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Ultimate speed limits to the growth of operator complexity

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In an isolated system, the time evolution of a given observable in the Heisenberg picture can be efficiently represented in Krylov space. In this representation, an initial operator becomes increasingly complex as time goes by, a feature that can be quantified by the Krylov complexity. We introduce a fundamental and universal limit to the growth of the Krylov complexity by formulating a Robertson uncertainty relation, involving the Krylov complexity operator and the Liouvillian, as generator of time evolution. We further show the conditions for this bound to be saturated and illustrate its validity in paradigmatic models of quantum chaos.

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Quantum speed limits (QSL) impose fundamental constraints on the pace at which a physical process can unfold. Since their conception[1,2], they have been formulated as bounds on the minimal time at which a distance between quantum states can be traversed. The freedom in the choice of the distance can be used to sharpen the discrimination between quantum states, and with it, the notion of the speed of evolution[3,4]. Additional efforts have been devoted to exploring the role of the underlying dynamics, generalizing early results from isolated systems to open[5–8] and classical processes[9,10]. The resulting speed limits have become a useful tool in various branches of physics, ranging from information processing[11] to many-body physics[12], quantum control[13], and quantum metrology[14]. However, traditional QSL are too conservative in estimating the relevant time scales in many processes, such as thermalization[15]. This has motivated the development of speed limits suited for specific measures and observables[16], as in the pioneering work by Mandelstam and Tamm[1]. In this sense, certain speed limits follow from generalized uncertainty relations such as those derived by Heisenberg and Robertson[17].

In parallel with the study of QSL, quantifying the complexity of a physical process is a central task for the advancement of fundamental physics and quantum technologies. Lloyd pointed out that the computational complexity of physical processes is limited by QSL[18]. Analogously, the circuit complexity of a quantum state[19], defined as the number of elementary operations required to generate it from a reference state, can be characterized in terms of conventional QSL[20–23]. A complementary approach for many-body quantum systems focuses on the buildup of complexity in the time evolution of an initial local observable, known as operator growth[24–28]. The intuition is that simple operators unitarily evolve into increasingly complex ones. Quantum information initially encoded in a few degrees of freedom is thus scrambled over the system in the course of evolution, making it impossible to recover it through local measurements and giving rise to thermalization. The unambiguous description of this scrambling process remains an open problem. One possibility is to probe it via an out-of-time-ordered correlator[29,30] that may be used to identify an analog of the Lyapunov exponent, providing a connection with classical chaos, e.g., the butterfly effect. Such quantum Lyapunov exponent obeys a universal upper bound[30], which helps refine the notion of maximal chaos, is saturated by black holes and is further tied to the eigenstate thermalization hypothesis[31,32]. A related approach, which we shall pursue in this work, is to study the dynamical evolution of operators in Krylov space, exploited in numerical techniques such as the recursion method[33]. In this context, operator growth is quantified by the so-called Krylov complexity, a measure of the delocalization of the time-dependent operator in the Krylov basis[34–38]. The authors of[34] made a conjecture on the universal operator growth, namely, that Krylov complexity can grow at most exponentially, and it does so in generic non-integrable systems. Remarkably, its growth rate upper bounds the Lyapunov exponent, establishing a connection with the bound on out-of-time-ordered correlators[30,39]. Further studies have shown that exponential operator growth is possible in free and integrable systems[40], while the role of the interaction graph in a quantum network has been explored in[41].

Here, we characterize the growth of Krylov's complexity by deriving a fundamental limit on its rate of change and by studying analytically the conditions under which this bound is saturated. Our results show that saturation, which is also found to correspond to a particular notion of minimum uncertainty, occurs whenever the dynamical evolution of the system has the underlying structure of a three-dimensional complexity algebra, which was introduced by[42]. In this setting, the unitary evolution of an operator can be represented as the displacement of generalized coherent states[42], which display classical-like behavior[43]. As demonstrated in several paradigmatic examples, the saturation of the growth rate may be possible in some chaotic systems, but quantum chaos is not required for it.

Results and discussion

Quantum dynamics in Krylov space. Consider an isolated quantum system in which the time evolution of an observable $\mathcal{O}$ is generated by a time-independent Hamiltonian H according to the Heisenberg equation of motion $\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)]$, setting $\hbar = 1$. The solution to this equation with the initial condition $\mathcal{O}(0) = \mathcal{O}$ is given by $\mathcal{O}(t) = e^{itH}\mathcal{O}e^{-itH}$. In terms of the Liouvillian superoperator given by $\mathcal{L} = [H, \cdot]$, the Taylor expansion of the time-evolving observable $\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathcal{L}^n \mathcal{O}$ shows that its dynamics is contained in the complex linear span of the operators $\{\mathcal{L}^n \mathcal{O}\}_{n=0}^{\infty}$. This span is completely determined by the Hamiltonian and the initial observable and is known as the Krylov space.

From now on, we consider the restriction of each operator and superoperator to the Krylov space. To highlight the vector space structure, we make use of the bracket notation $|A)$ when expressing operator A in an equation. We choose to equip the Krylov space with an inner product satisfying the properties

1. $(A|\mathcal{L}B) = (\mathcal{L}A|B), \forall\, A, B$.
2. $(A|\mathcal{L}A) = 0$, when A is Hermitian.

An example of a family of inner products satisfying these two properties is given by $(A|B) = \left\langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B \right\rangle_\beta$. The bracket $\langle\cdot\rangle_\beta$ denotes the thermal expectation value with respect to the equilibrium Gibbs state $e^{-\beta H}/Z$ and thus $(A|B)$ reduces to the Hilbert-Schmidt inner product when $\beta = 0$, up to a normalization factor. It follows from the second property of the inner product that the operators $\mathcal{O}$ and $\mathcal{L}\mathcal{O}$ are orthogonal. Let $b_0 = \|\mathcal{O}\|$ and $b_1 = \|\mathcal{L}\mathcal{O}\|$, where $\|\cdot\|$ is the norm induced by the inner product. By starting from the normalized vectors $\mathcal{O}_0 = \mathcal{O}/b_0$ and $\mathcal{O}_1 = \mathcal{L}\mathcal{O}/b_1$, we can construct an orthonormal basis $\{\mathcal{O}_n\}_{n=0}^{D-1}$ for the Krylov space by applying the Lanczos algorithm. This algorithm works as follows: given the first $n+1$ basis vectors, one constructs the orthogonal vector $|A_{n+1}) = \mathcal{L}|\mathcal{O}_n) - b_n|\mathcal{O}_{n-1})$, where $b_n = \|A_n\|$ and then normalize it to obtain $|\mathcal{O}_{n+1})$. We call the constructed basis *the Krylov basis*. It is possible that the Krylov dimension D is infinite, in which case the Lanczos algorithm never halts. We remark that the Lanczos algorithm is only guaranteed to construct an orthonormal basis if the Liouvillian is self-adjoint, i.e., the first property of the inner product is satisfied. Generally, the Lanczos algorithm involves a third term on the right-hand side of the equation for $|A_{n+1})$. This term is, however, always zero whenever the second property of the inner product is satisfied. Thus, with our chosen inner product, the action of the Liouvillian on the Krylov basis takes a specific form $\mathcal{L}|\mathcal{O}_n) = b_{n+1}|\mathcal{O}_{n+1}) + b_n|\mathcal{O}_{n-1})$. As pointed out in ref. [42], this motivates one to consider abstract raising and lowering operators that we denote by $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively. Their action on the Krylov basis is given by $\mathcal{L}_+|\mathcal{O}_n) = b_{n+1}|\mathcal{O}_{n+1})$ and $\mathcal{L}_-|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1})$. The Liouvillian can then be expressed as their sum.

It is further convenient to introduce the real-valued functions $\varphi_n(t)$, which appear in the expansion of $\mathcal{O}(t)$ as $|\mathcal{O}(t)) = \frac{1}{\|\mathcal{O}\|}\sum_{n=0}^{D-1} i^n \varphi_n(t)|\mathcal{O}_n)$. We will refer to these functions as *the amplitudes* of the observable. These amplitudes evolve according to the recursion relation $\partial_t\varphi_n(t) = b_{n-1}\varphi_{n-1}(t) - b_n\varphi_{n+1}(t)$ with the initial conditions $\varphi_0(0) = 1$ and $\varphi_n(0) = 0$ for $n > 0$. Thinking of the Krylov basis vectors as forming the sites of a one-dimensional lattice, b_n can be interpreted as a hopping amplitude, see, e.g.,[34,35]. In this sense, one can think of $\mathcal{O}$ as a one-dimensional discrete wave function that is initially localized and then spreads out over the lattice as time evolves. An increase in the population of the sites further away from the origin reflects a greater increase of complexity of the observable. In order to quantify this, it is natural to consider the Krylov complexity of $\mathcal{O}(t)$, defined to be

$$K(t) = \sum_{n=0}^{D-1} n\left|\varphi_n(t)\right|^2. \quad (1)$$

The main task of our work is to bound the growth of Krylov's complexity. Due to unitary dynamics, the norm of the evolution is preserved and the Krylov complexity is unchanged if one normalizes the operators studied. We will, therefore, without loss of generality, consider $\mathcal{O}$ to be normalized. By introducing the complexity operator $\mathcal{K} = \sum_{n=0}^{D-1} n|\mathcal{O}_n)(\mathcal{O}_n|$, which plays the role of the position operator in the Krylov lattice, it is possible to express Krylov complexity as the "expectation value" of $\mathcal{K}$ with respect to $\mathcal{O}(t)$. More precisely, if $\langle\mathcal{K}\rangle_t \equiv (\mathcal{O}(t)|\mathcal{K}\mathcal{O}(t))$ then $K(t) = \langle\mathcal{K}\rangle_t$.

Dispersion bound on Krylov complexity. If the Krylov space forms an inner product space in which $\mathcal{A}$ and $\mathcal{B}$ are self-adjoint superoperators, then there ought to exist a Robertson uncertainty relation given by $\Delta\mathcal{A}\Delta\mathcal{B} \geq \frac{1}{2}|\langle[\mathcal{A},\mathcal{B}]\rangle|$, where $\Delta\mathcal{A} = \sqrt{\langle\mathcal{A}^2\rangle - \langle\mathcal{A}\rangle^2}$ is the dispersion of $\mathcal{A}$ with respect to some state $|A)$. When the Krylov dimension is infinite, it is necessary that $|A)$ is contained in the intersection between the domains of $\mathcal{A}\mathcal{B}$ and $\mathcal{B}\mathcal{A}$, otherwise the inequality might not hold[44]. Letting $A = \mathcal{O}(t)$, $\mathcal{A} = \mathcal{L}$, $\mathcal{B} = \mathcal{K}$ and noting that $\Delta\mathcal{L} = b_1$, we can rewrite the uncertainty relation as

$$\left|\partial_t K(t)\right| \leq 2b_1\Delta\mathcal{K}. \quad (2)$$

In other words, the growth of Krylov complexity is upper bounded by a constant times the dispersion of the complexity operator. By defining a characteristic time scale $\tau_K = \Delta\mathcal{K}/\left|\partial_t K(t)\right|$, one obtains $\tau_K b_1 \geq 1/2$, which takes the form of a Mandelstam-Tamm bound, and emphasizes the role of $b_1 = \|\mathcal{L}\mathcal{O}\|$ as a norm of the generator of evolution in Krylov space. To avoid confusion with the uncertainty relation for observables, we will refer to this bound as the dispersion bound. We note that no bound tighter than (2) can be found by considering the more general Schrödinger uncertainty relation, as the extra term given by the anticommutator identically vanishes, as shown in Methods.

It is not self-evident that saturation of the dispersion bound can be achieved under the unitary dynamics of the observable. There are very specific relations between $\mathcal{L}$, $\mathcal{O}$ and $\mathcal{K}$ that need to hold: the Liouvillian is required to be tridiagonal in the eigenbasis of the complexity operator and the initial state of the observable is required to be parallel to the eigenvector with the lowest eigenvalue. The conditions for the saturation of the dispersion bound are thus highly constrained and differ from those known for saturation of a Robertson uncertainty relation in general. The required conditions admit a geometrical interpretation, elaborated in Methods. The bound is saturated if and only if the evolution curve moves along the gradient of the Krylov complexity. This requires that the dynamics is directed along the direction that maximizes the local growth of complexity; see Methods. The only exception involves extremal points in which any direction away from the extremal point leads to saturation. This is indeed the case for $t = 0$. Indeed, there exists Liouvillians of the form $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$ for which the tangent of the generated path will be parallel with the gradient for all times.

Saturation of the dispersion bound. Time evolutions saturating the dispersion bound are characterized by a unique algebraic structure. Define the superoperator $\mathcal{B} = \mathcal{L}_+ - \mathcal{L}_-$. Following[42], we consider their *simplicity hypothesis*: namely, the assumption that $\mathcal{L}$, $\mathcal{B}$ and the commutator $\tilde{\mathcal{K}} = [\mathcal{L},\mathcal{B}]$ close an algebra with respect to the Lie bracket. It was shown in[42] that this forces $\tilde{\mathcal{K}}$ to be related to the complexity operator via $\tilde{\mathcal{K}} = \alpha\mathcal{K} + \gamma$, where $\alpha, \gamma \in \mathbb{R}$. We show in Supplementary Note 2 that γ is a positive number and α is a real number satisfying the condition $\alpha \geq 0$ for infinite Krylov dimension and $\alpha = -\frac{2\gamma}{D-1}$ for finite Krylov dimension. Moreover, the only possible closure of the algebra is given by the commutation relations

$$[\mathcal{L},\mathcal{B}] = \tilde{\mathcal{K}}, \quad [\tilde{\mathcal{K}},\mathcal{L}] = \alpha\mathcal{B}, \quad [\tilde{\mathcal{K}},\mathcal{B}] = \alpha\mathcal{L}. \quad (3)$$

Given this algebra, the evolving observable can be interpreted as a curve of generalized coherent states evolving according to the displacement operator $D(\xi) = e^{\xi\mathcal{L}_+ - \bar{\xi}\mathcal{L}_-}$, where $\xi = it$. Moreover, the initial state is the highest weight state of the representation, which is annihilated by $\mathcal{L}_-$ by construction. Coherent states can be viewed as the states closest to the classical ones in the sense that they typically minimize an uncertainty relation. It is for example known that coherent states of the Harmonic oscillator saturate the Robertson uncertainty relation for the pair of observables of position and momentum. Building on this intuition, we could expect that the dispersion bound is saturated for the simplicity hypothesis. It turns out that this intuition is indeed correct. In fact, as we show in Supplementary Note 2, the dispersion bound is saturated if and only if the simplicity hypothesis holds. The saturation of the dispersion bound dictates the evolution of the Krylov complexity, where three different scenarios are possible, as shown in Fig. 1a. The growth of complexity at the speed limit is described by the differential equation

$$\partial_t^2 K(t) = \alpha K(t) + \gamma, \quad (4)$$

with the conditions that $K(0) = 0$ and $K(-t) = K(t)$. For finite Krylov dimension, saturation of the dispersion bound sets the complexity growing according to $K(t) = (D-1)\sin^2\omega t$, where $\omega = \sqrt{\frac{\gamma}{2(D-1)}}$. In this, case, the corresponding complexity algebra (3) reduces to the SU(2) algebra. By contrast, for infinite Krylov

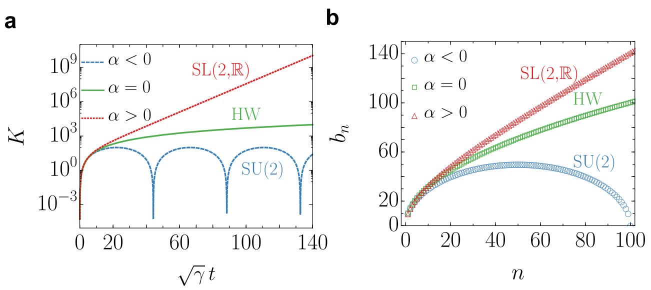

Fig. 1 Growth of Krylov complexity at the speed limit. Saturation of the dispersion bound occurs in three different scenarios, each of which is associated with a different complexity algebra that is specified by the sign of α. **a** Time-dependence of the Krylov complexity. **b** The corresponding growth of the Lanczos coefficients in the Krylov lattice. The plots are representative of the three different scenarios. The Krylov dimension in the SU(2) case is $D = 100$, and infinite in all other cases. In **b** we choose $\alpha = 4$ and -4 for the SL(2, $\mathbb{R}$) and SU(2) algebras, respectively, while α is always zero in the HW case. Finally, the parameter γ in **b** is chosen in each case such that the corresponding Lanczos coefficients share the same behavior near the origin of the Krylov lattice. Specifically, $\gamma = 202$, 200, and 198 for the SL(2, $\mathbb{R}$), HW and SU(2) algebras, respectively.

dimension there are two distinct scenarios for the complexity growth: for $\alpha > 0$ one finds $K(t) = \frac{2\gamma}{\alpha}\sinh^2\frac{\sqrt{\alpha}t}{2}$, while for $\alpha = 0$ the solution reads $K(t) = \frac{\gamma}{2}t^2$. The complexity algebra in these two cases reduces to $SL(2,\mathbb{R})$ and the Heisenberg–Weyl algebra (HW), respectively. Reference examples maximizing the Krylov complexity growth rate at all times are discussed in Supplementary Note 1. One such example with $\alpha > 1$ is the Sachdev-Ye-Kitaev (SYK) model[45], a paradigm of quantum chaos. However, the saturation of the bound does not require quantum chaos and can indeed be achieved by a single qubit, with $\alpha = 0$ (Supplementary Note 1). Together with the time-dependence of $K(t)$ and the complexity algebra, the value of α also determines the growth of the Lanczos coefficients in the Krylov lattice. As proven in Supplementary Note 2, the dispersion bound is saturated if and only if the Lanczos coefficients grow according to

$$b_n = \sqrt{\frac{1}{4}\alpha n(n-1) + \frac{1}{2}\gamma n}, \tag{5}$$

exhibiting three different scalings as a function of α, see Fig. 1b. That the simplicity hypothesis implies (5) has already been pointed out in[42]. For $\alpha > 1$ and large n, this dependence captures the linear growth $b_n = \sqrt{\alpha}n$ conjectured by Parker et al. to hold in generic non-integrable systems, maximizing the Krylov complexity growth[34].

Krylov complexity in generic systems. We next discuss the Krylov complexity growth in generic systems not fulfilling the simplicity hypothesis. We can use Eq. (5) to estimate when and at what time scale a generic system deviates from the bound. By expanding Krylov complexity up to fourth order, we find that $K(t) = b_1^2t^2 + \frac{1}{6}b_1^2(2b_2^2 - b_1^2)t^4 + O(t^6)$. Since we can always find a value on α and γ such that b_1 and b_2 satisfy (5), we conclude that the bound (2) is saturated up to the third order in time. By expanding the Krylov complexity up to sixth order, we find that the Lanczos coefficient b_3 will appear in the last term, and since we are not guaranteed to be able to find a value on α and γ such that b_1, b_2, and b_3 satisfy(5), we conclude that the system can only start deviating from the bound (2) as a result from fifth-order terms in the expansion. We can estimate this time scale by finding the value of t for which the third order coefficient of $\partial_t K(t)$ is equal to its fifth-order coefficient. We will call this time the *deviation time*, denoted by $\tau_{\rm d}$, and it is explicitly given by

$$\tau_{\rm d} = \sqrt{\frac{\frac{2}{3}b_1^2(2b_2^2 - b_1^2)}{\frac{1}{20}b_1^2(b_1^2+b_2^2) - \frac{1}{5}b_2^2(b_1^2+b_2^2+b_3^2) + \frac{1}{2}b_2^2b_3^2}}. \tag{6}$$

To get an understanding of the complexity growth in a generic setting, we next illustrate the Krylov dynamics of a system described by a random matrix Hamiltonian. Specifically, we consider the Krylov complexity of an ensemble $\mathcal{E}(H)$ of random matrix Hamiltonians, a paradigm of quantum chaos[46]. We sample the Hamiltonian matrices H from the Gaussian Orthogonal Ensemble GOE(d), where d is the dimension of the Hilbert space. We then calculate the Lanczos coefficients $\{b_n\}$ with partial re-orthogonalization[36,47]. Specifically, we consider samples of real matrices $H = (X + X^{\intercal})/2$, where all elements $x \in \mathbb{R}$ of X are pseudo-randomly generated with probability measure given by the normal distribution, $\exp(-x^2/(2\sigma^2))/(\sigma\sqrt{2\pi})$. In order to study the general behavior of Lanczos coefficients, we choose an initial observable, which is represented as the normalized vector $|\mathcal{O}) = (1/d, 1/d, \dots, 1/d)^T$, expressed in a fixed eigenbasis of the Liouvillian. However, the following results do not depend strongly on the choice of $\mathcal{O}$, provided it is dense in the eigenbasis of the Hamiltonian. Figure 2a shows the squares of the Lanczos coefficients for a single realization and the average $\langle\{b_n\}\rangle_{\mathcal{E}(H)}$ over 100 different Hamiltonians of dimension $d = 32$, sampled from GOE(d) with standard deviation $\sigma = 1$. Operator growth is displayed by the time-dependent amplitudes, which are found by solving the recursion relation and exhibit diffusion-like dynamics on the Krylov basis, shown for a single realization in Fig. 2b. The corresponding time evolution of Krylov complexity and its growth rate are shown in panels c and d, respectively. Hamiltonians sampled from GOE(d) behave as a generic system, given that the Lanczos coefficients do not, in general, grow according to (5), as shown in Fig. 2a. As a result, the growth rate starts deviating from the dispersion bound around the time scale $\tau_{\rm d}$ in Eq. (6), indicated by the vertical line in Fig. 2 c, d. In short, while GOE Hamiltonians provide a useful paradigm in the description of quantum chaotic systems, the dynamics generated by them do not maximize the growth of Krylov's complexity for $t > \tau_{\rm d}$.

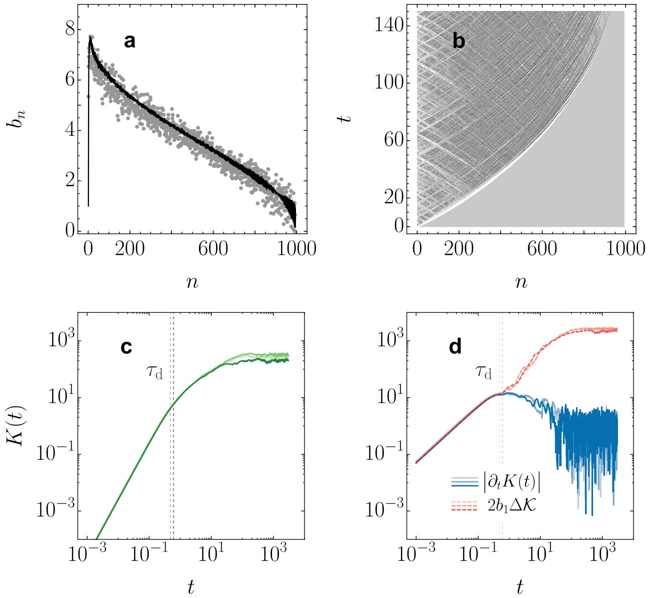

Fig. 2 Growth of Krylov complexity in a generic system. a Squares of the Lanczos coefficients for a single realization (gray points) and an average over 100 random Hamiltonian matrices (black line). **b** Operator growth in the Krylov lattice as displayed by the dynamics of the amplitudes $|\varphi_n(t)|^2$ for a single random matrix realization. **c** Krylov complexity (green solid lines) together with the deviation time (gray dashed line) for three independent random matrix realizations. **d** The corresponding absolute value of the growth rate of the Krylov complexity (blue solid lines), together with the dispersion bound (red dashed lines), Eq. (2). In all figures, the random Hamiltonian matrices are sampled from GOE(d) with standard deviation $\sigma = 1$, maximal Krylov dimension $D = 993$ and a uniform initial observable operator $\mathcal{O}$.

Our results establish the ultimate speed limit to operator growth in isolated quantum systems. Specifically, the dispersion bound governs the growth rate of Krylov complexity, playing the role of a Mandelstam-Tamm uncertainty relation in operator space. This bound is saturated by quantum systems in which the Liouvillian governing the time evolution fulfills a simple algebra. The latter arises naturally in certain quantum chaotic systems, such as the SYK model. However, other paradigmatic instances of quantum chaos, such as random matrix Hamiltonians, do not maximize the growth of Krylov complexity. Indeed, a saturation of the bound does not require quantum chaos and can be achieved, e.g., by a single qubit.

Methods

Vanishing of the anticommutator contribution in the Robertson uncertainty relation for $\mathcal{K}$ and $\mathcal{L}$. We establish a universal feature of Krylov complexity, valid for any physical system: namely, that its anticommutator with the Liouvillian $\mathcal{L}$ has vanishing expectation value over the evolved operator $|\mathcal{O}(t))$. The relevance of this result relies on the fact that this quantity enters the Schrödinger uncertainty principle for the two operators $\mathcal{K}$ and $\mathcal{L}$

$$4(\Delta\mathcal{K}\Delta\mathcal{L})^2 \geq \left|(\mathcal{O}(t)|[\mathcal{K},\mathcal{L}]|\mathcal{O}(t))\right|^2 + \left|(\mathcal{O}(t)|\{\mathcal{K},\mathcal{L}\}|\mathcal{O}(t))\right|^2, \quad (7)$$

from which one can bound the complexity rate $\partial_t K$. We have that

$$\left(\mathcal{O}(t)|[\mathcal{K},\mathcal{L}]|\mathcal{O}(t)\right) = 2i\,\mathrm{Im}\left(\mathcal{O}(t)|\mathcal{K}\mathcal{L}|\mathcal{O}(t)\right) \quad (8)$$

and

$$\left(\mathcal{O}(t)|\{\mathcal{K},\mathcal{L}\}|\mathcal{O}(t)\right) = 2\mathrm{Re}\left(\mathcal{O}(t)|\mathcal{K}\mathcal{L}|\mathcal{O}(t)\right), \quad (9)$$

where

$$\mathcal{K}\mathcal{L} = \sum_{n=0}^{D-1} b_{n+1}\left[n|\mathcal{O}_n)(\mathcal{O}_{n+1}| + (n+1)|\mathcal{O}_{n+1})(\mathcal{O}_n|\right]. \quad (10)$$

Let us now demonstrate that the anticomutator term in Eq. (9) is identically zero. By expanding $|\mathcal{O}(t))$ over the Krylov basis, we obtain

$$\left(\mathcal{O}(t)|\mathcal{K}\mathcal{L}|\mathcal{O}(t)\right) = \sum_{m,n,k=0}^{D-1} (-i)^m i^k \varphi_m \varphi_k b_{n+1}\left(n\delta_{mn}\delta_{n+1,k} + (n+1)\delta_{m,n+1}\delta_{nk}\right), \quad (11)$$

which, by performing the sums over k and n, yields

$$\begin{aligned}\left(\mathcal{O}(t)|\mathcal{K}\mathcal{L}|\mathcal{O}(t)\right) &= i\sum_{m=0}^{D-1} m\varphi_m(\varphi_{m+1}b_{m+1} - \varphi_{m-1}b_m) \\ &= -i\sum_{m=0}^{D-1} m\varphi_m \partial_t \varphi_m.\end{aligned} \quad (12)$$

Since the amplitudes φ_n and the coefficients b_n are real quantities, comparing Eqs. (9) and (12) we immediately conclude that

$$(\mathcal{O}(t)|\{\mathcal{K},\mathcal{L}\}|\mathcal{O}(t)) = 0 \quad \forall t. \quad (13)$$

Let us note that the key condition to obtain this result is the fact that the Liouvillian connects only states that are nearest neighbors on the Krylov lattice so that we are left with a purely imaginary phase $(-i)^m(i)^{m\pm1} = \pm i$. It is this peculiar property that allows the Liouvillian to be interpreted as a sum of generalized ladder operators $\mathcal{L}_\pm$ [42]. However, let us point out that here we are not making any assumption regarding the commutation rules between these operators: we are considering the structure of Krylov space in full generality.

Moreover, from Eq. (8), we immediately obtain the relation between the anticommutator $[\mathcal{K},\mathcal{L}]$ and the complexity rate $\partial_t K$:

$$\left(\mathcal{O}(t)|[\mathcal{K},\mathcal{L}]|\mathcal{O}(t)\right) = -2i\sum_{m=0}^{D-1} m\varphi_m\partial_t\varphi_m = -i\partial_t K. \quad (14)$$

Therefore, the Schrödinger uncertainty relation (7) can be recast as the dispersion bound (2) on the growth of Krylov complexity:

$$|\partial_t K| \leq 2b_1\Delta\mathcal{K}. \quad (15)$$

Geometrical interpretation of the saturation of the bound. For the geometrical interpretation of the saturation of the bound, we assume the Krylov space to be of finite dimension. However, the results could potentially be extended to infinite-dimensional Krylov spaces as well.

The Krylov space is isomorphic to a 2D-dimensional real vector space, and we can therefore consider the Euclidean metric g, given by the real part of the inner product. The evolution curve of $\mathcal{O}$ will then be restricted to the unit sphere of the Krylov space. This unit sphere forms a Riemannian manifold and we can consider the Krylov complexity as a function on this manifold defined by $K(\mathcal{A}) = (\mathcal{A}|\mathcal{K}\mathcal{A})$, for any element $|\mathcal{A})$ in the Krylov space with a unit norm. In this sense, when we write $K(t)$ we simply mean $K(\mathcal{O}(t))$ which is consistent with how we defined complexity for the evolution. The differential of Krylov complexity will be denoted by dK and its action on any tangent vector $\dot{\mathcal{A}}$ at $\mathcal{A}$ is given by $dK(\dot{\mathcal{A}}) = (\dot{\mathcal{A}}|\mathcal{A}) + (\mathcal{A}|\dot{\mathcal{A}})$. This differential together with the metric can be used to define the gradient of Krylov complexity. It follows from the theory of differential geometry that the gradient of Krylov complexity at $\mathcal{A}$, denoted by $\nabla K(\mathcal{A})$, is the unique vector satisfying the expression $g(\nabla K(\mathcal{A}), \dot{\mathcal{A}}) = dK(\dot{\mathcal{A}})$ for all tangent vectors $\dot{\mathcal{A}}$ at $\mathcal{A}$[48]. It can be checked that the gradient must then be given by $\nabla K(\mathcal{A}) = 2(\mathcal{K} - \langle\mathcal{K}\rangle)\mathcal{A}$, which indeed is tangent to the unit sphere at $\mathcal{A}$. The change of Krylov complexity along the curve $\mathcal{O}(t)$, generated by the Liouvillian, is given by $\partial_t K(t) = g(\nabla K(t), \partial_t \mathcal{O}(t))$, where $\nabla K(t)$ is the gradient at $\mathcal{O}(t)$. Applying the Cauchy-Schwarz inequality on the right-hand side gives us the inequality

$$\left|\partial_t K(t)\right| \leq \|\nabla K(t)\|\|\partial_t \mathcal{O}(t)\|. \quad (16)$$

The right-hand side of this inequality is exactly $2b_1\Delta\mathcal{K}$ and we note that it is saturated if and only if the tangent vector of $\mathcal{O}(t)$ is parallel to the gradient of Krylov complexity. We also note that the gradient is the zero vector at time zero and so the dispersion bound is always initially saturated.

The unitary orbit of $\mathcal{O}$ is the set of all points $U^\dagger \mathcal{O} U$, where U is a unitary operator. We emphasize that this is a proper subset of the unit sphere in Krylov space which, in contrast, is the set of all points $\mathcal{U}\mathcal{O}$, where $\mathcal{U}$ is a unitary superoperator. The gradient we have considered is with respect to the unit sphere, and it is therefore not obvious that this gradient will ever be tangential to the unitary orbit of $\mathcal{O}$. However, the gradient is indeed tangential to the unitary orbit at time zero and at all times, provided the simplicity algebra is fulfilled.

On the closure of the complexity algebra. Here we show the proof that the only possible closure of the complexity algebra introduced by[42] is given by Eq. (3). The (anti-Hermitian) operator $\mathcal{B} = \mathcal{L}_+ - \mathcal{L}_-$ "conjugated" to the Liouvillian can be expanded in Krylov space as

$$\mathcal{B} = \sum_{n=0}^{D-1} b_{n+1}\left[|\mathcal{O}_{n+1})(\mathcal{O}_n| - |\mathcal{O}_n)(\mathcal{O}_{n+1}|\right]. \quad (17)$$

We note that one can establish a formal analogy with the harmonic oscillator: $\mathcal{L}$ plays the role of the position of the harmonic oscillator, while $i\mathcal{B}$ corresponds to its momentum. However, in general, the commutator between $\mathcal{L}$ and $\mathcal{B}$ is not proportional to the identity, indeed:

$$\tilde{\mathcal{K}} = 2[\mathcal{L}_+, \mathcal{L}_-] = 2\sum_{n=0}^{D-1}\left(b_{n+1}^2 - b_n^2\right)|\mathcal{O}_n)(\mathcal{O}_n|, \quad (18)$$

where it is understood that b_0 has to be replaced with 0. Let us now investigate the conditions under which $\mathcal{L}$, $\mathcal{B}$ and $\tilde{\mathcal{K}}$ form a closed algebra with respect to the operation [,]: the so-called complexity algebra[42]. This happens if and only if the commutators $[\mathcal{L},\tilde{\mathcal{K}}]$ and $[\mathcal{B},\tilde{\mathcal{K}}]$ can be written as linear combinations of the operators $\mathcal{L}$, $\mathcal{B}$ and $\tilde{\mathcal{K}}$ themselves. These commutators can be expanded over the Krylov basis as follows:

$$[\mathcal{L},\tilde{\mathcal{K}}] = 2\sum_{n=0}^{D-1} f(n) b_{n+1}\left[|\mathcal{O}_{n+1})(\mathcal{O}_n| - |\mathcal{O}_n)(\mathcal{O}_{n+1}|\right], \quad (19)$$

$$[\mathcal{B},\tilde{\mathcal{K}}] = 2\sum_{n=0}^{D-1} f(n) b_{n+1}\left[|\mathcal{O}_{n+1})(\mathcal{O}_n| + |\mathcal{O}_n)(\mathcal{O}_{n+1}|\right], \quad (20)$$

where we have defined

$$f(n) = b_{n+1}^2 - b_n^2 - \left(b_{n+2}^2 - b_{n+1}^2\right) = \frac{\tilde{\mathcal{K}}_{nn} - \tilde{\mathcal{K}}_{n+1,n+1}}{2}. \quad (21)$$

Now, it is clear that the commutator (19) between $\mathcal{L}$ and $\tilde{\mathcal{K}}$ cannot contain any element of the complexity algebra other than $\mathcal{B} = \sum_{n=0}^{D-1} b_{n+1}[|\mathcal{O}_{n+1})(\mathcal{O}_n| - |\mathcal{O}_n)(\mathcal{O}_{n+1}|]$, while the commutator (20) can only contain $\mathcal{L} = \sum_{n=0}^{D-1} b_{n+1}[|\mathcal{O}_{n+1})(\mathcal{O}_n| + |\mathcal{O}_n)(\mathcal{O}_{n+1}|]$. Moreover, the only possibility for the algebra to be closed is that the discrete function $f(n)$ is a constant. By looking at Eq. (21), we conclude that $f(n)$ is constant if and only if

$$2\left(b_{n+1}^2 - b_n^2\right) = \alpha n + 2\gamma, \quad (22)$$

for some constants α and γ (the factors 2 are included for convenience). Again, b_0 has to be replaced with 0, so that Eq. (22) holds for $n \geq 1$, while $2b_1^2 = \alpha + 2\gamma$. Then, the function $f(n)$ takes the constant value $f = -\alpha/2$, so that the only possible closure of the complexity algebra is given by:

$$[\mathcal{L},\mathcal{B}] = \tilde{\mathcal{K}}, \quad [\tilde{\mathcal{K}},\mathcal{L}] = \alpha\mathcal{B}, \quad [\tilde{\mathcal{K}},\mathcal{B}] = \alpha\mathcal{L}. \quad (23)$$

Moreover, from Eq. (22), we immediately conclude that

$$\tilde{\mathcal{K}} = \alpha\mathcal{K} + \gamma. \quad (24)$$

Therefore, if $\alpha \neq 0$, the Krylov complexity is related to $\tilde{\mathcal{K}}$ by a shift. Conversely, if $\alpha = 0$, there is no simple relation between the Krylov complexity and the operator $\tilde{\mathcal{K}}$. In this case, $\tilde{\mathcal{K}}$ is proportional to the identity and the complexity algebra reduces to the Heisenberg–Weyl algebra[43], being $[\mathcal{L}_+, \mathcal{L}_-] = \gamma\mathbb{1}$.

Possible scenarios under the closure of the complexity algebra. As already discussed, if $\mathcal{L}$, $\mathcal{B}$ and their commutator $\tilde{\mathcal{K}}$ closes an algebra, then the only possible commutation relations are given by (23). This complexity algebra is then reduced to the Heisenberg–Weyl algebra whenever $\alpha = 0$. We next show that for the cases $\alpha < 0$ and $\alpha > 0$, the complexity algebra reduces to the SU(2) algebra and the SL(2, $\mathbb{R}$) algebra, respectively. Let us introduce the operators J_+ and J_-, which are defined by $\nu J_+ = \mathcal{L}_+$ and $\nu J_- = \mathcal{L}_-$, where ν is a strictly positive scaling parameter. We can then write $\mathcal{L} = \nu(J_+ + J_-)$ and $\mathcal{B} = \nu(J_+ - J_-)$. Let us also introduce the operator J_0 defined by $J_0 = -\frac{1}{2\nu^2}\tilde{\mathcal{K}}$. By substituting these operators into (23), one can rewrite the commutation relations as

$$[J_+, J_-] = J_0, \quad [J_0, J_\pm] = \mp\frac{\alpha}{2\nu^2}J_\pm. \quad (25)$$

By choosing the scaling parameter such that $2\nu^2 = \alpha$, we find that the algebra (23) is equivalent to

$$[J_+, J_-] = J_0, \quad [J_0, J_\pm] = \pm J_\pm \quad \alpha < 0 \quad \mathrm{SU}(2), \tag{26}$$

$$[J_+, J_-] = J_0, \quad [J_0, J_\pm] = \mp J_\pm \quad \alpha > 0 \quad \mathrm{SL}(2, \mathbb{R}). \tag{27}$$

What we have shown is that, whenever the simplicity hypothesis holds, then the algebra generated by $\mathcal{L}$, $\mathcal{B}$ and their commutator can always be reduced to either SU(2), SL(2, $\mathbb{R}$) or the Heisenberg–Weyl algebra, and for which of these it reduces to depends on the value of α.

Data availability

The datasets generated during and/or analysed during the current study are available from the corresponding author upon reasonable request.

Code availability

The codes generated and used during the current study are available from the corresponding author on reasonable request.

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Author contributions

N.H. and A.d.C. introduced the dispersion bound. N.H. provided its geometric interpretation and, together with N.C. studied the conditions for its saturation. N.H. provided the estimation of the deviation time. N.C. prepared Fig. 1 and analyzed the explicit models presented in the supplementary information. A.S.M.-R. found the differential equation for the Krylov complexity and performed the numerical analysis in GOE for Fig. 2. A.d.C. proposed the project, provided guidance, and supervised the work. All the authors participated in the analysis of the results and the writing of the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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