hilbert spaces. According to the authors: "the rise of quantum information science moved the emphasis from logics to information processing. The new field clearly showed that the mathematical principles of quantum theory imply an enormous amount of information-theoretic consequences ... The natural question is whether the implication can be reversed: is it possible to retrieve quantum theory from a set of purely informational principles?" (Chiribella et al. 2011, p. 1). They then say:

> In this paper we provide a complete derivation of finite dimensional quantum theory based on purely operational principles. Our principles do not refer to abstract properties of the mathematical structures that we use to represent states, transformations, or measurements, but only to the way in which states, transformations, and measurements combine with each other. More specifically, our principles are of *informational* nature: they assert basic properties of information processing, such as the possibility or impossibility to carry out certain tasks by manipulating physical systems. In this approach the rules by which information can be processed determine the physical theory, in accordance with Wheeler's program "it from bit," for which he argued that "all things physical are information-theoretic in origin" [Wheeler 1990]. Note [however, that] our axiomatization of quantum theory is relevant, as a rigorous result, also for those who do not share Wheeler's ideas on the informational origin of physics. In particular, in the process of deriving quantum theory we provide alternative proofs for many key features of the Hilbert space formalism, such as the spectral decomposition of self-adjoint operators or the existence of projections. The interesting feature of these proofs is that they are obtained by manipulation of the principles, without assuming Hilbert spaces from the start. (Chiribella et al. 2011, p. 1)

[5]Among the key predecessors are Zeilinger's article (1999), Christopher Fuchs's work—which, however, "mutated" into a somewhat different program, that of quantum Bayesianism or QBism (Fuchs et al. 2014)—and Hardy (2001). Hardy's paper was the first rigorous derivation of that type. For further connections to quantum-informational approaches in quantum modeling beyond physics from a somewhat different overall perspective, and some extension of and alternative to them, see Khrennikov (2016) and further references there. (The present chapter does not claim that a sufficient understanding of QM from such principles has been achieved. This remains an open question, especially when dealing with continuous variables, where the application of the principles of quantum information is more complex as well.)

One of the principles advanced by the authors, the purification principle, plays a particularly, indeed uniquely, important role in their program, as an essentially quantum principle (the rest of their principles define the statistical theories in general):

> The main message of our work is simple: within a standard class of theories of information processing, quantum theory is uniquely identified by a single postulate: *purification*. The purification postulate, introduced in (Chiribella et al. 2010), expresses a distinctive feature of quantum theory, namely that the ignorance about a part is always compatible with the maximal knowledge of the whole. The key role of this feature was noticed already in 1935 by Schrödinger in his discussion about entanglement [Schrödinger 1935b], of which he famously wrote "I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." In a sense, our work can be viewed as the concrete realization of Schrödinger's claim: the fact that every physical state can be viewed as the marginal of some pure state of a compound system is indeed the key to single out quantum theory within a standard set of possible theories. It is worth stressing, however, that the purification principle assumed in this paper includes a requirement that was not explicitly mentioned in Schrödinger's discussion: if two pure states of a composite system AB have the same marginal on system A, then they are connected by some reversible transformation on system B. In other words, we assume that all purifications of a given mixed state are equivalent under local reversible operations." (Chiribella et al. 2011, p. 2)

The authors also speak of "the purification postulate," and they refer to the remaining informational principles as "axioms," because "as opposed to the purification 'postulate,' ... they are not at all specific [to] quantum theory" (Chiribella et al. 2011, p. 3). While these terminological distinctions are somewhat tenuous, they do not affect the authors' argument. Besides, as will be seen presently, the authors qualify these terms and state their strictly operational principles later in their article (Chiribella et al. 2011, p. 6).

The purification principle is a new principle, although it has its genealogy in the previous operational approaches mentioned above, which, in particular, equally stress the significance of quantum entanglement. The principle could be related to the RWR principle, combined with complementarity, which implies that "the ignorance about a part [one of the two complementary parts] is always compatible with the maximal knowledge of the whole." Indeed, Bohr saw the EPR experiment (the background for Schrödinger's claim and for his concept of entanglement, the term he introduced in German (*Verschränkung*) and English) as a manifestation of complementarity and the RWR principle (Bohr 1935; 1987, vol. 2, pp. 59–62).[6]

While having an essential and even unique role in the authors' operational derivation of the finite-dimensional quantum theory, the purification postulate or principle is not sufficient to do so. The authors need five additional axioms, which I shall state below. This is not surprising. In Heisenberg's derivation, the grounding quantum principles—the suspension of the description of quantum objects and processes (the proto-RWR principle) and the quantum-probability (QP) principle—were not sufficient to derive QM either. He needed the correspondence principle, to which he gave a mathematical form. What is remarkable, however, is that one needs only one "postulate" to distinguish classical and quantum information theory. A similar situation transpires in Hardy's paper mentioned above, where this difference turns not only on a single "axiom," but also on the use of a single word, "continuity," technically a single feature of the situation, "the *continuity* of a reversible transformation between any two pure states" (Hardy 2001, p. 2; emphasis added). Both principles also reflect the apparently uncircumventable roles of complex numbers and the tensor product in QM.

There are instructive specific parallels between the authors' and Heisenberg's approaches, in particular between Heisenberg's proto-RWR principle and the purification principle. The QP principle present in both cases, given that D'Ariano et al. (rightly) see QM an "operational-probabilistic theory" of a special type, is defined by the purification postulate. As they write, "the operational-probabilistic framework combines the operational language of circuits with the toolbox of probability theory: on the one hand experiments are described by circuits resulting from the connection of physical devices, on the other hand each device in the circuit can have classical outcomes and the theory provides the probability distribution of outcomes when the devices are connected to form closed circuits (that is, circuits that start with a preparation and end with a measurement)" (Chiribella et al. 2011, p. 3). This is close to Heisenberg's and Bohr's view of the quantum-mechanical situation, keeping in mind the difference defined by the concept of "circuit" (not found in Heisenberg or Bohr). As explained earlier, Heisenberg found his formalism by using the mathematical correspondence principle, not exactly the first principle, because it depended on the equations of classical mechanics in the classical limit where h could be neglected. Heisenberg needed new variables

[6] Bub, one among only a few commentators on QM as a principle theory, uses its principle character in order to account for the EPR-type experiments and quantum entanglement (2000).

because the classical variables (as functions of real variables) do not give Bohr's frequency rules for hydrogen spectra. Heisenberg discovered that these rules are satisfied by, in general, noncommuting matrix variables with complex coefficients related to amplitudes, from which one derives, in essence by means of Born's rule for this case, the probability distributions for transitions between stationary states defining these spectra.

By contrast, D'Ariano et al. arrive at the architecture of quantum theory in a more first-principle-like way, in particular, independently of classical physics. This is accomplished by using the rules governing the structure of operational devices, rules that are more empirical, albeit not completely, because they are given a mathematical representation or expression, as they must be, in accordance with the authors' and the present view, or *principle*. This principle entails the necessity of establishing a rigorous mathematical expression for the physical architecture considered or indeed the fundamental physical principles of quantum theory: "the rules summarized in this section define the operational language of circuits, which has been discussed in detail in a series of inspiring works by Coecke" (Chiribella et al. 2011, p. 4; Coecke 2010). This may indeed be a more natural way to give the fundamental structures and principles of quantum theory their mathematical expression. D'Ariano et al. need five additional axioms for their derivation of quantum theory:

> In addition to the purification postulate, our derivation of quantum theory is based on five informational axioms. The reason why we call them "axioms," as opposed to the purification "postulate," is that they are not at all specific of quantum theory. These axioms represent standard features of information processing that everyone would, more or less implicitly, assume. They define a class of theories of information processing that includes, for example, classical information theory, quantum information theory, and quantum theory with superselection rules. The question whether there are other theories satisfying our five axioms and, in case of a positive answer, the full classification of these theories is currently an open problem. Here we informally illustrate the five axioms, leaving the detailed description to the remaining part of the paper:
>
> (1) *Causality*: the probability of a measurement outcome at a certain time does not depend on the choice of measurements that will be performed later.[7]
> (2) *Perfect distinguishability*: if a state is not completely mixed (i.e., if it cannot be obtained as a mixture from any other state), then there exists at least one state that can be perfectly distinguished from it.

[7]This principle is different from that of classical causality (already by virtue of its probabilistic character), while being consistent with locality.

(3) *Ideal compression:* every source of information can be encoded in a suitable physical system in a lossless and maximally efficient fashion. Here *lossless* means that the information can be decoded without errors and *maximally efficient* means that every state of the encoding system represents a state in the information source.
(4) *Local distinguishability*: if two states of a composite system are different, then we can distinguish between them from the statistics of local measurements on the component systems.
(5) *Pure conditioning*: if a pure state of system AB undergoes an atomic measurement on system A, then each outcome of the measurement induces a pure state on system B. (Here *atomic measurement* means a measurement that cannot be obtained as a coarse graining of another measurement.) (Chiribella et al. 2011, p. 3)

Importantly, "all these axioms are satisfied by classical information theory" (ibid.). The authors also "make precise the usage of the expression 'operational principle' in the context of [their] paper:"

By [an] operational principle we mean here a principle that can be stated using only the operational-probabilistic language, i.e., using only

(1) the notions of system, test, outcome, probability, state, effect, transformation;
(2) their specifications: atomic, pure, mixed, completely mixed; and
(3) more complex notions constructed from the above terms (e.g., the notion of "reversible transformation").

The distinction between operational principles and principles referring to abstract mathematical properties, mentioned in the Introduction, should now be clear: for example, a statement like "the pure states of a system cannot be cloned" is a valid operational principle, because it can be analyzed in basic operational-probabilistic terms as "for every system A there exists no transformation C with input system A and output system AA such that C $|\varphi) = |\varphi)|\varphi)$ for every pure state φ of A. On the contrary [by contrast?], a statement like the state space of a system with two perfectly distinguishable states is a three-dimensional sphere is not a valid operational principle, because there is no way to express what it means for a state space to be a three-dimensional sphere in terms of basic operational notions. The fact that a state spate is a sphere may be eventually derived from operational principles, but cannot be assumed as a starting point." (Chiribella et al. 2011, p. 6)

This distinction is important, although, as emphasized throughout this chapter, operational principles, too, must be given a proper mathematical expression in the formalism of the theory.

5 Fundamental Principles and Q-Models Beyond Physics

As I stated from the outset, most currently used models in those fields where mathematical modeling applies, such as biology, cognitive psychology, and economics, are either classical-like statistical models or models based in chaos and complexity theories, all of which are C-models. These models presuppose the processes considered in these fields to be continuous and causal, and idealize these processes accordingly. Most of these processes are too complex to track, which requires recourse to probabilistic or statistical predictions, similarly to classical statistical physics or chaos and complexity theories. Essentially, we never deal in these fields with rigorously individual processes of the type we deal with, probabilistically, in QM, where this individuality is defined by h, which has no equivalent outside quantum physics. Some C-models of these types work reasonably well in these fields. However, there also appear to be phenomena when considering which C-models do not appear to work: their predictions fail, as did those of classical statistical models in quantum phenomena, which compelled Planck to introduce quantum theory, eventually developed into QM.

Beginning with Tversky and Kahneman's work in the 1970s and 1980s (e.g., Tversky and Kahneman 1974), it has been primarily the presence of probabilistic data akin to those encountered in quantum physics that suggested the possibility or even necessity of using Q-models in order to predict properly these probabilities in mathematical cognitive psychology and economics. Economic behavior, too, involves psychological factors of the type analyzed by Tversky and Kahneman. Kahneman was eventually awarded a Nobel Prize in economics (Tversky died a few years earlier and thus was not eligible). Until recently, however, these factors have not been considered in mathematical economic modeling or elsewhere in economics. One might say that the recourse to Q-models is motivated in these cases by a form of the QP principle, especially by the non-additive character of quantum probabilities, coupled with quantum-like noncommutativity, insofar as the order of events (responses to the questions asked) statistically depend on the order in which these questioned are asked. One does not have, however, either rigorous quantum discreteness or individuality of the processes leading to the events considered. But then, at this stage of history, if one needs a model that is able to predict quantum-like probabilities or, more generally, address an experimental situation analogous to that of QM outside physics, one need not invent an appropriate new formalism, either from principle or by way of mathematical experimentation. One already has mathematical models, those of QM, that could be used. In these cases Q-models (thus far mostly finite-dimensional) are used to predict probabilities and correlations involved in the corresponding phenomena or experiments in cognitive psychology or economics, without much concern for the fundamental principles apart from those, such as noncommutativity or the rules for Q-probabilities, derived from the quantum-mechanical formalism (e.g., Dzhafarov and Kujala 2012; Pothos and Busemeyer 2013). Whether these Q-models are required or C-models suffice remains an open question, although it is difficult to assume that C-models could give us either noncommutativity or, correlatively, Q-probabilities. It may, however, be possible to construct a C-model, underlying the Q-model, a model that would have a descriptive potential (cf., e.g., Khrennikov 2012). It is also possible to use a Bohmian model in the same way, without being concerned with those of its features, such as nonlocality, that pose problems for physics.

The question remains, however, to what degree the use of Q-models in these fields could be brought in accord with the quantum principles considered here, either along more conventional quantum-mechanical lines or those of quantum information theory, even if only interpretatively, rather than in order to derive such Q-models from these principles. Thus, does the QD principle, correlative to the QP principle in QM, also find its place in quantum-like theories in other domains? And if so, would the RWR principle also be applicable? Bohr thought so in the case of biology and psychology. In the case of biology he argued as follows:

> The existence of life must be considered as an elementary fact that cannot be explained, but must be taken as a starting point in biology, in a similar way as the quantum of action, which appears as *an irrational element* from the point of view of the classical mechanical physics, taken together with the existence of elementary particles, forms the foundation of atomic physics. The asserted impossibility of a physical or chemical explanation of the function peculiar to life would in this sense be analogous to the insufficiency of the mechanical analysis for the understanding of the stability of atoms. (Bohr 1931, p. 458; emphasis added)

In other words, the ultimate nature of biological processes may be indescribable and even inconceivable, and thus subject to the RWR principle. At least, this nature would not be ("mechanically") described by a theory, similarly to Heisenberg's proto-RWR principle. Either way, once the theory suspends accounting for the connections between the phenomena considered, these phenomena are unavoidably discrete and our predictions concerning them are unavoidably probabilistic, and are likely to follow (non-additive) rules of quantum probability, although this last question is, again, open. Bohr's invocation of "an irrational element" is of some interest, and I shall comment on it below. It is also important, and was one of Bohr's points, that this approach may be adopted even if the nature of biological processes is not physically quantum in the sense of life being a physically quantum effect. In this case, it would be indescribable or inconceivable in Bohr's interpretation. At stake here, however, are *parallel*, rather than connected, situations that may require the same mathematical models.[8]

As explained above, in both cognitive psychology and economics, we do encounter situations in which we deal with probabilities and correlations akin to those encountered in QM, which compel the use of Q-models for predicting these probabilities. But would discreteness and hence the QD principle, even if not the RWR principle, accompany this use? There are reasons to believe that such could be the case, at least in economics, and I would surmise, in cognitive psychology, as well, in both cases, as also noted above, along with the proto-RWR principle, à la Heisenberg. Recent work by Haven and Khrennikov, especially their article "Quantum-Like Tunnelling and Levels or Arbitrage," provides an instructive example for my, admittedly, hypothetical argument (Haven and Khrennikov 2013a).[9] The phenomenon of quantum tunneling refers to the quantum phenomenon where a particle tunnels through a barrier that it would not be able to surmount if it behaved classically; it is a quantum phenomenon par excellence. The process itself cannot be observed. We only deal with effects of this process, specifically with the fact that there is a nonzero probability that a particle can be found beyond the barrier, which is to say that the corresponding measurement will register an event of the impact of this particle on the measuring instrument beyond the barrier. Thus, we deal with

[8] Some arguments for such connections have been advanced, most prominently by Penrose, from his first major work on the subject (1995) on. While Penrose's theory primarily concerns the neurobiology of the brain and consciousness, he also links his argument to the possibility that life itself is a quantum effect.

[9] See Haven and Khrennikov (2013b) for a more general discussion of quantum-like modeling in economics and other social sciences.

two discrete phenomena, connected by probabilistic predictions concerning the second on the basis of the first.

Haven and Khrennikov argue that the phenomena of arbitrage in finance could be mathematically modeled by a Q-model of the type used in the quantum-mechanical account of tunneling. In economics and finance, "arbitrage" is the practice of taking advantage of a price difference between two or more markets, striking a combination of matching deals that capitalize upon the imbalance, the profit being the difference between the market prices. An arbitrage is a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state; in simple terms, it is the possibility of a risk-free profit at zero cost. Ideally, an arbitrage is risk free. In practice, there are always risks in arbitrage, sometimes minor (such as fluctuation of prices decreasing profit margins) and sometimes major (such as devaluation of a currency or derivative). In most ideal models, an arbitrage involves taking advantage of differences in price of a *single* asset or *identical* cash flows.

The main point here is that, if arbitrage can be modeled analogously to quantum tunneling in physics, one might expect features analogous to those in actual quantum tunneling, which is paradigmatic of the features of quantum phenomena. Haven and Khrennikov are primarily concerned with the use of Q-models in predicting the probabilities involved, the QP principle, rather than with discreteness or the epistemology accompanying them, the QD and the RWR (or the proto-RWR) principles. Nevertheless, they offer some considerations concerning discreteness:

> We believe that the equivalent of quantum discreteness in this paper corresponds to the idea that each act of arbitrage is a discrete event corresponding to the detection of a quantum system after it passed ... the barrier. In reality arbitrage opportunities do not occur on a continuous time scale. They appear at discrete time spots and often experience very short lives. We would like to argue that it is the tunnelling effect which is closely associated to the occurrence of arbitrage. This argument is linked to Proposition 5 below [which gives a necessary mathematical condition of the existence of arbitrage]. We also mentioned the wave function in the discussion above, and quantum discreteness is narrowly linked with quantum probabilities. (Haven and Khrennikov 2013a, p. 4095)

This remark allows for an interpretation of the phenomenon along the lines of the quantum principles considered here, although there is space for debate as to whether arbitrage requires such an interpretation. Haven and Khrennikov themselves do not appear to subscribe to all of these principles, especially to the

RWR principle. However, whether we can or cannot mathematically *describe* the economic processes themselves involved in arbitrage, rather than only *predict* certain probabilities involved, is secondary for the moment. The proto-RWR principle suffices: one is not concerned with this description any more than Heisenberg was concerned with describing the behavior of the electron in the hydrogen atom in deriving his formalism.[10] One is only concerned with predicting the probabilities of certain future events of arbitrage given the preceding situation and the data involved, analogously dealing even with the individual event of arbitrage, although in this case the underlying dynamics, even if indescribable, is multiple, rather than individual, as it is in quantum physics.[11]

The situation appears to be more complex in the case of Tversky and Kahneman's analysis in psychological decision-making and subsequent uses of Q-models in cognitive modeling (Pothos and Busemeyer 2013). As noted above, the main reasons for using models there are, correlatively, the noncommutative principle and non-additive probabilities, both found in Q-models. However, the underlying dynamics of the cognitive or psychological processes leading to the situations in question are causal or quasi-causal, and unlike the behavior of quantum objects, may be open to analysis within the domain demarcated by the theory. This is because one might expect that there are psychological and social reasons for this quantum-like decision-making, and in a way the task of psychology is to understand and explain these reasons, although the research using Q-models generally renounces this task. This situation, again, raises the question of the nature of relationships between the QP, QD, and (if it is used) RWR principles in this field, specifically, whether they are linked in the way they are in QM, and whether the corresponding experiments could be interpreted accordingly. Under the RWR principle, we cannot explain the reasons for the peculiar behavior of quantum objects and the corresponding features of quantum phenomena, and, as just suggested, one could surmise that this may be the case in certain economic phenomena, where Q-models, including infinite-dimensional ones, could be used. Overall, economics requires manifold models, just as physics does—be they classical, classical statistical, chaos-theoretical, relativistic, or quantum.

The same might be expected in psychology. The question, however, is, again, the possible role of the RWR principle in some psychological situations. I would surmise that such situations are possible and are likely to emerge. Thus

[10] Indeed, as I indicated, elsewhere Khrennikov argued for a classical-like model at the ultimate level of the constitution of nature (2012).

[11] In this regard the situation may be more analogous to quantum field theory.

far, however, Q-models in cognitive psychology only use the QP principle; in other words, these Q-models do not appear to involve either the QD or the RWR principle. I would venture that our brains may work, at least sometimes, in accordance with all three principles, thus without relying on hidden causality but only on the quantum-like play of probabilities and correlations. One can speak of a Bayesian brain. But, as against rational Bayesian agents, associated with the term Bayesian in cognitive psychology (against which association Q-models of cognitive psychology are advanced as well), this kind of Bayesian brain need not always function rationally. It may, accordingly, be possible to have a Q-model that would allow one to predict the outcomes of decision-making situations as governed, on Bayesian lines, by the information involved in making them, although we cannot have access to all of this information, even the most crucial one, in accordance with the RWR principle.[12] Nor can those who make these decisions have this access, given the role of the unconscious in making them, and that this unconscious is not causal in its functioning in the way, for example, Freud saw it, although his thinking was ultimately more complex on this point.

Bohr's repeated invocation of "an irrational element," as in the passage on the parallel between biology and quantum physics cited above, is instructive in this context. The idea and the very language of irrationality have been used against Bohr by his critics and have troubled some of his advocates, in my view as a result of misunderstanding his thinking. This "irrationality" is not any "irrationality" of QM itself, which Bohr always saw as a rational theory (e.g., Bohr 1987, vol. 1, p. 48; 1987, vol. 2, p. 63). It is a rational theory of something that may, in a certain sense, be irrational—inaccessible to rational thinking or even thinking in general. Bohr's point is misunderstood in part by virtue of overlooking the difference between the rationality of a theory and the irrationality of what this theory (rationally) deals with. If, as he says, "the quantum of action [h], ... appears as *an irrational element* from the point of view of the classical physics," it cannot be accounted for by the latter. In other words, it cannot be rationally incorporated into the scheme of classical mechanics.

Although Tversky and Kahneman's and related arguments are epistemologically different from that of Bohr, they too are often seen as bringing into consideration and incorporating "irrational" elements into psychological or economic decision-making. This decision-making replaces or supplements

[12]This suggestion need not depend on the applicability of a Bayesian (such as QBist), as against a frequentist or statistical, approach to QM itself.

rational Bayesian agents of traditional economics, agents who use probabilistic reasoning subject to updating their estimates on the basis of new information, with the partially "irrational" Bayesian agents. This irrationality may be divided into three types, with sometimes uncertain borderlines between them. The first type is actually a form of rationality, but of a kind different from that presumed to be dominant (say that of maximizing one's monetary benefits), and this alternative rationality may be unconscious. The second type of this irrationality would be something that could be explained but that defies it as anything that could be seen as rational. It is true that this irrationality may eventually reveal itself to be the irrationality of the first type.[13] Finally, the third type of irrationality may be seen in terms of Bohr's interpretation, insofar as classical decision theory cannot incorporate it, while quantum-like decision theory can make it part of its predictive scheme without explaining it. In the first place, such a scheme does not, again, describe how things happen, but only predicts the probabilities involved in the decision-making by the agent considered. In this way, QD, QP, and RWR principles can be brought together in this domain.

However, given that in this case one customarily uses finite-dimensional Q-models (in economics one might also need the infinite dimensional one) and that a quantum informational approach may be especially fitting, we may want to *gauge* this use against the scheme of D'Ariano et al. as considered above.[14] As I noted above, at this stage, in mathematical modeling in psychology we need not be concerned with deriving the necessary mathematical model from fundamental principles. We already know that the finite-dimensional Q-models, used in cognitive psychology, follow from them (from six principles and the operational framework defined within this scheme). This gauging, however, even if now used as an interpretation of such a Q-model, would allow us to make important conclusions about the nature of the phenomena being considered in relation to the QD, QP, and RWR principles, as considered earlier, because these relationships obtain in quantum theory. The key point here is the role of certain fundamental principles behind a given model, even if this model is already available, because otherwise we don't really have a rigorous theory or even a rigorous model. The recourse to such principles gives

[13] Some, such as Freud, would challenge this type of "irrationality," because they would presume a hidden, unconscious form of rationality, as just considered. Indeed Freud does so by way of comparing it in this regard with the then (1915) emerging quantum physics (Freud 2008, p. 115). However, one could in turn challenge Freud on this point, as in effect he himself did, against his own grain, on the same occasion.

[14] I am grateful to G. Mauro D'Ariano for directing my attention to this line of inquiry and insightful suggestions.

us a deeper understanding of a given field than only finding a workable model to use, although the latter should not be undervalued either.

The first step would be to check the five axioms, which "represent standard features of information processing that everyone would, more or less implicitly, assume. They define a class of theories of information processing that includes, for example, classical information theory, quantum information theory, and quantum theory with superselection rules. The question whether there are other theories satisfying our five axioms and, in case of a positive answer, the full classification of these theories is currently an open problem" (Chiribella et al. 2011, p. 3). One might expect that these axioms would be satisfied by decision-making, classical or quantum-like, in psychology and economics (see pp. 336–337 above). As "all these axioms are satisfied by classical information theory," the question now is of course whether the purification postulate applies in those situations where Q-models are used. The postulate states that "the ignorance about a part is always compatible with the maximal knowledge of the whole," and further that "if two pure states of a composite system AB have the same marginal on system A, then they are connected by some reversible transformation on system B. In other words ... all purifications of a given mixed state are equivalent under local reversible operations" (Chiribella et al. 2011, p. 2). It would be surprising if this or analogous postulate did not apply once Q-models were applicable (e.g., Asano et al. 2012), but a rigorous verification of this postulate, or of the five axioms, would be a necessary task.

The operational framework merits a brief, but essential, reflection as well. As I explained, D'Ariano et al. arrive at the required mathematical architecture in a first-principle-like way, by using the rules governing the structure of operational devices, "circuits," via Coecke's work on monoidal categories and linear logic (Chiribella et al. 2011, p. 4; Coecke 2010, p. 1). Admittedly, in cognitive decision-making we deal with human subjects, and establishing the corresponding quantum mathematical architecture for these "circuits" is a formidable task, but it is all the more exciting and important for that. Indeed, we already have important related work along the lines of category theory in this field (Abramsky and Brandenburger 2011). I suspect that biology and neuroscience are likely to pursue these lines of thinking as well.

There is plenty of work to be done along these lines in all these fields. But as QM, in the hands of Heisenberg and Dirac (one could hardly find better hands), taught us, the principle approach is likely to serve as exceptionally helpful guidance and bring a rich harvest of new and deeper understanding in these fields, just as it did in quantum theory.

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Index

A
adaptive dynamics, 95–118, 322, 325–7
affective evaluation, 267–86
affective priming, 267, 285
affective state (final), 271, 279
affective state (initial), 279
Allais paradox, 141
ambiguity, 95, 103, 111
amplitude, 21, 24, 25, 27–30, 40, 44, 47–9, 70, 126, 149, 210, 213, 234, 253, 256–8, 271, 304, 323, 325, 326, 346
anti-discounting, 261, 263
arbitrage, 33, 351, 352
Aumann, R.J., 70, 141
axioms, 65, 121, 122, 126, 127, 141, 142, 159, 164, 198, 200, 201, 209, 270, 345, 346, 355

B
basis, 3, 6, 11, 13, 16, 20, 27, 30, 33, 34, 49, 50, 82, 98, 111, 112, 128, 133, 136, 151, 157, 181, 201, 216, 234, 236, 237, 243, 251, 268, 270, 302, 306, 312, 326, 336, 337, 351, 354
basis (orthonormal), 3, 6, 11, 16, 49, 50, 98, 111, 180, 234, 236
bath, 146, 150, 151
Bayes formula, 5, 147, 200
Bayes' nets, viii
Bell's inequality, vi, 102, 104, 178–80, 188, 323
Bohmian mechanics, ix, 303
Boolean algebra, 16n8, 195
Boolean logic, 10, 16, 40
Born rule, vi, 67, 68, 84, 149, 150, 171
Bose-Einstein distribution, 160, 182
bra, 10

C
call option, 25–7, 30
Cauchy-Bunyakovsky-Schwartz inequality, 8
causal efficacy, 295–301, 313
causal graphical models, viii, 252–5

Notes: Page numbers followed by "n" refer to endnotes.

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causal reasoning, 251–63
channels, 42, 98, 242, 298
classical causality, 338–40, 346n7
Clauser-Horne-Shimony-Holt (CHSH) inequality, 104, 179, 206
C-models, 335, 336, 348, 349
cognition, 1, 2, 9, 12, 13, 75–91, 110, 157–91, 195–223, 231, 263, 267, 270, 271, 310, 322–6, 327, 340
cognition (quantum), 183, 188, 190, 191, 195–223
cognitive dissonance, 231, 232, 236, 239–41, 247, 269
cognitive process (constructive), 175, 205, 207, 209, 268, 270–2, 283
cognitive systems, 13, 78, 79, 102, 208, 267, 270–2, 276, 286, 327
complementarity principle, 341, 342
completely positive maps, 327
completeness, 3
complex numbers, 2, 2n1, 4, 6, 69, 165, 233n4, 325, 336, 345
complex probability amplitudes, 323
concept combinations, 163–74, 176, 177, 187, 190
conjugate symmetry, 2
conjunction, 141, 157–60, 163, 164, 166–72, 174–6, 204, 252, 263
consciousness, 71, 293–314
constructive influence, 267–86
constructive theories, 336–9
context, 32n9, 52, 95–7, 102, 105–8, 110, 111, 114–18, 121, 130, 131, 134, 148, 150, 159–63, 183, 185, 199n7, 201, 204, 208, 209, 217n19, 222, 231, 232, 236, 239, 240, 242, 308–11, 347, 353
context (unbiased), 115, 116
contextuality, 97, 103, 105–7, 110, 148, 149, 157, 161–3, 188, 196, 197, 201, 201n9, 204–9, 215–17, 219, 222, 248
Copenhagen interpretation, 60, 76, 303, 323, 337n2
Cramer-Rao inequality, 22, 33

D
de Broglie, 30, 303
decision making, 1, 2, 12, 13, 75–91, 110, 126, 127, 134, 142, 144, 150–2, 196, 197, 201, 223, 236, 241–3, 247, 248, 252, 263, 268–72, 321–6, 352–5
decoherence, 41, 144, 151
determinism, 205, 338n4
determinism (non), 161, 205
Dirac notation, 3
disjunction, 141, 146–8, 157, 158, 164–6, 168, 169, 174–6, 204, 263
dynamic consistency, 263

E
economics, 1, 2, 19–34, 39–53, 79, 91, 122, 124, 141, 147, 196, 232, 242, 247, 248, 321, 322, 335, 336, 348–50, 350n9, 351, 352, 354, 355
eigentypes, 243
eigenvalues, 4, 5, 5n4, 6, 11, 45, 46, 100, 113, 234, 236, 256
Einstein, A., 206, 303, 336, 337, 339, 340
Einstein-Podolsky-Rosen (EPR), 206, 344, 345n6
Ellsberg (paradox), 141
emergence, 53, 142n1, 144, 148, 160, 165, 166, 168, 169, 171, 174, 175, 188, 190, 238
entanglement, 41, 43, 52, 143, 145, 148, 150, 151, 160, 161, 178–82, 188, 270, 303, 323, 344
Euclidean space, 125, 135, 142

F
Fermi-Dirac distribution, 182
Feynman, R.P., 200, 218, 222, 323
finance, 1, 79, 91, 141, 321, 322, 351
Fisher information, 20, 22, 23, 30, 31, 33, 34, 40
Fock space, 160, 164–75, 177, 190
Fokker-Planck, 29, 30
free choice paradigm, 268, 269

G
game model (under-determination), 60–2
game theory, 40–3, 48, 50, 52, 53, 59–72, 230, 248
Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation, 144, 326

H
Hamiltonian, 9, 12, 13, 100
Heisenberg, 7–10, 229, 336, 337, 339–42, 345, 350, 352, 355
Heisenberg uncertainty principle, 7–10
Hilbert space, 2, 8, 10, 11, 15, 16, 42, 44, 46, 49, 50, 61n3, 63, 71, 79, 98–112, 121–37, 142, 148–50, 160, 163, 165, 166, 171, 174, 178, 180, 182, 189, 190, 196, 216, 221, 223, 229, 230, 233, 234, 236, 239, 244, 271, 275, 325, 336, 341–3
hyperbolic quantum mechanics, 149, 325

I
incomplete knowledge, 230, 236
information, 2, 11–12, 19–34, 40, 62, 81, 98, 129, 142, 146, 186, 230, 252, 268, 293, 297–305, 307–10, 310–12, 321–5, 336, 342–8
information (active), 294, 297–300, 305–14
information (incomplete), 230, 233n5
informational interpretation, 323
interference, 41, 67–70, 122, 129, 132, 148–50, 160, 165–7, 169, 173, 174, 177, 188, 197, 204, 213, 215, 216, 236, 237, 246, 263, 270, 271, 304, 305, 307, 325
interference (deliberational), 67–70
interference (hyperbolic), 148
interference (trigonometric), 148

K
ket, 10, 126, 136
Kochen-Specker theorem, 61n3
Kolmogorov, 16, 16n8, 95–7, 103–18, 121, 122, 122n2, 142, 143, 146, 147, 159, 164, 165, 170, 171, 176, 178, 197, 200, 200n8, 219, 270
Kolmogorov axioms, 103, 270
Kolmogorov probability theory, 95, 105, 106

L
Lacy's model of survey responses, 135–6
Lagrangian, 20, 23, 31
lattice, 16, 198, 233n6
Leggett-Garg inequality, 97
lifting, 97–101, 112
lifting (isometric), 99
Lüder's postulate, 5n5

M
Markov, 122, 127, 128, 144, 150, 151, 204, 242, 245, 252, 255, 260, 261, 263, 326
Markov condition (local), 252, 260, 263
matrices, 2, 4, 89, 98, 150, 341, 342
matrices (density), 4, 6, 42, 65, 151

matrices (Hermitian), 2, 2n1, 3–7, 11, 12, 14, 46, 84, 85, 112, 151, 216, 256, 324, 325
Maxwell-Boltzmann distribution, 182
measurements (indirect), 76–8, 82
measurements (self), 76–8, 323
measurements (von Neumann-Lüders), 77, 78, 81, 82
mechanics (Bohmian), 303
mechanics (classical), 9, 52, 205, 207, 336, 337, 338n4, 341, 345, 349, 353
mechanics (quantum), 1, 19, 40, 41, 48, 50–2, 60, 61, 70, 75, 91, 104, 148, 148n4, 149, 160, 163, 165n5, 196, 208, 218, 229, 233, 248, 307, 321, 325, 335, 339–42, 344, 345, 349, 351
mixing law, 80–1, 86
mixing law (of probabilities), 81, 86
momentum, 7–10, 60, 217, 218

N

Nash equilibrium, 51, 59, 61–4, 65n7, 66, 67, 70, 71
noncommutativity, 7, 10, 231, 232, 239, 245, 247, 261, 342, 348, 349
non-Kolmogorov probability theory, 95, 97, 110
nonlocality, 189, 205–8, 216n18, 324, 349
nonreductive physicalism, 301, 302
nonseparability, 130–2, 143–50, 152

O

observables, 1–4, 5n5, 7, 9–11, 15, 29, 46, 79, 81–3, 98, 99, 103, 104, 142, 148, 197, 216, 231, 235–7, 239, 241, 256–8, 259–61, 263, 324, 325, 327
ontology, 293–314
open quantum systems, 144, 150, 321, 322, 324–7
operator, 2–4, 4n3, 5–8, 11–15, 42, 46, 47, 49, 52, 79–81, 83, 86, 86n4, 88, 89, 98, 99, 111, 112, 150–2, 165, 167, 168, 172, 182, 216, 223, 223n5, 237, 256, 324, 325
operator (adjoint), 14, 88, 99
operator (anti-commutator), 8
operator (commutator), 8
operator (density), 4, 4n3, 5, 14, 42, 79–81, 86, 86n4, 98, 111, 150–2, 182, 223, 223n5, 256
operator (Hermitian), 2, 3, 5–7, 11, 14, 46, 112, 151, 216, 256, 324, 325
operator (linear), 15, 79, 86, 86n4, 89, 98, 151, 182
operator (mental energy), 13
operator (momentum), 8
operator (position), 8
operator (projection), 3, 4, 14, 46, 165, 167, 168, 172
operator (unitary), 12, 13, 47, 49, 52, 83, 112, 182, 237
order effect, 128–30, 132, 141, 143, 146, 204, 208, 259–60, 263, 271, 274
oscillators (neural), 195–223
overextension, 158–60, 164, 170, 173

P

Pauli, 43, 285, 323n3
Penrose, R., 12n7, 307, 322, 350n8
perception, 95–7, 102, 103, 110–18, 246, 263, 296, 326, 327
perception (bistable), 95, 97, 103, 110–18
phase space, 9
Planck constant, 7, 12–14, 45, 45n4
politics, 91, 142, 143, 151, 321
position, 7–10, 20, 44, 60, 91, 104, 142, 205, 217, 218, 246, 278, 280
positive definiteness, 2–4, 152

positive operator valued measures, 14–15, 101, 324
potential function, 9, 25, 26, 29, 30, 32, 33
potentiality, 161, 162, 308
preferences, 51, 52, 122, 124, 130–5, 141–52, 201, 229–48, 268, 269
preferences (nonseparable), 130–2, 145
preferences (separable), 130, 145, 145n3
principle theories, 336–42, 345n6
principle thinking, 336–9
probability (conditional), 5, 146, 147, 200, 235, 239, 253, 254, 256, 258, 260, 262
probability (context dependent), 95–7, 102, 106, 111
probability (joint), 96, 97, 100–6, 111–3, 200, 206, 208, 216, 217, 217n19, 218, 220, 221, 254, 260
probability (law of classical), 143
probability (law of total), 122, 122n2, 143, 146, 322
probability (negative), 195–223
probability amplitude, 21, 24, 25, 27, 29, 40, 44, 47, 70, 126, 253, 256–8, 323, 325, 326
projection, 3–5, 5n5, 6, 7n6, 14, 46, 50, 77, 80, 81, 84, 85, 101, 112, 113, 126, 165, 167, 168, 172, 241, 256, 262, 275, 276, 278, 279, 343
projectors (orthogonal), 3, 5, 15, 16, 65, 67, 82, 165, 167, 168, 172, 325
psychology, 9, 40, 76, 78, 81, 91, 105, 110, 127, 147, 174, 189, 190, 196, 229, 229n1, 230, 231, 242, 247, 251, 271, 286, 321, 322, 324, 335, 335n1, 336, 348–50, 352–5
purification principle, 344, 345
put option, 26

Q

Q-models, 335, 336, 338, 348–55
quantization, 9, 19–34, 47, 52, 341
quantum adaptive dynamics, 7, 112–14, 322, 325–7
quantum auctions, 42n1, 47–50, 53
quantum Bayes net, 253, 255–8, 259–61, 263
quantum biophysics, 322–5
quantum cognition, 183, 188, 190, 191, 195–223
quantum cognition (neural model), 209–17
quantum field theory, 166, 335n1, 339, 352n11
quantum game theory, 40–3, 48, 53
quantum information, 2, 4, 8, 10–12, 12n7, 48, 83, 90, 98, 307, 309, 313, 314, 321–5, 327, 336, 339, 342, 343, 343n5, 345, 346, 349, 354, 355
quantum information theory, 2, 4, 10–12, 90, 323, 336, 339, 342, 343, 345, 346, 349, 355
quantum-like, 1, 12n7, 20, 39–41, 53, 77, 79, 82, 91, 99, 141, 148, 148n4, 151, 191, 214–17, 229–48, 309, 348–50, 350n9, 352–5
quantum-like paradigm, v
quantum logic, 10, 15–16, 71, 160, 167, 169, 174, 175, 233n6
quantum master equation, 143, 144, 150, 151, 326, 326n6
quantum measurement, 75–8, 91, 100, 159, 161, 162
quantum probability, 5–7, 91, 97, 141, 148, 159, 161, 166, 188, 236, 251, 253, 256, 258, 261–3, 267–86, 322, 338, 350
quantum strategy, 42, 43
quantum theory (ontological interpretation), 294, 302–5, 309
quantum tunneling, 350, 351
quantum wave function, v, vi
qubit, 4, 11, 12, 12n7, 49, 49n6, 150

R

rational, 59–71, 121, 122, 161, 195, 198, 201, 202, 221, 223, 231, 239, 245n13, 270, 353, 354
rationality, 65, 70, 142, 195, 232, 245, 248, 353, 354, 354n13
reality without realism principle (proto-RWR), 340, 345, 350–2
reality without realism principle (RWR), 338, 340–2, 344, 345, 349–54
reciprocity, 261–2, 263
reciprocity (law of), 261–2
reductive physicalism, 302
Robertson inequality, 9, 10

S

Savage, L.J., 40, 122, 201, 202
scalar product, 2, 3, 8, 13, 49, 79, 137, 165, 325
Schröder's stair, 95–7, 102, 107, 110
Schrödinger, 2n2, 7, 9, 12–14, 20–32, 34, 87, 127, 150, 151, 303, 305, 325, 326n6, 339, 340, 344
Schrödinger inequality, 7, 9
Schrödinger-like, 20, 24, 25, 32
self-perception theory, 246, 247, 269
Shafir, E., 122n1, 202, 204
social science, 1, 2, 13, 19, 33, 34, 52, 76–8, 91, 134, 142, 151, 196, 229n1, 231, 239–41, 294, 309, 321, 322
spin, 182, 189, 206, 207, 218, 282
state (collapse of), 281
state (information), 7n6, 83, 324
state context property, 160
states, 1, 3–4, 4n3, 6, 8, 11, 13, 14, 78–82, 80–5, 85, 87, 88, 91, 99, 143–6, 148–51, 150, 151, 233n5, 325, 344
states (belief), 1, 8, 13, 78, 80–2, 85, 87, 88, 91, 143–6, 148–51
states (decision), 80–5, 87
states (mixed), 4, 79, 150, 151
states (pure), 3–4, 4n3, 6, 11, 14, 79, 99, 150, 151, 233n5, 325, 344
superoperators, 79, 87
superoperators (completely positive), 42, 89, 327
superposition, 6, 7, 11, 49, 67, 71, 82, 83, 110, 111, 143, 144, 148, 150, 151, 160, 165–9, 171, 174, 188, 230, 230n2, 233–5, 243, 245n13, 246, 270–2, 274, 279, 286, 322, 323
sure-thing principle, 40, 122, 201, 263

T

tensor product, 11, 79, 83, 89, 99, 101, 111, 166, 171, 182, 345
Tsirelson's bound, 182, 189
Tversky, A., 122n1, 147, 195, 196, 202, 204, 229–48, 282, 348, 352, 353
type-indeterminacy, 229
type space, 233, 236, 241

U

uncertainty, 6–10, 19, 21, 33, 34, 52, 83, 142–4, 148, 152, 195, 202, 230, 235, 236, 239, 241, 246, 269, 279, 338, 342
underextension, 158, 160, 164
unitary dynamics, 276
unitary interaction, 77, 81–5, 87, 88

V

von Neumann equation, 14
von Neumann-Lüders approach, 77, 78, 81, 82
von Neumann projection postulate, 5n5

voters, 132, 133, 141–52
voters (preferences), 141–52

W
wave function, 12, 70, 71, 77, 149, 303, 310, 332, 351
wave function (collapse of), 77
wave function (constructive approach), 229–48
Wigner, 217, 218