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Lower and upper bounds for entanglement of Rényi-α entropy

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Entanglement Rényi-α entropy is an entanglement measure. It reduces to the standard entanglement of formation when α tends to 1. We derive analytical lower and upper bounds for the entanglement Rényi-α entropy of arbitrary dimensional bipartite quantum systems. We also demonstrate the application our bound for some concrete examples. Moreover, we establish the relation between entanglement Rényi-α entropy and some other entanglement measures.

Quantum entanglement is one the most remarkable features of quantum mechanics and is the key resource central to much of quantum information applications. For this reason, the characterization and quantification of entanglement has become an important problem in quantum-information science[1]. A number of entanglement measures have been proposed for bipartite states such as the entanglement of formation (EOF)[2], concurrence[3], relative entropy[4], geometric entanglement[5], negativity[6] and squashed entanglement[7,8]. Among them EOF is one of the most famous measures of entanglement. For a pure bipartite state $|\psi\rangle_{AB}$ in the Hilbert space, the EOF is given by

$$E_F(|\psi\rangle_{AB}) = S(\rho_A), \tag{1}$$

where $S(\rho_A) := -\mathrm{Tr}\rho_A \log \rho_A$ is the von Neumann entropy of the reduced density operator of system A. Here "log" refers to the logarithm of base two. The situation for bipartite mixed states ρ_{AB} is defined by the convex roof

$$E_F(\rho_{AB}) = \min \sum_i p_i E_F(|\psi_i\rangle_{AB}), \tag{2}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$ with $\sum_i p_i = 1$ and $p_i > 0$. The EOF provides an upper bound on the rate at which maximally entangled states can be distilled from ρ and a lower bound on the rate at which maximally entangled states needed to prepare copies of ρ[9]. For two-qubit systems, an elegant formula for EOF was derived by Wootters in ref. 3. However, for the general highly dimensional case, the evaluation of EOF remains a nontrivial task due to the the difficulties in minimization procedures[10]. At present, there are only a few analytic formulas for EOF including the isotropic states[11], Werner states[12] and Gaussian states with certain symmetries[13]. In order to evaluate the entanglement measures, many efforts have also been devoted to the study of lower and upper bounds of different entanglement measures[14–32]. Especially, Chen *et al.*[18] derived an analytic lower bound of EOF for an arbitrary bipartite mixed state, which established a bridge between EOF and two strong separability criteria. Based on this idea, there are several improved lower and upper bounds for EOF presented in refs 33–36. While the entanglement of formation is the most common measure of entanglement, it is not the unique measure. There are other measures such as entanglement Rényi-α entropy (ERαE) which is the generalization of the entanglement of formation. The ERαE has a continuous spectrum parametrized by the non-negative real parameter α. For a bipartite pure state $|\psi\rangle_{AB}$, the ERαE is defined as[37]

$$E_\alpha(|\psi\rangle_{AB}) := S_\alpha(\rho_A) := \frac{1}{1-\alpha}\log(\mathrm{tr}\rho_A^\alpha), \tag{3}$$

where $S_\alpha(\rho_A)$ is the Rényi-α entropy. Let $\mu_1, \cdots, \mu_m$ be the eigenvalues of the reduced density matrix ρ_A of $|\psi\rangle_{AB}$. We have

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$$S_\alpha(\rho_A) = \frac{1}{1-\alpha}\log\left(\sum_i \mu_i^{\ \alpha}\right) := H_\alpha(\vec{\mu}), \tag{4}$$

where $\vec{\mu}$ is called the Schmidt vector $(\mu_1, \mu_2, \cdots, \mu_m)$. The Rényi-$\alpha$ entropy is additive on independent states and has found important applications in characterizing quantum phases with differing computational power[38], ground state properties in many-body systems[39], and topologically ordered states[40,41]. Similar to the convex roof in (2), the ERαE of a bipartite mixed state ρ_{AB} is defined as

$$E_\alpha(\rho_{AB}) = \min\sum_i p_i E_\alpha(|\psi_i\rangle_{AB}). \tag{5}$$

It is known that the Rényi-α entropy converges to the von Neumann entropy when α tends to 1. So the ERαE reduces to the EOF when α tends to 1. Further ERαE is not increased under local operations and classical communications (LOCC)[37]. So the ERαE is an entanglement monontone, and becomes zero if and only if ρ_{AB} is a separable state.

An explicit expression of ERαE has been derived for two-qubit mixed state with $\alpha \geq (\sqrt{7}-1)/2 \simeq 0.823$[37,42]. Recently, Wang *et al.*[42] further derived the analytical formula of ERαE for Werner states and isotropic states. However, the general analytical results of ERαE even for the two-qubit mixed state with arbitrary parameter α is still a challenging problem.

The aim of this paper is to provide computable lower and upper bounds for ERαE of arbitrary dimensional bipartite quantum systems, and these results might be utilized to investigate the monogamy relation[43–46] in high-dimensional states. The key step of our work is to relate the lower or upper bounds with the concurrence which is relatively easier to dealt with. We also demonstrate the application of these bounds for some examples. Furthermore, we derive the relation of ERαE with some other entanglement measures.

Lower and upper bounds for entanglement of Rényi-α entropy. For a bipartite pure state with Schmidt decomposition $|\psi\rangle = \sum_{i=1}^m \sqrt{\mu_i}|ii\rangle$, the concurrence of $|\psi\rangle$ is given by $c(|\psi\rangle) := \sqrt{2(1-\mathrm{Tr}\rho_A^2)} = \sqrt{2(1-\sum_{i=1}^m \mu_i^{\ 2})}$. The expression $1-\mathrm{Tr}\rho_A^2$ is also known as the mixedness and linear entropy[47,48]. The concurrence of a bipartite mixed state ρ is defined by the convex roof $c(\rho) = \min\sum_i p_i c(|\psi_i\rangle)$ for all possible pure state decompositions of $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. A series of lower and upper bounds for concurrence have been obtained in refs 19,24,25. For example, Chen *et al.*[19] provides a lower bound for the concurrence by making a connection with the known strong separability criteria[49,50], i.e.,

$$c(\rho) \geq \sqrt{\frac{2}{m(m-2)}}(\max(\|\rho^{T_A}\|, \|\mathcal{R}(\rho)\|) - 1), \tag{6}$$

for any $m\otimes n(m\leq n)$ mixed quantum system. The $\|\cdot\|$ denotes the trace norm and T_A denotes the partial transpose. Another important bound of squared concurrence used in our work is given by refs 24,25.

$$\mathrm{Tr}(\rho\otimes\rho V_i) \leq [C(\rho)]^2 \leq \mathrm{Tr}(\rho\otimes\rho K_i), \tag{7}$$

with $V_1 = 4(P_-^{(1)} - P_+^{(1)})\otimes P_-^{(2)}, V_2 = 4P_-^{(1)}\otimes(P_-^{(2)} - P_+^{(2)}), K_1 = 4(P_-^{(1)}\otimes I^{(2)}), K_2 = 4(I^{(1)}\otimes P_-^{(2)})$ and $P_-^{(i)}(P_+^{(i)})$ is the projector on the antisymmetric (symmetric) subspace of the two copies of the ith system. These bounds can be directly measured and can also be written as

$$\mathrm{Tr}(\rho\otimes\rho V_1) = 2(\mathrm{Tr}\rho^2 - \mathrm{Tr}\rho_A^2), \tag{8}$$

$$\mathrm{Tr}(\rho\otimes\rho V_2) = 2(\mathrm{Tr}\rho^2 - \mathrm{Tr}\rho_B^2), \tag{9}$$

$$\mathrm{Tr}(\rho\otimes\rho K_1) = 2(1 - \mathrm{Tr}\rho_A^2), \tag{10}$$

$$\mathrm{Tr}(\rho\otimes\rho K_2) = 2(1 - \mathrm{Tr}\rho_B^2). \tag{11}$$

Below we shall derive the lower and upper bounds of ERαE based on these existing bounds of concurrence. Different states may have the same concurrence. Thus the value of $H_\alpha(\vec{\mu})$ varies with different Schmidt coefficients μ_i for fixed concurrence. We define two functions

$$R_U(c) = \max\left\{H_\alpha(\vec{\mu})\middle|\sqrt{2\left(1-\sum_{i=1}^m \mu_i^{\ 2}\right)} \equiv c\right\}, \tag{12}$$

$$R_L(c) = \min\left\{H_\alpha(\vec{\mu})\middle|\sqrt{2\left(1-\sum_{i=1}^m \mu_i^{\ 2}\right)} \equiv c\right\}. \tag{13}$$

The derivation of them is equivalent to finding the maximal and minimal of $H_\alpha(\vec{\mu})$. Notice that the definition of $H_\alpha(\vec{\mu})$, it is equivalent to find the maximal and minimal of $\sum_{i=1}^m \mu_i^{\ \alpha}$ under the constraint $\sqrt{2(1-\sum_{i=1}^m \mu_i^{\ 2})} \equiv c$

since the logarithmic function is a monotonic function . With the method of Lagrange multipliers we obtain the necessary condition for the maximum and minimum of $\sum_{i=1}^{m}\mu_i^{\alpha}$ as follows

$$\alpha\mu_i^{\alpha-1} = 2\lambda_1\mu_i - \lambda_2, \tag{14}$$

where λ_1, λ_2 denote the Lagrange multipliers. This equation has maximally two nonzero solutions γ and δ for each μ_i. Let n_1 be the number of entries where $\mu_i = \gamma$ and n_2 be the number of entries where $\mu_i = \delta$. Thus the derivation is reduced to maximizes or minimizes the function

$$R_{n_1n_2}(c) = \frac{1}{1-\alpha}\log(n_1\gamma^{\alpha} + n_2\delta^{\alpha}), \tag{15}$$

under the constrains

$$n_1\gamma + n_2\delta = 1,\ 2(1 - n_1\gamma^2 - n_2\delta^2) = c^2, \tag{16}$$

where $n_1 + n_2 = d \le m$. From Eq. (16) we obtain two solutions of γ

$$\gamma_{n_1n_2}^{\pm} = \frac{n_1 \pm \sqrt{n_1^2 - n_1(n_1+n_2)[1 - n_2(1 - c^2/2)]}}{n_1(n_1+n_2)}, \tag{17}$$

$$\delta_{n_1n_2}^{\pm} = \frac{1 - n_1\gamma_{n_1n_2}^{\pm}}{n_2}, \tag{18}$$

with $\max\{\sqrt{2(n_1-1)/n_1}, \sqrt{2(n_2-1)/n_2}\} \le c \le \sqrt{2(d-1)/d}$. Because $\gamma_{n_2n_1}^{-} = \delta_{n_1n_2}^{+}$, $\delta_{n_2n_1}^{-} = \gamma_{n_1n_2}^{+}$, we should only consider the case for $\gamma_{n_1n_2}^{+}$. When $n_2 = 0$, γ can be uniquely determined by the constrains thus we omit this case.

When $m = 3$, the solution of Eq. (15) is $R_{12}(c)$ and $R_{21}(c)$ for $1 < c \le 2/\sqrt{3}$. After a direct calculation we find $R_{12}(c)$ and $R_{21}(c)$ are both monotonically function of the concurrence c, and $R_{12}(2/\sqrt{3}) = R_{21}(2/\sqrt{3})$. In order to compare the value of $R_{12}(c)$ and $R_{21}(c)$ we only need to compare the value of them at the endpoint $c = 1$. For convenience we divide the problem into three cases. If $0 < \alpha < 2$, then $R_{12}(1) > R_{21}(1)$; If $\alpha = 2$, then $R_{12}(1) = R_{21}(1)$; If $\alpha > 2$, then $R_{12}(1) < R_{21}(1)$. Thus we conclude that the maximal and minimal function of $H_\alpha(\vec{\mu})$ is given by $R_{21}(c)$ and $R_{12}(c)$ respectively for $\alpha > 2$. When $\alpha < 2$, the maximal and minimal function of $H_\alpha(\vec{\mu})$ is $R_{12}(c)$ and $R_{21}(c)$ respectively. When $\alpha = 2$, we can check that the two functions $R_{21}(c)$ and $R_{12}(c)$ always have the same value for $1 < c \le 2/\sqrt{3}$. In the general case for $m = d$, numerical calculation shows the following results

(i) When $\alpha > 2$,

$$R_L(c) = \frac{\log[(\gamma_{1,d-1}^{+})^{\alpha} + (d-1)^{1-\alpha}(1 - \gamma_{1,d-1}^{+})^{\alpha}]}{1-\alpha}, \tag{19}$$

$$R_U(c) = \frac{\log[(\gamma_{1,d-1}^{-})^{\alpha} + (d-1)^{1-\alpha}(1 - \gamma_{1,d-1}^{-})^{\alpha}]}{1-\alpha}, \tag{20}$$

with $\sqrt{2(d-2)/(d-1)} < c \le \sqrt{2(d-1)/d}$, $1 \le d \le m-1$ and $\gamma_{1,d-1}^{\pm} = (2 \pm \sqrt{2(d-1)[d(2-c^2)-2]})/2d$.

(ii) When $\alpha < 2$,

$$R_L(c) = \frac{\log[(\gamma_{1,d-1}^{-})^{\alpha} + (d-1)^{1-\alpha}(1 - \gamma_{1,d-1}^{-})^{\alpha}]}{1-\alpha}, \tag{21}$$

$$R_U(c) = \frac{\log[(\gamma_{1,d-1}^{+})^{\alpha} + (d-1)^{1-\alpha}(1 - \gamma_{1,d-1}^{+})^{\alpha}]}{1-\alpha}. \tag{22}$$

(iii) When $\alpha = 2$, these lower and upper bounds give the same value.

We use the denotation co(g) to be the convex hull of the function g, which is the largest convex function that is bounded above by g, and ca(g) to be the smallest concave function that is bounded below by g. Using the results presented in Methods, we can prove the main result of this paper.

Theorem. For any $m \otimes n (m \le n)$ mixed quantum state ρ, its ERαE satisfies

$$co[R_L(\underline{C})] \le E_\alpha(\rho) \le ca[R_U(\overline{C})], \tag{23}$$

where

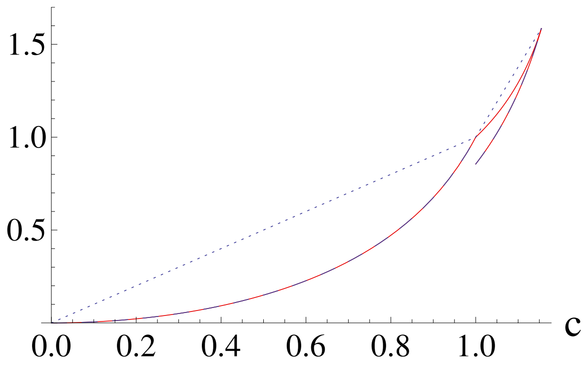

Figure 1. The plot of lower bound (dashed line) and upper bound (dotted line) for $\alpha = 3$, m = 3. The upper bound consists of two segments and the lower bound consists of three segments. The solid line corresponds to R_{11}, R_{12} and R_{21}.

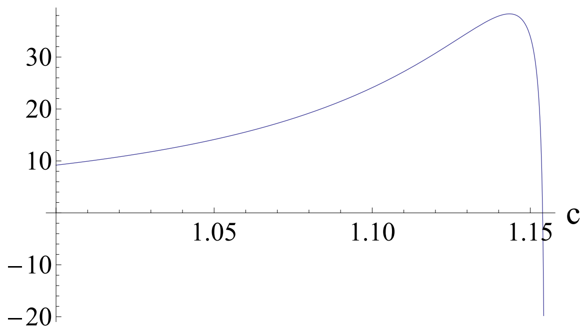

Figure 2. The plot of the second derivative of R_{12} for $1 < c \le 2/\sqrt{3}$.

$$\overline{C} = \min\left\{\sqrt{2(1-\mathrm{Tr}\rho_A^2)},\ \sqrt{2(1-\mathrm{Tr}\rho_B^2)}\right\}, \tag{24}$$

and

$$\begin{aligned}\underline{C}^2 \;=\; &\max\{0,\ 2/m(m-1)(\|\rho^{T_A}\|-1)^2,\ 2/m(m-1)(\|\mathcal{R}(\rho)\|-1)^2,\\ &2(\mathrm{Tr}\rho^2-\mathrm{Tr}\rho_A^2),\quad 2(\mathrm{Tr}\rho^2-\mathrm{Tr}\rho_B^2)\}.\end{aligned} \tag{25}$$

Next we consider how to calculate the expressions of $co(R_L(c))$ and $ca(R_U(c))$. As an example, we only consider the case $m = 3$. In order to obtain $co(R_L(c))$, we need to find the largest convex function which bounded above by $R_L(c)$. We first set the parameter $\alpha = 3$, then we can derive

$$\begin{aligned} R_L(c) &= \begin{cases} R_{11}, & 0 < c \le 1\\ R_{12}, & 1 < c \le 2/\sqrt{3},\end{cases}\\ R_U(c) &= \begin{cases} R_{11}, & 0 < c \le 1\\ R_{21}, & 1 < c \le 2/\sqrt{3}.\end{cases}\end{aligned} \tag{26}$$

We plot the function R_{11}, R_{12} and R_{21} in Fig. 1 which illustrates our result. It is direct to check that $R''_{11} \ge 0$, therefore $co(R_{11}) = R_{11}$ for $0 < c \le 1$. The second derivative of R_{12} is not convex near $c = 2/\sqrt{3}$ as shown in Fig. 2. In order to calculate $co(R_{12})$, we suppose $l_1(c) = k_1(c - 2/\sqrt{3}) + \log 3$ to be the line crossing through the point $[2/\sqrt{3}, R_{12}(2/\sqrt{3})]$. Then we solve the equations $l_1(c) = R_{12}(c)$ and $dl_1(c)/dc = dR_{12}(c)/dc = k_1$ and the solution is $k_1 = 5.2401$, $c = 1.1533$. Combining the above results, we get

$$co(R_L(c)) = \begin{cases} R_{11}(0 < c \le 1)\\ R_{12}(1 < c \le 1.1533)\\ 5.2401(c - 2/\sqrt{3}) + \log 3\\ (1.1533 < c \le 2/\sqrt{3}).\end{cases} \tag{27}$$

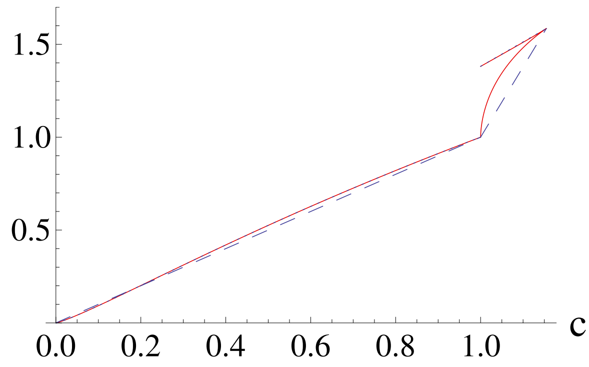

Figure 3. The plot of lower bound (dashed line) and upper bound (dotted line) for α = 0.6, m = 3. The upper bound consists of two segments and the lower bound also consists of two segments. The solid line corresponds to R_{11}, R_{12} and R_{21}.

Similarly, we can calculate that $R''_{11} \geq 0$ and $R''_{21} \geq 0$, thus $co(R_U(c))$ is the broken line connecting the following points: $[0,\ 0]$, $[1,\ \log 2]$, $[2/\sqrt{3},\ \log 3]$. In Fig. 3 we have plotted the lower and upper bounds with dashed and dotted line respectively.

Then we choose the parameter $\alpha = 0.6$, and we get

$$R_L(c) = \begin{cases} R_{11}(0 < c \leq 1) \\ R_{21}(1 < c \leq 2/\sqrt{3}), \end{cases} \tag{28}$$

$$R_U(c) = \begin{cases} R_{11}(0 < c \leq 1) \\ R_{12}(1 < c \leq 2/\sqrt{3}). \end{cases} \tag{29}$$

Since $R''_{11} \leq 0$, $R''_{21} \leq 0$, we have that $co(R_L(c))$ is the broken line connecting the points: $[0,\ 0]$, $[1,\ \log 2]$, $[2/\sqrt{3},\ \log 3]$. In order to obtain $ca(R_U(c))$, we need to find the smallest concave function which bounded below by $R_U(c)$. We find $R''_{11} \leq 0$, $R''_{12} \geq 0$, therefore $ca(R_U(c))$ is the curve consisting R_{11} for $0 < c \leq 1$ and the line connecting points $[1, R_{12}(1)]$ and $[2/\sqrt{3}, R_{12}(2/\sqrt{3})]$ for $1 < c \leq 2/\sqrt{3}$. As shown in Fig. 3, the lower and upper bound both consists of two segments in this case.

Generally, we can get the expression of $co(R_L(c))$ and $ca(R_U(c))$ for other parameters α and m using similar method.

Examples.

In the following, we give two examples as applications of the above results.

Example 1. We consider the $d \otimes d$ Werner states

$$\rho_f = \frac{1}{d^3 - d}[(d - f)I + (df - 1)\mathcal{F}], \tag{30}$$

where $-1 \leq f \leq 1$ and $\mathcal{F}$ is the flip operator defined by $\mathcal{F}(\phi \otimes \psi) = \psi \otimes \phi$. It is shown in ref. 51 that the concurrence $C(\rho_f) = -f$ for $f < 0$ and $C(\rho_f) = 0$ for $f \geq 0$. According to the theorem we obtain that $1/(1 - \alpha)\log[((1 + \sqrt{1 - f^2})/2)^\alpha + ((1 - \sqrt{1 - f^2})/2)^\alpha] \leq E_a(r_f) \leq -f$ for $-1 \leq f \leq 0$ when $m = 3$.

Example 2. The second example is the $3 \otimes 3$ isotropic state $\rho = (x/9)I + (1 - x)|\psi\rangle\langle\psi|$, where $|\psi\rangle = (a,\ 0,0,0,1/\sqrt{3},\ 0,\ 0,\ 0,1/\sqrt{3})^t/\sqrt{a^2 + 2/3}$ with $0 \leq a \leq 1$. We choose $x = 0.1$, it is direct to calculate that

$$C_1 = \sqrt{2(Tr\rho^2 - Tr\rho_A^2)} = \sqrt{2(Tr\rho^2 - Tr\rho_B^2)} = \frac{2\sqrt{6.53 + 41.46a^2 - 1.71a^4}}{3(2 + 3a^2)}, \tag{31}$$

$$C_2 = \frac{1}{\sqrt{3}}(\|\rho^{T_A}\| - 1) = \frac{2(5 + 6.9a^2 - 0.9a^4 + 9.353a(2 + 3a^2))}{3(2 + 3a^2)^2}, \tag{32}$$

$$C_3 = \frac{1}{\sqrt{3}}(\|R(\rho)\| - 1) = \frac{0.346 + 1.2a}{0.667 + a^2}, \tag{33}$$

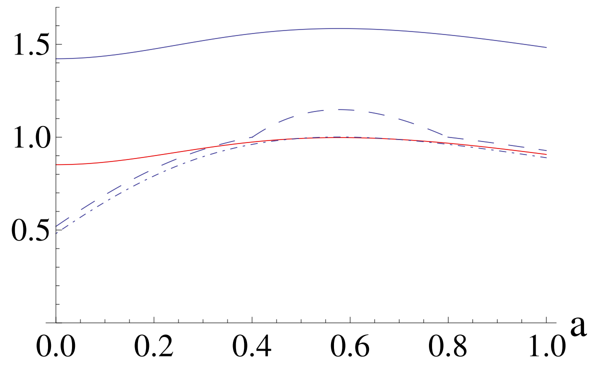

Figure 4. Lower and upper bounds of Eα(ρ) for α = 0.6 where we have set x = 0.1. Red solid line is obtained by C_1, the dash-dotted and dashed line is obtained by C_2 and C_3, respectively. The blue solid line is the upper bound of $E_\alpha(\rho)$.

$$\overline{C} = \sqrt{2(1 - Tr\rho_A^2)} = \sqrt{2(1 - Tr\rho_B^2)} = \frac{\sqrt{6(6.38 + 33.72a^2 + 3.42a^4)}}{3(2 + 3a^2)}. \tag{34}$$

When $\alpha = 0.6$, we can calculate the lower and upper bounds and the results is shown in Fig. 4. The solid red line corresponds to the lower bound of E_α by choosing the lower bound of concurrence is C_1, and the dash-dotted and dashed line correspond to the cases when we choose the lower bound of concurrence is C_2 and C_3, respectively. We can choose the maximum value of the three curves as the lower bound of E_α. The blue solid line is the upper bound of E_α.

Relation with other entanglement measures. In this section we establish the relation between ERαE and other well-known entanglement measures, such as the entanglement of formation, the geometric measure of entanglement[52], the logarithmic negativity and the G-concurrence.

Entanglement of formation. Let ρ be a bipartite pure state with Schmidt coefficients $(\mu_1, \mu_2, \ldots)$. We investigate the derivative of ERαE w.r.t. α as follows.

$$\begin{aligned} \frac{dE_\alpha(\rho)}{d\alpha} &= \frac{1}{(1-\alpha)^2}\left(\sum_j \frac{\mu_j^\alpha}{\sum_k \mu_k^\alpha} \log \mu_j^{1-\alpha} + \log\sum_k \mu_k^\alpha\right) \\ &\le \frac{1}{(1-\alpha)^2}\left(\log \frac{\sum_j \mu_j}{\sum_k \mu_k^\alpha} + \log\sum_k \mu_k^\alpha\right) \\ &= 0. \end{aligned} \tag{35}$$

The inequality follows from the concavity of logarithm function. The last equality follows from the fact $\sum_j \mu_j = 1$. Hence the ERαE is monotonically non-increasing with $\alpha \ge 0$. Since it becomes the von Neumann entropy when α tends to one, we have

$$E_\alpha(\rho) \ge E_F(\rho) \ge E_\beta(\rho) \tag{36}$$

where $0 \le \alpha \le 1$ and $\beta \ge 1$. Using the convex roof, one can show that (36) also holds for mixed bipartite states ρ.

Geometric measure of entanglement. The geometric measure (GM) of entanglement measures the closest distance between a quantum state and the set of separable states[52]. The GM has many operational interpretations, such as the usability of initial states for Grovers algorithm, the discrimination of quantum states under LOCC and the additivity and output purity of quantum channels, see the introduction of ref. 48 for a recent review on GM. For pure state $|\psi\rangle$ we define $G_l(\psi) = -\log \max |\langle\varphi|\psi\rangle|^2$, where the maximum runs over all product states $|\varphi\rangle$. it is easy to see that $\max |\langle\varphi|\psi\rangle|^2$ is equal to the square of the maximum of Schmidt coefficients of $|\psi\rangle$. For mixed states ρ we define

$$G_l^c(\rho) := \min\sum_i p_i G_l(|\psi_i\rangle), \tag{37}$$

where the minimum runs over all decompositions of $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$[48]. We construct the linear relation between the GM and ERαE as follows.

Lemma. If $\alpha > 1$ then

$$\frac{\alpha}{2(\alpha-1)} G_l^c(\rho) \ge E_\alpha(\rho). \tag{38}$$

If $\alpha = 1$ and ρ is a pure state then

$$G_1^c(\rho) \le E_\alpha(\rho). \tag{39}$$

If $\alpha < 1$ then

$$E_\alpha(\rho) + \frac{\alpha}{2(1-\alpha)} G_1^c(\rho) \le \frac{1}{1-\alpha}\log d, \tag{40}$$

where d is the minimum dimension of $\mathcal{H}_A$ and $\mathcal{H}_B$. The details for proving the lemma can be seen from Methods.

logarithmic negativity. In this subsection we consider the logarithmic negativity[53]. It is the lower bound of the PPT entanglement cost[53], and an entanglement monotone both under general LOCC and PPT operations[54]. The logarithmic negativity is defined as

$$LN(\rho) = \log\|\rho^{T_A}\|. \tag{41}$$

Suppose $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ is the optimal decomposition of ERαE $E_\alpha(\rho)$, and the pure state $|\psi_i\rangle$ has the standard Schmidt form $|\psi_i\rangle = \sum_j \sqrt{\mu_{i,j}}|a_{i,j}, b_{i,j}\rangle$. For $1/2 \le \alpha \le (2n-1)/2n$ and $n > 1$, we have

$$\begin{aligned} n \times LN(\rho) &= n\log\|\rho^{T_A}\| \ge n\log\sum_i p_i\|(|\psi_i\rangle\langle\psi_i|)^{T_A}\| \ge n\sum_i p_i \log\|(|\psi_i\rangle\langle\psi_i|)^{T_A}\| \\ &= 2n\sum_i p_i \log\left(\sum_j \sqrt{\mu_{i,j}}\right) \ge 2n\sum_i p_i \log\sum_j \mu_{i,j}^{\alpha} \ge \frac{1}{1-\alpha}\sum_i p_i \log\sum_j \mu_{i,j}^{\alpha} \\ &= \sum_i p_i E_\alpha(|\psi_i\rangle) = E_\alpha(\rho) \end{aligned} \tag{42}$$

where the first inequality is due to the property proved in ref. 54, the second inequality is due to the concavity of logarithm function, and in the last inequality we have used the inequality $2n \ge 1/(1-\alpha)$ for $1/2 \le \alpha \le (2n-1)/2n$, $n \ge 1$.

G-concurrence. The G-concurrence is one of the generalizations of concurrence to higher dimensional case. It can be interpreted operationally as a kind of entanglement capacity[55,56]. It has been shown that the G-concurrence plays a crucial role in calculating the average entanglement of random bipartite pure states[57] and demonstration of an asymmetry of quantum correlations[58]. Let $|\psi\rangle$ be a pure bipartite state with the Schmidt decomposition $|\psi\rangle = \sum_{i=1}^{d}\sqrt{\mu_i}|ii\rangle$. The G-concurrence is defined as the geometric mean of the Schmidt coefficients[55,56]

$$G(|\psi\rangle) := d(\mu_1\mu_2\cdots\mu_d)^{1/d}. \tag{43}$$

For $\alpha > 1$, we have

$$\begin{aligned} E_\alpha(|\psi\rangle) &= \frac{1}{1-\alpha}\log\sum_i \mu_i^{\alpha} \\ &\le \frac{1}{1-\alpha}\log\left(d(\mu_1^{\alpha}\cdots\mu_d^{\alpha})^{\frac{1}{d}}\right) \\ &= \frac{1}{(1-\alpha)}[\alpha\log d + \log(\mu_1\cdots\mu_d)^{\frac{\alpha}{d}} \\ &- (\alpha-1)\log d] \\ &= \frac{\alpha}{1-\alpha}\log G(|\psi\rangle) + \log d. \end{aligned} \tag{44}$$

For $0 < \alpha < 1$, we have

$$E_\alpha(|\psi\rangle) \ge \frac{\alpha}{1-\alpha}\log G(|\psi\rangle) + \log d. \tag{45}$$

Discussion and Conclusion

Entanglement Rényi-α entropy is an important generalization of the entanglement of formation, and it reduces to the standard entanglement of formation when α approaches to 1. Recently, it has been proved[59] that the squared ERαE obeys a general monogamy inequality in an arbitrary N-qubit mixed state. Correspondingly, we can construct the multipartite entanglement indicators in terms of ERαE which still work well even when the indicators based on the concurrence and EOF lose their efficacy. However, the difficulties in minimization procedures restrict the application of ERαE. In this work, we present the first lower and upper bounds for the ERαE of arbitrary dimensional bipartite quantum systems based on concurrence, and these results might provide an alternative method to investigate the monogamy relation in high-dimensional states. We also demonstrate the application our bound for some examples. Furthermore, we establish the relation between ERαE and some other entanglement measures. These lower and upper bounds can be further improved for other known bounds of concurrence[60,61]. After completing this manuscript, we became aware of a recently related paper by Leditzky *et al.* in which they also obtained another lower bound of ERαE in terms of Rényi conditional entropy[62].

Methods

Proof of the theorem. Suppose $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ is the optimal decomposition of ERαE $E_\alpha(\rho)$, and the concurrence of $|\psi_j\rangle$ is denoted as c_j. Thus we have

$$\begin{aligned} E_\alpha(\rho) &= \sum_j p_j E_\alpha(|\psi_j\rangle) = \sum_j p_j H_\alpha(\vec{\mu}) \\ &\geq \sum_j p_j co(R_L(c_j)) \geq co\left[R_L\left(\sum_j p_j c_j\right)\right] \\ &\geq co[R_L(\underline{C})], \end{aligned} \tag{46}$$

where the first inequality is due to the definition of $co(g)$; in the second inequality we have used the monotonically increasing and convex properties of $co(R_L(c_j))$ as a function of concurrence c_j; and in the last inequality we have used the lower bound of concurrence. On the other hand, we have

$$\begin{aligned} E_\alpha(\rho) &= \sum_j p_j E_\alpha(|\psi_j\rangle) = \sum_j p_j H_\alpha(\vec{\mu}) \\ &\leq \sum_j p_j ca(R_U(c_j)) \leq ca\left[R_U\left(\sum_j p_j c_j\right)\right] \\ &\leq ca[R_U(\overline{C})], \end{aligned} \tag{47}$$

where the first inequality is due to the definition of $ca(g)$; the second inequality is due to the monotonically increasing and concave properties of $ca(R_U(c_j))$ as a function of concurrence c_j; and in the last inequality we have used the upper bound of concurrence. Thus we have completed the proof of the theorem.

Proof of the lemma. Suppose the minimum in (37) is reached at $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Let the Schmidt decomposition of $|\psi_i\rangle$ be $|\psi_i\rangle = \sum_j \sqrt{\mu_{i,j}} |a_{i,j}, b_{i,j}\rangle$ where $\mu_{i,1}$ is the maximum Schmidt coefficient. For $\alpha > 1$, we have

$$\begin{aligned} \frac{\alpha}{2(\alpha-1)} G_1^c(\rho) &= \frac{-\alpha}{2(\alpha-1)} \sum_i p_i \log \mu_{i,1}^2 \\ &= \frac{-1}{\alpha-1} \sum_i p_i \log \mu_{i,1}^\alpha \\ &\geq \frac{-1}{\alpha-1} \sum_i p_i \log\left(\sum_j \mu_{i,j}^\alpha\right) \\ &= \sum_i p_i E_\alpha(|\psi_i\rangle) \\ &\geq E_\alpha(\rho). \end{aligned} \tag{48}$$

We have proved (38). For $\alpha = 1$, let μ_i be the Schmidt coefficients of ρ, we have

$$\begin{aligned} E_\alpha(\rho) &= S(\rho) = -\sum_i \mu_i \log \mu_i \\ &\geq -\sum_i \mu_i \log \max_j\{\mu_j\} = -\log \max_j\{\mu_j\} \\ &= G_1^c(\rho). \end{aligned} \tag{49}$$

We have proved (39). For $\alpha < 1$, we have

$$\begin{aligned} E_\alpha(\rho) + \frac{\alpha}{2(1-\alpha)} G_1^c(\rho) &= E_\alpha(\rho) - \frac{\alpha}{2(1-\alpha)} \sum_i p_i \log \mu_{i,1}^2 \\ &= E_\alpha(\rho) - \frac{1}{1-\alpha} \sum_i p_i \log \mu_{i,1}^\alpha \\ &= E_\alpha(\rho) - \frac{1}{1-\alpha} (\sum_i p_i \log(d\mu_{i,1}^\alpha) - \log d) \\ &\leq E_\alpha(\rho) - \frac{1}{1-\alpha} \sum_i p_i \log\left(\sum_j \mu_{i,j}^\alpha\right) + 11 - \alpha \log d \\ &\leq \frac{1}{1-\alpha} \log d. \end{aligned} \tag{50}$$

The inequality holds because the pure state $|\psi_i\rangle$ is in the $d \times d$ space. So we have proved (40).

References

1. Neilsen, M. A. & Chuang, I. L. *Quantum Computation and Quantum Information*. (Cambridge University Press: New York, 2000).
2. Bennett, C. H., DiVincenzo, D. P., Smolin, J. A. & Wootters, W. K. Mixed-state entanglement and quantum error correction. *Phys. Rev. A* **54**, 3824 (1996).
3. Wootters, W. K. Entanglement of formation of an arbitrary state of two qubits. *Phys. Rev. Lett.* **80**, 2245 (1998).
4. Vedral, V., Plenio, M. B., Rippin, M. A. & Knight, P. L. Quantifying entanglement. *Phys. Rev. Lett.* **78**, 2275 (1997).
5. Wei, T. C. & Goldbart, P. M. Geometric measure of entanglement and applications to bipartite and multipartite quantum states. *Phys. Rev. A* **68**, 042307 (2003).
6. Vidal, G. & Werner, R. F. Computable measure of entanglement. *Phys. Rev. A* **65**, 032314 (2002).
7. Christandl, M. & Winter, A. Squashed entanglement: An additive entanglement measure. *J. Math. Phys.* **45**, 829 (2003).
8. Yang, D. *et al.* Squashed entanglement for multipartite states and entanglement measures based on the mixed convex roof. *IEEE. Tran. Info. Theory* **55**, 3375 (2009).
9. Hayden, P., Horodecki, M. & Terhal, B. M. The asymptotic entanglement cost of preparing a quantum state. *J. Phys. A* **34**, 6891 (2001).
10. Huang, Y. Computing quantum discord is NP-complete. *New J. Phys.* **16**, 033027 (2014).
11. Terhal, B. M. & Vollbrecht, K. G. H. Entanglement of formation for isotropic states. *Phys. Rev. Lett.* **85**, 2625 (2000).
12. Vollbrecht, K. G. H. & Werner, R. F. Entanglement measures under symmetry. *Phys. Rev. A* **64**, 062307 (2001).
13. Giedke, G., Wolf, M. M., Kruger, O., Werner, R. F. & Cirac, J. I. Entanglement of formation for symmetric Gaussian states. *Phys. Rev. Lett.* **91**, 107901 (2003).
14. Vidal, G., Dur, W. & Cirac, J. I. Entanglement cost of bipartite mixed states. *Phys. Rev. Lett.* **89**, 027901 (2002).
15. Fei, S. M. & Li-Jost, X. A class of special matrices and quantum entanglement. *Rep. Math. Phys.* **53**, 195 (2004).
16. Gerjuoy, E. Lower bound on entanglement of formation for the qubit-qudit system. *Phys. Rev. A* **67**, 052308 (2003).
17. Mintert, F., Kus, M. & Buchleitner, A. Concurrence of mixed bipartite quantum states in arbitrary dimensions. *Phys. Rev. Lett.* **92**, 167902 (2004).
18. Chen, K., Albeverio, S. & Fei, S. M. Entanglement of formation of bipartite quantum states. *Phys. Rev. Lett.* **95**, 210501 (2005).
19. Chen, K., Albeverio, S. & Fei, S. M. Concurrence of arbitrary dimensional bipartite quantum states. *Phys. Rev. Lett.* **95**, 040504 (2005).
20. Osborne, T. J. Entanglement measure for rank-2 mixed states. *Phys. Rev. A* **72**, 022309 (2005).
21. Mintert, F., Kus, M. & Buchleitner, A. Concurrence of mixed multipartite quantum states. *Phys. Rev. Lett.* **95**, 260502 (2005).
22. Fei, S. M. & Li-Jost, X. R function related to entanglement of formation. *Phys. Rev. A* **73**, 024302 (2006).
23. Datta, A., Flammia, S. T., Shaji, A. & Caves, C. M. Constrained bounds on measures of entanglement. *Phys. Rev. A* **75**, 062117 (2007).
24. Mintert, F. & Buchleitner, A. Observable entanglement measure for mixed quantum states. *Phys. Rev. Lett.* **98**, 140505 (2007).
25. Zhang, C. J., Gong, Y. X., Zhang, Y. S. & Guo, G. C. Observable estimation of entanglement for arbitrary finite-dimensional mixed states. *Phys. Rev. A* **78**, 042308 (2008).
26. Ma, Zhihao & Bao, Minli Bound of concurrence. *Phys. Rev. A* **82**, 034305 (2010).
27. Li, X. S., Gao, X. H. & Fei, S. M. Lower bound of concurrence based on positive maps. *Phys. Rev. A* **83**, 034303 (2011).
28. Zhao, M. J., Zhu, X. N., Fei, S. M. & Li-Jost, X. Lower bound on concurrence and distillation for arbitrary-dimensional bipartite quantum states. *Phys. Rev. A* **84**, 062322 (2011).
29. Sabour, A. & Jafarpour, M. Probability interpretation, an equivalence relation, and a lower bound on the convex-roof extension of negativity. *Phys. Rev. A* **85**, 042323 (2012).
30. Chen, Z. H., Ma, Z. H., Guhne, O. & Severini, S. Estimating entanglement monotones with a generalization of the Wootters formula. *Phys. Rev. Lett.* **109**, 200503 (2012).
31. Nicacio, F. & de Oliveira, M. C. Tight bounds for the entanglement of formation of Gaussian states. *Phys. Rev. Lett.* **89**, 012336 (2014).
32. Nicacio, F. & de Oliveira, M. C. Tight bounds for the entanglement of formation of Gaussian states. *Phys. Rev. A* **89**, 012336 (2014).
33. Li, M. & Fei, S. M. Measurable bounds for entanglement of formation. *Phys. Rev. A* **82**, 044303 (2010).
34. Zhang, C. J., Yu, S. X., Chen, Q. & Oh, C. H. Observable estimation of entanglement of formation and quantum discord for bipartite mixed quantum states. *Phys. Rev. A* **84**, 052112 (2011).
35. Zhu, X. N. & Fei, S. M. Improved lower and upper bounds for entanglement of formation. *Phys. Rev. A* **86**, 054301 (2012).
36. Zhang, C., Yu, S., Chen, Q., Yuan, H. & Oh, C. H. Evaluation of entanglement measures by a single observable. arXiv:1506.01484.
37. Kim, J. S. & Sanders, B. C. Monogamy of multi-qubit entanglement using Rényi entropy. *J. Phys. A: Math. Theor* **43**, 445305 (2010).
38. Cui, J. *et al.* Quantum phases with differing computational power. *Nature Commun* **3**, 812 (2012).
39. Franchini, F. *et al.* Local convertibility and the quantum simulation of edge states in many-body systems. *Phys. Rev. X* **4**, 041028 (2014).
40. Flammia, S. T., Hamma, A., Hughes, T. L. & Wen, X. G. Topological entanglement Rényi entropy and reduced density matrix structure. *Phys. Rev. Lett.* **103**, 261601 (2009).
41. Halasz, G. B. & Hamma, A. Topological Rényi Entropy after a Quantum Quench. *Phys. Rev. Lett.* **110**, 170605 (2013).
42. Wang, Y. X., Mu, L. Z., Vedral, V. & Fan, H. Entanglement Rényi-α entropy. *Phys. Rev. A* **93**, 022324 (2016).
43. Coffman, V., Kundu, J. & Wootters, W. K. Distributed entanglement. *Phys. Rev. A* **61**, 052306 (2000).
44. Osborne, T. J. & Verstraete, F. General monogamy inequality for bipartite qubit entanglement. *Phys. Rev. Lett.* **96**, 220503 (2006).
45. Bai, Y. K., Xu, Y. F. & Wang, Z. D. General monogamy relation for the entanglement of formation in multiqubit systems. *Phys. Rev. Lett.* **113**, 100503 (2014).
46. Bai, Y. K., Xu, Y. F. & Wang, Z. D. Hierarchical monogamy relations for the squared entanglement of formation in multipartite systems. *Phys. Rev. A* **90**, 062343 (2014).
47. Ghosh, S., Kar, G., Sen, A. & Sen, U. Mixedness in the Bell violation versus entanglement of formation. *Phys. Rev. A* **64**, 044301 (2001).
48. Chen, L., Aulbach, M. & Hajdusek, M. Comparison of different definitions of the geometric measure of entanglement. *Phys. Rev. A* **89**, 042305 (2014).
49. Peres, A. Separability criterion for density matrices. *Phys. Rev. Lett.* **77**, 1413 (1996).
50. Chen, K. & Wu, L. A. A matrix realignment method for recognizing entanglement. *Quantum Inf. Comput.* **3**, 193 (2003).
51. Chen, K., Albeverio, S. & Fei, S. M. Concurrence-based entanglement measure for Werner States. *Rep. Math. Phys.* **58**, 325 (2006).
52. Wei, T. -C. & Goldbart, P. M. Geometric measure of entanglement and applications to bipartite and multipartite quantum states. *Phys. Rev. A* **68**, 042307 (2003).
53. Audenaert, K., Plenio, M. B. & Eisert, J. Entanglement cost under positive-partial-transpose-preserving operations. *Phys. Rev. Lett.* **90**, 027901 (2003).
54. Plenio, M. B. Logarithmic negativity: a full entanglement monotone that is not convex. *Phys. Rev. Lett.* **95**, 090503 (2005).
55. Gour, G. Family of concurrence monotones and its applications. *Phys. Rev. A* **71**, 012318 (2005).
56. Gour, G. Mixed-state entanglement of assistance and the generalized concurrence. *Phys. Rev. A* **72**, 042318 (2005).
57. Cappellini, V., Sommers, H.-J. & Zyczkowski, K. Distribution of G concurrence of random pure states. *Phys. Rev. A* **74**, 062322 (2006).
58. Horodecki, K., Horodecki, M. & Horodecki, P. Are quantum correlations symmetric? arXiv:quant-ph/0512224.

59. Song, W., Bai, Y. K., Yang, M., Yang, M. & Cao, Z. L. General monogamy relation of multiqubit systems in terms of squared Rényi-α entanglement. *Phys. Rev. A* **93**, 022306 (2016).
60. Ma, Z. H., Chen, Z. H. & Chen, J. L. Detecting the concurrence of an unknown state with a single observable, arXiv:1104.1006.
61. Vicente, J. Ide Lower bounds on concurrence and separability conditions. *Phys. Rev. A* **75**, 052320 (2007).
62. Leditzky, F., Rouze, C. & Datta, N. Data processing for the sandwiched Rényi divergence: a condition for equality. arXiv:1604.02119.

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Author Contributions

W. Song and L. Chen carried out the calculations. W. Song and L. Chen conceived the idea. All authors contributed to the interpretation of the results and the writing of the manuscript. All authors reviewed the manuscript.

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