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TOMOGRAPHY FOR SEVERAL PARTICLES WITH ONE RANDOM VARIABLE

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Abstract

The tomographic map of the quantum state of a system with several degrees of freedom, which depends on one random variable, analogous to the rotated and scaled center-of-mass of the system, is constructed. The time-evolution equation of the tomogram for this map is given in the explicit form. The properties of the map such as the transition probabilities between different states and relation to the star-product formalism are elucidated. An example of a multimode oscillator is considered in detail. Identical particles are discussed within the framework of the proposed tomography scheme.

Keywords: tomography, probability, computer simulation, star-product, identical particles.

1. Introduction

Recently [1] a new formulation of quantum mechanics was suggested. This new formulation uses a nonnegatively defined probability distribution function to describe quantum states, which is called the marginal distribution [2, 3] or tomogram. This function can be considered an analog of known quasidistribution functions like the nonnegative Husimi Q-function [4] or the Sudarshan–Glauber P-function [5, 6]. The tomographic approach was initially developed for one-mode systems; in this case, the quantum state is described by the density matrix $\rho(q',q'')$ [7, 8] or by the symplectic tomogram $w(X,\mu,\nu)$ [9]. Here the density matrix is a function of two variables, and the tomogram is a function of three variables. The seeming overcompleteness of the tomographic description is balanced by the fact that the quantum tomogram is a homogeneous function [10, 11].

In the most general case, the state of a system with N degrees of freedom is described by the density matrix $\rho(\mathbf{q}',\mathbf{q}'')$, which is a function of $2N$ variables. What is then the generalization of the quantum tomogram? Will it depend on $3N$ variables or $2N+1$ variables? Such a generalization was developed for a tomogram depending on $3N$ variables (usual a symplectic tomography) [12]. In this paper, we propose a tomographic map with only one random variable (i.e., $2N+1$ variables totally) and discuss its properties in detail.

The aim of our work is to elaborate the tomographic approach to describe multimode quantum states using the tomogram, which depends on one random variable and several real parameters. We also elucidate the relation of this tomographic approach to the star-product formalism which was used for the standard symplectic tomography in [13].

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Quantum tomography is popular nowadays due to a number of reasons. It was developed not only for continuous variables like position but also for discrete spin variables [14–20]. First, the tomographic representation operates with values that are directly measured in experiments, for example, in experiments with nonclassical and coherent states of light [20–22]. Second, the tomogram is nonnegative, and this attracts the attention of those who deal with computer simulations. Many problems in this field result in the use of alternating-sign (or even complex) values to describe the quantum state (for example, the sign problem in Fermi-system simulations). In fact, the tomographic map has already been used in quantum simulations [24]. These circumstances along with the tomography applications in quantum computations and entanglement (see, e.g., [25]), as well as in information theory and signal analysis [26], show that the development of a convenient and simple tomographic map for the case of many particles (or, which is the same, many modes) is a task of great significance.

This paper is organized as follows.

In Sec. 2. we present the definition of a tomographic scheme with one random variable, elucidate some of its useful properties, and derive the equations describing quantum evolution, stationary states, and quantum transitions for the proposed tomography map. The rules for average-value calculations and star-product formalism are considered in Sec. 3.. Some examples of the state description using the approach developed are given in Sec. 4., and the symmetry of the map with respect to particles permutations is discussed in Sec. 5. The results are summarized in Sec. 6.

2. Tomogram of a System with Several Degrees of Freedom

2.1. Definition of the Tomographic Maps

In this section, we present the equations connecting the quantum tomograms (for the usual symplectic scheme and a new scheme with only one random variable) vis-a-vis the standard quantum mechanics formulation. We also develop here the evolution equations for tomograms.

Throughout the paper, the notation is as follows:

We consider a system of N particles in d dimensions; the number of degrees of freedom is Nd. Vectors are written as $\mathbf{a}$, and everywhere vectors with Nd components are used unless otherwise indicated. The notation $\mathbf{e}$ is used for vectors with all components equal to 1 ($e_i = 1$). The scalar product of vectors is designated as

$$a = \mathbf{bc} \qquad (\text{meaning} \quad a = \sum_i b_i c_i),$$

$\mathbf{a} = \mathbf{b} \circ \mathbf{c}$ denotes the componentwise product of vectors ($a_i = b_i c_i$). The tomogram for the usual symplectic scheme is designated as $w_1(\mathbf{X}, \mu, \nu)$ ($\mathbf{X}, \mu$, and ν with Nd components each); the tomogram with one random variable is written as $w_2(X, \mu, \nu)$ (now μ and ν have Nd components each and X is one real variable). We also use the Planck's constant $\hbar = 1$ everywhere.

There exists the Wigner formulation of quantum mechanics [27] where the system's state is described by the real Wigner function $F^W(\mathbf{q}, \mathbf{p})$ defined in the phase space. It is more convenient to use the Wigner representation to develop the tomographic scheme; therefore, let us start with the equations connecting the Wigner function with the density matrix:

$$F^W(\mathbf{q},\mathbf{p}) = \int \rho\left(\mathbf{q}+\frac{\mathbf{u}}{2},\mathbf{q}-\frac{\mathbf{u}}{2}\right) e^{-i\mathbf{p}\mathbf{u}} \frac{d\mathbf{u}}{(2\pi)^{Nd}}\,, \tag{1}$$

$$\rho(\mathbf{q}',\mathbf{q}'') = \int F^W\left(\frac{\mathbf{q}'+\mathbf{q}''}{2},\mathbf{p}\right) e^{i\mathbf{p}(\mathbf{q}'-\mathbf{q}'')} d\mathbf{p}. \tag{2}$$

Given the connection between the tomogram and the Wigner function, it is always possible to reconstruct the relation of the tomogram to the density matrix using Eqs. (1) and (2).

Let us now define the usual symplectic tomography map developed in [13, 28].

The tomogram $w_1(\mathbf{X},\mu,\nu)$ and Wigner function $F^W(\mathbf{q},\mathbf{p})$ are connected as follows:

$$w_1(\mathbf{X},\mu,\nu) = \int F^W(\mathbf{q},\mathbf{p}) \exp\Big[-i\mathbf{k}(\mathbf{X}-\mu\circ\mathbf{q}-\nu\circ\mathbf{p})\Big] \frac{d\mathbf{k}\,d\mathbf{q}\,d\mathbf{p}}{(2\pi)^{Nd}}\,, \tag{3}$$

$$F^W(\mathbf{q},\mathbf{p}) = \int \exp\Big[-i\mathbf{e}(\mu\circ\mathbf{q}+\nu\circ\mathbf{p}-\mathbf{X})\Big] w_1(\mathbf{X},\mu,\nu) \frac{d\mathbf{X}\,d\mu\,d\nu}{(2\pi)^{2Nd}}\,. \tag{4}$$

One can see that the exponent and integration over $\mathbf{k}$ in (3) can be rewritten as the product of $3Nd$ delta-functions $\delta(X_i-\mu_i q_i-\nu_i p_i)$. Considering Eq. (3) as a definition of $w_1(\mathbf{X},\mu,\nu)$, Eq. (4) is verified simply by replacing $w_1(\mathbf{X},\mu,\nu)$ by Eq. (3).

The transition from the symplectic scheme with $3Nd$ variables to $2Nd+1$ variables is very simple. The componentwise product in Eq. (3) is replaced by the scalar product and instead of the set of random variables

$$X_i = \mu_i q_i + \nu_i p_i$$

we get one random variable

$$X = \mu\mathbf{q} + \nu\mathbf{p}.$$

The corresponding connection between the tomogram $w_2(X,\mu,\nu)$ and the Wigner function $F^W(\mathbf{q},\mathbf{p})$ can be written as follows:

$$w_2(X,\mu,\nu) = \int F^W(\mathbf{q},\mathbf{p}) \exp\Big[-ik(X-\mu\mathbf{q}-\nu\mathbf{p})\Big] \frac{dk\,d\mathbf{q}\,d\mathbf{p}}{(2\pi)}\,, \tag{5}$$

$$F^W(\mathbf{q},\mathbf{p}) = \int \exp\Big[-i(\mu\mathbf{q}+\nu\mathbf{p}-X)\Big] w_2(X,\mu,\nu) \frac{dX\,d\mu\,d\nu}{(2\pi)^{2Nd}}\,. \tag{6}$$

Since the Wigner function is connected by invertible maps with both probability distributions w_1 and w_2, it is obvious that these tomograms contain the same information on the quantum state. In fact, one has

$$w_2(X,\mu,\nu) = \int w_1(\mathbf{Y},\mu,\nu)\,\delta\Big(X-\sum_{j=1}^{Nd} Y_j\Big)\,d\mathbf{Y}, \tag{7}$$

$$w_1(\mathbf{X},\mu,\nu) = \int w_2(Y,\mathbf{k}\circ\mu,\mathbf{k}\circ\nu)\,e^{i(Y-\mathbf{k}\mathbf{X})}\,d\mathbf{k}\,dY\,. \tag{8}$$

The quantum-mechanical average of a function, which can take only positive values, is also positive. The integral of some function over the phase space with the Wigner function is the average value; therefore, because the product of the delta-functions $\delta(X_i - \mu_i q_i - \nu_i p_i)$ in (3) cannot be negative, the symplectic tomogram w_1 is nonnegatively defined. The same is valid for w_2, because for this function one considers $\delta(X - \mu\mathbf{q} - \nu\mathbf{p})$, which is also nonnegative.

The form of the random variables $\mathbf{X}$ (for w_1) and X (for w_2) gives the possible interpretation of μ and ν — they are the parameters of scaling and rotation of the reference frame in the phase space [1]. Then the random variables are the positions of particles, measured in the scaled and rotated reference frame

$$X_i = \mu_i q_i + \nu_i p_i \qquad \text{for} \quad w_1,$$

or the sum of such positions

$$X = \mu\mathbf{q} + \nu\mathbf{p} \qquad \text{for} \quad w_2.$$

The variable X (for w_2) plays a role analogous to the center-of-mass coordinate.

2.2. Properties of the Tomographic Maps

The Wigner function is normalized:

$$\begin{aligned}\int F^W(\mathbf{q},\mathbf{p})\, d\,q d\mathbf{p} &= \int \rho\left(\mathbf{q}+\frac{\mathbf{u}}{2}, \mathbf{q}-\frac{\mathbf{u}}{2}\right)\, e^{-i\mathbf{pu}}\,\frac{d\mathbf{u}\, d\mathbf{q}\, d\mathbf{p}}{(2\pi)^{Nd}} \\ &= \int \rho\left(\mathbf{q}+\frac{\mathbf{u}}{2}, \mathbf{q}-\frac{\mathbf{u}}{2}\right)\delta(\mathbf{u})\, d\mathbf{u}\, d\mathbf{q} = 1,\end{aligned} \tag{9}$$

where we choose normalization for the density matrix:

$$\mathrm{Tr}\,(\hat{\rho}) = 1.$$

Then the tomograms w_1 and w_2 are normalized in $\mathbf{X}$ (for w_1) or X (for w_2) variables:

$$\int w_1(\mathbf{X},\mu,\nu) d\mathbf{X} = \int F^W(\mathbf{q},\mathbf{p})\,\delta(\mathbf{k})\, e^{i\mathbf{k}(\mu\circ\mathbf{q}+\nu\circ\mathbf{p})} d\mathbf{k}\, d\mathbf{q}\, d\mathbf{p} = 1, \tag{10}$$

$$\int w_2(X,\mu,\nu) dX = \int F^W(\mathbf{q},\mathbf{p})\,\delta(k)\, e^{ik(\mu\mathbf{q}+\nu\mathbf{p})} dk\, d\mathbf{q}\, d\mathbf{p} = 1. \tag{11}$$

As mentioned in Sec. 1, although the 1D tomogram depends on three variables, instead of two for the density matrix, the completeness of the physical description is the same for both formulations due to the fact that the tomogram is a homogeneous function:

$$w(\lambda X, \lambda\mu, \lambda\nu) = \frac{w(X,\mu,\nu)}{|\lambda|}$$

for any real λ.

Similar properties take place for more degrees of freedom.

Consider the definitions (3) and (5) and multiply all variables in w_1 or w_2 by a real number λ:

$$\begin{aligned} w_1(\lambda\mathbf{X},\lambda\mu,\lambda\nu) &= \int F^W(\mathbf{q},\mathbf{p}) \exp\Big[-i\lambda\mathbf{k}(\mathbf{X}-\mu\circ\mathbf{q}-\nu\circ\mathbf{p})\Big]\frac{d\mathbf{k}\, d\mathbf{q}\, d\mathbf{p}}{(2\pi)^{Nd}} \\ &= \int F^W(\mathbf{q},\mathbf{p}) \exp\Big[-i\mathbf{k}(\mathbf{X}-\mu\circ\mathbf{q}-\nu\circ\mathbf{p})\Big]\frac{d\mathbf{k}\, d\mathbf{q}\, d\mathbf{p}}{(2\pi|\lambda|)^{Nd}} \\ &= \frac{w_1(\mathbf{X},\mu,\nu)}{|\lambda|^{Nd}}\,, \end{aligned} \tag{12}$$

where we simply made the change of variables $\lambda\mathbf{k} \to \mathbf{k}$.

For w_2, the vector $\mathbf{k}$ becomes a single variable k and we obtain

$$\begin{aligned} w_2(\lambda X, \lambda\mu, \lambda\nu) &= \int F^W(\mathbf{q},\mathbf{p}) \exp\Big[-i\lambda k(X-\mu\mathbf{q}-\nu\mathbf{p})\Big] \frac{dk\, d\mathbf{q}\, d\mathbf{p}}{(2\pi)} \\ &= \int F^W(\mathbf{q},\mathbf{p}) \exp\Big[-ik(X-\mu\mathbf{q}-\nu\mathbf{p})\Big] \frac{dk\, d\mathbf{q}\, d\mathbf{p}}{(2\pi|\lambda|)} \\ &= \frac{w_2(X,\mu,\nu)}{|\lambda|}. \end{aligned} \tag{13}$$

We can also consider componentwise scaling using a real vector λ instead of a single number. Then we obtain the following property of w_1:

$$\begin{aligned} w_1(\lambda\circ\mathbf{X}, \lambda\circ\mu, \lambda\circ\nu) &= \int F^W(\mathbf{q},\mathbf{p}) e^{-i\lambda\circ\mathbf{k}(\mathbf{X}-\mu\circ\mathbf{q}-\nu\circ\mathbf{p})} \frac{d\mathbf{k}\, d\mathbf{q}\, d\mathbf{p}}{(2\pi)^{Nd}} \\ &= \int F^W(\mathbf{q},\mathbf{p}) e^{-i\mathbf{k}(\mathbf{X}-\mu\circ\mathbf{q}-\nu\circ\mathbf{p})} \frac{d\mathbf{k}\, d\mathbf{q}\, d\mathbf{p}}{\prod_{j=1}^{Nd}|\lambda_j|(2\pi)^{Nd}} \\ &= \prod_{j=1}^{Nd} |\lambda_j|^{-1} w_1(\mathbf{X},\mu,\nu), \end{aligned} \tag{14}$$

The properties (12)–(14) make obvious the relations

$$w_1(\mathbf{X},\mu,\nu) = \prod_{j=1}^{Nd}|X_j|^{-1} w_1\left(\mathbf{e}, \frac{\mu}{\mathbf{X}}, \frac{\nu}{\mathbf{X}}\right) = \prod_{j=1}^{Nd}|\mu_j|^{-1} w_1\left(\frac{\mathbf{X}}{\mu}, \mathbf{e}, \frac{\nu}{\mu}\right) = \prod_{j=1}^{Nd}|\nu_j|^{-1} w_1\left(\frac{\mathbf{X}}{\nu}, \frac{\mu}{\nu}, \mathbf{e}\right), \tag{15}$$

where $\mathbf{a} = \mathbf{b}/\mathbf{c}$ means $a_j = b_j/c_j$, and

$$w_2(X,\mu,\nu) = |X|^{-1} w_2\left(1, \frac{\mu}{X}, \frac{\nu}{X}\right). \tag{16}$$

Note that for $Nd = 1$, Eqs. (12) and (14) are equivalent, and the tomographic maps w_1 and w_2 become the same in this case.

For the pure state with the wave function $\Psi(\mathbf{q})$, the symplectic tomogram w_1 was expressed in terms of the modulus squared of the fractional Fourier transform of the wave function in [29]. The tomogram w_2 for the pure state is given by

$$w_2(X,\mu,\nu) = \int d\mathbf{Y}\, \frac{\delta\Big(X - \sum_{j=1}^{Nd} Y_j\Big)}{(2\pi)^{Nd}\prod_{j=1}^{Nd}|\nu_j|} \left| \Psi(\mathbf{q}) \exp\left[i\left(\mathbf{q}\frac{\mathbf{Y}}{\nu} - \frac{\mathbf{q}\circ\mathbf{q}}{2}\frac{\mathbf{Y}}{\nu}\right)\right] d\mathbf{q}\right|^2. \tag{17}$$

2.3. Evolution Equations

Now we discuss the evolution equation for w_1 and w_2.

We start with the most general evolution equation for the density matrix:

$$i\,\frac{\partial\rho(\mathbf{q}',\mathbf{q}'')}{\partial t} = \Big[\hat{H}, \rho(\mathbf{q}',\mathbf{q}'')\Big]. \tag{18}$$

Here and throughout the paper, we omit the dependence on time t but imply that all functions describing the state (density matrix, Wigner function, tomograms) depend on time as a parameter. We consider the Hamiltonian

$$\hat{H}=\sum_i\frac{\hat{p}_i^2}{2m_i}+V(\mathbf{q}).$$

To derive the evolution equation for tomograms, we consider the Moyal evolution equation for the Wigner function [16, 30, 31]:

$$\frac{\partial F^W}{\partial t}+\frac{\mathbf{p}}{\mathbf{m}}\frac{\partial F^W}{\partial\mathbf{q}}+i\left[V\left(\mathbf{q}+\frac{i}{2}\frac{\partial}{\partial\mathbf{p}}\right)-V\left(\mathbf{q}-\frac{i}{2}\frac{\partial}{\partial\mathbf{p}}\right)\right]F^W=0, \tag{19}$$

where $\mathbf{p}/\mathbf{m}$ means the vector with components p_i/m_i (the equation holds for the case of different masses for different particles and directions) and the operators in the potential V designate the analytical expansion of the potential and use of the products of corresponding operators. This equation can be easily obtained by applying the transform (1) to Eq. (18).

To derive the evolution equation for the tomograms, one applies the transforms (3) and (5) to the evolution equation for the Wigner function (19). By expanding the potential in Eq. (19), one can see that transforms of the quantities $\mathbf{q}F^W$, $\partial F^W/\partial\mathbf{q}$, $\mathbf{p}F^W$, and $\partial F^W/\partial\mathbf{p}$ have to be considered. Let us derive the evolution equation for w_2. The transform (5) of $\mathbf{q}F^W$ reads

$$\begin{aligned}\int\mathbf{q}F^W(\mathbf{q},\mathbf{p})\exp\Big[-ik(X-\mu\mathbf{q}-\nu\mathbf{p})\Big]\frac{dk\,d\mathbf{q}\,d\mathbf{p}}{2\pi} \quad &= \quad -i\,\frac{\partial}{\partial\mu}\int\frac{F^W(\mathbf{q},\mathbf{p})}{k}\\ &\times\exp\Big[-ik(X-\mu\mathbf{q}-\nu\mathbf{p})\Big]\frac{dk\,d\mathbf{q}\,d\mathbf{p}}{2\pi}\,. \end{aligned}\tag{20}$$

Consider the operator $(\partial/\partial X)^{-1}$, which gives the antiderivative of the function it operates on. Then we have

$$i\,\frac{e^{-ikX}}{k}=\frac{i}{k}\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial X}e^{-ikX}=\left(\frac{\partial}{\partial X}\right)^{-1}e^{-ikX}, \tag{21}$$

and Eq. (20) becomes

$$\int\mathbf{q}F^W(\mathbf{q},\mathbf{p})\exp\Big[-ik(X-\mu\mathbf{q}-\nu\mathbf{p})\Big]\frac{dk\,d\mathbf{q}\,d\mathbf{p}}{2\pi}=-\frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial X}\right)^{-1}w_2(X,\mu,\nu). \tag{22}$$

Using the same simple operations we obtain the transformation rules for the terms of Eq. (19), which we formally designate by arrow:

$$\mathbf{q}F^W(\mathbf{q},\mathbf{p})\to-\frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial X}\right)^{-1}w_2(X,\mu,\nu), \tag{23}$$

$$\frac{\partial F^W(\mathbf{q},\mathbf{p})}{\partial\mathbf{q}}\to\mu\frac{\partial}{\partial X}w_2(X,\mu,\nu), \tag{24}$$

$$\mathbf{p}F^W(\mathbf{q},\mathbf{p})\to-\frac{\partial}{\partial\nu}\left(\frac{\partial}{\partial X}\right)^{-1}w_2(X,\mu,\nu), \tag{25}$$

$$\frac{\partial F^W(\mathbf{q},\mathbf{p})}{\partial\mathbf{p}}\to\nu\frac{\partial}{\partial X}w_2(X,\mu,\nu). \tag{26}$$

Successive application of rules (23)–(26) allows us to transform all powers of $\mathbf{q}$, $\mathbf{p}$ and corresponding derivatives in Eq. (19). As a result, we obtain the evolution equation for the quantum tomogram w_2 of one random variable:

$$\frac{\partial w_2}{\partial t} - \frac{\mu}{\mathbf{m}}\frac{\partial w_2}{\partial \nu} + i\left[V\left(-\frac{\partial}{\partial \mu}\left(\frac{\partial}{\partial X}\right)^{-1} + \frac{i}{2}\nu\frac{\partial}{\partial X}\right) - V\left(-\frac{\partial}{\partial \mu}\left(\frac{\partial}{\partial X}\right)^{-1} - \frac{i}{2}\nu\frac{\partial}{\partial X}\right)\right] w_2 = 0. \tag{27}$$

The same operations can be used to obtain the evolution equation for symplectic tomogram w_1. We do not give its derivation here because it is very similar to that for w_2 and has already been considered, e.g., in [28, 32]. The equation has the form

$$\frac{\partial w_1}{\partial t} - \frac{\mu}{\mathbf{m}}\circ\frac{\partial w_1}{\partial \nu} + i\left[V\left(-\frac{\partial}{\partial \mu}\circ\left(\frac{\partial}{\partial \mathbf{X}}\right)^{-1} + \frac{i}{2}\nu\circ\frac{\partial}{\partial \mathbf{X}}\right) - V\left(-\frac{\partial}{\partial \mu}\circ\left(\frac{\partial}{\partial \mathbf{X}}\right)^{-1} - \frac{i}{2}\nu\circ\frac{\partial}{\partial \mathbf{X}}\right)\right] w_1 = 0. \tag{28}$$

From the form of Eqs. (27) and (28), we see that the variables μ, ν, which, as we mentioned earlier, can be interpreted as the parameters of the scaling and rotated reference frame, appear as dynamical variables in the evolution equations for quantum tomograms.

2.4. Stationary States

For the stationary state with certain energy, we can turn from the time-dependent Schrödinger equation (18) to the eigenvalue equation

$$\hat{H}\hat{\rho}_E = \hat{\rho}_E\hat{H} = E\hat{\rho}_E. \tag{29}$$

Applying the transform (1) we obtain the following rules of transition from the equation for the density matrix to the equation for the Wigner function:

$$\begin{aligned} \frac{\partial^2 \rho(\mathbf{q},\mathbf{q}')}{\partial \mathbf{q}^2} &\to \left(\frac{1}{4}\frac{\partial^2}{\partial \mathbf{q}^2} - \mathbf{p}^2 + i\mathbf{p}\frac{\partial}{\partial \mathbf{q}}\right) F^W(\mathbf{q},\mathbf{p}), V(\mathbf{q})\rho(\mathbf{q},\mathbf{q}') \\ &\to V\left(\mathbf{q} + \frac{i}{2}\frac{\partial}{\partial \mathbf{p}}\right) F^W(\mathbf{q},\mathbf{p}). \end{aligned} \tag{30}$$

After that, using (23)–(26), we have the eigenvalue equation for the tomogram w_2 with one random variable

$$\begin{aligned} &\sum_{j=1}^{Nd}\left[\frac{1}{2m_j}\frac{\partial^2}{\partial \nu_j^2}\left(\frac{\partial}{\partial X}\right)^{-2} - \frac{1}{8m_j}\mu_j^2\frac{\partial^2}{\partial X^2}\right] w_2 + \operatorname{Re} V\left(\frac{i}{2}\nu\frac{\partial}{\partial X} - \frac{\partial}{\partial \mu}\left(\frac{\partial}{\partial X}\right)^{-1}\right) w_2 = E w_2, \\ &-\sum_{j=1}^{Nd}\frac{\mu_j}{2m_j}\frac{\partial w_2}{\partial \nu_j} = \operatorname{Im} V\left(\frac{i}{2}\nu\frac{\partial}{\partial X} - \frac{\partial}{\partial \mu}\left(\frac{\partial}{\partial X}\right)^{-1}\right) w_2. \end{aligned} \tag{31}$$

The corresponding equation for w_1 is almost the same, the only difference being that X in the jth term of the sum must be replaced by the jth component of vector $\mathbf{X}$.

2.5. Quantum Transitions

In general, there is a possibility for transition between quantum states.

Consider two states and designate them a and b. The probability of transition from state a to state b is

$$P_{ab} = \mathrm{Tr}\,(\hat{\rho}_a\hat{\rho}_b) = \int \rho_a(\mathbf{q}',\mathbf{q}'')\rho_b(\mathbf{q}'',\mathbf{q}')d\mathbf{q}'\,d\mathbf{q}''.$$

In terms of the Wigner formalism, this can be rewritten as follows:

$$P_{ab} = (2\pi)^{Nd}\int F^{W(a)}(\mathbf{q},\mathbf{p})F^{W(b)}(\mathbf{q},\mathbf{p})\,d\mathbf{q}\,d\mathbf{p}. \tag{32}$$

Recalling the connection of the Wigner function with tomograms w_1 (4) and w_2 (6), we easily get the following expressions for P_{ab} in the tomography approach:

$$\begin{aligned}\int w_1^a(\mathbf{X},\mu,\nu)\,w_1^b(\mathbf{Y},-\mu,-\nu)\,e^{i\mathbf{e}(\mathbf{X}+\mathbf{Y})}\,\frac{d\mathbf{X}\,d\mathbf{Y}\,d\mu\,d\nu}{(2\pi)^{Nd}} \quad &= \quad \int w_2^a(X,\mu,\nu)\,w_2^b(Y,-\mu,-\nu)\\ &\times e^{i(X+Y)}\,\frac{dX\,dY\,d\mu\,d\nu}{(2\pi)^{Nd}}.\end{aligned} \tag{33}$$

2.6. Tomographic Map in Temperature-Dependent Processes

The tomographic representation can be introduced without changes for systems with temperature $T \neq 0$. In this case, we consider the "imaginary time" $\beta = 1/T$ (measuring T in units of energy). β enters as a parameter in the density matrix, which is now defined by the equation

$$-\frac{\partial\rho(\mathbf{q}',\mathbf{q}'',\beta)}{\partial\beta} = \hat{H}_{\mathbf{q}'}\rho(\mathbf{q}',\mathbf{q}'',\beta), \tag{34}$$

where the index $\mathbf{q}'$ in $\hat{H}_{\mathbf{q}'}$ shows that the Hamiltonian acts only on that variable.

Now the transition to the tomograms w_1 or w_2 is straightforward. We just use the same rule as in the derivation of evolution equations (27), (28) and eigenvalue equation (31). Then the "evolution equation in imaginary time" β for w_2 is given by

$$\begin{aligned}-\frac{\partial w_2}{\partial\beta} &= \sum_{j=1}^{Nd}\left[\frac{1}{2m_j}\frac{\partial^2}{\partial\nu_j^2}\left(\frac{\partial}{\partial X}\right)^{-2} - \frac{1}{8m_j}\mu_j^2\frac{\partial^2}{\partial X^2}\right]w_2 + \mathrm{Re}\,V\left(\frac{i\nu}{2}\frac{\partial}{\partial X} - \frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial X}\right)^{-1}\right)w_2,\\ -\sum_{j=1}^{Nd}\frac{\mu_j}{2m_j}\frac{\partial w_2}{\partial\nu_j} &= \mathrm{Im}\,V\left(\frac{i}{2}\nu\frac{\partial}{\partial X} - \frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial X}\right)^{-1}\right)w_2.\end{aligned} \tag{35}$$

The equation for w_1 is obtained from (35) replacing X by the corresponding components of the vector $\mathbf{X}$.

The initial condition for Eq. (34) reads

$$\rho(\mathbf{q}',\mathbf{q}'',\beta=0) = \delta(\mathbf{q}'-\mathbf{q}'').$$

It corresponds to the constant Wigner function [see Eq. (1)]. Using Eqs.(3)–(5) we see that both tomograms w_1 and w_2 for $\beta = 0$ must have the delta-function form, equal zero everywhere, besides the point $\mu,\nu = 0$, and be constant in the $\mathbf{X}$ (or X) direction in that point.

3. Calculation of Average Values and Star-Product Formalism

3.1. General Rules

In any application of the theory being developed, when we try to describe the physical reality, the quantities we deal with are measurements of some devices. Such measurements are described in quantum theory by the average values of the operators, corresponding to a certain physical quantity. Using the density matrix to describe the system state, we can obtain the average value of some operator $\hat{A}$ as follows:

$$\langle A\rangle = \mathrm{Tr}\,(\hat{\rho}\hat{A}), \tag{36}$$

where we choose

$$\mathrm{Tr}\,(\hat{\rho}) = 1.$$

In connection with this formula, it is worth rewriting the basic relations of the tomogram w_2 and density operator $\hat{\rho}$ in an invariant form. One can check that

$$w_2(X,\mu,\nu) = \langle \delta(X-\mu\hat{\mathbf{q}}-\nu\hat{\mathbf{p}})\rangle\,, \tag{37}$$

$$\hat{\rho} = \int w_2(X,\mu,\nu)\, e^{i(X-\mu\hat{\mathbf{q}}-\nu\hat{\mathbf{p}})}\,\frac{d\mathbf{X}\,d\mu\,d\nu}{(2\pi)^{Nd}}\,. \tag{38}$$

What will be the expression for the average values if we work in the quantum tomography approach? Here it is again convenient to begin with the Wigner–Moyal formulation of quantum mechanics. Within its framework, to calculate the average value, one deals with the Weyl symbol $A^W(\mathbf{q},\mathbf{p})$ [33] of the operator $A(\hat{\mathbf{q}},\hat{\mathbf{p}})$ (for review, see [34, 35]):

$$\langle A\rangle = \int A^W(\mathbf{q},\mathbf{p})F^W(\mathbf{q},\mathbf{p})\,d\mathbf{q}\,d\mathbf{p}, \tag{39}$$

where the Weyl symbol is given by

$$A^W(\mathbf{q},\mathbf{p}) = \int \mathrm{Tr}\left(A(\hat{\mathbf{q}},\hat{\mathbf{p}})e^{i\xi\hat{\mathbf{q}}+i\eta\hat{\mathbf{p}}}\right)e^{-i\xi\mathbf{q}-i\eta\mathbf{p}}\,\frac{d\xi\,d\eta}{(2\pi)^{2Nd}}\,. \tag{40}$$

Consider at first the symplectic tomographic map w_1. Expressing the Wigner function in Eq. (39) through the tomogram w_1 [as was done in (4)], we get (see also [36])

$$\begin{aligned}\langle A\rangle &= \int A^W(\mathbf{q},\mathbf{p})e^{-i\mathbf{e}(\mu\circ\mathbf{q}+\nu\circ\mathbf{p}-\mathbf{X})}w_1(\mathbf{X},\mu,\nu)\,\frac{d\mathbf{X}\,d\mu\,d\nu\,d\mathbf{q}\,d\mathbf{p}}{(2\pi)^{2Nd}}\\ &= \int e^{i\sum_{j=1}^{Nd}X_j}w_1(\mathbf{X},\mu,\nu)A(\mu,\nu)\,d\mathbf{X}\,d\mu\,d\nu,\end{aligned} \tag{41}$$

where we have used

$$\mathbf{e}\mathbf{X} = \sum_j 1\cdot X_j = \sum_j X_j$$

and the Fourier transform of the Weyl symbol $A(\mu,\nu)$

$$A(\mu,\nu) = \int A^W(\mathbf{q},\mathbf{p})e^{-i(\mu\mathbf{q}+\nu\mathbf{p})}\,\frac{d\mathbf{q}\,d\mathbf{p}}{(2\pi)^{2Nd}}\,. \tag{42}$$

Here we have used the properties of a componentwise product:

$$\mathbf{a}(\mathbf{b}\circ\mathbf{c}) = \sum_j a_j(b_jc_j) = \mathbf{b}(\mathbf{a}\circ\mathbf{c}) = \mathbf{c}(\mathbf{b}\circ\mathbf{a}),$$

which result in

$$\mathbf{e}(\mathbf{b}\circ\mathbf{c}) = \mathbf{b}(\mathbf{e}\circ\mathbf{c}) = \mathbf{bc}.$$

Note that in Eq. (41) $3Nd$ variables become $2Nd+1$ variables, and $\mathbf{X}$ appears there only as a sum of the components. Only the average values of operators are of concern in the measurable physical reality; therefore, it is natural to consider $\mathbf{X}$ in the form it appears in the expression for average values. But the sum of the components

$$X_j = \mu_j q_j + \nu_j p_j$$

is exactly the same as the variable

$$X = \mu\mathbf{q} + \nu\mathbf{p},$$

which appears in w_2. So it seems that the tomographic map with one random variable contains the same physical information in a more economic way than the usual symplectic tomographic map. This conclusion is also confirmed by the fact that for description of the system we need only $2Nd$ variables ($\mathbf{q}', \mathbf{q}''$ for the density matrix or $\mathbf{q}, \mathbf{p}$ for the Wigner function), while the symplectic tomography operates with $3Nd$ variables. Turning to the one-random-variable tomography map w_2, we decrease the number of dynamical variables to only $2Nd+1$, and recalling the property (13), we face the real and nonnegative probability distribution function of $2Nd$ variables, which completely describes the quantum state.

The expression for average values in the one-random-variable tomography formulation is obtained in the same way as for w_1, using the connection between the Wigner function and w_2 (6):

$$\begin{aligned}\langle A\rangle &= \int A^W(\mathbf{q},\mathbf{p})e^{-i(\mu\mathbf{q}+\nu\mathbf{p}-X)}w_2(X,\mu,\nu)\,\frac{dX\,d\mu\,d\nu\,d\mathbf{q}\,d\mathbf{p}}{(2\pi)^{2Nd}} \\ &= \int e^{iX}w_2(X,\mu,\nu)A(\mu,\nu)\,dX\,d\mu\,d\nu. \end{aligned} \tag{43}$$

Here we recall that due to the property (16) the tomogram $w_2(X,\mu,\nu)$ is known for any X if it is known for only one $X_0 \neq 0$:

$$w_2(X,\mu,\nu) = \frac{X_0}{X}\,w_2\left(X_0, \frac{X_0}{X}\,\mu, \frac{X_0}{X}\,\nu\right). \tag{44}$$

However, in the general case, X cannot be extracted either from the evolution equation (27) or from the expression for average values (43). It is the unusual form of these equations, where X is "entangled" with other variables, that does not allow one to deal with w_2 as a function of only $2Nd$ variables, and this is the price of its nonnegativity. And still, considering w_2, we do work with the nonnegative function of $2Nd$ variables — having determined, for example, $w_2(1,\mu,\nu)$ for all μ,ν from $-\infty$ to $+\infty$, we can easily find w_2 for every X.

3.2. Operators Depending on Coordinates or Momenta Only

Equations (41) and (43) take an interesting form if the operator under consideration depends on coordinates $\hat{\mathbf{q}}$ or momenta $\hat{\mathbf{p}}$ only. In this case, the Weyl symbols have the same form as the corresponding operators in the coordinate or momentum representation. The operator $A(\hat{\mathbf{q}})$ is $A(\mathbf{x})$ in the $\mathbf{x}$-coordinate representation; then its Weyl symbol $A^W(\mathbf{q},\mathbf{p})$ is equal to $A(\mathbf{q})$. The same is valid for the momenta-dependent operator — $B(\hat{\mathbf{p}})$ is $B(\mathbf{y})$ in the $\mathbf{y}$-momentum representation and

$$B^W(\mathbf{q},\mathbf{p}) = B(\mathbf{p}).$$

We consider an operator $A(\hat{\mathbf{q}})$ depending on coordinates only.

For momenta-dependent operators, all equations are the same provided μ is replaced by ν, and vice versa, because the pairs $\mathbf{q},\mu$ and $\mathbf{p},\nu$ enter the equations connecting the tomograms w_1 and w_2 with the Wigner function absolutely symmetrically.

For both w_1 and w_2, integration over ν in Eqs. (41) and (43) gives the delta-function $\delta(\nu)$. In the symplectic tomography formulation, recalling the property (14), $\langle A\rangle$ is expressed as follows:

$$\begin{aligned}\langle A\rangle &= \int A^W(\mathbf{q})\, e^{-i\left(\mu\mathbf{q}-\sum_{j=1}^{Nd} X_j\right)}\, w_1\Big(\mathbf{X},\mu,\nu=0\Big)\,\frac{d\mathbf{X}\,d\mu\,d\mathbf{q}}{(2\pi)^{Nd}}\\ &= \int A^W(\mathbf{q})\, e^{-i\mu\Big(\mathbf{q}-\mathbf{X}/\mu\Big)}\, w_1\Big(\mathbf{X}/\mu,\mathbf{e},\nu=0\Big)\,\frac{d\mathbf{X}\,d\mu\,d\mathbf{q}}{\prod_{j=1}^{Nd}|\mu_j|(2\pi)^{Nd}}\\ &= \int A^W(\mathbf{q})\, e^{-i\mu(\mathbf{q}-\mathbf{X})}\, w_1\Big(\mathbf{X},\mu=\mathbf{e},\nu=0\Big)\,\frac{d\mathbf{X}\,d\mu\,d\mathbf{q}}{(2\pi)^{Nd}}\\ &= \int A^W(\mathbf{q})\,\delta(\mathbf{q}-\mathbf{X})\, w_1\Big(\mathbf{X},\mu=\mathbf{e},\nu=0\Big)\,d\mathbf{X}\,d\mathbf{q}\\ &= \int A^W(\mathbf{X}) w_1\Big(\mathbf{X},\mu=\mathbf{e},\nu=0\Big)\,d\mathbf{X}.\end{aligned} \tag{45}$$

In view of Eqs. (1) and (3), we have

$$w_1(\mathbf{X},\mu=\mathbf{e},\nu=0) = \int F^W(\mathbf{q},\mathbf{p})\,\delta(\mathbf{X}-\mathbf{q})\,d\mathbf{q}\,d\mathbf{p} = \int F^W(\mathbf{X},\mathbf{p})\,d\mathbf{p} = \rho(\mathbf{X},\mathbf{X}), \tag{46}$$

i.e., $w_1(\mathbf{X},\mu=\mathbf{e},\nu=0)$ is the particle density in the coordinate space. Then Eq. (45) looks quite natural.

For the tomogram w_2 with one random variable, we obtain

$$\langle A\rangle = \int A^W(\mathbf{q})\, e^{-i(\mu\mathbf{q}-X)}\, w_2(X,\mu,\nu=0)\,\frac{dX\,d\mu\,d\mathbf{q}}{(2\pi)^{Nd}}. \tag{47}$$

This expression is not reduced to the simple form as it takes place for w_1. Nevertheless, some improvement can appear in certain cases. For example, one often operates with one-particle and one-dimension operators. Then, quite generally, we can consider an operator $A(\hat{q}_1)$. The corresponding average value $\langle A\rangle$ reads

$$\begin{aligned}&\int A^W(q_1)e^{-i(\mu_1 q_1-X)}\delta(\tilde{\mu})w_2(X,\mu,0)\,\frac{dX\,d\mu\,dq_1}{2\pi}\\ &= \int A^W(q_1)e^{-i\mu_1(q_1-X/\mu_1)}\, w_2\left(\frac{X}{\mu_1},\mu_1=1,\tilde{\mu}=0,0\right)\frac{dX\,d\mu_1\,dq_1}{2\pi|\mu_1|}\\ &= \int A^W(q_1)\,\delta(q_1-X)w_2\Big(X,\mu_1=1,\tilde{\mu}=0,0\Big)\,dX\,dq_1\\ &= \int A^W(X)\, w_2\Big(X,\mu_1=1,\tilde{\mu}=0,0\Big)\,dX,\end{aligned} \tag{48}$$

where $\tilde{\mu}$ designates all μ_j except the specified, i.e., here all except μ_1.

3.3. Star-Products

There is a possibility to develop quantum mechanics without operators, using some functions instead and the so-called star products of these functions. Within the framework of star-product formalism (see [13] and references therein), one associates any operator with its "symbol," i.e., the function depending on a specific set of parameters and corresponding to certain rules of the operators' ordering. Any state-describing function (such as the Wigner function, symplectic tomogram w_1, etc.) is the symbol of the density operator $\hat{\rho}$, and the product of operators is replaced by the star-product of their symbols. For example, the Wigner–Moyal formulation of quantum mechanics corresponds to the Wigner–Weyl ordering of the operators, Weyl symbols (40) replace the operators, and the Wigner function is the Weyl symbol of $\hat{\rho}$.

The tomography representation can be developed with the help of the star-product formalism too. Here we give the corresponding expressions for the one-random-variable tomography; for symplectic tomography the same can be found in the literature.

One obtains the symbol of any operator $\hat{A}$ for the tomographic map with one random variable and reconstructs this operator as follows:

$$w_2^A(X,\mu,\nu) = \mathrm{Tr}\left(\hat{A}\delta(X-\mu\hat{\mathbf{q}}-\nu\hat{\mathbf{p}})\right), \tag{49}$$

$$\hat{A} = \int w_2^A(X,\mu,\nu)\, e^{i(X-\mu\hat{\mathbf{q}}-\nu\hat{\mathbf{p}})}\, \frac{d\mathbf{X}\, d\mu\, d\nu}{(2\pi)^{Nd}}. \tag{50}$$

In particular, these expressions give the connection between the tomogram w_2 (37) and the density operator $\hat{\rho}$ (38).

The star-product of two symbols can be defined through the kernel of the corresponding integral expression:

$$(w^A * w^B)(y) = \int w^A(y'')w^B(y')K(y'',y',y)\, dy''\, dy', \tag{51}$$

where, for the tomography with one random variable, $y = \{X,\mu,\nu\}$ and the kernel is given by

$$K(y'',y',y) = \mathrm{Tr}\left(\hat{D}(y'')\hat{D}(y')\hat{U}(y)\right), \tag{52}$$

$$\hat{D}(y) = (2\pi)^{-Nd} \exp\left[i\left(X-\mu\hat{\mathbf{q}}-\nu\hat{\mathbf{p}}\right)\right], \tag{53}$$

$$\hat{U}(y) = \delta\left(X-\mu\hat{\mathbf{q}}-\nu\hat{\mathbf{p}}\right). \tag{54}$$

The analytic expression for the kernel has the following form:

$$\begin{aligned} K(y'',y',y) \quad = \quad & \int e^{-i(kX-X'-X'')}\, \delta(\mu''+\mu'-k\mu)\, \delta(\nu''+\nu'-k\nu) \\ & \times \exp\left\{-i\left[\mu''\nu' - k\left(\mu''\nu'+\mu'\nu\right) + \frac{\mu'\nu'+\mu''\nu''+k^2\mu\nu}{2}\right]\right\} \frac{dk}{(2\pi)^{Nd+1}}. \end{aligned} \tag{55}$$

4. Examples

In this section, we introduce several examples of tomographic maps (symplectic or with one random variable) for many-particle quantum states. For simplicity, here we do not regard the symmetry over the particles' permutations.

4.1. Gaussian States

Quite simple is the case where the system state is pure and the wave function has the Gaussian form. This can be the ground state of the system of independent oscillators, as well as the coherent or squeezed state, or any many-dimensional Gaussian wave packet. Such a wave packet can be created due to the parametric excitation of the multimode vacuum state of an electromagnetic field [37], e.g., within the framework of the nonstationary Casimir effect [38].

Consider first the pure state with the wave function

$$\Psi(\mathbf{q}) = \prod_{j=1}^{Nd} \psi_j(q_j),$$

where

$$\psi_j(q) = \left(\frac{A_j}{\pi}\right)^{1/4} \exp\left[-\frac{A_j}{2}(q-x_j)^2 - iy_j q\right]. \tag{56}$$

The only mathematical principle we need here is the fact that the Fourier transform of a Gaussian is Gaussian. Then, using Eq. (1), we immediately obtain the Wigner function as a product of $F_j^W(q_j, p_j)$, where

$$F_j^W(q,p) = e^{-A_j(q-x_j)^2} e^{-B_j(p-y_j)^2} \,\frac{(A_j B_j)^{1/2}}{\pi}\,, \tag{57}$$

and for states (56) $B_j = 1/A_j$.

For the set of parameters $\mathbf{x}, \mathbf{y}, \mathbf{A}, \mathbf{B}$, by applying Fourier transformations (3), (5) to (57) we have w_1:

$$w_1^{\text{Gauss}}(\mathbf{X}, \mu, \nu) = \prod_{j=1}^{Nd} \frac{1}{\sqrt{\pi C_j}}\, e^{-(X_j - \mu_j x_j - \nu_j y_j)^2/C_j}, \tag{58}$$

where

$$C_j = \frac{\mu_j^2}{A_j} + \frac{\nu_j^2}{B_j}\,.$$

The tomogram w_2 has the form

$$w_2^{\text{Gauss}}(X, \mu, \nu) = \frac{1}{\sqrt{\pi C}}\, e^{-(X - \mu\mathbf{x} - \nu\mathbf{y})^2/C}, \tag{59}$$

where

$$C = \sum_{j=1}^{Nd} C_j.$$

Thermal density matrix of independent oscillators is also Gaussian, but it is not a product of wave functions, since the state is not pure. Still it is a product of density matrices of individual oscillators (see, e.g., [39]):

$$\rho_j(q, q') = \sqrt{\frac{2A_j(B_j - 1)}{\pi}}\, e^{-A_j[B_j(q^2 + q'^2) - 2qq']}, \tag{60}$$

where

$$A_j = \frac{m\omega_j}{2\sinh(\omega_j \beta)} \qquad \text{and} \qquad B = \cosh(\omega_j \beta).$$

Omitting the straightforward calculations, we obtain the tomogram w_2 in the following form:

$$w_2^{(\beta)}(X,\mu,\nu) = \frac{e^{-X^2/D}}{\sqrt{\pi D}}\,, \tag{61}$$

where

$$D = \sum_{j=1}^{Nd}\left(\frac{\mu_j^2}{2A_j(B_j-1)} + 2\nu_j^2 A_j(B_j+1)\right). \tag{62}$$

The symplectic tomogram w_1 in this case is the product of functions like (61), where X is replaced by X_j and the sum in the expression for D is substituted for the jth term of the sum.

4.2. Fock States

One can also consider the Fock states of light (eigenstates in the photon-number representation). Such functions correspond to the ground or excited state of a multimode oscillator. The state is labeled by the vector $\mathbf{n}$ of integer numbers, and the wave function has the form

$$\Psi(\mathbf{q}) = \prod_{j=1}^{Nd}\frac{e^{-q_j^2/2}H_{n_j}(q_j)}{\pi^{1/4}\sqrt{2^{n_j}n_j!}}\,, \tag{63}$$

where H_m is a Hermite polynomial of mth order. To obtain the tomograms for such state, we use the following facts. First, the coherent state of an oscillator is described by the Gaussian wave function and, correspondingly, by the Gaussian tomogram [both w_1 and w_2, see Eqs. (58) and (59)]. The coherent state is labeled by the complex vector

$$\alpha = \mathbf{a} + i\mathbf{b},$$

and the parameters of the Gaussian wave function in the coordinate representation (56) are

$$x_j = \sqrt{2}a_j \qquad \text{and} \qquad y_j = -\sqrt{2}b_j.$$

Second, the wave function of the coherent state (for simplicity, one dimension is considered here) is expanded in the basis of Fock states as follows:

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,, \tag{64}$$

which is connected with the expression for generating a function of Hermite polynomials:

$$e^{-\alpha^2+2\alpha q} = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}H_n(q). \tag{65}$$

One can express the tomogram w_1 of the coherent state in terms of Hermite polynomials. The wave function of the coherent state can be given as a series of wave functions of Fock states. Noting that the

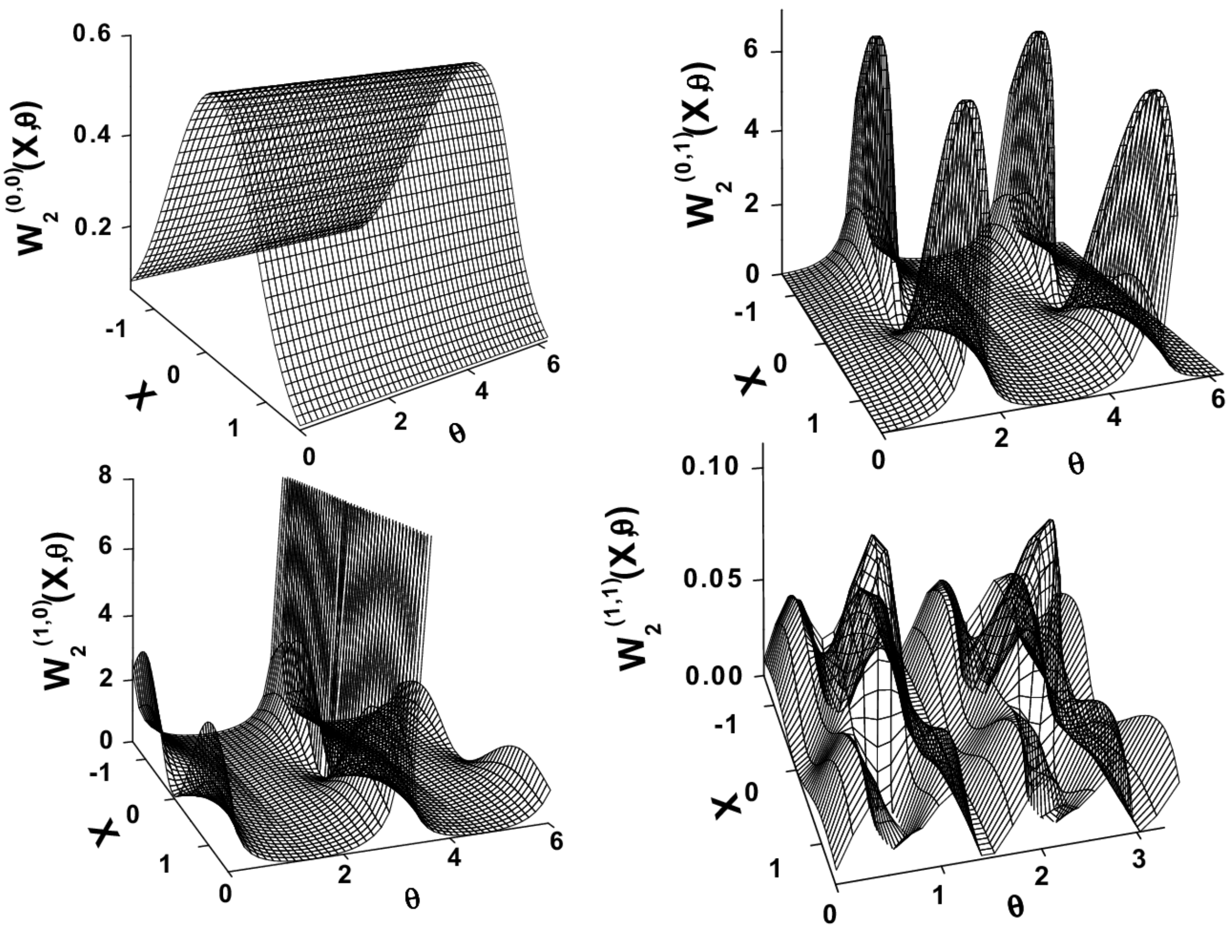

Fig. 1. Tomograms w_2 for Fock states, $Nd = 2$. X and θ are dimensionless parameters. States with $\{n_1, n_2\}$ equal to $\{0,0\}$, $\{0,1\}$, $\{1,0\}$ and $\{1,1\}$ are presented (see text for details).

tomogram of the Fock state is the product of one-dimensional tomograms, we have

$$w_1^{\mathbf{n}}(\mathbf{X}, \mu, \nu) = \prod_{j=1}^{Nd} \frac{H_{n_j}^2 \left(\frac{X_j}{\sqrt{\mu_j^2+\nu_j^2}} \right) e^{-X_j^2/(\mu_j^2+\nu_j^2)}}{2^{n_j} n_j! \sqrt{\pi(\mu_j^2+\nu_j^2)}}, \tag{66}$$

$$w_2^{\mathbf{n}}(X, \mu, \nu) = \int \delta\Big(X - \sum_{j=1}^{Nd} X_j \Big) \prod_{j=1}^{Nd} \frac{H_{n_j}^2 \left(\frac{X_j}{\sqrt{\mu_j^2+\nu_j^2}} \right) e^{-X_j^2/(\mu_j^2+\nu_j^2)}}{2^{n_j} n_j! \sqrt{\pi(\mu_j^2+\nu_j^2)}} \, d\mathbf{X}. \tag{67}$$

For example, for $N = 2, d = 1$ and states with n_1, n_2 equal to 0 or 1 [denoted (n_1, n_2)], the tomograms

$w_2(X,\mu_1,\mu_2,\nu_1,\nu_2)$ have the form

$$w_2^{(0,0)} = \frac{e^{-X^2/C}}{\sqrt{\pi C}}\,, \tag{68}$$

$$w_2^{(0,1)} = \sqrt{\frac{C_2}{\pi C_1}}\frac{(2C_2X^2+C_1C_2+C_1^2)e^{-X^2/C}}{C^{5/2}}\,, \tag{69}$$

$$w_2^{(1,1)} = \frac{4\,C_1^2C_2^2\,e^{-X^2/C}}{\sqrt{\pi}C^{5/2}}\left(\frac{X^4}{C^2}+\frac{X^2}{C}\frac{C_1^2+C_2^2-4C_1C_2}{C_1C_2}+\frac{3}{4}\right), \tag{70}$$

where

$$C_1 = \mu_1^2+\nu_1^2, \qquad C_2 = \mu_2^2+\nu_2^2, \qquad C = C_1+C_2.$$

In Fig. 1 we present the tomograms w_2 for these states. Due to the homogeneity of w_2 we can reduce the number of variables to one. Therefore, we can set

$$\mu_1^2+\nu_1^2+\mu_2^2+\nu_2^2 = 1.$$

Variables μ and ν enter Eqs. (68)–(70) only as $\mu_1^2+\nu_1^2$ and $\mu_2^2+\nu_2^2$, so we choose

$$\mu_1^2+\nu_1^2 = \cos^2\theta \qquad \text{and} \qquad \mu_2^2+\nu_2^2 = \sin^2\theta.$$

Then tomogram $w_2^{n_1,n_2}$ is the function of two variables (X and θ) and Fig. 1 shows its dependence on these variables. The tomograms are plotted for the first excited states of a two-mode oscillator, namely, the ground state $n_1=n_2=0$, the first excited state in one mode $n_1=0, n_2=1$ or $n_1=1, n_2=0$, and the first excited state in both modes $n_1=1, n_2=1$.

5. Particle Permutations and Corresponding Symmetry Properties of Quantum Tomograms

While considering real physical systems, we must usually take into account the identity of the particles, which constitute the system. Such consideration of exchange imposes restrictions concerning the possible form of the functions describing the state. These restrictions are reflected by the symmetry properties of the state-describing functions with regards to the particle permutations. In this section, we discuss the corresponding properties of tomographic maps, especially of the tomogram with one random variable (permutation symmetry for the symplectic tomography has already been developed to some extent in [40]).

Further we use the following notation.

A vector without index $\mathbf{a}$ has Nd components, a vector with index $\mathbf{a}_j$ denotes the set of some values, corresponding to the jth particle, and consists of d components.

A vector with tilde $\tilde{\mathbf{a}}$ denotes the collection of all components of $\mathbf{a}$, except those that are specified in the same expression. For example, $\tilde{\mathbf{q}}$ in the expression $\psi(\mathbf{q}_j,\tilde{\mathbf{q}})$ is the vector of all coordinates, except the coordinates of the jth particle.

For particles obeying Fermi or Bose statistics, we have the following symmetry properties concerning the particle permutations:

$$\rho(\mathbf{q}_j,\mathbf{q}_i,\tilde{\mathbf{q}};\mathbf{q}'_i,\mathbf{q}'_j,\tilde{\mathbf{q}}') = \rho(\mathbf{q}_i,\mathbf{q}_j,\tilde{\mathbf{q}};\mathbf{q}'_j,\mathbf{q}'_i,\tilde{\mathbf{q}}') = \pm\rho(\mathbf{q}_i,\mathbf{q}_j,\tilde{\mathbf{q}};\mathbf{q}'_i,\mathbf{q}'_j,\tilde{\mathbf{q}}'), \tag{71}$$

where the upper sign (plus) is for Bose systems, and the lower sign (minus) is for Fermi systems. Note that the "entire" particle permutation (two particles exchange both q and q' variables) corresponds to the sign conservation for both Fermi and Bose statistics:

$$\rho(\mathbf{q}_j, \mathbf{q}_i, \tilde{\mathbf{q}}; \mathbf{q}'_j, \mathbf{q}'_i, \tilde{\mathbf{q}}') = \rho(\mathbf{q}_i, \mathbf{q}_j, \tilde{\mathbf{q}}; \mathbf{q}'_i, \mathbf{q}'_j, \tilde{\mathbf{q}}'). \tag{72}$$

In the expressions for obtaining the Wigner function from the density matrix (1) and both tomograms w_1 and w_2 from the Wigner function (3), (5), we can exchange the integration variables ($\mathbf{u}_j \leftrightarrow \mathbf{u}_i$, etc.); then we immediately arrive at

$$F^W(\mathbf{q}_j, \mathbf{q}_i, \tilde{\mathbf{q}}; \mathbf{p}_j, \mathbf{p}_i, \tilde{\mathbf{p}}) = F^W(\mathbf{q}_i, \mathbf{q}_j, \tilde{\mathbf{q}}; \mathbf{p}_i, \mathbf{p}_j, \tilde{\mathbf{p}}), \tag{73}$$

$$w_1(\mathbf{X}_j, \mathbf{X}_i, \tilde{\mathbf{X}}; \mu_j, \mu_i, \tilde{\mu}; \nu_j, \nu_i, \tilde{\nu}) = w_1(\mathbf{X}_i, \mathbf{X}_j, \tilde{\mathbf{X}}; \mu_i, \mu_j, \tilde{\mu}; \nu_i, \nu_j, \tilde{\nu}), \tag{74}$$

$$w_2(X; \mu_j, \mu_i, \tilde{\mu}; \nu_j, \nu_i, \tilde{\nu}) = w_2(X; \mu_i, \mu_j, \tilde{\mu}; \nu_i, \nu_j, \tilde{\nu}). \tag{75}$$

We see that there is no distinction between Fermi and Bose statistics when the particles exchange "entirely," i.e., q and q' in the density matrix, q and p in the Wigner function, X, μ, ν in w_1 or μ, ν in w_2 are permuted simultaneously. The distinction appears when not all the variables corresponding to the particles considered are permuted. When we use the density matrix, Fermi and Bose statistics differ only in sign ± 1, which appears after the permutation of either $\mathbf{q}_i, \mathbf{q}_j$ or $\mathbf{q}'_i, \mathbf{q}'_j$. For the Wigner function and both tomograms, this difference is expressed using some integral transforms (see corresponding formulas for the symplectic tomography in [40]).

6. Conclusion

To conclude, we summarize the main results of our work. We studied in detail a version of the tomographic map of the density matrix and Wigner function for which the quantum state of the multimode system is associated with a fair probability distribution function. This function depends on one random variable X and $2Nd$ real parameters (real Nd-vectors μ and ν) and it determines the quantum state completely. This means that, provided this probability distribution function is known, one can reconstruct the Wigner function of the system state and the corresponding density operator. The random variable X can be interpreted as the system's center-of-mass coordinate considered in a specifically rotated and scaling reference frame in the complete phase space of the system. The real parameters (vectors μ and ν) determine this rotated and scaling reference frame.

It is interesting that the information contained in the introduced tomogram (w_2) is the same as that contained in the symplectic tomogram (w_1), which depends on a larger number of variables. This corresponds to the fact that the tomograms have high symmetry properties. By means of the symmetry operations, one can reconstruct the dependence of the function on a larger number of variables starting from the initial function with a smaller number of variables.

We have constructed the quantum evolution equations and energy level equations for the introduced center-of-mass tomogram. We established also the relation to the star-product formalism and calculated the kernel of the star-product. Example of the multimode oscillator and the symmetry properties of the tomogram for identical particles (fermions and bosons) were discussed.

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