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PROPAGATORS IN THE MOYAL AND TOMOGRAPHIC REPRESENTATIONS OF STATES

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Abstract

The tomographic map and density operator description of quantum states are reviewed. The connection between the tomogram and the probability distribution function is discussed. The relation of the Green function of the Moyal equation for the Wigner function to the Green function of the Schrödinger (von Neumann) equation in the tomographic representation is studied. An example of a quantum system with a quadratic Hamiltonian is considered.

Keywords: probability representation, tomographic symbol, density operator, propagator, Wigner function.

1. Introduction

The state of a physical system can be determined if one knows all the physical properties of the system, which makes it possible to describe completely the system at a given time and predict its behavior in the future. Take, for example, the spin-1/2 system. Let $\hat{\mathbf{s}}$ be the spin operator and $\mathbf{n}_i$, $i = 1, 2, 3$ be three linearly independent unit vectors. We can see that, if one knows the probability distribution for eigenvalues of the operators $\hat{\mathbf{s}}\mathbf{n}_i$, $i = 1, 2, 3$, then one can reconstruct the density operator corresponding to the quantum state of the system [1]. In other words, if $\wp_i(\xi),\ \xi = \pm\hbar/2$ is the probability distribution function of eigenvalues of the operator $\hat{\mathbf{s}}\mathbf{n}_i$, then a class of such probability distribution functions corresponding to three linearly independent unit vectors $\mathbf{n}_i$ can describe the system's quantum state. Generally, the state of a quantum system at an arbitrary time can be completely determined by a class of probability distribution functions of the appropriate number of physical quantities involved.

In the standard formulation of quantum mechanics using a vector from an appropriate Hilbert space and the set of orthonormal basis vectors in this space, one can describe the state of a quantum system. Nevertheless, a description of quantum states by means of positive probability distributions for the system's observable, similar to the classical description of states in classical statistical mechanics, is desirable. To construct a classical-like formalism for quantum mechanics, different kinds of quasiprobability distribution functions such as the Wigner function [2], Husimi Q-function [3], and Sudarshan–Glauber P-function [4, 5] are introduced.

Wigner used the Weyl map to represent a Hermitian density operator by a real function. Thus, the state of a quantum system can be represented by the Weyl transformation of the corresponding density

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operator, which is called the Wigner function. Also physical quantities can be represented by the Weyl transformation of the corresponding Hermitian operators. Moyal found the time evolution differential equation of the Wigner function, called the Moyal equation. Though the Weyl formulation of quantum mechanics is useful when considering concrete problems [6, 7], the Wigner function can take negative values and cannot be interpreted as a probability distribution function.

Indeed, due to the Heisenberg uncertainty relation for the position and momentum of the quantum systems a joint probability distribution function in the phase space does not exist. So for a long time, people think that it is impossible to formulate quantum mechanics using the notion of probability and avoiding the notion of wave function. But recently it was found that in the optical-tomography scheme [8, 9] it is possible to reconstruct the Wigner function of photons using the probability distribution function involved. An extension of the optical-tomography method [10, 11] leads to introducing the tomographic symbol of a quantum state or tomogram. Using, for the description of quantum states, the tomographic symbol of the density operator (symplectic tomogram), which is the standard positive probability distribution, a new probability distribution of quantum mechanics was suggested [11, 12].

For a one-dimensional quantum system, the physical content of the tomographic symbol is as follows.

We introduce an overcomplete set of basis ket-vectors such as $|X,\mu,\nu\rangle$, in which X, μ, and ν are three real variables. The wave functions in the position representation of these ket-vectors are

$$\langle q|X,\mu,\nu\rangle = \frac{1}{\sqrt{2\pi i\nu}}\exp\left(\frac{i\mu}{2\nu}q^2 - \frac{iX}{\nu}q + \frac{i\mu}{2\nu}X^2\right).$$

If the quantum system is in the state $|\psi\rangle$, then the state tomographic symbol reads

$$w(X,\mu,\nu) = |\langle\psi|X,\mu,\nu\rangle|^2.$$

The possibility of tomograms to represent a quantum state is based on using the appropriate overcomplete set of basis ket-vectors.

The time evolution of quasiprobabilities mentioned above can be determined using the corresponding Green functions. The Green functions of quantum-evolution equations for quasidistributions (or propagators) have been studied. Nevertheless, the connection of the Green function of the Moyal equation for the Wigner function with the Green function of the Schrödinger (von Neumann) equation in the tomographic representation has not been considered until now. The aim of this paper is to study this connection in detail.

The paper is organized as follows.

In Sec. 2, we review the wave function (or equivalently the ket-vector) formalism of quantum mechanics. In Sec. 3, we present the concept of density operator and the von Neumann equation in quantum mechanics. Section 4 is devoted to a review of the Wigner function, the Moyal equation, and the relation between the propagator of the von Neumann equation and the propagator of the Moyal equation. In Sec. 5, we introduce the tomographic symbol of the density operator as some map of the Wigner function and review some of its properties. Furthermore, in this section we introduce a set of inverses for this map. In Sec. 6, we study the properties of the Fourier transform of the tomographic symbol. Also in this section we obtain the direct and inverse maps that relate the Wigner function to the Fourier transform of the tomographic symbol. In Sec. 7, using the results of Sec. 6, we find a correspondence rule between operators acting on the Wigner functions and those acting on tomographic symbols. Moreover, we rederive the Schrödinger equation in the tomographic representation. In Sec. 8, we study in detail the propagator of the Schrödinger equation in the tomographic representation and its relation to the propagator of the Moyal equation. In Sec. 9, we test our results on some well-known problems.

2. The Wave-Function Formalism of Quantum Mechanics

Let the state of a one-dimensional quantum system be represented by the ket-vector $|\psi\rangle$. Then the Schrödinger equation for this quantum system with the Hamiltonian $\hat{H}(\hat{q},\hat{p},t)$ reads

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(\hat{q},\hat{p},t)|\psi(t)\rangle, \tag{1}$$

where

$$\hat{H}(\hat{q},\hat{p},t) = \frac{\hat{p}^2}{2m} + V(\hat{q},t).$$

Generally, $\hat{q}$ and $\hat{p}$ are the position-like and momentum-like operators, respectively, with the commutator

$$[\hat{q},\hat{p}] = i\hbar,$$

and $V(\hat{q},t)$ is the potential-energy-like operator. Since this equation is linear and of first order in time and $\hat{H}(\hat{q},\hat{p},t)$ is a Hermitian operator, then there exists a unitary time-evolution operator $\hat{U}(t)$ such that

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle.$$

In the position basis, this equation reads

$$\psi(q,t) = \int G(q,q',t)\psi(q',0)\,dq', \tag{2}$$

where

$$G(q,q',t) = \langle q|\hat{U}(t)|q'\rangle$$

is called the Green function of the Schrödinger equation. In view of (1) and the arbitrariness of $\psi(q',0)$, the Green function must satisfy the following differential equation:

$$i\hbar\,\frac{\partial}{\partial t}\,G(q,q',t) = H\left(q,\,\frac{\hbar}{i}\frac{\partial}{\partial q}\,,t\right)G(q,q',t)$$

with the given initial condition

$$G(q,q',0) = \delta(q-q').$$

This is the well-known wave-function formalism of quantum mechanics.

3. The Density-Operator Formalism of Quantum Mechanics

To describe the time evolution of both pure and mixed quantum states, one can use the density operator, which is defined as follows:

$$\hat{\rho} = \sum_j w_j|\varphi_j\rangle\langle\varphi_j|, \tag{3}$$

where w_j's are the fractional populations of normalized states $|\varphi_j\rangle$ of the quantum system. If an observable $\hat{A}$ is given, then its expectation value in the quantum state corresponding to the density operator $\hat{\rho}$ is given by

$$\langle\hat{A}\rangle_{\hat{\rho}} = \mathrm{Tr}\,[\hat{\rho}\hat{A}].$$

The time evolution of the density operator $\hat{\rho}$ can be determined using the Schrödinger equation (1) and is given by the von Neumann equation as follows:

$$\frac{\partial \hat{\rho}}{\partial t} + \frac{i}{\hbar}[\hat{H}, \hat{\rho}] = 0.$$

Using the representation of the operators $\hat{q}$ and $\hat{p}$ in the position basis, we obtain

$$\langle q|\hat{H}(\hat{q},\hat{p},t)|\psi\rangle = H\left(q,\, \frac{\hbar}{i}\frac{\partial}{\partial q}\,,\, t\right)\langle q|\psi\rangle$$

and

$$\langle \psi|\hat{H}(\hat{q},\hat{p},t)|q'\rangle = \langle \psi|q'\rangle H\left(q',\, i\hbar\,\frac{\overleftarrow{\partial}}{\partial q'}\,,\, t\right),$$

in which $|\psi\rangle$ is an arbitrary state.

Writing down the von Neumann equation in the position basis, we have

$$\frac{\partial}{\partial t}\,\rho(q,q',t) + \frac{i}{\hbar}\left[H\left(q,\, \frac{\hbar}{i}\frac{\partial}{\partial q}\,,\, t\right) - H\left(q',\, i\hbar\,\frac{\partial}{\partial q'}\,,\, t\right)\right]\rho(q,q',t) = 0. \tag{4}$$

This equation is linear and of first order in time, so there exists a propagator, which we denote as $K(q,q',s,s',t)$, such that

$$\rho(q,q',t) = \int K(q,q',s,s',t)\rho(s,s',0)\,ds\,ds'. \tag{5}$$

Moreover, this propagator satisfies the differential equation (4) with the initial condition

$$K(q,q',s,s',0) = \delta(q-s)\,\delta(q'-s').$$

In view of (2) and (3), the relation between propagators of the Schrödinger equation and the von Neumann equation reads

$$K(q,q',s,s',t) = G(q,s,t)G^*(q',s',t), \tag{6}$$

and this is valid for pure and mixed quantum states.

4. The Wigner-Function Formalism of Quantum Mechanics

The Weyl transformation of an operator $\hat{A}$ by definition is

$$a(q,p) = \int \left\langle q + \frac{u}{2}|\hat{A}|q - \frac{u}{2}\right\rangle \exp\left(-\frac{ipu}{\hbar}\right)\,du, \tag{7}$$

where we use matrix elements of the operator $\hat{A}$ in the position basis. The Weyl transformation of a Hermitian operator is a real function but not everywhere positive. The operator $\hat{A}$ can be reconstructed from its Weyl transformation in the following way:

$$\hat{A} = \frac{1}{2\pi\hbar}\int a(q,p)\hat{\Delta}(q,p)\,dq\,dp, \tag{8}$$

in which the operator $\hat{\Delta}(q,p)$ is

$$\hat{\Delta}(q,p) = \int \left| q - \frac{u}{2} \right\rangle \left\langle q + \frac{u}{2} \right| \exp\left(-\frac{ipu}{\hbar}\right) du. \tag{9}$$

In view of (7) and (9), one can find that the Weyl transformation of an operator is its symbol [10] with respect to the class of operators $\hat{\Delta}(q,p)$, i.e.,

$$a(q,p) = \mathrm{Tr}\left(\hat{A}\hat{\Delta}(q,p)\right).$$

Operators $\hat{\Delta}(q,p)$ satisfy the following orthogonallity relationship:

$$\mathrm{Tr}\left(\hat{\Delta}(q,p)\hat{\Delta}(q',p')\right) = 2\pi\hbar\,\delta(q-q')\,\delta(p-p'). \tag{10}$$

The Wigner function is defined in terms of the density matrix as follows:

$$W(q,p) = \int \rho\left(q + \frac{u}{2}\,,\, q - \frac{u}{2}\right) \exp\left(-\frac{ipu}{\hbar}\right) du, \tag{11}$$

so it is the Weyl transformation of the density operator. This relation is invertible and we have

$$\rho(q,q') = \frac{1}{2\pi\hbar} \int W\left(\frac{q+q'}{2}\,,\, p\right) \exp\left(\frac{ip(q-q')}{\hbar}\right) dp. \tag{12}$$

The Weyl transformations of Hermitian operators corresponding to observables and Wigner functions associated with quantum states form an alternative classical-like approach to quantum mechanics. For example, using (8) and (10), we can prove that the expectation value of an observable $\hat{A}$ in a state $\hat{\rho}$ is

$$\langle \hat{A} \rangle = \frac{1}{2\pi\hbar} \int a(q,p) W(q,p)\, dq\, dp, \tag{13}$$

which (interpreting the Wigner function as the quasiprobability distribution) has a well-known classical-like form.

Corresponding to a differential operator, which acts on density matrices, there is another operator, which mimics its role for Wigner functions. In view of (12), one can derive the following correspondence rule, which will be helpful later on,

$$\begin{aligned}
&q\rho(q,q') \longleftrightarrow \left(q + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) W(q,p), \qquad && \frac{\partial}{\partial q}\rho(q,q') \longleftrightarrow \left(\frac{1}{2}\frac{\partial}{\partial q} + \frac{ip}{\hbar}\right) W(q,p), \\
&q'\rho(q,q') \longleftrightarrow \left(q - \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) W(q,p), \qquad && \frac{\partial}{\partial q'}\rho(q,q') \longleftrightarrow \left(\frac{1}{2}\frac{\partial}{\partial q} - \frac{ip}{\hbar}\right) W(q,p).
\end{aligned} \tag{14}$$

Using (4) and the above correspondence rule, we can obtain the Moyal equation

$$\frac{\partial}{\partial t} W(q,p,t) = -\frac{p}{m}\frac{\partial}{\partial q} W(q,p,t) + \frac{2}{\hbar}\,\mathrm{Im}\left[V\left(q + \frac{i\hbar}{2}\,\frac{\partial}{\partial p}\right)\right] W(q,p,t), \tag{15}$$

which describes the time evolution of the Wigner function. Just like the von Neumann equation, the Moyal equation is linear and of first order in time, so one can write down

$$W(q,p,t)=\int \Gamma(q,p,q',p',t)W(q',p',0)\,dq'\,dp'. \tag{16}$$

The propagator $\Gamma(q,p,q',p',t)$ satisfies the same differential equation as the Wigner function, i.e.,

$$\frac{\partial}{\partial t}\Gamma(q,p,q',p',t)=-\frac{p}{m}\frac{\partial}{\partial q}\Gamma(q,p,q',p',t)+\frac{2}{\hbar}\,\mathrm{Im}\left[V\left(q+\frac{i\hbar}{2}\frac{\partial}{\partial p}\right)\right]\Gamma(q,p,q',p',t), \tag{17}$$

with the following initial condition:

$$\Gamma(q,p,q',p',0)=\delta(q-q')\,\delta(p-p'). \tag{18}$$

Propagators of the von Neumann equation and the Moyal equation are correlated. In view of (5), (16), (11), and (12), the following relationships are straightforward:

$$\Gamma(q,p,q',p',t)=\frac{1}{2\pi\hbar}\int K\left(q+\frac{u}{2}\,,\,q-\frac{u}{2}\,,\,q'+\frac{u'}{2}\,,\,q'-\frac{u'}{2}\,,\,t\right)e^{-i(pu-p'u')/\hbar}\,du\,du' \tag{19}$$

and

$$K(q,q',r,r',t)=\frac{1}{2\pi\hbar}\int \Gamma\left(\frac{q+q'}{2}\,,\,p,\,\frac{r+r'}{2}\,,\,p',\,t\right)e^{i[p(q-q')-p'(r-r')]/\hbar}\,dp\,dp'. \tag{20}$$

5. The Tomographic-Symbol Formalism of Quantum Mechanics

The tomographic symbol of a quantum state can be defined in terms of the Wigner function as follows:

$$w(X,\mu,\nu)=\frac{1}{2\pi\hbar}\int W(q,p)\,\delta(X-\mu q-\nu p)\,dq\,dp. \tag{21}$$

This formula has the same form as (13). Indeed, one can show that

$$\mathrm{Tr}\left[\delta(X-\mu\hat{q}-\nu\hat{p})\hat{\Delta}(q,p)\right]=\delta(X-\mu q-\nu p),$$

where $\hat{q}$ and $\hat{p}$ are the coordinate and momentum operators and

$$[\hat{q},\hat{p}]=i\hbar.$$

Thus the tomographic symbol of a quantum state is

$$w(X,\mu,\nu)=\langle\delta(X-\mu\hat{q}-\nu\hat{p})\rangle_{\hat{\rho}}=\mathrm{Tr}\left[\delta(X-\mu\hat{q}-\nu\hat{p})\hat{\rho}\right]. \tag{22}$$

There is another (more physical) definition of the tomographic symbol of a quantum state. We introduce the Hermitian operator

$$\hat{X}_{\mu,\nu}=\mu\hat{q}+\nu\hat{p},$$

in which μ and ν are real numbers. This operator has a normalized set of eigenvectors $|X\rangle$, $-\infty < X < \infty$, such that

$$\hat{X}_{\mu,\nu}|X\rangle = X|X\rangle.$$

These eigenvectors form an orthonormal and complete set of basis vectors, i.e.,

$$\langle X|X'\rangle = \delta(X - X'), \qquad \int |X\rangle\langle X|\, dX = \hat{1}.$$

We can show that

$$\delta(X - \mu\hat{q} - \nu\hat{p}) = |X\rangle\langle X|,$$

so, in view of (22), we have

$$w(X,\mu,\nu) = \langle X|\hat{\rho}|X\rangle.$$

This formula shows that the tomographic symbol of a quantum state takes real and positive values. Furthermore, from (21) we can prove that the tomographic symbol is a homogeneous function of order (-1), i.e.,

$$w(\lambda X, \lambda\mu, \lambda\nu) = \frac{1}{|\lambda|}\, w(X,\mu,\nu). \tag{23}$$

Relation (21) defines a linear map, which assigns to an arbitrary Wigner function a tomographic symbol. We denote this map by ϕ, i.e.,

$$W(q,p) \overset{\phi}{\longmapsto} w(X,\mu,\nu).$$

Indeed, there is not any one-to-one map between the set of two-variable functions and the set of three-variable functions. This means that map ϕ does not have an inverse map. Nevertheless, one can find a left inverse $\tilde{\phi}$ such that

$$w(X,\mu,\nu) \overset{\tilde{\phi}}{\longmapsto} W(q,p) \qquad \text{and} \qquad \tilde{\phi}\phi = 1.$$

As one can verify, there is not only one but also a set of left inverse maps $\tilde{\phi}_{k_0}$, which can be defined by the following formula:

$$W(q,p) = \frac{\hbar}{2\pi} \int k_0^2 \exp\Big(ik_0[X - \mu q - \nu p]\Big) w(X,\mu,\nu)\, dX\, d\mu\, d\nu, \tag{24}$$

where k_0 is any nonzero real number. Since the tomographic symbol w of a quantum state is a homogeneous function of order (-1), then $[\tilde{\phi}_{k_0} w](q,p)$ is independent of k_0. We will derive this formula in the next section using the Fourier transform of the tomographic symbol.

6. Properties of the Fourier Transform of Tomographic Symbol

We introduce the Fourier transform of the tomographic symbol with respect to the variable X as follows:

$$\tilde{w}(k,\mu,\nu) = \int w(X,\mu,\nu) \exp(ikX)\, dX, \tag{25}$$

so inversely we have

$$w(X,\mu,\nu) = \frac{1}{2\pi} \int \tilde{w}(k,\mu,\nu) \exp(-ikX)\, dk. \tag{26}$$

Taking the Fourier transform of both sides of Eq. (21) we find

$$\tilde{w}(k,\mu,\nu)=\frac{1}{2\pi\hbar}\int W(q,p)\exp\Big[ik(\mu q+\nu p)\Big]\,dq\,dp. \tag{27}$$

Relation (27) defines the map ψ

$$W(q,p)\overset{\psi}{\longmapsto}\tilde{w}(k,\mu,\nu).$$

From (27) we find that $\tilde{w}(k,\mu,\nu)$ satisfies the following condition:

$$\tilde{w}(k,\mu,\nu)=\tilde{w}(1,k\mu,k\nu), \tag{28}$$

which, in view of (25) and (26), is a necessary and sufficient condition for homogeneity of tomographic symbol (23). From (28) we understand that for a fixed but arbitrary value of k, say k_0, the function $\tilde{w}(k_0,\mu,\nu)$ contains all the physically significant information contained in the tomographic symbol and its Fourier transform. Indeed, it is a direct consequence of homogeneity of the tomographic symbol.

For $k=k_0$, we can obtain the inverse of relation (27)

$$W(q,p)=\frac{\hbar k_0^2}{2\pi}\int\exp\Big[-ik_0(\mu q+\nu p)\Big]\tilde{w}(k_0,\mu,\nu)\,d\mu\,d\nu. \tag{29}$$

As a consequence of the homogeneity condition (28), this relation gives a unique result for all values of k_0. Also relation (29) suggests introducing a set of maps $\tilde{\psi}_{k_0}$ as follows:

$$[\tilde{\psi}_{k_0}\tilde{w}](q,p)=\frac{\hbar k_0^2}{2\pi}\int\exp\Big[-ik_0(\mu q+\nu p)\Big]\,\delta(k-k_0)\tilde{w}(k,\mu,\nu)\,dk\,d\mu\,d\nu. \tag{30}$$

Evidently for all real values $k_0\neq 0$, we have

$$\tilde{\psi}_{k_0}\psi=1_{\text{Wigner}},$$

so the $\tilde{\psi}_{k_0}$'s form a set of left inverses of ψ. In view of (25), we can write $\tilde{w}(k,\mu,\nu)$ in terms of the tomographic symbol and find relation (24). This relation defines a set of left inverse maps $\tilde{\phi}_{k_0}$.

For the special case where $\tilde{w}(k,\mu,\nu)$ is obtained by (27) and satisfies homogeneity condition (28),

$$\tilde{P}_{k_0}=\psi\tilde{\psi}_{k_0}$$

acts as a unit operator. But if in (30) we consider $\tilde{w}(k,\mu,\nu)$ as an arbitrary three-variable function, one can show that

$$\tilde{P}_{k_0}=\psi\tilde{\psi}_{k_0}\neq 1.$$

In view of (30) and (27), we can obtain the kernel of map

$$\tilde{P}_{k_0}=\psi\tilde{\psi}_{k_0},$$

as follows:

$$K_{\tilde{P}_{k_0}}(k,\mu,\nu,k',\mu',\nu')=k_0^2\,\delta(k'-k_0)\,\delta(\mu k-\mu' k_0)\,\delta(\nu k-\nu' k_0). \tag{31}$$

Using this kernel we can show that

$$[\tilde{P}_{k_0}]^2=\tilde{P}_{k_0},$$

which means that $\tilde{P}_{k_0}$ is a projection operator.

For a fixed value of k_0 and a given Wigner function W, there exists a class of three-variable functions

$$C_{W,k_0} = \{\eta(k,\mu,\nu)|\tilde{\psi}_{k_0}\eta = W\}.$$

Evidently, for an arbitrary value of k_0,

$$\tilde{w} = \psi[W]$$

is an unique element of this class, which satisfies the homogeneity condition (28). Also for an arbitrary element $\eta \in C_{W,k_0}$, we find that

$$\tilde{P}_{k_0}[\eta] = \psi[W] = \tilde{w},$$

which means that $\tilde{P}_{k_0}$ projects every element of C_{W,k_0} into

$$\psi[W] = \tilde{w}.$$

7. Map of Operators

Let $\ell = \ell\,(q,p,\partial/\partial q,\partial/\partial p)$ denote an operator that acts on the Wigner functions, and $\mathcal{L} = \mathcal{L}\,(k,\mu,\nu,\partial/\partial k,\partial/\partial\mu,\partial/\partial\nu)$ be another operator that acts on the Fourier transform of tomographic symbols. For a given Wigner function W and fixed value of k_0, there are many operators such as $\mathcal{L}$ which satisfy the relationship

$$\ell[W] = \tilde{\psi}_{k_0}\mathcal{L}[\tilde{w}]. \tag{32}$$

Only one of these operators, say L, gives the Fourier transform of the tomographic symbol corresponding to $\ell[W]$, i.e.,

$$[L\tilde{w}](k,\mu,\nu) = [\psi\ell W](k,\mu,\nu).$$

To find L, we use the following method.

First find an arbitrary operator $\mathcal{L}$ that satisfies the condition (32). Then evidently $\mathcal{L}[\tilde{w}] \in C_{\ell W,k_0}$, so using the projection operator $\tilde{P}_{k_0}$, we can write

$$L\tilde{w} = \tilde{P}_{k_0}\mathcal{L}\tilde{w}. \tag{33}$$

As an example, we work out the operator L corresponding to

$$\ell W(q,p) = qW(q,p).$$

Setting $k_0 = 1$ in (29) we have

$$W(q,p) = \frac{\hbar}{2\pi}\int \tilde{w}(1,\mu,\nu)\exp\Big[-i(\mu q + \nu p)\Big]\,d\mu\,d\nu.$$

Mulitplying both sides by q and integrating the right-hand side by parts, we obtain

$$qW(q,p) = \frac{\hbar}{2\pi}\int \left(-i\frac{\partial}{\partial\mu}\tilde{w}(1,\mu,\nu)\right)\exp\Big[-i(\mu q + \nu p)\Big]\,d\mu\,d\nu.$$

This equation shows that

$$\mathcal{L}\tilde{w}(k,\mu,\nu) = -i\frac{\partial}{\partial\mu}\tilde{w}(1,\mu,\nu)$$

satisfies the condition (32). Now, in view of (33) and using the projection operator with $k_0 = 1$, we can write

$$L\tilde{w}(k,\mu,\nu) = -i\int K_{\tilde{P}_1}(k,\mu,\nu,k',\mu',\nu')\,\frac{\partial}{\partial\mu'}\tilde{w}(1,\mu',\nu')\,dk'\,d\mu'\,d\nu'.$$

Finally, we will find that

$$L\tilde{w}(k,\mu,\nu) = -\frac{i}{k}\frac{\partial}{\partial\mu}\tilde{w}(k,\mu,\nu),$$

which evidently satisfies the homogeneity condition (28).

The right-hand side of the last formula is just the Fourier transform of the function $-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}w(X,\mu,\nu)$. This function is obviously homogeneous of order (-1) and indeed is the tomographic symbol of $qW(q,p)$. Taking the inverse Fourier transform of both sides of the last equation, we find that

$$-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}w(X,\mu,\nu) = \frac{1}{2\pi}\int qW(q,p)\,\delta(X-\mu q-\nu p)\,dq\,dp. \tag{34}$$

By a similar method, we find the following correspondence relations between operators that act on the Wigner functions and those that mimic their roles for the tomographic symbols,

$$\begin{aligned} q &\longleftrightarrow -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}\,, \qquad \frac{\partial}{\partial q}\longleftrightarrow \mu\frac{\partial}{\partial X}\,,\\ p &\longleftrightarrow -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu}\,, \qquad \frac{\partial}{\partial p}\longleftrightarrow \nu\frac{\partial}{\partial X}\,. \end{aligned} \tag{35}$$

Using these relations and the Moyal equation, one can find the differential equation for the time evolution of tomographic symbols, i.e., the Schrödinger (von Neumann) equation in the tomographic representation

$$\frac{\partial w(X,\mu,\nu)}{\partial t} = \frac{\mu}{m}\frac{\partial w(X,\mu,\nu)}{\partial\nu} + \frac{2}{\hbar}\,\mathrm{Im}\left\{V\left[-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu} + i\frac{\hbar\nu}{2}\frac{\partial}{\partial X}\right]\right\}w(X,\mu,\nu). \tag{36}$$

8. Propagators

Since Eq. (36) is linear and of first order in time, then there exists at least one propagator such that

$$w(X,\mu,\nu,t) = \int \Pi(X,\mu,\nu,X',\mu',\nu',t)\,w(X',\mu',\nu',0)\,dX\,d\mu\,d\nu. \tag{37}$$

Indeed, we can show that there is a class of such propagators and all of them satisfy the above equation. This is because the tomographic symbol of a quantum state contains overcomplete information on the quantum state. We will show that there is a propagator for Eq. (36) called the classical propagator $\Pi^{\mathrm{cl}}(X,\mu,\nu,X',\mu',\nu',t)$ that not only holds in (37) but also satisfies the following initial condition:

$$\Pi^{\mathrm{cl}}(X,\mu,\nu,X',\mu',\nu',0) = \delta(X-X')\,\delta(\mu-\mu')\,\delta(\nu-\nu'). \tag{38}$$

It is remarkable that this relation cannot be derived from (37) at the initial time $t = 0$, because tomographic symbols are restricted to homogeneous functions of order (-1).

Let $W(q,p,t)$ represent the Wigner function of a quantum state at time t; then we can rewrite relation (16) in matrix-like notation as follows:

$$W(t) = \Gamma(t)W(0).$$

Also considering relations (27) and (30) in the matrix-like notation, from the last equation we can write

$$\tilde{w}(t) = \psi W(t) = \psi\Gamma(t)W(0) = \psi\Gamma(t)\tilde{\psi}_{k_1}\tilde{w}(0),$$

in which k_1 is an arbitrary constant or a function of some unknown parameters. We can rewrite the last equation as follows:

$$\begin{aligned}\tilde{w}(k,\mu,\nu,t) &= [\psi\Gamma(t)\tilde{\psi}_{k_1}\tilde{w}(0)](k,\mu,\nu)\\ &= \int dk'\,d\mu'\,d\nu'\,\tilde{w}(k',\mu',\nu',0)\left(\frac{1}{4\pi^2}\int dq\,dp\,dq'\,dp'\,k_1^2\,\delta(k'-k_1)\right.\\ &\left.\times e^{ik(\mu q+\nu p)}\,\Gamma(q,p,q',p',t)\,e^{-ik_1(\mu' q'+\nu' p')}\right). \end{aligned} \tag{39}$$

So we have found a set of propagators $\tilde{\Pi}_{k_1}(k,\mu,\nu,k',\mu',\nu',t)$ such that

$$\tilde{w}(k,\mu,\nu,t) = \int dk'\,d\mu'\,d\nu'\,\tilde{\Pi}_{k_1}(k,\mu,\nu,k',\mu',\nu',t)\,\tilde{w}(k',\mu',\nu',0),$$

in which

$$\begin{aligned}\tilde{\Pi}_{k_1}(k,\mu,\nu,k',\mu',\nu',t) &= \frac{1}{4\pi^2}\int dq\,dp\,dq\,dp'\,k_1^2\delta(k'-k_1)\\ &\times e^{ik(\mu q+\nu p)}\,\Gamma(q,p,q',p',t)\,e^{-ik_1(\mu' q'+\nu' p')}. \end{aligned} \tag{40}$$

All the propagators corresponding to different choices of k_1 can equivalently describe the time evolution of the Fourier transform of the tomographic symbol. In view of the initial condition (18), we obtain

$$\tilde{\Pi}_{k_1}(k,\mu,\nu,k',\mu',\nu',0) = k_1^2\,\delta(k'-k_1)\,\delta(\mu k-\mu' k_1)\,\delta(\nu k-\nu' k_1),$$

which shows that only the choice $k_1 = k_1(k) = k$ leads to a propagator that satisfies the initial condition (38), i.e.,

$$\tilde{\Pi}^{\rm cl}(k,\mu,\nu,k',\mu',\nu',t) = \frac{k^2}{4\pi^2}\delta(k-k')\int dq\,dp\,dq'\,dp'\,e^{ik(\mu q+\nu p)}\,\Gamma(q,p,q',p',t)\,e^{-ik(\mu' q'+\nu' p')}. \tag{41}$$

From (25) and (26) one can verify that

$$\Pi^{\rm cl}(X,\mu,\nu,X',\mu',\nu',t) = \frac{1}{2\pi}\int \tilde{\Pi}^{cl}(k,\mu,\nu,k',\mu',\nu',t)\exp\Big[i(k'X'-kX)\Big]\,dk\,dk',$$

so we can write the classical propagator of the Schrödinger equation in the tomographic representation in terms of the propagator of the Moyal equation as follows:

$$\begin{aligned}\Pi^{\rm cl}(X,\mu,\nu,X',\mu',\nu',t) &= \frac{1}{(2\pi)^3}\int k^2\exp\Big[ik(X'-X+\mu q+\nu p-\mu' q'-\nu' p')\Big]\\ &\times\Gamma(q,p,q',p',t)\,dq\,dp\,dq'\,dp'\,dk. \end{aligned} \tag{42}$$

Invoking (35) and (17) we can show that the classical propagator satisfies the Schrödinger equation in the tomographic representation, i.e.,

$$\frac{\partial \Pi^{\text{cl}}(X,\mu,\nu,X',\mu',\nu',t)}{\partial t} = \frac{\mu}{m}\frac{\partial \Pi^{\text{cl}}(X,\mu,\nu,X',\mu',\nu',t)}{\partial \nu}$$
$$+\frac{2}{\hbar}\,\text{Im}\left(V\left[-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \mu}+i\frac{\hbar\nu}{2}\frac{\partial}{\partial X}\right]\right)\Pi^{\text{cl}}(X,\mu,\nu,X',\mu',\nu',t),$$

with the initial condition (38). Also the classical propagator, being a function of X,μ,ν and X',μ',ν', is a homogeneous function of order (-3). For the sake of completeness, we find the propagator of the Moyal equation in terms of the classical propagator. Using (24) with an arbitrary k_0, say $k_0=1$, and (21) and (37), we obtain

$$\Gamma(q,p,q',p',t) = \frac{1}{(2\pi)^2}\int \exp\Big[i(X-\mu q-\nu p)\Big]$$
$$\times \Pi^{\text{cl}}(X,\mu,\nu,X',\mu',\nu',t)\,\delta(X'-\mu'q'-\nu'p')\,dX\,d\mu\,d\nu\,dX'\,d\mu'\,d\nu'. \tag{43}$$

9. Some Examples

As an example, we study a quantum system with the quadratic Hamiltonian

$$\hat{H} = \frac{1}{2}\tilde{Q}BQ + \tilde{C}Q,$$

in which B is a 2×2 symmetric real matrix.The matrices C and Q are as follows:

$$C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix},$$

where c_1 and c_2 are real numbers and $\hat{p}$ and $\hat{q}$ are well-known momentum-like and position-like operators with $[\hat{q},\hat{p}]=i$.

In this section, we are working using units such that $\hbar=1$.

This system has two integrals of motion $\hat{I}_1$ and $\hat{I}_2$ [13, 14], which are linear in $\hat{q}$ and $\hat{p}$

$$\begin{bmatrix} \hat{I}_1 \\ \hat{I}_2 \end{bmatrix} = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) \end{bmatrix}\begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix} + \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix}.$$

Here $\Lambda=[\lambda_{ij}(t)]$ is a real and symmetric matrix and $\Delta=[d_i(t)]$ is a real vector. They satisfy the following system of differential equations with the given initial condition:

$$\begin{aligned} &\frac{d}{dt}\Lambda = i\Lambda B\sigma_y, &&\Lambda(0)=1, \\ &\frac{d}{dt}\Delta = i\Lambda\sigma_y C, &&\Delta(0)=0, \end{aligned} \tag{44}$$

in which σ_y is the second Pauli matrix.

We know that the Wigner function changes with time as follows [15]:

$$W(q,p,t) = W(I_1, I_2, 0),$$

so the propagator of the Moyal equation reads

$$\Gamma(Q, Q', t) = \delta^2(\Lambda Q + \Delta - Q').$$

We used here the two-dimensional Dirac delta-function.

Employing (42) one can work out the propagator of the Schrödinger equation in the tomographic representation $\Pi^{\mathrm{cl}}(X,\mu,\nu,X',\mu',\nu',t)$ in terms of the above propagator of the Moyal equation $\Gamma(Q,Q',t)$, i.e.,

$$\begin{aligned}\Pi^{cl}(X,\mu,\nu,X',\mu',\nu',t) &= \frac{1}{(2\pi)^3}\int k^2 \exp\Big(ik[(X'-X)+\tilde{N}Q-\tilde{N}'Q']\Big)\\ &\quad\times\delta^2(\Lambda Q+\Delta-Q')\,dq\,dp\,dq'\,dp'\,dk, \end{aligned} \tag{45}$$

where

$$N = \begin{bmatrix} \nu \\ \mu \end{bmatrix}, \qquad N' = \begin{bmatrix} \nu' \\ \mu' \end{bmatrix}.$$

Integrating first in $dq'\,dp'$, then in $dq\,dp$, and finally in dk results in

$$\Pi^{\mathrm{cl}}(X,\mu,\nu,X',\mu',\nu',t) = \delta(X - X' + \tilde{N}'\Delta)\,\delta^2(\tilde{N} - \tilde{N}'\Lambda). \tag{46}$$

As a special case, let us derive the free-particle propagator for the tomogram. The Hamiltonian operator of a free particle with unit mass reads

$$\hat{H} = \frac{\hat{p}^2}{2}.$$

One can solve the system of equations (44) for the Hamiltonian of a free particle and find the matrices

$$\Lambda = [\lambda_{ij}(t)] \qquad \text{and} \qquad \Delta = [d_i(t)].$$

These matrices will be

$$\Lambda^{\mathrm{free}} = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}, \qquad \Delta^{\mathrm{free}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Setting these matrices in formula (46) we have

$$\Pi^{\mathrm{free}}(X,\mu,\nu,X',\mu',\nu',t) = \delta(X-X')\,\delta(\nu'-\nu-\mu t)\,\delta(\mu-\mu'), \tag{47}$$

which is in agreement with the result obtained in [16].

Another known problem with a well-known answer is finding the classical propagator for the harmonic oscillator. Working in units such that $\hbar = m = \omega_0 = 1$, the Hamiltonian operator is

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2}.$$

Solving the system of differential equations (44) for this Hamiltonian, we can obtain the following matrices:

$$\Lambda^{\mathrm{osc}} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \qquad \Delta^{\mathrm{osc}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Consequently, in view of formula (46), the classical propagator for the harmonic oscillator reads

$$\begin{aligned} \Pi^{\mathrm{osc}}(X,\mu,\nu,X',\mu',\nu',t) &= \delta(X-X')\,\delta(\nu\cos t+\mu\sin t-\nu') \\ &\quad \times\delta(\mu\cos t+\nu\sin t-\mu'), \end{aligned}$$

which is also derived in [16, 17].

10. Conclusions

The state of a quantum system can be equivalently described by the ket-vector, density operator, Wigner function, or tomographic symbol. The density operator and Wigner function are related by an invertible map. But the map relating Wigner functions and tomographic symbols is not one-to-one and does not have an inverse.

In view of the homogeneity of tomographic symbols, it is possible to find a set of left inverses for this map. All of these left inverses acting on a tomogram consistently give a unique Wigner function.

Since the tomographic symbol of a quantum state contains redundant information on the quantum state, there are many propagators that give the time evolution of tomograms.

The classical propagator is the one that at the initial time has a three-dimensional delta-function form. Also we showed that the classical propagator is a homogeneous function and satisfies the Schrödinger equation in the tomographic representation.

There are invertible maps between propagators of the von Neumann equation, the Moyal equation, and the Schrödinger equation in the tomographic representation.

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