SURFACE INTEGRAL AND GAUSS-OSTROGRADSKIJ THEOREM FROM THE VIEWPOINT OF APPLICATIONS

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Dedicated to Professor Jaroslav Kurzweil

Abstract. Making use of a surface integral defined without use of the partition of unity, trace theorems and the Gauss-Ostrogradskij theorem are proved in the case of three-dimensional domains Ω with a Lipschitz-continuous boundary for functions belonging to the Sobolev spaces $H^{1,p}(\Omega)$ $(1 \leqslant p < \infty)$. The paper is a generalization of the previous author's paper which is devoted to the line integral.

Keywords: variational problems, surface integral, trace theorems, Gauss-Ostrogradskij theorem

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Introduction

In [Ne] (and similarly in [KJF]) a surface integral is defined with help of the partition of unity in a rather complicated and unnatural way. This approach is then used with advantage in the proofs of trace theorems and the Gauss-Ostrogradskij theorem. Using a Gauss-Ostrogradskij theorem proved in this way for deriving the variational formulation of the boundary value problem

$$-\Delta u = f \text{ in } \Omega, \quad u\big|_{\Gamma_1} = 0, \quad \frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q,$$

where $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, we obtain the following problem: Find

$$u \in V := \{v \in H^1(\Omega) : v\big|_{\Gamma_1} = 0\}$$

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such that

$$\iiint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \iiint_\Omega vf\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iint_{\Gamma_2} vq\,\mathrm{d}\sigma \quad \forall v \in V.$$

In this variational problem the surface integral is defined with help of the partition of unity and thus this formulation is not suitable for practical computations.

In this paper we define a surface integral in a quite natural way (this means, *without use* of the partition of unity) and prove the Gauss-Ostrogradskij theorem and the necessary trace theorems. Of course, the theory of a surface integral is also developed.

In some books the proof of the Gauss-Ostrogradskij theorem reduces to a list of wishes (or a list of unproved assertions—see, e.g., [PF]) and even in textbooks of a high standard the proof of this theorem suffers from confusion (see, e.g., [Fi3]). In books of the top standard the notion of the surface integral is either artificial (see the already mentioned references [Ne] and [KJF]) or difficult to understand and unnecessarily general (as far as the applications are concerned) (see, e.g., [Si]).

In the famous book of Saks (see [Sa]) the expression for the area of a surface $z = F(x,y)$ is derived, where $F(x,y)$ is a continuous function and the expression $\sqrt{[F_x(x,y)]^2 + [F_y(x,y)]^2}$ can be integrable only in the sense of Lebesgue. Our approach, which accentuates the viewpoint of practical applications, is different. We will consider surfaces $z = f(x,y)$ for which the expression $\sqrt{f_x^2 + f_y^2}$ is Riemann integrable (at least in an improper sense) because only such surfaces appear in applications. These surfaces form parts of boundaries of domains Ω in which variational problems considered in applications are formulated. The solutions of these problems belong, in general, to $H^1(\Omega)$. Thus the traces of these solutions are elements of spaces $L_2(\Omega)$. (As to the notation of Sobolev spaces, it is the same as in [KJF].)

The outline of the paper follows: We will define a surface integral over surfaces which appear in applications. This definition will be done without help of the partition of unity. First we will define a surface integral in a Riemann sense and prove its properties. Then we extend the definition to the case when the integrand is the trace of a function from $H^{1,p}(\Omega) \equiv W^{1,p}(\Omega)$ $(p \in \langle 1, \infty))$. In connection with it we prove the corresponding trace theorems. The second part of the paper is focused on various formulations of the Gauss-Ostrogradskij theorem.

It should be noted that the conception expressed in Convention 6.7 and Definition 10.1 is new and quite suitable for applications. Moreover, it avoids problems which are unnecessary from the practical viewpoint, solved with difficulties by some mathematicians.

1. Regular part of a surface

In this section we deal with parts of surfaces over which we shall integrate. Unless stated otherwise we consider an arbitrary but fixed Cartesian coordinate system $\langle O, x, y, z\rangle$, where O is the origin of the system, $O = [0,0,0]$.

1.1. Definition. a) We say that a set $\bar{S}$ is *a part of a surface*, which is *regular with respect to the coordinate plane* (x,y), if the points $[x,y,z] \in \bar{S}$ satisfy

$$z = f(x,y), \quad [x,y] \in \bar{S}_{xy}, \tag{1.1}$$

where $\bar{S}_{xy}$ is a simply connected two-dimensional bounded closed domain lying in the plane (x,y), which is bounded by a simple piecewise smooth closed curve ∂S_{xy}, and $f\colon \bar{S}_{xy} \to \mathbb{R}^1$ is a real function continuous on $\bar{S}_{xy}$, which has continuous first partial derivatives $f_x \equiv \frac{\partial f}{\partial x}$, $f_y \equiv \frac{\partial f}{\partial y}$ in S_{xy} (where the symbol S_{xy} denotes the interior of $\bar{S}_{xy}$, i.e., $S_{xy} = \bar{S}_{xy} - \partial S_{xy}$). The closed domain $\bar{S}_{xy}$ is called the orthogonal projection of the part $\bar{S}$ onto the plane (x,y).

The set of points $[x,y,z]$, for which

$$z = f(x,y), \quad [x,y] \in \partial S_{xy},$$

is called *the boundary of the part* $\bar{S}$ and is denoted ∂S.

b) Similarly we say that a set $\bar{S}$ is a part of a surface, which is regular with respect to the coordinate plane (x,z) (or (y,z)), if the points $[x,y,z] \in \bar{S}$ satisfy

$$y = g(x,z), \quad [x,z] \in \bar{S}_{xz}, \tag{1.2}$$

or

$$x = h(y,z), \quad [y,z] \in \bar{S}_{yz}, \tag{1.3}$$

where the closed domains $\bar{S}_{xz}$, $\bar{S}_{yz}$ and the functions $g\colon \bar{S}_{xz} \to \mathbb{R}^1$, $h\colon \bar{S}_{yz} \to \mathbb{R}^1$ have analogous properties as the closed domain $\bar{S}_{xy}$ and the function $f\colon \bar{S}_{xy} \to \mathbb{R}^1$. The closed two-dimensional domains $\bar{S}_{xz}$ and $\bar{S}_{yz}$ are called orthogonal projections of the part $\bar{S}$ onto the planes (x,z) and (y,z).

c) We will often use the notion "part" instead of the notion "part of a surface".

d) We say that $\bar{S}$ is *a regular part* if $\bar{S}$ is regular at least with respect to one coordinate plane.

1.2. Definition. a) We say that a part $\bar{S}$ is *Lipschitz-regular with respect to the plane* (x, y), if $\bar{S}$ is regular with respect to (x, y) and the derivatives f_x, f_y are bounded in S_{xy} (i.e., in the interior of $\bar{S}_{xy}$).

b) Similarly we define that the part $\bar{S}$ is Lipschitz-regular with respect to (x, z) or (y, z).

c) We say that a part $\bar{S}$ is *Lipschitz-regular* if $\bar{S}$ is Lipschitz-regular at least with respect to one coordinate plane.

1.3. Definition. a) We say that a part $\bar{S}$ is *strongly regular with respect to the plane* (x, y) if $\bar{S}$ is Lipschitz-regular with respect to (x, y) and if the derivatives f_x, f_y can be continuously extended from S_{xy} onto $\bar{S}_{xy}$.

b) Similarly we define a part which is strongly regular with respect to (x, z) or (y, z).

c) We say that a part $\bar{S}$ is *strongly regular* if $\bar{S}$ is strongly regular at least with respect to one coordinate plane.

1.4. Example. a) Let $\bar{S}_{xy} = \{[x, y]: x^2 + y^2 \leqslant r^2, r > 0\}$. The set $\bar{S}$ of points $[x, y, z]$ satisfying (1.1) with $f(x, y) = \sqrt{r^2 - x^2 - y^2}$ is a part of a sphere which is regular (but not Lipschitz-regular) with respect to the coordinate plane (x, y).

b) Let us divide the part $\bar{S}$ described in a) into 16 parts by the coordinate planes and the planes $z = r/2$, $y = x$ and $y = -x$. Each of these 16 parts is strongly regular.

1.5. Example. Let the set $\bar{S}_{xy}$ be the same as in Example 1.4 and let $f(x, y) = \sqrt{x^2 + y^2}$. Then the set $\bar{S}$ of points $[x, y, z]$ satisfying (1.1) is a part of a cone. The coordinate plane (x, z) (or (y, z)) divides $\bar{S}$ into two parts which are Lipschitz-regular. However, neither of them is strongly regular because the derivatives

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \quad f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

cannot be continuously extended to the point $[0, 0]$.

1.6. Theorem. a) *Let a part $\bar{S}$ be regular with respect to (x, y) and (x, z). Then*

$$f(x, g(x, z)) = z \quad \forall [x, z] \in \bar{S}_{xz}, \tag{1.4}$$

$$g(x, f(x, y)) = y \quad \forall [x, y] \in \bar{S}_{xy}. \tag{1.5}$$

b) *Let a part $\bar{S}$ be regular with respect to (x, y) and (y, z). Then*

$$f(h(y, z), y) = z \quad \forall [y, z] \in \bar{S}_{yz}, \tag{1.6}$$

$$h(y, f(x, y)) = x \quad \forall [x, y] \in \bar{S}_{xy}. \tag{1.7}$$

c) *Let a part $\bar{S}$ be regular with respect to (x,z) and (y,z). Then*

$$g(h(y,z),z) = y \quad \forall [y,z] \in \bar{S}_{yz}, \tag{1.8}$$

$$h(g(x,z),z) = x \quad \forall [x,z] \in \bar{S}_{xy}. \tag{1.9}$$

Proof. We shall prove a). The orthogonal projections of the two closed two-dimensional domains $\bar{S}_{xy}$ and $\bar{S}_{xz}$ onto the axis x are identical; this projection is a segment which will be denoted by $\langle a,b \rangle$. Let $x_0 \in \langle a,b \rangle$ be an arbitrary but fixed point. Let $\bar{z}$ be any fixed point such that $[x_0,\bar{z}] \in \bar{S}_{xz}$, and let us set $\bar{y} = g(x_0,\bar{z})$. As $[x_0,\bar{y},\bar{z}] \in \bar{S}$ (by virtue of the fact that $\bar{S}$ is regular with respect to (x,z)) we have (by virtue of the fact that $\bar{S}$ is regular with respect to (x,y)) $[x_0,\bar{y}] \in \bar{S}_{xy}$, and simultaneously $\bar{z} = f(x_0,\bar{y})$. Thus $z = f(x_0,y)$ and $y = g(x_0,z)$ are mutually inverse functions in one variable. This implies

$$z = f(x_0, g(x_0,z)) \quad \text{and} \quad y = g(x_0, f(x_0,y)).$$

As $x_0 \in \langle a,b \rangle$ is an arbitrary point these relations imply (1.4) and (1.5). The proof of b) and c) is similar. □

1.7. Theorem. *At every interior point $[x,y,z] \in S$ of a regular part $\bar{S}$ there exists a tangent plane to $\bar{S}$ (and thus also a normal to $\bar{S}$).*

Proof. Let the part $\bar{S}$ be regular with respect to (x,y). The assertion of Theorem 1.7 follows then from the following facts (see, for example, [Fil, pp. 433–442]):

I. The plane $z = f(x,y)$ has at the point $M_0[x_0,y_0,z_0]$, where $z_0 = f(x_0,y_0)$, a tangent plane, which is not parallel to the axis z, if and only if the function $f(x,y)$ is differentiable at the point $[x_0,y_0]$.

II. The function $f(x,y)$ is differentiable at the point $[x_0,y_0]$ if the partial derivatives $f_x(x,y)$, $f_y(x,y)$ exist in some neighbourhood of the point $[x_0,y_0]$ and are continuous at this point. □

1.8. Definition (Orientation of the normal to a regular part). Let a part $\bar{S}$ be regular with respect to (ξ,η), where (ξ,η) denotes one of the planes (x,y), (x,z), (y,z). In case the part $\bar{S}$ is regular with respect to more than one coordinate plane we choose for (ξ,η) one of them. Let ζ be that of the coordinate axes x, y, z which is different from ξ a η. Let $\bar{V}$ be a domain whose boundary ∂V is a cylinder directed by the curve $\partial S_{\xi\eta}$ and with surface straight-lines which are parallel to the axis ζ. The part $\bar{S}$ divides the domain $\bar{V}$ into two parts, which will be denoted by the symbols $\bar{V}_1$ a $\bar{V}_2$ (the numbering is either arbitrary or depends on the problem with which the orientation of the normal is connected). At each point $[x, y, z] \in \bar{S}$ the unit normal $\mathbf{n}(x, y, z)$ to $\bar{S}$ will be oriented from $\overline{V}_1$ into $\overline{V}_2$.

If $\bar{S}$ is a part of the boundary $\partial\Omega$ of a domain $\bar{\Omega}$ (which has no cuts) then the definition of the orientation of the normal is simpler: At all points $[x, y, z] \in \bar{S}$ the unit normal $\mathbf{n}(x, y, z)$ is directed either from $\bar{\Omega}$ (we speak about the outer normal), or into Ω (we speak about the inner normal).

The angle which is made by $\mathbf{n}(x, y, z)$ with the axis x (or y, or z) will be denoted by $\alpha(x, y, z)$ (or $\beta(x, y, z)$, or $\gamma(x, y, z)$); briefly α (or β, or γ). Thus we have

$$\mathbf{n}(x, y, z) = (\cos\alpha(x, y, z),\ \cos\beta(x, y, z),\ \cos\gamma(x, y, z)). \tag{1.10}$$

1.9. Theorem. a) *If a part $\bar{S}$ is regular with respect to (x, y) and is described by relation* (1.1) *then at each point $[x, y, z] \in S$ (where S denotes the interior of $\bar{S}$) the oriented unit normal satisfies*

$$\mathbf{n}(x, y, z) = \frac{\varepsilon_z}{\sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}}(-f_x(x, y),\ -f_y(x, y),\ 1),$$

or more briefly

$$\mathbf{n} = \frac{\varepsilon_z}{\sqrt{1 + f_x^2 + f_y^2}}(-f_x, -f_y, 1), \tag{1.11}$$

where $\varepsilon_z = 1$ if $\gamma(x, y, z) < \frac{\pi}{2}$ for all points $[x, y, z] \in S$, and $\varepsilon_z = -1$ if $\gamma(x, y, z) > \frac{\pi}{2}$ for all points $[x, y, z] \in S$. (Relation (1.11) *implies that we cannot have $\gamma(x, y, z) = \frac{\pi}{2}$ at the points $[x, y, z] \in S$ and that ε_z is the same for all interior points of the part $\bar{S}$.)*

b) *If a part $\bar{S}$ is regular with respect to (x, z) and is described by relation* (1.2) *then at each point $[x, y, z] \in S$ the oriented unit normal satisfies*

$$\mathbf{n} = \frac{\varepsilon_y}{\sqrt{1 + g_x^2 + g_z^2}}(-g_x, 1, -g_z), \tag{1.12}$$

where $\varepsilon_y = 1$ if $\beta(x, y, z) < \frac{\pi}{2}$ for all points $[x, y, z] \in S$, and $\varepsilon_y = -1$ if $\beta(x, y, z) > \frac{\pi}{2}$ for all points $[x, y, z] \in S$.

c) *If a part $\bar{S}$ is regular with respect to (y, z) and is described by relation* (1.3) *then at each point $[x, y, z] \in S$ the oriented unit normal satisfies*

$$\mathbf{n} = \frac{\varepsilon_x}{\sqrt{1 + h_y^2 + h_z^2}}(1, -h_y, -h_z), \tag{1.13}$$

where $\varepsilon_x = 1$ *if* $\alpha(x,y,z) < \frac{\pi}{2}$ *for all points* $[x,y,z] \in S$, *and* $\varepsilon_x = -1$ *if* $\alpha(x,y,z) > \frac{\pi}{2}$ *for all points* $[x,y,z] \in S$.

Proof. We prove a). Let $[x_0, y_0] \in S_{xy}$ be an arbitrary point. According to Theorem 1.7 and its proof, there exists a total differential at the point $[x_0, y_0]$

$$\mathrm{d}z = f_x(x_0, y_0)\,\mathrm{d}x + f_y(x_0, y_0)\,\mathrm{d}y. \tag{1.14}$$

It is well-known (see, for example, [Fil, p. 442]) that the equation of the tangent plane $\tau(P_0)$ to $\bar{S}$ at the point $P_0 = [x_0, y_0, z_0]$ will be obtained if the differentials $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ in (1.14) are substituted by the differences $x - x_0$, $y - y_0$ a $z - z_0$. Hence

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0. \tag{1.15}$$

The left-hand side of (1.15) can be interpreted as the scalar product of the vectors

$$(x - x_0, y - y_0, z - z_0), \tag{1.16}$$

$$(-f_x(x_0, y_0), -f_y(x_0, y_0), 1). \tag{1.17}$$

In (1.16), x, y, z are coordinates of an arbitrary point $[x,y,z] \in \tau(P_0)$. Thus the vector (1.16) is parallel to the plane $\tau(P_0)$. Relation (1.15) then implies that the vector (1.17) is perpendicular to the plane $\tau(P_0)$. As the z-coordinate of the vector (1.17) is positive this vector makes an acute angle with the z-axis. If $\varepsilon_z = 1$ in (1.11) then the orientations of the parallel vectors (1.11) and (1.17) are identical; if $\varepsilon_z = -1$ then the orientations of these vectors are opposite. Assertion a) is proved. □

2. Functions continuous on a regular part

In Section 1 we have described sets over which we shall integrate; in this section we will describe functions which will be integrated on these sets in the Riemann sense.

2.1. Definition. a) Let $\bar{S}$ be a regular part. Let $F\colon \bar{S} \to \mathbb{R}^1$ be a real function defined on $\bar{S}$, i.e., to each point $[x,y,z] \in \bar{S}$ just one real number $F(x,y,z)$ corresponds (this means that we exclude the cases $F(x,y,z) = +\infty$ and $F(x,y,z) = -\infty$). We say that the function $F\colon \bar{S} \to \mathbb{R}^1$ is *continuous at a point* $[x_0, y_0, z_0] \in \bar{S}$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all points $[x,y,z] \in \bar{S}$ satisfying the inequality

$$[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2} < \delta \tag{2.1}$$

we have

$$|F(x,y,z) - F(x_0,y_0,z_0)| < \varepsilon. \tag{2.2}$$

b) Let $\bar{S}$ be a regular part. We say that a function $F: \bar{S} \to \mathbb{R}^1$ is *continuous on* $\bar{S}$ if it is continuous at each point of the part $\bar{S}$.

2.2. Theorem. a) *Let a part $\bar{S}$ be regular with respect to (x,y) and let a function $F: \bar{S} \to \mathbb{R}^1$ be continuous on $\bar{S}$. Then the function $\varphi: \bar{S}_{xy} \to \mathbb{R}^1$ defined at each point $[x,y] \in \bar{S}_{xy}$ by the relation*

$$\varphi(x,y) := F(x,y,f(x,y))$$

is continuous on $\bar{S}_{xy}$.

b) *Let a part $\bar{S}$ be regular with respect to (x,z) and let a function $F: \bar{S} \to \mathbb{R}^1$ be continuous on $\bar{S}$. Then the function $\psi: \bar{S}_{xz} \to \mathbb{R}^1$ defined at each point $[x,z] \in \bar{S}_{xz}$ by the relation*

$$\psi(x,z) := F(x,g(x,z),z)$$

is continuous on $\bar{S}_{xz}$.

c) *Let a part $\bar{S}$ be regular with respect to (y,z) and let a function $F: \bar{S} \to \mathbb{R}^1$ be continuous on $\bar{S}$. Then the function $\chi: \bar{S}_{yz} \to \mathbb{R}^1$ defined at each point $[y,z] \in \bar{S}_{yz}$ by the relation*

$$\chi(y,z) := F(h(y,z),y,z)$$

is continuous on $\bar{S}_{yz}$.

Proof. a) Let $\varepsilon > 0$ be given. Let us consider an arbitrary point $[x_0,y_0] \in \bar{S}_{xy}$. This point is in a one-to-one correspondence with the point

$$[x_0,y_0,z_0] = [x_0,y_0,f(x_0,y_0)] \in \bar{S}.$$

For the points $[x,y,z] \in \bar{S}$ appearing in Definition 2.1a we have $[x,y,z] \in U$, where $U = S \cap K(\delta;[x_0,y_0,z_0])$ with

$$K(\delta;[x_0,y_0,z_0]) := \{[x,y,z]: (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 < \delta^2\}.$$

As the part $\bar{S}$ is regular with respect to (x,y), the orthogonal projection U_{xy} of the set U onto the plane (x,y) satisfies $\operatorname{meas}_2 U_{xy} > 0$ and U_{xy} is a simply connected set. Thus there exists such a $\varrho > 0$ that

$$D_{xy} := \kappa(\varrho;[x_0,y_0]) \cap S_{xy} \subset U_{xy}, \quad \operatorname{meas}_2 D_{xy} > 0,$$

where $\kappa(\varrho; [x_0, y_0]) = \{[x, y] : (x - x_0)^2 + (y - y_0)^2 < \varrho^2\}$.

For the given $\varepsilon > 0$ we have found such a $\varrho > 0$ that the points $[x, y] \in D_{xy}$, i.e. points $[x, y] \in S_{xy}$ for which

$$[(x - x_0)^2 + (y - y_0)^2]^{1/2} < \varrho,$$

satisfy

$$\begin{aligned}|\varphi(x, y) - \varphi(x_0, y_0)| &= |F(x, y, f(x, y)) - F(x_0, y_0, f(x_0, y_0))| \\ &= |F(x, y, z) - F(x_0, y_0, z_0)| < \varepsilon.\end{aligned}$$

Assertions b), c) can be proved similarly. □

2.3. Corollary. *A function $F : \bar{S} \to \mathbb{R}^1$, where $\bar{S}$ is a regular part of a surface, is bounded, if it is continuous on $\bar{S}$ (i.e., there exists such a constant $M > 0$ that $|F(x, y, z)| \leqslant M$ for all points $[x, y, z] \in \bar{S}$).*

In applications we use also the notion of a function which is bounded and piecewise continuous on a regular part of a surface. This notion is a simple and straightforward generalization of Definition 2.1.

3. A surface integral of the first kind over a strongly regular part of a surface

In this section the notion of a surface integral of the first kind will be introduced formally. The meaning of this notion will be explained in the next section.

3.1. Notation. Let a function $F : \bar{S} \to \mathbb{R}^1$ be continuous on a regular part $\bar{S}$.

a) If the part $\bar{S}$ is strongly regular with respect to (x, y) then we set

$$I^S_{xy}(F) := \iint_{\bar{S}_{xy}} F(x, y, f(x, y))\sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.1}$$

b) If the part $\bar{S}$ is strongly regular with respect to (x, z) then we set

$$I^S_{xz}(F) := \iint_{\bar{S}_{xz}} F(x, g(x, z), z)\sqrt{1 + g_x^2(x, z) + g_z^2(x, z)}\,\mathrm{d}x\,\mathrm{d}z. \tag{3.2}$$

c) If the part $\bar{S}$ is strongly regular with respect to (y, z) then we set

$$I^S_{yz}(F) := \iint_{\bar{S}_{yz}} F(h(y, z), y, z)\sqrt{1 + h_y^2(y, z) + h_z^2(y, z)}\,\mathrm{d}y\,\mathrm{d}z. \tag{3.3}$$

Remark. By the assumptions of Notation 3.1, the integrals on the right-hand sides of (3.1)–(3.3) are Riemann integrals in $\mathbb{R}^2$.

3.2. Theorem. a) *Let a part $\bar{S}$ be strongly regular with respect to (x, y) and (x, z). Then*

$$I^S_{xy}(F) = I^S_{xz}(F).$$

b) *Let a part $\bar{S}$ be strongly regular with respect to (x, y) and (y, z). Then*

$$I^S_{xy}(F) = I^S_{yz}(F).$$

c) *Let a part $\bar{S}$ be strongly regular with respect to (x, z) and (y, z). Then*

$$I^S_{xz}(F) = I^S_{yz}(F).$$

P r o o f. We shall prove assertion a). Let us define mapping $\Phi\colon \bar{S}_{xz} \to \mathbb{R}^2$ by the relations

$$x = x, \ y = g(x, z), \quad [x, z] \in \bar{S}_{xz}. \tag{3.4}$$

Let us verify that the mapping Φ satisfies the assumptions of the theorem on substitution in a two-dimensional integral.

Functions (3.4) are continuously differentiable in S_{xz}.

The relations

$$[x, y, z] = [x, y, f(x, y)] = [x, g(x, z), z] \in \bar{S}, \quad [x, y] \in \bar{S}_{xy}, \quad [x, z] \in \bar{S}_{xz}$$

imply that

$$\Phi(\bar{S}_{xz}) = \bar{S}_{xy}, \quad \Phi(S_{xz}) = S_{xy}$$

and that the mapping $\Phi\colon \bar{S}_{xz} \to \bar{S}_{xy}$ is bijective.

By virtue of (3.4), the Jacobian of the mapping $\Phi\colon \bar{S}_{xz} \to \bar{S}_{xy}$ satisfies

$$J(x, z) = g_z(x, z), \quad [x, z] \in \bar{S}_{xz}.$$

By the assumptions of Theorem 3.2a the function $J(x, z)$ is bounded on $\bar{S}_{xz}$. Now we verify that

$$J(x, z) \neq 0 \quad \forall [x, z] \in S_{xz}. \tag{3.5}$$

According to Theorem 1.9 and the assumptions of Theorem 3.2, we have at each point $[x, y, z] \in \bar{S}$

$$\mathbf{n} = \frac{\varepsilon_z}{\sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}}(-f_x(x, y), -f_y(x, y), 1)$$
$$= \frac{\varepsilon_y}{\sqrt{1 + g_x^2(x, z) + g_z^2(x, z)}}(-g_x(x, z), 1, -g_z(x, z)).$$

Comparing the z-components of the two vectors, we see that relation (3.5) is satisfied.

Further, the domains $\bar{S}_{xz}$ and $\bar{S}_{xy}$ are measurable (in both the Jordan and the Lebesgue sense).

As $\operatorname{meas}_2 \partial S_{xy} = \operatorname{meas}_2 \partial S_{xz} = 0$ and as the integrands on both sides of relation (3.6) are continuous and bounded functions on $\bar{S}_{xy}$ or $\bar{S}_{xz}$, the theorem for the transformation of two-dimensional integral gives (we use also (1.4))

$$
\begin{aligned}
(3.6) \qquad & \iint_{\bar{S}_{xy}} F(x,y,f(x,y))\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\bar{S}_{xz}} F(x,g(x,z),z)\sqrt{1+[f_x(x,g(x,z))]^2+[f_y(x,g(x,z))]^2}\,|g_z(x,z)|\,\mathrm{d}x\,\mathrm{d}z.
\end{aligned}
$$

Now we transform the integrand on the right-hand side of (3.6). Differentiating relation (1.4) with respect to x and z, we obtain

$$
(3.7) \qquad f_x(x,g(x,z)) + f_y(x,g(x,z))g_x(x,z) = 0
$$

and

$$
(3.8) \qquad f_y(x,g(x,z))g_z(x,z) = 1,
$$

respectively. Relation (3.8) yields

$$
(3.9) \qquad f_y(x,g(x,z)) = \frac{1}{g_z(x,z)}.
$$

Relations (3.7) and (3.9) imply

$$
(3.10) \qquad f_x(x,g(x,z)) = -\frac{g_x(x,z)}{g_z(x,z)}.
$$

From (3.9) and (3.10) we obtain

$$
\begin{aligned}
&\sqrt{1+[f_x(x,g(x,z))]^2+[f_y(x,g(x,z))]^2}\,|g_z(x,z)| \\
&= \sqrt{1+[g_x(x,z)]^2+[g_z(x,z)]^2}, \quad [x,z]\in \bar{S}_{xz}.
\end{aligned}
$$

Inserting this result into the right-hand side of (3.6), we obtain, according to (3.1) and (3.2), the relation $I^S_{xy}(F) = I^S_{xz}(F)$.

Assertions b), c) can be proved similarly. □

3.3. Corollary. *If a part $\bar{S}$ is strongly regular with respect to all three coordinate planes then $I^S_{xy}(F) = I^S_{xz}(F) = I^S_{yz}(F)$.*

Theorem 3.2 enables us to formulate the following definition:

3.4. Definition. a) If a part $\bar{S}$ is strongly regular with respect to (x,y) and $F\colon \bar{S} \to \mathbb{R}^1$ is a function continuous on $\bar{S}$ then the term $I_{xy}^S(F)$ is called the *surface integral (of the first kind) of the function* F over the part $\bar{S}$. We use the notation

$$\iint_{\bar{S}} F(x,y,z)\,\mathrm{d}\sigma := I_{xy}^S(F). \tag{3.11}$$

b) If a part $\bar{S}$ is strongly regular with respect to (x,z) and $F\colon \bar{S} \to \mathbb{R}^1$ is a function continuous on $\bar{S}$ then the term $I_{xz}^S(F)$ is called the *surface integral (of the first kind) of the function* F over the part $\bar{S}$. We use the notation

$$\iint_{\bar{S}} F(x,y,z)\,\mathrm{d}\sigma := I_{xz}^S(F). \tag{3.12}$$

c) If a part $\bar{S}$ is strongly regular with respect to (y,z) and $F\colon \bar{S} \to \mathbb{R}^1$ is a function continuous on $\bar{S}$ then the term $I_{yz}^S(F)$ is called the *surface integral (of the first kind) of the function* F over the part $\bar{S}$. We use the notation

$$\iint_{\bar{S}} F(x,y,z)\,\mathrm{d}\sigma := I_{yz}^S(F). \tag{3.13}$$

Remark. a) Theorem 3.2 enables us to use the same symbol on the left-hand sides of relations (3.11)–(3.13).

b) Instead of the long expression "a surface integral of the first kind" we will often use the notion "a surface integral".

4. Geometrical and analytical meaning of a surface integral

4.1. Definition (A measure of a strongly regular part of a surface). The definition contained in this subsection is standard (see, e.g., [ŠT, p. 198]): Let a part $\bar{S}$ be strongly regular with respect to (x,y). Let us embed the closed domain $\bar{S}_{xy}$ into an arbitrary square Q the sides of which are parallel to the coordinate axes x, y. Let $\mathcal{D}_n$ be a square net consisting of n^2 squares of the same size with mutually disjoint interiors and sides parallel to the axes x, y. Let the union of these closed squares cover the square Q. (The net $\mathcal{D}_n$ will be obtained by dividing the sides of the square Q into n parts of the same length and connecting the opposite points by lines parallel to the coordinate axes.) Let r_n be the number of squares belonging to $\mathcal{D}_n$ the intersection of which with $\bar{S}_{xy}$ has a positive two-dimensional measure. Let us denote these squares by the symbols $\Delta s_1^{(n)}, \Delta s_2^{(n)}, \dots, \Delta s_{r_n}^{(n)}$. In each square $\Delta s_k^{(n)}$ let us choose arbitrarily a point $[x_k^{(n)}, y_k^{(n)}]$ which lies also in S_{xy}. The corresponding point $P_k^{(n)} = [x_k^{(n)}, y_k^{(n)}, z_k^{(n)}]$, where $z_k^{(n)} = f(x_k^{(n)}, y_k^{(n)})$, lies on S. By Theorem 1.7 at the point $P_k^{(n)}$ there exists a tangent plane $\tau(P_k^{(n)})$ of the part $\bar{S}$. Let $\Delta\sigma_k^{(n)}$ be that part of $\tau(P_k^{(n)})$, the orthogonal projection of which onto the plane (x, y) is equal to $\Delta s_k^{(n)}$. Let $|\Delta\sigma_k^{(n)}|$ denote the two-dimensional measure of the parallelogram $\Delta\sigma_k^{(n)}$. If there exists a finite limit

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \left|\Delta\sigma_k^{(n)}\right| \tag{4.1}$$

and has the property that it does not depend on the choice of the points $[x_k^{(n)}, y_k^{(n)}]$, then this limit will be called the *measure of the strongly regular part* $\bar{S}$ and will be denoted by $|\bar{S}|$.

Let us note that in case the part $\bar{S}$ is strongly regular with respect to (x, z) or (y, z) the definition of $|\bar{S}|$ is analogous.

4.2. Theorem. *Let a part $\bar{S}$ be strongly regular. Then*

$$|\bar{S}| = \iint_{\bar{S}} \mathrm{d}\sigma. \tag{4.2}$$

P r o o f. Let $\bar{S}$ be strongly regular with respect to (x, y). The acute angle which is made by the nonoriented normal at the point $P_k^{(n)}$ and the z-axis, will be denoted by $\omega_k^{(n)}$. We have

$$\cos\omega_k^{(n)} = |\cos\gamma(x_k^{(n)}, y_k^{(n)}, z_k^{(n)})|,$$

where, according to (1.11),

$$|\cos\gamma(x_k^{(n)}, y_k^{(n)}, z_k^{(n)})| = \frac{1}{\sqrt{1 + [f_x(x_k^{(n)}, y_k^{(n)})]^2 + [f_y(x_k^{(n)}, y_k^{(n)})]^2}}.$$

The same acute angle is made by the tangent plane $\tau(P_k^{(n)})$ and the plane (x, y). Hence

$$|\Delta\sigma_k^{(n)}| = \frac{|\Delta s_k^{(n)}|}{\cos\omega_k^{(n)}} = \sqrt{1 + [f_x(x_k^{(n)}, y_k^{(n)})]^2 + [f_y(x_k^{(n)}, y_k^{(n)})]^2}\, |\Delta s_k^{(n)}|,$$

where $|\Delta s_k^{(n)}|$ is the measure of the square $\Delta s_k^{(n)}$. The expression

$$|S_n| = \sum_{k=1}^{r_n} \sqrt{1 + [f_x(x_k^{(n)}, y_k^{(n)})]^2 + [f_y(x_k^{(n)}, y_k^{(n)})]^2}\, |\Delta s_k^{(n)}|$$

is, according to Definition 4.1, an approximate measure of the part $\bar{S}$. On the other hand, the expression $|S_n|$ is an integral sum corresponding to the two-dimensional Riemann integral

$$\iint_{\bar{S}_{xy}} \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\,\mathrm{d}x\,\mathrm{d}y,$$

where the integrand $\sqrt{1 + f_x^2 + f_y^2}$ is an integrable function in the sense of Riemann (because the part $\bar{S}$ is strongly regular with respect to (x,y)). Thus

$$\lim_{n\to\infty} |S_n| = \iint_{\bar{S}} \mathrm{d}\sigma,$$

which was to be proved. In case the part $\bar{S}$ is strongly regular with respect to (x,z) or (y,z) the proof is similar. □

The analytical meaning of the surface integral is introduced in the following theorem.

4.3. Theorem. *Let a part $\bar{S}$ be strongly regular and let $F\colon \bar{S} \to \mathbb{R}^1$ be a continuous function on $\bar{S}$. Let $\Delta\sigma_k^{(n)}$ have an analogous meaning as in 4.1 and 4.2. Then the sequence $\{I_n(F)\}$ of integral sums*

$$I_n(F) = \sum_{k=1}^{r_n} F(x_k^{(n)}, y_k^{(n)}, z_k^{(n)})|\Delta\sigma_k^{(n)}| \tag{4.3}$$

converges to the surface integral $\iint_{\bar{S}} F(x,y,z)\,\mathrm{d}\sigma$ independently of the choice of points $P_k^{(n)} = [x_k^{(n)}, y_k^{(n)}, z_k^{(n)}]$.

Proof. We prove Theorem 4.3 in the case when $\bar{S}$ is strongly regular with respect to (x,z). Then we can write the integral sum (4.3) in the form

$$I_n(F) = \sum_{k=1}^{r_n} F(x_k^{(n)}, g(x_k^{(n)}, z_k^{(n)}), z_k^{(n)})\sqrt{1 + [g_x(x_k^{(n)}, z_k^{(n)})]^2 + [g_z(x_k^{(n)}, z_k^{(n)})]^2}|\Delta s_k^{(n)}|,$$

where the square $\Delta s_k^{(n)}$ lies in the plane (x,z). As the function $\Psi\colon \bar{S}_{xz} \to \mathbb{R}^1$, where

$$\Psi(x,z) = F(x, g(x,z), z)\sqrt{1 + [g_x(x,z)]^2 + [g_z(x,z)]^2},$$

is bounded and continuous on $\bar{S}_{xz}$, we obtain from here and from the results of Section 3 the assertion of Theorem 4.3. □

A very important application of the surface integral of the first kind appears in the formulations of the so called weak solutions of boundary value problems of mathematical physics.

For example, in the case of three-dimensional elasticity problems we meet a surface integral of the form

$$\sum_{i=1}^{3}\iint_{\bar{S}} p_i(s,y,z)u_i(x,y,z)\,\mathrm{d}\sigma, \tag{4.4}$$

where (p_1,p_2,p_3) is the pressure vector prescribed on part $\bar{S}$ and (u_1,u_2,u_3) is the displacement vector. The physical meaning of integral (4.4) is the work of the external pressure which acts on the part $\bar{S}$.

5. A surface integral over a regular part of a surface

A generalization of Sections 3 and 4 to the case of a regular part of a surface needs an improper two-dimensional Riemann integral.

5.1. Definition. a) Let a part $\bar{S}$ be regular with respect to (x,y) and let $F\colon \bar{S}\to\mathbb{R}^1$ be a function continuous on $\bar{S}$. We say that *the function F can be integrated over the part $\bar{S}$ with respect to the variables x, y* if there exists a real number $\widetilde{I}^S_{xy}(F)$ with the following property: For every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,F)>0$ such that every part $\bar{D}\subset S$, for which

$$\operatorname{meas}_2(\bar{S}_{xy}-\bar{D}_{xy})\leqslant\delta, \tag{5.1}$$

satisfies

$$\left|\iint_{\bar{D}_{xy}} F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y-\widetilde{I}^S_{xy}(F)\right|\leqslant\varepsilon. \tag{5.2}$$

b) Let a part $\bar{S}$ be regular with respect to (x,z) and let $F\colon \bar{S}\to\mathbb{R}^1$ be a function continuous on $\bar{S}$. We say that *the function F can be integrated over the part $\bar{S}$ with respect to the variables x, z* if there exists a number $\widetilde{I}^S_{xz}(F)$ with the following property: For every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,F)>0$ such that every part $\bar{D}\subset S$, for which

$$\operatorname{meas}_2(\bar{S}_{xz}-\bar{D}_{xz})\leqslant\delta, \tag{5.3}$$

satisfies

$$\left|\iint_{\bar{D}_{xz}} F(x,g(x,z),z))\sqrt{1+g_x^2(x,z)+g_z^2(x,z)}\,\mathrm{d}x\,\mathrm{d}z-\widetilde{I}^S_{xz}(F)\right|\leqslant\varepsilon. \tag{5.4}$$

c) Let a part $\bar{S}$ be regular with respect to (y,z) and let $F\colon \bar{S} \to \mathbb{R}^1$ be a function continuous on $\bar{S}$. We say that *the function F can be integrated over the part $\bar{S}$ with respect to the variables y, z* if there exists a number $\widetilde{I}^S_{yz}(F)$ with the following property: For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, F) > 0$ such that every part $\bar{D} \subset S$, for which

$$\operatorname{meas}_2(\bar{S}_{yz} - \bar{D}_{yz}) \leqslant \delta, \tag{5.5}$$

satisfies

$$\left| \iint_{\bar{D}_{yz}} F(h(y,z),y,z)\sqrt{1 + h_y^2(y,z) + h_z^2(y,z)}\,\mathrm{d}y\,\mathrm{d}z - \widetilde{I}^S_{yz}(F) \right| \leqslant \varepsilon. \tag{5.6}$$

Let us note that every part $\bar{D}$ satisfying the inclusion $\bar{D} \subset S$ is strongly regular. Hence the integrals appearing on the left-hand sides of inequalities (5.2), (5.4) and (5.6) are Riemann integrals.

5.2. Definition. Let (s,t) be one of the pairs (x,y), (x,z) and (y,z). Let $\bar{S}$ be regular with respect to (s,t). We say that *we can integrate over the part $\bar{S}$ with respect to s, t* if we can integrate an arbitrary function $F\colon \bar{S} \to R^1$, which is continuous on $\bar{S}$, over $\bar{S}$ with respect to s, t.

5.3. Theorem. a) *Let a part $\bar{S}$ be regular with respect to both (x,y) and (x,z) and let $F\colon \bar{S} \to \mathbb{R}^1$ be a function continuous on $\bar{S}$. If we can integrate the function F over $\bar{S}$ with respect to x, y (or with respect to x, z) then we can integrate the function F over $\bar{S}$ with respect to x, z (or with respect to x, y) and we have*

$$\widetilde{I}^S_{xy}(F) = \widetilde{I}^S_{xz}(F). \tag{5.7}$$

b) *Let a part $\bar{S}$ be regular with respect to both (x,y) and (y,z) and let $F\colon \bar{S} \to \mathbb{R}^1$ be a function continuous on $\bar{S}$. If we can integrate the function F over $\bar{S}$ with respect to x, y (or with respect to y, z) then we can integrate the function F over $\bar{S}$ with respect to y, z (or with respect to x, y) and we have*

$$\widetilde{I}^S_{xy}(F) = \widetilde{I}^S_{yz}(F). \tag{5.8}$$

c) *Let a part $\bar{S}$ be regular with respect to both (x,z) and (y,z) and let $F\colon \bar{S} \to \mathbb{R}^1$ be a function continuous on $\bar{S}$. If we can integrate the function F over $\bar{S}$ with respect to x, z (or with respect to y, z) then we can integrate the function F over $\bar{S}$ with respect to y, z (or with respect to x, z) and we have*

$$\widetilde{I}^S_{xz}(F) = \widetilde{I}^S_{yz}(F). \tag{5.9}$$

P r o o f. a) Let it be possible to integrate the function F over $\bar{S}$ with respect to x, y. Let $\varepsilon > 0$ be arbitrary but fixed. By Definition 5.1a there exists $\delta_1 = \delta_1(\varepsilon, F) > 0$ such that every part $\bar{D} \subset S$, for which

$$\operatorname{meas}_2(\bar{S}_{xy} - \bar{D}_{xy}) \leqslant \delta_1, \tag{5.10}$$

satisfies

$$\left| \iint_{\bar{D}_{xy}} F(x, y, f(x, y))\sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}\, \mathrm{d}x\, \mathrm{d}y - \widetilde{I}_{xy}^S(F) \right| \leqslant \frac{\varepsilon}{2}. \tag{5.11}$$

Let $\bar{D}^* \subset S$ be a fixed part satisfying (5.10), (5.11). Let us set

$$\delta_2^* := \operatorname{meas}_2(\bar{S}_{xz} - \bar{D}_{xz}^*)$$

and let us choose arbitrarily a part $\bar{K}$ which satisfies the inclusions

$$\bar{D}^* \subset \bar{K} \subset S. \tag{5.12}$$

The part $\bar{K}$ is, according to the assumptions of Theorem 5.3 and Definitions 1.1, 1.3, strongly regular with respect to both (x, y) and (x, z) and we have

$$\begin{aligned} \operatorname{meas}_2(\bar{S}_{xy} - \bar{K}_{xy}) &\leqslant \operatorname{meas}_2(\bar{S}_{xy} - \bar{D}_{xy}^*) \leqslant \delta_1, \\ \operatorname{meas}_2(\bar{S}_{xz} - \bar{K}_{xz}) &\leqslant \operatorname{meas}_2(\bar{S}_{xz} - \bar{D}_{xz}^*) = \delta_2^*. \end{aligned}$$

By Theorem 3.4a (where we substitute the symbol $\bar{S}$ by $\bar{K}$) we have

$$\begin{aligned} &\iint_{\bar{K}_{xy}} F(x, y, f(x, y))\sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}\, \mathrm{d}x\, \mathrm{d}y \\ &\quad = \iint_{\bar{K}_{xz}} F(x, g(x, z), z)\sqrt{1 + g_x^2(x, z) + g_z^2(x, z)}\, \mathrm{d}x\, \mathrm{d}z. \end{aligned}$$

Let us subtract from both sides of this relation the expression $\widetilde{I}_{xy}^S(F)$ (which does exist, according to the assumption formulated at the beginning of this proof) and let us take the absolute values. Using (5.11) and (5.12) we obtain

$$\left| \iint_{\bar{K}_{xz}} F(x, g(x, z), z)\sqrt{1 + g_x^2(x, z) + g_z^2(x, z)}\, \mathrm{d}x\, \mathrm{d}z - \widetilde{I}_{xy}^S(F) \right| \leqslant \frac{\varepsilon}{2}. \tag{5.13}$$

Let us set

$$\delta_2^{**} := \varepsilon / \Big\{ 2 \max_{[x,z] \in \bar{D}_{xz}^*} |F(x, g(x, z), z)| \sqrt{1 + g_x^2(x, z) + g_z^2(x, z)} \Big\} \tag{5.14}$$

and define

$$\delta_2 := \min(\delta_2^*, \delta_2^{**}). \tag{5.15}$$

Let $\overline{M} \subset S$ be an arbitrary part satisfying

$$\operatorname{meas}_2(\bar{S}_{xz} - \overline{M}_{xz}) \leqslant \delta_2. \tag{5.16}$$

and let us set

$$\overline{K}_{xz} := \overline{M}_{xz} \cup \overline{D}^*_{xz}, \quad \Delta_{xz} = \overline{D}^*_{xz} - \overline{M}_{xz}. \tag{5.17}$$

Relation $(5.17)_1$ means that we have set $\overline{K} := \overline{M} \cup \overline{D}^*$, which is not in contradiction with (5.12).

The part $\overline{M}$ is by assumptions of Theorem 5.3 (which include the regularity of $\bar{S}$ with respect to (x, y) a (x, z)) and Definitions 1.1, 1.3 strongly regular with respect to both (x, y) and (x, z) and it holds

$$\begin{aligned} &\left| \iint_{\overline{M}_{xz}} F(x, g(x,z), z) \sqrt{1 + g_x^2(x,z) + g_z^2(x,z)}\, \mathrm{d}x\, \mathrm{d}z - \widetilde{I}^S_{xy}(F) \right| \\ &\quad \leqslant \left| \iint_{\overline{K}_{xz}} F(x, g(x,z), z) \sqrt{1 + g_x^2(x,z) + g_z^2(x,z)}\, \mathrm{d}x\, \mathrm{d}z - \widetilde{I}^S_{xy}(F) \right| \\ &\qquad + \left| \iint_{\Delta_{xz}} F(x, g(x,z), z) \sqrt{1 + g_x^2(x,z) + g_z^2(x,z)}\, \mathrm{d}x\, \mathrm{d}z \right|, \end{aligned} \tag{5.18}$$

because, according to (5.17), the integral over $\overline{M}_{xz}$ is equal to the difference of integrals over $\overline{K}_{xz}$ and Δ_{xz}. The first expression on the right-hand side of (5.18) is, according to (5.13), less or equal to $\varepsilon/2$. As $\Delta_{xz} \subset S_{xz} - \overline{M}_{xz}$, it holds by (5.16)

$$\operatorname{meas}_2 \Delta_{xz} \leqslant \delta_2. \tag{5.19}$$

Further we have, according to $(5.17)_2$,

$$\Delta_{xz} \subset \overline{D}^*_{xz}. \tag{5.20}$$

Relations (5.14), (5.15), (5.19) and (5.20) imply

$$\begin{aligned} &\left| \iint_{\Delta_{xz}} F(x, g(x,z), z) \sqrt{1 + g_x^2(x,z) + g_z^2(x,z)}\, \mathrm{d}x\, \mathrm{d}z \right| \\ &\quad \leqslant \max_{[x,z] \in \overline{D}^*_{xz}} \left| F(x, g(x,z), z) \sqrt{1 + g_x^2(x,z) + g_z^2(x,z)} \right| \operatorname{meas}_2 \Delta_{xz} \leqslant \frac{\varepsilon}{2}. \end{aligned} \tag{5.21}$$

Combining inequality (5.18) with (5.13) and (5.21) we obtain

$$\left|\iint_{\overline{M}_{xz}} F(x,g(x,z),z)\sqrt{1+g_x^2(x,z)+g_z^2(x,z)}\,\mathrm{d}x\,\mathrm{d}z - \widetilde{I}_{xy}^S(F)\right| \leqslant \varepsilon. \tag{5.22}$$

For the given $\varepsilon > 0$ we have found $\delta_2 > 0$ which depends on ε, F and $\overline{S}$ and is such that relation (5.22) holds for all parts $\overline{M} \subset S$ satisfying (5.16). As $\varepsilon > 0$ is arbitrary, we obtain from here and from Definition 5.1b that the function F can be integrated over the part $\overline{S}$ with respect to x, z and (5.7) holds. □

5.4. Theorem. *Let a part $\overline{S}$ be strongly regular with respect to (s,t), where (s,t) is one of the pairs (x,y), (x,z), (y,z). Then we can integrate over $\overline{S}$ with respect to (s,t) (in the sense of Definition* 5.2) *and we have*

$$\widetilde{I}_{st}^S(F) = \iint_{\overline{S}} F(x,y,z)\,\mathrm{d}\sigma, \tag{5.23}$$

where $F\colon \overline{S} \to \mathbb{R}^1$ is an arbitrary function continuous on $\overline{S}$.

Proof. Theorem 5.4 will be proved in the case that $(s,t) = (x,y)$. Let us choose an arbitrary $\varepsilon > 0$ and set

$$\delta := \varepsilon \Big/ \max_{[x,y]\in\overline{S}_{xy}} \left|F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\right|.$$

For every part $\overline{D} \subset S$ satisfying (5.1) we then have (see relations (3.11) and (3.1))

$$\begin{aligned}
&\left|\iint_{\overline{D}_{xy}} F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y - \iint_{\overline{S}} F(x,y,z)\,\mathrm{d}\sigma\right| \\
&\quad= \left|\iint_{\overline{S}_{xy}-\overline{D}_{xy}} F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\quad\leqslant \max_{[x,y]\in\overline{S}_{xy}} \left|F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\right| \operatorname{meas}_2(\overline{S}_{xy}-\overline{D}_{xy}) \leqslant \varepsilon.
\end{aligned}$$

Thus, taking into account Definitions 5.1 and 5.2, we can see that relation (5.23) is satisfied. □

5.5. Definition. Let a part $\overline{S}$ be regular with respect to (s,t), where (s,t) is one of the pairs (x,y), (x,z), (y,z), and let $F\colon \overline{S} \to \mathbb{R}^1$ be a function continuous on $\overline{S}$. If the function F can be integrated over $\overline{S}$ with respect to s, t then we set

$$\iint_{\overline{S}} F(x,y,z)\,\mathrm{d}\sigma := \widetilde{I}_{st}^S(F).$$

Theorem 5.4 shows that Definition 5.5 is an extension of Definition 3.6 (i.e., the definition of the surface integral over a strongly regular part of a surface) to the case of a surface integral over a regular part of a surface, and Theorem 5.3 guarantees correctness of this definition.

In Section 6 a practical criterion is introduced enabling us to decide whether it is possible to integrate over a given part of a surface (or over a given surface)—see Theorem 6.2.

5.6. Definition. We say that *we can integrate over a part* $\bar{S}$ if $\bar{S}$ is regular at least with respect to one of the three coordinate planes (let us denote it by (s,t)) and if we can integrate over $\bar{S}$ with respect to s, t.

Theorem 4.2 can be extended only by definition:

5.7. Definition. Let us assume that we can integrate over a part $\bar{S}$. Then we set

$$|\bar{S}| := \iint_{\bar{S}} \mathrm{d}\sigma.$$

5.8. Notation. For the sake of brevity we set

$$\begin{aligned}
\sigma(x,y) &:= \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)},\\
\sigma(x,z) &:= \sqrt{1 + g_x^2(x,z) + g_z^2(x,z)},\\
\sigma(y,z) &:= \sqrt{1 + h_y^2(y,z) + h_z^2(y,z)}.
\end{aligned}$$

The same symbol σ for three different functions is introduced in order to be able to use the neutral variables s, t.

6. The surface integral over a union of Lipschitz-regular parts of a surface

6.1. Theorem. a) *Let $\bar{S}$ be a Lipschitz-regular part with respect to (x,y). Then we can integrate over $\bar{S}$ and for an arbitrary function $F\colon \bar{S} \to \mathbb{R}^1$, which is continuous on $\bar{S}$, we have*

$$\iint_{\bar{S}} F(x,y,z)\,\mathrm{d}\sigma = \iint_{S_{xy}} F(x,y,f(x,y))\sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{6.1}$$

b) *In the case that $\bar{S}$ is a Lipschitz-regular part with respect to the plane (x,z) (or (y,z)) the assertion of the theorem is analogous.*

P r o o f. By the assumptions of Theorem 6.1 the integral on the right-hand side of (6.1) is a proper Riemann integral. Let us choose an arbitrary $\varepsilon > 0$ and set

$$\delta = \varepsilon / \{\sup_{S_{xy}}(|F(x,y,f(x,y))|\sigma(x,y))\}.$$

Let $\overline{D} \subset S$ be an arbitrary part for which

$$\operatorname{meas}_2(\overline{S}_{xy} - \overline{D}_{xy}) \leqslant \delta.$$

Then we have

$$\begin{aligned}
&\left| \iint_{S_{xy}} F(x,y,f(x,y))\sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{\overline{D}_{xy}} F(x,y,f(x,y))\sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad \leqslant \sup_{S_{xy}} (|F(x,y,f(x,y))|\sigma(x,y)) \iint_{S_{xy}-\overline{D}_{xy}} \mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant \sup_{S_{xy}} (|F(x,y,f(x,y))|\sigma(x,y))\,\delta = \varepsilon.
\end{aligned}$$

From here we obtain, according to Definition 5.1,

$$\widetilde{I}^S_{xy}(F) = \iint_{S_{xy}} F(x,y,f(x,y))\sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

If $\overline{S}$ is not regular with respect to the other two coordinate planes then relation (6.1) is proved, according to Definition 5.6. If in addition the part $\overline{S}$ is regular with respect to (x,z) or (y,z) then we prove in the same way as in Theorem 5.3 that $I^S_{xz}(F) = \widetilde{I}^S_{xy}(F)$ or $\widetilde{I}^S_{yz}(F) = \widetilde{I}^S_{xy}(F)$. From here and Definition 5.5 we obtain relation (6.1). □

Now we formulate a sufficient condition for integration over a regular part $\overline{S}$.

6.2. Theorem. *Let a regular part $\overline{S}$ satisfy*

$$\overline{S} = \bigcup_{i=1}^{m} \overline{S}^i, \quad S^i \cap S^j = \emptyset \quad (i \neq j;\ i,j = 1,\ldots,m), \tag{6.2}$$

where $\overline{S}^1,\ldots,\overline{S}^m$ are Lipschitz-regular parts. Then we can integrate over $\overline{S}$ and we have

$$\iint_{\overline{S}} F(x,y,z)\,\mathrm{d}\sigma = \sum_{i=1}^{m} \iint_{\overline{S}^i} F(x,y,z)\,\mathrm{d}\sigma, \tag{6.3}$$

where $F\colon \overline{S} \to \mathbb{R}^1$ is an arbitrary function continuous on $\overline{S}$.

P r o o f. By Theorem 6.1 we can integrate over the parts $\bar{S}^i$; thus the right-hand side of (6.3) has sense. Let, for example, the part $\bar{S}$ be regular with respect to (x, y). Let us choose an arbitrary $\varepsilon > 0$. We have to prove that there exists such a $\delta > 0$ that for every part $\bar{D} \subset S$, which satisfies

$$\operatorname{meas}_2(\bar{S}_{xy} - \bar{D}_{xy}) \leqslant \delta,$$

we have

$$\left| \iint_{\bar{D}_{xy}} F(x,y,z)\,\mathrm{d}\sigma - \sum_{i=1}^{m} \iint_{\bar{S}^i} F(x,y,z)\,\mathrm{d}\sigma \right| \leqslant \varepsilon. \tag{6.4}$$

The proof of inequality (6.4) meets with a small technical difficulty: Namely, we have

$$\bar{D}_{xy} = \bigcup_{i=1}^{m} \bar{D}_{xy} \cap \bar{S}^i_{xy},$$

where the closed domains $\bar{D}_{xy} \cap \bar{S}^i_{xy}$ are not subsets of the domains S^i_{xy}. Thus the proofs introduced in Section 5 must be modified. Details are laborious but not surprising; thus we omit them. □

Now we extend the preceding results by definition:

6.3. Definition. Let a surface $\bar{S}$ and Lipschitz-regular parts $\bar{S}^1, \ldots, \bar{S}^n$ satisfy

$$\bar{S} = \bigcup_{i=1}^{n} \bar{S}^i, \; S^i \cap S^j = \emptyset \quad (i \neq j;\; i,j = 1,\ldots,n). \tag{6.5}$$

Let $F\colon \bar{S} \to \mathbb{R}^1$ be a function defined almost everywhere on $\bar{S}$ and such that it is continuous and bounded in S^i $(i = 1, \ldots, n)$ and continuously extendible from S^i on $\bar{S}^i$ $(i = 1, \ldots, n)$. Then we set

$$\iint_{\bar{S}} F(x,y,z)\,\mathrm{d}\sigma := \sum_{i=1}^{n} \iint_{\bar{S}^i} F(x,y,z)\,\mathrm{d}\sigma. \tag{6.6}$$

It has to be proved that this definition does not lead to a contradiction.

6.4. Theorem. *Let it be possible to express a surface $\bar{S}$ in the form*

$$\bar{S} = \bigcup_{i=1}^{m} \bar{D}^i, \quad D^j \cap D^k = \emptyset \quad (j \neq k;\; j,k = 1,\ldots,m)$$

and also in the form

$$\bar{S} = \bigcup_{j=1}^{n} \overline{K}^j, \quad K^i \cap K^k = \emptyset \quad (i \neq k;\ i,k = 1,\dots,n),$$

where $\overline{D}^1,\dots,\overline{D}^m$ and $\overline{K}^1,\dots,\overline{K}^n$ are Lipschitz-regular parts. Let $F\colon \bar{S} \to R^1$ be a function defined almost everywhere on $\bar{S}$ and such that it is continuous and bounded in $D^1,\dots,D^m, K^1,\dots,K^n$ and continuously extendible from D^i to $\overline{D}^i$ $(i = 1,\dots,m)$ and from K^j to $\overline{K}^j$ $(j = 1,\dots,n)$. Then

$$\sum_{i=1}^{m} \iint_{\overline{D}^i} F(x,y,z)\,\mathrm{d}\sigma = \sum_{j=1}^{n} \iint_{\overline{K}^j} F(x,y,z)\,\mathrm{d}\sigma. \tag{6.7}$$

P r o o f. As the intersection $\overline{D}^i \cap \overline{K}^j$ of Lipschitz-regular parts $\overline{D}^i$, $\overline{K}^j$ is either a Lipschitz-regular part, or the union of a finite number of mutually disjoint Lipschitz-regular parts, the evident relations

$$\overline{D}^i = \bigcup_{j=1}^{n} \overline{D}^i \cap \overline{K}^j, \quad \bigcup_{i=1}^{m} \overline{D}^i = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \overline{D}^i \cap \overline{K}^j,$$
$$\overline{K}^j = \bigcup_{i=1}^{m} \overline{K}^j \cap \overline{D}^i, \quad \bigcup_{j=1}^{n} \overline{K}^j = \bigcup_{j=1}^{n} \bigcup_{i=1}^{m} \overline{K}^j \cap \overline{D}^i$$

imply

$$\bar{S} = \bigcup_{i=1}^{m} \overline{D}^i = \bigcup_{j=1}^{n} \overline{K}^j = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \overline{D}^i \cap \overline{K}^j = \bigcup_{k=1}^{p} \overline{M}^k,$$

where $M^i \cap M^j = \emptyset$ $(i,j = 1,\dots,p;\ p \geqslant mn)$ and $\overline{M}^1,\dots,\overline{M}^p$ are Lipschitz-regular parts. From here and Theorem 6.2 we immediately obtain (6.7). □

6.5. Corollary. *Definition* 6.3 *does not lead to a contradiction.*

It is natural to expect that the surface integral is also additive. Let us prove it.

6.6. Theorem (additivity of the surface integral). *Let $\bar{S}$ be a regular part of a surface and $F_1 : \bar{S} \to \mathbb{R}^1$, $F_2 : \bar{S} \to \mathbb{R}^1$ two functions continuous on $\bar{S}$. Let the integrals on the right-hand side of (6.8) exist. Then*

$$\iint_{\bar{S}} (F_1(x,y,z) + F_2(x,y,z))\,\mathrm{d}\sigma = \iint_{\bar{S}} F_1(x,y,z)\,\mathrm{d}\sigma + \iint_{\bar{S}} F_2(x,y,z)\,\mathrm{d}\sigma. \tag{6.8}$$

P r o o f. Let the part $\bar{S}$ be regular with respect to (x, y). Let us choose an arbitrary $\varepsilon > 0$. Then, according to Definition 5.1, there exist $\delta_i > 0$ $(i = 1, 2)$ such that for an arbitrary part $\overline{D} \subset S$, which satisfies the inequality

$$\operatorname{meas}_2(\bar{S}_{xy} - \overline{D}_{xy}) \leqslant \delta_i, \quad (i = 1 \text{ or } i = 2),$$

we have

$$\left| \iint_{\overline{D}_{xy}} F_1(x, y, f(x, y))\sigma(x, y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{\bar{S}} F_1(x, y, z)\,\mathrm{d}\sigma \right| \leqslant \varepsilon,$$

or

$$\left| \iint_{\overline{D}_{xy}} F_2(x, y, f(x, y))\sigma(x, y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{\bar{S}} F_2(x, y, z)\,\mathrm{d}\sigma \right| \leqslant \varepsilon.$$

Let us choose an arbitrary part $\overline{D} \subset S$ satisfying the inequality

$$\operatorname{meas}_2(\bar{S}_{xy} - \overline{D}_{xy}) \leqslant \min(\delta_1, \delta_2).$$

Then the additivity of the Riemann two-dimensional integral and the preceding inequalities imply

$$\left| \iint_{\overline{D}_{xy}} (F_1(x, y, f(x, y)) + F_2(x, y, f(x, y)))\sigma(x, y)\,\mathrm{d}x\,\mathrm{d}y - \right.$$
$$\left. - \iint_{\bar{S}} F_1(x, y, z)\,\mathrm{d}\sigma - \iint_{\bar{S}} F_2(x, y, z)\,\mathrm{d}\sigma \right| \leqslant 2\varepsilon,$$

which proves relation (6.8). □

6.7. Convention. In what follows we will consider only surfaces satisfying the assumption of Definition 6.3 expressed by relation (6.5).

7. A PARAMETRIC REPRESENTATION OF A SURFACE INTEGRAL

A parametric representation is convenient for computation of some surface integrals.

7.1. Definition. We say that a part (or surface) $\bar{S}$ has a smooth parametric representation if every point $[x, y, z] \in \bar{S}$ can be expressed in the form

(7.1) $$x = X(u, v), \quad y = Y(u, v), \quad z = Z(u, v), \quad [u, v] \in \overline{M},$$

and the following conditions are satisfied:

a) $\overline{M}$ is a closed and bounded simply connected two-dimensional domain;

b) the functions $X\colon \overline{M} \to \mathbb{R}^1$, $Y\colon \overline{M} \to \mathbb{R}^1$, $Z\colon \overline{M} \to \mathbb{R}^1$ are continuous;

c) the partial derivatives $X_u, X_v, Y_u, Y_v, Z_u, Z_v$ exist at each point $[u,v] \in \overline{M}$ and the functions $X_u\colon \overline{M} \to \mathbb{R}^1, \dots, Z_v\colon \overline{M} \to \mathbb{R}^1$ are continuous;

d) each point $[u,v] \in \overline{M}$ is mapped onto a point $[x,y,z] \in \bar{S}$;

e) if two points $[u_1,v_1] \neq [u_2,v_2]$ satisfy

$$X(u_1,v_1) = X(u_2,v_2), \quad Y(u_1,v_1) = Y(u_2,v_2), \quad Z(u_1,v_1) = Z(u_2,v_2),$$

then both the points $[u_1,v_1]$, $[u_2,v_2]$ lie on the boundary ∂M of the domain $\overline{M}$;

f) if $\bar{S}$ is not a closed surface (i.e., a boundary of a three-dimensional domain) then the boundary ∂S of the surface $\bar{S}$ is the image of a subset of the boundary ∂M of $\overline{M}$.

7.2. Lemma. *Let a part $\bar{S}$ have a smooth parametric representation* (7.1).

a) *If $\bar{S}$ is regular with respect to (x,y), i.e., it can be expressed in the form*

$$z = f(x,y), \quad [x,y] \in \bar{S}_{xy}, \tag{7.2}$$

then

$$Z(u,v) = f(X(u,v), Y(u,v)) \quad \forall [u,v] \in \overline{M}, \tag{7.3}$$

$$\bar{S}_{xy} = \{[x,y]\colon x = X(u,v),\ y = Y(u,v),\ [u,v] \in \overline{M}\}. \tag{7.4}$$

b) *If $\bar{S}$ is regular with respect to (x,z), i.e., it can be expressed in the form*

$$y = g(x,z), \quad [x,z] \in \bar{S}_{xz}, \tag{7.5}$$

then

$$Y(u,v) = g(X(u,v), Z(u,v)) \quad \forall [u,v] \in \overline{M}, \tag{7.6}$$

$$\bar{S}_{xz} = \{[x,z]\colon x = X(u,v),\ z = Z(u,v),\ [u,v] \in \overline{M}\}. \tag{7.7}$$

c) *If $\bar{S}$ is regular with respect to (y,z), i.e., it can be expressed in the form*

$$x = h(y,z), \quad [y,z] \in \bar{S}_{yz}, \tag{7.8}$$

then

$$X(u,v) = h(Y(u,v), Z(u,v)) \quad \forall [u,v] \in \overline{M}, \tag{7.9}$$

$$\bar{S}_{yz} = \{[y,z]\colon y = Y(u,v),\ z = Z(u,v),\ [u,v] \in \overline{M}\}. \tag{7.10}$$

P r o o f. We shall prove a). According to the assumptions, the coordinates of each point $[x, y, z] \in \bar{S}$ satisfy both the relations (7.2) and (7.1) and in the case (7.1) for every point $[u, v] \in M$ there exists just one point $[x, y, z] \in S$. Hence, inserting (7.1) in (7.2), we obtain relation (7.3).

Relations (7.1) and (7.3) enable us to write all points $[x, y, z] \in \bar{S}$ in the form

$$[x, y, z] = [X(u, v), Y(u, v), f(X(u, v), Y(u, v))], \quad [u, v] \in \overline{M}.$$

From it we can see that the orthogonal projection of $\bar{S}$ onto the plane (x, y) is the set appearing on the right-hand side of (7.4). By (7.2) the orthogonal projection of $\bar{S}$ onto the plane (x, y) is $\bar{S}_{xy}$. Thus relation (7.4) is satisfied. □

7.3. N o t a t i o n. We set

$$H(u, v) := \sqrt{[A(u, v)]^2 + [B(u, v)]^2 + [C(u, v)]^2}, \tag{7.11}$$

where

$$A(u, v) := \begin{vmatrix} Y_u(u, v) & Z_u(u, v) \\ Y_v(u, v) & Z_v(u, v) \end{vmatrix}, \tag{7.12}$$

$$B(u, v) := -\begin{vmatrix} X_u(u, v) & Z_u(u, v) \\ X_v(u, v) & Z_v(u, v) \end{vmatrix}, \tag{7.13}$$

$$C(u, v) := \begin{vmatrix} X_u(u, v) & Y_u(u, v) \\ X_v(u, v) & Y_v(u, v) \end{vmatrix}. \tag{7.14}$$

7.4. Lemma. a) *Let $\bar{S}$ be regular with respect to (x, y) and let it have a smooth parametric representation* (7.1). *Then*

$$C(u, v) \neq 0 \quad \forall [u, v] \in M \tag{7.15}$$

and the cosine of the angle, which is made by the normal to $\bar{S}$ and the z-axis at an arbitrary interior point of S, satisfies

$$\cos \gamma(x, y, z) = \varepsilon_z |C(u, v)| / H(u, v), \tag{7.16}$$

where ε_z is defined in Theorem 1.9 *and $[x, y, z]$ corresponds to $[u, v] \in M$ according to* (7.1).

b) *Let $\bar{S}$ be regular with respect to (x, z) and let it have a smooth parametric representation* (7.1). *Then*

$$B(u, v) \neq 0 \quad \forall [u, v] \in M \tag{7.17}$$

and the cosine of the angle, which is made by the normal to $\overline{S}$ and the y-axis at an arbitrary interior point of S, satisfies

$$\cos\beta(x,y,z) = \varepsilon_y|B(u,v)|/H(u,v), \tag{7.18}$$

where ε_y is defined in Theorem 1.9 and $[x,y,z]$ corresponds to $[u,v] \in M$ according to (7.1).

c) *Let $\overline{S}$ be regular with respect to (y,z) and let it have a smooth parametric representation (7.1). Then*

$$A(u,v) \neq 0 \quad \forall [u,v] \in M \tag{7.19}$$

and the cosine of the angle, which is made by the normal to $\overline{S}$ and the x-axis at an arbitrary interior point of S, satisfies

$$\cos\alpha(x,y,z) = \varepsilon_x|A(u,v)|/H(u,v), \tag{7.20}$$

where ε_x is defined in Theorem 1.9 and $[x,y,z]$ corresponds to $[u,v] \in M$ according to (7.1).

Proof. By (1.11) and (7.1) we have

$$\cos\gamma(x,y,z) = \frac{\varepsilon_z}{\sqrt{1 + [f_x(X(u,v),Y(u,v))]^2 + [f_y(X(u,v),Y(u,v))]^2}}. \tag{7.21}$$

In order to obtain (7.16), we must conveniently express the term appearing under the sign of the square root on the right-hand side of (7.21). In the course of computation we also prove (7.15).

Taking into account the assumptions of assertion a) we can see that relation (7.3) holds. According to the rule for differentiation of a composite function, relation (7.3) implies

$$Z_u(u,v) = f_x(X(u,v),Y(u,v))X_u(u,v) + f_y(X(u,v),Y(u,v))Y_u(u,v),$$
$$Z_v(u,v) = f_x(X(u,v),Y(u,v))X_v(u,v) + f_y(X(u,v),Y(u,v))Y_v(u,v).$$

Inserting these relations into both (7.12) and (7.13), we obtain after an easy computation in which we use (7.14):

$$A(u,v) = -f_x(X(u,v),Y(u,v))C(u,v), \tag{7.22}$$
$$B(u,v) = -f_y(X(u,v),Y(u,v))C(u,v). \tag{7.23}$$

It will be proved in Section 8 that no point $[u,v] \in M$ satisfies $A(u,v) = B(u,v) = C(u,v) = 0$. This fact and relations (7.22), (7.23) imply (7.15): Let us choose an arbitrary point $[u_0,v_0] \in M$. If $C(u_0,v_0) \neq 0$, then we need not prove (7.15) for this point. If $A(u_0,v_0) \neq 0$ then (7.22) implies $C(u_0,v_0) \neq 0$ (because both the expressions f_x and C on the right-hand side are finite). If $B(u_0,v_0) \neq 0$ then the relation $C(u_0,v_0) \neq 0$ follows from (7.23).

As (7.15) holds, we can divide both relations (7.22) and (7.23) by $C(u,v)$. If we insert such modified relations (7.22), (7.23) in (7.21), then after a small rearrangement in which we use (7.11), relation (7.16) follows. □

7.5. Theorem. *Let a part $\bar{S}$ be Lipschitz-regular and let it have a smooth parametric representation* (7.1). *Let $F\colon \bar{S} \to \mathbb{R}^1$ be a function continuous on $\bar{S}$. Then*

$$\iint_{\bar{S}} F(x,y,z)\,\mathrm{d}\sigma = \iint_M F(X(u,v),Y(u,v),Z(u,v))H(u,v)\,\mathrm{d}u\,\mathrm{d}v, \tag{7.24}$$

where the function $H\colon M \to \mathbb{R}^1$ is given by (7.11)–(7.14).

Proof. The proof of (7.24) is a modification of the proof of Theorem 3.2; it is again based on the theorem on transformation of a two-dimensional integral. If $\bar{S}$ is Lipschitz-regular with respect to (x,y) then we have

$$\iint_{\bar{S}} F(x,y,z)\,\mathrm{d}\sigma = \iint_{S_{xy}} F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y. \tag{7.25}$$

The integral on the right-hand side of (7.25) will be transformed by means of the substitution

$$x = X(u,v), \quad y = Y(u,v), \quad [u,v] \in \overline{M}. \tag{7.26}$$

By (7.14) we can write

$$J(u,v) = C(u,v) \quad \forall [u,v] \in M;$$

hence by Lemma 7.4a

$$J(u,v) \neq 0 \quad \forall [u,v] \in M.$$

The other assumptions of the theorem on substitution in an integral can be easily verified and we have

$$\iint_{S_{xy}} F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \iint_M F(X(u,v),Y(u,v),Z(u,v))$$
$$\times\sqrt{1+f_x^2(X(u,v),Y(u,v))+f_y^2(X(u,v),Y(u,v))}\,|C(u,v)|\,\mathrm{d}u\,\mathrm{d}v.$$

By (7.22), (7.23) and (7.11) we have

$$\sqrt{1+f_x^2(X(u,v),Y(u,v))+f_y^2(X(u,v),Y(u,v))}|C(u,v)| = H(u,v)$$

and we obtain (7.24). □

7.6. Corollary. *The value of a surface integral does not depend on the form of the parametric representation of the surface.*

7.7. Theorem. *Let a surface $\bar{S}$ satisfy relation* (6.5) *and let it have a smooth parametric representation* (7.1). *Let the parts $\bar{S}^i$ have smooth parametric representations*

$$x = X(u,v), \quad y = Y(u,v), \quad z = Z(u,v), \quad [u,v] \in \overline{M}_i,$$

where the functions $x = X(u,v)$, $y = Y(u,v)$, $z = Z(u,v)$ are the same as in (7.1), *and let closed domains $\overline{M}_i$ satisfy the relation*

$$\overline{M} = \bigcup_{i=1}^{n} \overline{M}_i, \quad M_i \cap M_j = \emptyset \quad (i \neq j;\ i,j = 1,\ldots,n).$$

Then we have

$$\iint_{\bar{S}} F(x,y,z)\,\mathrm{d}\sigma = \iint_M F(X(u,v),Y(u,v),Z(u,v))H(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

Proof. According to Theorem 7.5, we have

$$\iint_{\bar{S}^i} F(x,y,z)\,\mathrm{d}\sigma = \iint_{M_i} F(X(u,v),Y(u,v),Z(u,v))H(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

Summing up these relations from $i = 1$ to $i = n$, we obtain the assertion of Theorem 7.7; we use the assumptions of the theorem and the preceding results. □

Let us consider a smooth parametric representation (7.1) of a regular part $\bar{S}$. Let us choose an arbitrary point $[x_0, y_0, z_0] \in S$. There is just one point $[u_0, v_0] \in M$ which corresponds to this point in transformation (7.1). We shall prove that the equation of the tangent plane of $\bar{S}$ at the point $[x_0, y_0, z_0]$ can be written in the form

$$A(u_0, v_0)(x - x_0) + B(u_0, v_0)(y - y_0) + C(u_0, v_0)(z - z_0) = 0, \tag{8.1}$$

where $A(u_0, v_0)$, $B(u_0, v_0)$, $C(u_0, v_0)$ are the values of the determinants $A(u, v)$, $B(u, v)$, $C(u, v)$ given by relations (7.12)–(7.14) at the point $[u_0, v_0]$.

In order to prove it let us write for the first time the equation of the tangent plane at the point $[x_0, y_0, z_0]$ in the form

$$k_1(x - x_0) + k_2(y - y_0) + k_3(z - z_0) = 0, \tag{8.2}$$

where the coefficients k_1, k_2, k_3 should be determined. Let us consider the so called v_0-curve on the part $\bar{S}$, whose parametric representation can be obtained if we set $v = v_0$ in (7.1):

$$x = X(u, v_0), \quad y = Y(u, v_0), \quad z = Z(u, v_0), \quad [u, v_0] \in \overline{M}.$$

Let us consider an arbitrary point $[x^*, y^*, z^*] \in S$ lying on the v_0-curve. Then the parametric equations of the straight line determined by this point and the point $[x_0, y_0, z_0]$ are of the form

$$\begin{aligned} x &= x_0 + \frac{X(u^*, v_0) - X(u_0, v_0)}{u^* - u_0} t, \quad t \in (-\infty, \infty), \\ y &= y_0 + \frac{Y(u^*, v_0) - Y(u_0, v_0)}{u^* - u_0} t, \quad t \in (-\infty, \infty), \\ z &= z_0 + \frac{Z(u^*, v_0) - Z(u_0, v_0)}{u^* - u_0} t, \quad t \in (-\infty, \infty). \end{aligned}$$

If $[x^*, y^*, z^*] \to [x_0, y_0, z_0]$ then $u^* \to u_0$ and we obtain from the preceding relations the parametric equations of the tangent line to the v_0-curve at the point $[x_0, y_0, z_0]$:

$$\begin{aligned} x = x_0 + X_u(u_0, v_0)t,\ & y = x_0 + Y_u(u_0, v_0)t, \\ z = x_0 + Z_u(u_0, v_0)t, & \quad t \in (-\infty, \infty). \end{aligned} \tag{8.3}$$

Similarly we obtain the parametric equations of the tangent line to the u_0-curve at the point $[x_0, y_0, z_0]$:

$$x = x_0 + X_v(u_0, v_0)t, \ y = x_0 + Y_v(u_0, v_0)t, \qquad (8.4)$$
$$z = x_0 + Z_v(u_0, v_0)t, \quad t \in (-\infty, \infty).$$

As (8.3), (8.4) are parametric equations of straight lines, none of the vectors

$$(X_u(u_0, v_0), Y_u(u_0, v_0), Z_u(u_0, v_0)), \qquad (8.5)$$

$$(X_v(u_0, v_0), Y_v(u_0, v_0), Z_v(u_0, v_0)), \qquad (8.6)$$

can have all components equal to zero. Further, as the straight lines $u = u_0$, $v = v_0$ intersect, the v_0-curve and u_0-curve intersect too; hence, the tangent lines (8.3), (8.4) are noncollinear and thus determine the tangent plane (8.2). Hence we obtain that the vectors (8.5), (8.6) are noncollinear.

We express $x - x_0$, $y - y_0$, $z - z_0$ from equations (8.3) and insert in (8.2). Assuming $[x, y, z] \neq [x_0, y_0, z_0]$, i.e., $t \neq 0$, we obtain dividing (8.2) by the parameter t:

$$k_1 X_u(u_0, v_0) + k_2 Y_u(u_0, v_0) + k_3 Z_u(u_0, v_0) = 0. \qquad (8.7)$$

From equations (8.4) and (8.2) we similarly obtain

$$k_1 X_v(u_0, v_0) + k_2 Y_v(u_0, v_0) + k_3 Z_v(u_0, v_0) = 0. \qquad (8.8)$$

We can see from (8.7) a (8.8) that the numbers k_1, k_2, k_3 can be interpreted as components of a vector different from the zero vector,

$$(k_1, k_2, k_3) \neq (0, 0, 0), \qquad (8.9)$$

which is orthogonal to noncollinear vectors (8.5), (8.6). Thus we can set

$$(k_1, k_2, k_3) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ X_u(u_0, v_0) & Y_u(u_0, v_0) & Z_u(u_0, v_0) \\ X_v(u_0, v_0) & Y_v(u_0, v_0) & Z_v(u_0, v_0) \end{vmatrix},$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the unit vectors which are parallel to the positive directions of the axes x, y and z, respectively. The determinant on the right-hand side expresses the vector product of vectors (8.5) and (8.6). From here and from (7.12)–(7.14) we obtain that

$$k_1 = A(u_0, v_0), \quad k_2 = B(u_0, v_0), \quad k_3 = C(u_0, v_0). \qquad (8.10)$$

Inserting (8.10) in (8.2), we obtain equation (8.1).

Relations (8.9) and (8.10) imply the following lemma, which we have used in the proof of Lemma 7.4:

8.1. Lemma. *At any point $[u,v] \in M$ which corresponds to the point $[x,y,z] \in S$, the relations*

$$A(u,v) = 0, \quad B(u,v) = 0, \quad C(u,v) = 0$$

cannot hold simultaneously.

9. Invariance of the surface integral with respect to a transformation between two Cartesian coordinate systems

Till now we have considered the surface integral only in one (arbitrarily chosen) Cartesian coordinate system.

9.1. Theorem (on the invariance of the surface integral). *Let (x,y,z) and (ξ,η,ζ) be two Cartesian coordinate systems related by the transformation*

$$\begin{aligned} x &= x_0 + a_1\xi + a_2\eta + a_3\zeta, \\ y &= y_0 + b_1\xi + b_2\eta + b_3\zeta, \\ z &= z_0 + c_1\xi + c_2\eta + c_3\zeta \end{aligned} \tag{9.1}$$

where $[x_0,y_0,z_0]$ is the origin of the system (ξ,η,ζ) in the system (x,y,z). Let a part $\bar{S}$ be strongly regular to both the coordinate planes (x,y) and (ξ,η). Let $\bar{S}$ be expressed in the system (x,y,z) by

$$z = f(x,y), \quad [x,y] \in \bar{S}_{xy}, \tag{9.2}$$

where $f \in C^1(\bar{S}_{xy})$, and in the system (ξ,η,ζ) by

$$\zeta = \varphi(\xi,\eta), \quad [\xi,\eta] \in \bar{S}_{\xi\eta}, \tag{9.3}$$

where $\varphi \in C^1(\bar{S}_{\xi\eta})$. If $F\colon \bar{S} \to \mathbb{R}^1$ is a function continuous on $\bar{S}$ with values $F(x,y,z)$, $[x,y,z] \in \bar{S}$ then

$$\begin{aligned} &\iint_{\bar{S}_{xy}} F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{\bar{S}_{\xi\eta}} \tilde{F}(\xi,\eta,\varphi(\xi,\eta))\sqrt{1+\varphi_\xi^2(\xi,\eta)+\varphi_\eta^2(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned} \tag{9.4}$$

where

$$\widetilde{F}(\xi,\eta,\zeta) := F(x_0 + a_1\xi + a_2\eta + a_3\zeta, \dots, z_0 + c_1\xi + c_2\eta + c_3\zeta). \tag{9.5}$$

P r o o f. A) It is well-known that the determinant of transformation (9.1) satisfies the relation

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \pm 1, \tag{9.6}$$

where we take the sign + (sign −) if both systems (x,y,z) and (ξ,η,ζ) have the same (opposite) orientation. In the case of the same orientation each entry of the matrix of transformation (9.1) is equal to its complement. In the case of the opposite orientation each entry is equal to its complement with the opposite sign.

It is also well-known that the lines (columns) of (9.6) form a system of three orthonormal vectors.

B) Let $P \in \overline{S}$ be an arbitrary but fixed point. By (9.2) we can write its coordinates in the system (x,y,z) in the form $[x,y,f(x,y)]$ and by (9.3) in the system (ξ,η,ζ) in the form $[\xi,\eta,\varphi(\xi,\eta)]$. By (9.1) these coordinates satisfy the relations

$$x = \varrho(\xi,\eta) := x_0 + a_1\xi + a_2\eta + a_3\varphi(\xi,\eta), \tag{9.7}$$

$$y = \sigma(\xi,\eta) := y_0 + b_1\xi + b_2\eta + b_3\varphi(\xi,\eta), \tag{9.8}$$

$$f(x,y) = z_0 + c_1\xi + c_2\eta + c_3\varphi(\xi,\eta). \tag{9.9}$$

First we prove that the mapping $\Phi\colon \overline{S}_{\xi\eta} \to \mathbb{R}^2$, which is given by relations (9.7) and (9.8), is injective and that $\Phi(\overline{S}_{\xi\eta}) = \overline{S}_{xy}$. As the part $\overline{S}$ is regular with respect to both (x,y) and (ξ,η), every point $P \in \overline{S}$ is the image of just one point $[x,y] \in \overline{S}_{xy}$ and of just one point $[\xi,\eta] \in \overline{S}_{\xi\eta}$ and the coordinates of these points $[x,y]$, $[\xi,\eta]$ satisfy relations (9.7), (9.8). From here both properties of the mapping Φ follow.

As the part $\overline{S}$ is strongly regular with respect to $\overline{S}_{\xi\eta}$, the functions $\varrho\colon \overline{S}_{\xi\eta} \to \mathbb{R}^1$, $\sigma\colon \overline{S}_{\xi\eta} \to \mathbb{R}^1$ are continuous and bounded on $\overline{S}$ together with their first partial derivatives. The Jacobian of transformation (9.7), (9.8) satisfies

$$\begin{aligned} J(\xi,\eta) \equiv \widetilde{C}(\xi,\eta) &:= \begin{vmatrix} \varrho_\xi(\xi,\eta) & \sigma_\xi(\xi,\eta) \\ \varrho_\eta(\xi,\eta) & \sigma_\eta(\xi,\eta) \end{vmatrix} \\ &= (a_1 + a_3\varphi_\xi(\xi,\eta))(b_2 + b_3\varphi_\eta(\xi,\eta)) - (b_1 + b_3\varphi_\xi(\xi,\eta))(a_2 + a_3\varphi_\eta(\xi,\eta)) \\ &= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} - \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \varphi_\xi(\xi,\eta) + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \varphi_\eta(\xi,\eta). \end{aligned}$$

Thus, according to part A), we can write

$$J(\xi,\eta) \equiv \widetilde{C}(\xi,\eta) = \pm c_3 \mp c_1\varphi_\xi(\xi,\eta) \mp c_2\varphi_\eta(\xi,\eta). \tag{9.10}$$

As the part $\bar{S}$ is strongly regular with respect to (ξ,η) and is expressed by (9.3), we obtain from (9.10) that the Jacobian $J(\xi,\eta)$ is bounded on $\bar{S}_{\xi\eta}$. In part C) of this proof we will show that

$$J(\xi,\eta) \neq 0 \quad \forall[\xi,\eta] \in S_{\xi\eta}. \tag{9.11}$$

Finally, the domains $\bar{S}_{xy}$ and $\bar{S}_{\xi\eta}$ are measurable and the function $F(x,y,f(x,y))$ is continuous and bounded on $\bar{S}_{xy}$. Thus all assumptions of the theorem on transformation of the two-dimensional Riemann integral are satisfied for the integral appearing on the left-hand side of relation (9.4).

For the sake of brevity let us set

$$\tau(\xi,\eta) := f(\varrho(\xi,\eta),\sigma(\xi,\eta)). \tag{9.12}$$

Then, according to (9.9),

$$\tau(\xi,\eta) = z_0 + c_1\xi + c_2\eta + c_3\varphi(\xi,\eta) \tag{9.13}$$

and by the rule of differentiation of a composite function

$$\begin{aligned}\widetilde{A}(\xi,\eta) := \begin{vmatrix} \sigma_\xi(\xi,\eta) & \tau_\xi(\xi,\eta) \\ \sigma_\eta(\xi,\eta) & \tau_\eta(\xi,\eta) \end{vmatrix} &= \begin{vmatrix} \sigma_\xi & f_x(\varrho,\sigma)\varrho_\xi + f_y(\varrho,\sigma)\sigma_\xi \\ \sigma_\eta & f_x(\varrho,\sigma)\varrho_\eta + f_y(\varrho,\sigma)\sigma_\eta \end{vmatrix} \\ &= -f_x(\varrho(\xi,\eta),\sigma(\xi,\eta)) \begin{vmatrix} \varrho_\xi(\xi,\eta) & \sigma_\xi(\xi,\eta) \\ \varrho_\eta(\xi,\eta) & \sigma_\eta(\xi,\eta) \end{vmatrix};\end{aligned}$$

similarly

$$\widetilde{B}(\xi,\eta) := -\begin{vmatrix} \varrho_\xi(\xi,\eta) & \tau_\xi(\xi,\eta) \\ \varrho_\eta(\xi,\eta) & \tau_\eta(\xi,\eta) \end{vmatrix} = -f_y(\varrho(\xi,\eta),\sigma(\xi,\eta)) \begin{vmatrix} \varrho_\xi(\xi,\eta) & \sigma_\xi(\xi,\eta) \\ \varrho_\eta(\xi,\eta) & \sigma_\eta(\xi,\eta) \end{vmatrix}.$$

Using the definition of $\widetilde{C}(\xi,\eta)$, we obtain

$$\widetilde{A}(\xi,\eta) = -f_x(\varrho(\xi,\eta),\sigma(\xi,\eta))\widetilde{C}(\xi,\eta), \tag{9.14}$$

$$\widetilde{B}(\xi,\eta) = -f_y(\varrho(\xi,\eta),\sigma(\xi,\eta))\widetilde{C}(\xi,\eta), \tag{9.15}$$

and as $J(\xi,\eta) \equiv \widetilde{C}(\xi,\eta)$ the theorem on transformation of the two-dimensional Riemann integral yields

$$
\begin{aligned}
(9.16)\quad &\int_{\bar{S}_{xy}} F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\bar{S}_{\xi\eta}} F(\varrho(\xi,\eta),\sigma(\xi,\eta),f(\varrho(\xi,\eta),\sigma(\xi,\eta)) \\
&\quad\times \sqrt{1+f_x^2(\varrho(\xi,\eta),\sigma(\xi,\eta))+f_y^2(\varrho(\xi,\eta),\sigma(\xi,\eta))}\,|\widetilde{C}(\xi,\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \iint_{\bar{S}_{\xi\eta}} F(\varrho(\xi,\eta),\sigma(\xi,\eta),\tau(\xi,\eta))\sqrt{\widetilde{A}^2(\xi,\eta)+\widetilde{B}^2(\xi,\eta)+\widetilde{C}^2(\xi,\eta)}.\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}
$$

The definition of the determinants $\widetilde{A}(\xi,\eta)$, $\widetilde{B}(\xi,\eta)$ and part A) yield (similarly as (9.10), with the use of (9.7), (9.8), (9.13))

$$(9.17)\qquad \widetilde{A}(\xi,\eta) = \pm a_3 \mp a_1\varphi_\xi(\xi,\eta) \mp a_2\varphi_\eta(\xi,\eta).$$

$$(9.18)\qquad \widetilde{B}(\xi,\eta) = \pm b_3 \mp b_1\varphi_\xi(\xi,\eta) \mp b_2\varphi_\eta(\xi,\eta).$$

Relations (9.17), (9.18) and (9.10) give (for the sake of brevity we omit the arguments ξ, η)

$$
\begin{aligned}
\widetilde{A}^2 &= a_3^2 + a_1^2\varphi_\xi^2 + a_2^2\varphi_\eta^2 - 2a_1a_3\varphi_\xi - 2a_2a_3\varphi_\eta + 2a_1a_2\varphi_\xi\varphi_\eta, \\
\widetilde{B}^2 &= b_3^2 + b_1^2\varphi_\xi^2 + b_2^2\varphi_\eta^2 - 2b_1b_3\varphi_\xi - 2b_2b_3\varphi_\eta + 2b_1b_2\varphi_\xi\varphi_\eta, \\
\widetilde{C}^2 &= c_3^2 + c_1^2\varphi_\xi^2 + c_2^2\varphi_\eta^2 - 2c_1c_3\varphi_\xi - 2c_2c_3\varphi_\eta + 2c_1c_2\varphi_\xi\varphi_\eta.
\end{aligned}
$$

Hence by the last assertion of part A) we obtain

$$\widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 = 1 + \varphi_\xi^2 + \varphi_\eta^2$$

and relation (9.16) can be rewritten to the form

$$
\begin{aligned}
(9.19)\quad &\int_{\bar{S}_{xy}} F(x,y,f(x,y))\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\bar{S}_{\xi\eta}} F(\varrho(\xi,\eta),\sigma(\xi,\eta),\tau(\xi,\eta))\sqrt{1+\varphi_\xi^2(\xi,\eta)+\varphi_\eta^2(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}
$$

By (9.5), (9.7), (9.8) and (9.13) we have

$$F(\varrho(\xi,\eta),\sigma(\xi,\eta),\tau(\xi,\eta)) = \widetilde{F}(\xi,\eta,\varphi(\xi,\eta))\,;$$

thus relation (9.19) is identical with relation (9.4).

C) It remains to prove relation (9.11). As

$$x = \varrho(\xi, \eta), \quad y = \sigma(\xi, \eta), \quad z = \tau(\xi, \eta), \quad [\xi, \eta] \in \bar{S}_{\xi\eta}$$

is a smooth parametric representation of the part $\bar{S}$, it follows from Lemma 8.1 that at any point $[\xi, \eta] \in S_{\xi\eta}$ the relations

$$\widetilde{A}(\xi, \eta) = 0, \quad \widetilde{B}(\xi, \eta) = 0, \quad \widetilde{C}(\xi, \eta) = 0$$

cannot hold simultaneously. Using this result and (9.14), (9.15), we can prove in the same way as in Lemma 7.4 that

$$J(\xi, \eta) \equiv \widetilde{C}(\xi, \eta) \neq 0 \quad \forall [\xi, \eta] \in S_{\xi\eta}.$$

Theorem 9.1 is proved. □

9.2. Remark. Using symbolism analogous to that of Section 3, we can write relation (9.4) in the form

$$I^S_{xy}(F) = I^S_{\xi\eta}(\widetilde{F}). \tag{9.20}$$

Relation (9.20) is one of the nine possibilities, which can be formally expressed by the relation

$$I^S_{uv}(F) = I^S_{\sigma\tau}(\widetilde{F}), \tag{9.21}$$

where (u, v) is one of the pairs (x, y), (x, z), (y, z) and (σ, τ) one of the pairs (ξ, η), (ξ, ζ), (η, ζ).

9.3. Theorem. *Let $\langle O, x, y, z\rangle$, $\langle O', \xi, \eta, \zeta\rangle$ be two arbitrary Cartesian coordinate systems. Let the part $\bar{S}$ be strongly regular in both systems and let F, $\widetilde{F}$ be functions from Theorem 9.1. Then*

$$\iint_S F(x, y, z)\, \mathrm{d}\sigma = \iint_S \widetilde{F}(\xi, \eta, \zeta)\, \mathrm{d}\widetilde{\sigma} \tag{9.22}$$

where the meaning of the symbols is obvious.

Proof. Relation (9.22) is an immediate consequence of Theorem 3.2, Definition 3.4 and relations (9.21). □

9.4. Remark. It is not difficult to see that Theorem 9.3 holds also in the case when the part $\bar{S}$ is only regular in both systems provided it is possible to integrate over $\bar{S}$ at least in one of the systems. (Then we can integrate over $\bar{S}$ also in the other system and relation (9.22) holds.)

9.5. Remark. On the contrary, the invariance of the three-dimensional integral over a domain Ω with respect to transformation (9.1) is an immediate consequence of the theorem on transformation of an integral:

$$\iiint_\Omega F(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_\Omega \widetilde{F}(\xi,\eta,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{9.23}$$

On the right-hand side of (9.23) one usually writes another symbol; for example, if we write transformation (9.1) in the form $(x,y,z)^T = \mathcal{A}(\xi,\eta,\zeta)^T$ then it is more convenient to write

$$\iiint_\Omega F(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\mathcal{A}^{-1}(\Omega)} \widetilde{F}(\xi,\eta,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{9.24}$$

In both cases we integrate over the same domain but we see it in two different ways.

In the case of a surface integral the situation is different: Both integrals in relation (9.22) are symbols the precise meaning of which is given in Definition 3.4 and by Theorem 3.2.

10. Trace theorems

10.1. Definition. a) We say that a domain Ω has a piecewise smooth boundary $\partial\Omega$ if $\partial\Omega$ satisfies the assumptions of Definition 6.3 concerning $\bar{S} = \partial\Omega$ and if $\partial\Omega$ has an outer normal at almost all points.[2]

b) We say that a domain Ω has an S-continuous boundary if Ω has a continuous boundary in the sense of Nečas (see [Ne, pp. 14–15]) and if Ω has a piecewise smooth boundary.[3]

c) We say that a domain Ω has an S-Lipschitz continuous boundary if Ω has a Lipschitz continuous boundary $\partial\Omega$ in the sense of Nečas and Hlaváček (see [NH, p. 17]) and if Ω has a piecewise smooth boundary $\partial\Omega$.[4]

Restricting our considerations to S-Lipschitz continuous boundaries we shall be able to prove both the trace theorems and the Gauss-Ostrogradskij theorem without use of the partition of unity.

In the case of Sobolev spaces we will use the same notation as in [KJF]. In what follows we will need these three well-known theorems:

[2] This assumption guarantees that the domain Ω has no "cuts".
[3] There are domains which have a continuous boundary but do not have a piecewise smooth boundary.
[4] There are domains which have a Lipschitz continuous boundary but do not have a piecewise smooth boundary.

10.2. Lemma (density theorem). *Let a domain Ω have a continuous boundary in the sense of Nečas. Then the set $C^\infty(\bar{\Omega})$ is dense in the Sobolev space $H^{k,p}(\Omega)$ for every $p \in \langle 1, \infty)$ and $k \in \mathbb{N}$. Moreover, $H^{k,p}(\Omega) \leftrightarrows W^{k,p}(\Omega)$.*

For the proof see [KJF, pp. 271–272].

10.3. Lemma (extension theorem). *Let a domain $\Omega \subset \mathbb{R}^N$ have a Lipschitz continuous boundary. Then there exists a bounded linear operator $E\colon H^1(\Omega) \to H^1(\mathbb{R}^N)$ such that*

$$E(u)\big|_\Omega = u \quad \text{on } \Omega \quad \forall u \in H^1(\Omega).$$

For the proof see [Ne, p. 80].

10.4. Lemma (imbedding theorem). *Let a domain $\Omega \subset \mathbb{R}^N$ have a Lipschitz continuous boundary, let $N \geqslant 2$, $p \in \langle 1, N)$. Put $q^* = Np/(N-p)$. Then*

$$H^{1,p}(\Omega) \subset L_q(\Omega) \quad \text{algebraically and topologically}$$

provided $q \in \langle 1, q^* \rangle$ (i.e., $1 \geqslant \frac{1}{q} \geqslant \frac{1}{p} - \frac{1}{N}$).

For the proof see [KJF, pp. 282–286].

10.5. Definition. Let a surface $\bar{S}$ satisfy (6.5) with Lipschitz-regular parts $\bar{S}^1, \ldots, \bar{S}^n$. Let

$$w = \omega_i(s,t), \quad [s,t] \in \bar{S}^i_{st}$$

be the explicit expression of the part $\bar{S}^i$, where s, t, w are the so called neutral variables. Let the corresponding function from Notation 5.8 be denoted by $\sigma_i(s,t)$, i.e.,

$$\sigma_i(s,t) = \sqrt{1 + \left[\frac{\partial \omega_i}{\partial s}(s,t)\right]^2 + \left[\frac{\partial \omega_i}{\partial t}(s,t)\right]^2}.$$

Let $u\colon \bar{S} \to \mathbb{R}^1$ be such a function that $u(s,t,\omega_i(s,t)) \in L_p(S^i_{st})$ where $1 \leqslant p < \infty$. Then we say that $u \in L_p(S)$ and define the norm

$$\|u\|_{L_p(S)} := \left(\sum_{i=1}^n \iint_{S^i_{st}} |u(s,t,\omega_i(s,t))|^p \sigma_i(s,t) \,\mathrm{d}s\,\mathrm{d}t \right)^{1/p}. \tag{10.1}$$

It is easy to see that the expression $\|u\|_{L_p(S)}$ defined by (10.1) satisfies all three axioms of a norm. Moreover, according to the results of Section 9, expression (10.1) does not depend on the choice of a Cartesian coordinate system.

10.6. Theorem. *Let a surface $\bar{S}$ satisfy relation* (6.5) *with Lipschitz-regular parts $\bar{S}^1, \ldots, \bar{S}^n$. Let $u \in L_p(S)$, where $1 \leqslant p < \infty$. Then the expression*

$$I_{p,S}(u) = \Big(\sum_{i=1}^{n} \iint_{S^i_{st}} |u(s,t,\omega_i(s,t)|^p \,\mathrm{d}s\,\mathrm{d}t \Big)^{1/p} \tag{10.2}$$

defines a norm which is equivalent to the norm $\|u\|_{L_p(S)}$, i.e., there exist two positive constants C_1, C_2 such that

$$C_1 I_{p,S}(u) \leqslant \|u\|_{L_p(S)} \leqslant C_2 I_{p,S}(u) \quad \forall u \in L_p(S). \tag{10.3}$$

Proof. The assumptions of Theorem 10.7 imply

$$1 \leqslant \sigma_i(s,t) \leqslant K_i \quad (i = 1, \ldots, n),$$

where K_i are constants. These inequalities immediately imply

$$[I_{p,S}(u)]^p \leqslant \|u\|^p_{L_p(S)} \leqslant \max_{i=1,\ldots,n} K_i [I_{p,S}(u)]^p \quad \forall u \in L_p(S). \tag{10.4}$$

Relations (10.4) yield inequalities (10.3) with $C_1 = 1$ and $C_2 = \Big(\max_{i=1,\ldots,n} K_i\Big)^{1/p}$. □

10.7. Corollary. *Let a surface $\bar{S}$ satisfy relation* (6.5) *with Lipschitz-regular parts $\bar{S}^1, \ldots, \bar{S}^n$. Then the normed space $L_p(S)$, where $1 \leqslant p < \infty$, is a Banach space.*

Proof. According to Definition 10.5, the set of elements belonging to $L_p(S)$ equipped with the norm $I_{p,S}(u)$ is a Banach space. This fact and Theorem 10.6 yield the assertion. □

Now we can start our considerations concerning the trace theorems. First we prove a generalization of [KJF, Theorem 6.4.1]:

10.8. Theorem (trace theorem). *Let a domain $\Omega \subset \mathbb{R}^N$ have an S-Lipschitz continuous boundary. Let $1 \leqslant p < N = 3$, $q = (Np-p)/(N-p) = 2p/(3-p)$. Then there exists a uniquely determined continuous linear mapping $\gamma\colon H^{1,p}(\Omega) \to L_q(\partial\Omega)$ such that $\gamma u = u\big|_{\partial\Omega}$ for all $u \in C^\infty(\bar{\Omega})$.*

Proof. A) First we prove that for any function $u \in C^\infty(\bar{G})$ we have

$$\|u\|_{L_q(\partial\Omega)} \leqslant C \|u\|_{H^{1,p}(G)}, \tag{10.5}$$

where G is a simply connected bounded domain such that

$$\bar{\Omega} \subset G \tag{10.6}$$

with

$$2\beta := \operatorname{dist}(\partial\Omega, \partial G) > 0. \tag{10.7}$$

Let $u \in C^\infty(\bar{G})$ and let

$$v(s,t,\omega_i(s,t)) = |u(s,t,\omega_i(s,t))|^{(Np-p)/(N-p)}. \tag{10.8}$$

Then

$$v(s,t,\omega_i(s,t)) = -\int_{\omega_i(s,t)}^{\tau} \frac{\partial v}{\partial \xi}(s,t,\xi)\,\mathrm{d}\xi + v(s,t,\tau), \tag{10.9}$$

where

$$\omega_i(s,t) \leqslant \tau \leqslant \omega_i(s,t) + \beta. \tag{10.10}$$

Inserting (10.8) into (10.9) and taking into account (10.10), we obtain, after calculating the derivative,

$$\begin{aligned} |u(s,t,\omega_i(s,t))|^{(Np-p)/(N-p)} &\leqslant |u(s,t,\tau)|^{(Np-p)/(N-p)} \\ &+ \frac{Np-p}{N-p} \int_{\omega_i(s,t)}^{\omega_i(s,t)+\beta} |u(s,t,\xi)|^{(Np-N)/(N-p)} \left|\frac{\partial u}{\partial \xi}(s,t,\xi)\right| \mathrm{d}\xi. \end{aligned} \tag{10.11}$$

Let us integrate (10.11) with respect to τ over the interval $\langle \omega_i(s,t), \omega_i(s,t)+\beta\rangle$ and let us use the definition of q. Then we find that

$$\begin{aligned} \beta|u(s,t,\omega_i(s,t))|^q \leqslant &\int_{\omega_i(s,t)}^{\omega_i(s,t)+\beta} |u(s,t,\tau)|^q\,\mathrm{d}\tau \\ &+ q\beta \int_{\omega_i(s,t)}^{\omega_i(s,t)+\beta} |u(s,t,\xi)|^{(Np-N)/(N-p)} \left|\frac{\partial u}{\partial \xi}(s,t,\xi)\right| \mathrm{d}\xi. \end{aligned}$$

Let us multiply this inequality by the inequality

$$\sigma_i(s,t) \leqslant K_i$$

and integrate the result over S^i_{st}. We obtain

$$\beta\|u\|^q_{L_q(S^i)} \leqslant K_i \iiint_{V_i} |u(s,t,\tau)|^q \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\tau$$
$$(10.12) \qquad + q\beta K_i \iiint_{V_i} |u(s,t,\xi)|^{(Np-N)/(N-p)} \left|\frac{\partial u}{\partial \xi}(s,t,\xi)\right| \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\xi,$$

where

$$(10.13) \qquad \overline{V}_i \subset G.$$

Let us write shortly $\mathrm{d}V = \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\xi$ and let us use the Hölder inequality for the second integral on the right-hand side of (10.12):

$$(10.14) \quad \iiint_{V_i} |u(s,t,\xi)|^{(Np-N)/(N-p)} \left|\frac{\partial u}{\partial \xi}(s,t,\xi)\right| \mathrm{d}V$$
$$\leqslant \left(\iiint_{V_i} \left|\frac{\partial u}{\partial \xi}(s,t,\xi)\right|^p \mathrm{d}V\right)^{1/p} \left(\iiint_{V_i} |u(s,t,\xi)|^{\frac{Np-N}{N-p}\cdot\frac{p}{p-1}} \mathrm{d}V\right)^{\frac{p-1}{p}}.$$

Denoting the right-hand side of (10.14) shortly by RHS and using the notation $q^* = Np/(N-p)$ introduced in Lemma 10.4, we obtain by (10.13) and then by Lemma 10.4

$$(10.15) \qquad RHS \leqslant \|u\|_{H^{1,p}(G)} \|u\|^{\frac{Np-N}{N-p}}_{L_{q^*}(G)} \leqslant C\|u\|^q_{H^{1,p}(G)},$$

because $1 + \frac{Np-N}{N-p} = \frac{Np-p}{N-p} = q$. Relations (10.12)–(10.15) yield

$$(10.16) \qquad \|u\|_{L_q(\partial\Omega)} \leqslant C_1\|u\|_{L_q(G)} + C_2\|u\|_{H^{1,p}(G)}.$$

As $q = \frac{Np-p}{N-p} < \frac{Np}{N-p}$ we can use Lemma 10.4 to obtain from (10.16) inequality (10.5).

B) Let us choose $v \in H^{1,p}(\Omega)$ arbitrarily. Then, according to Lemma 10.3, there exists $\widetilde{v} \in H^{1,p}(G)$ such that $\widetilde{v} = v$ on Ω. By Lemma 10.2 there exists a sequence $\{u_n\} \subset C^\infty(\overline{G})$ such that

$$(10.17) \qquad u_n \to \widetilde{v} \quad \text{in } H^{1,p}(G).$$

By (10.5)

$$(10.18) \qquad \|\gamma u_n\|_{L_q(S)} \leqslant C\|u_n\|_{H^{1,p}(G)}$$

and by (10.5) and (10.17)

$$\|\gamma u_n - \gamma u_m\|_{L_q(S)} \leqslant C\|u_n - u_m\|_{H^{1,p}(G)} \to 0 \quad \text{as } m, n \to \infty. \tag{10.19}$$

By (10.19), $\{\gamma u_n\}$ is a Cauchy sequence in $L_q(S)$ and as $L_q(S)$ is a Banach space (see Corollary 10.7), there exists a $\gamma v \in L_q(S)$ such that

$$\gamma u_n \to \gamma v \quad \text{in } L_q(S). \tag{10.20}$$

Using (10.17) and (10.20), we obtain from (10.18):

$$\|\gamma v\|_{L_q(S)} \leqslant C\|\widetilde{v}\|_{H^{1,p}(G)}. \tag{10.21}$$

Inequality (10.21) and Lemma 10.3 imply

$$\|\gamma v\|_{L_q(S)} \leqslant C\|v\|_{H^{1,p}(\Omega)} \quad \forall v \in H^{1,p}(\Omega), \tag{10.22}$$

which was to be proved. □

10.9. Theorem (trace theorem). *Let a domain $\Omega \subset \mathbb{R}^N$ $(N = 3)$ have an S-Lipschitz continuous boundary.*

a) *Let $p \geqslant N$. Then for any $q \geqslant 1$ there exists a unique continuous linear mapping $\gamma\colon H^{1,p}(\Omega) \to L_q(\partial\Omega)$ such that $\gamma u = u|_{\partial\Omega}$ for all $u \in C^\infty(\overline{\Omega})$.*

b) *Let $p = 2$. Then for $q = 2$ there exists a unique continuous linear mapping $\gamma\colon H^{1,2}(\Omega) \to L_2(\partial\Omega)$ such that $\gamma u = u|_{\partial\Omega}$ for all $u \in C^\infty(\overline{\Omega})$.*

Proof. a) Because of its shortness we reproduce the proof of [KJF, Theorem 6.4.2] and correct simultaneously a misprint appearing in this proof.

Let $q \geqslant 1$ be an arbitrary fixed number. Since the function $\nu(t) = 2t/(3-t)$ increases from 1 to ∞ on the interval $\langle 1, 3)$, there exists a $\bar{p} \in \langle 1, 3)$ such that $q = \nu(\bar{p})$. According to Theorem 10.9, there exists a uniquely defined linear mapping $\mathcal{M}\colon H^{1,\bar{p}}(\Omega) \to L_q(\partial\Omega)$, $\mathcal{M}u = u|_{\partial\Omega}$ if $u \in C^\infty(\overline{\Omega})$. Composing it with the identity mapping I from $H^{1,p}(\Omega)$ into $H^{1,\bar{p}}(\Omega)$, we can see that $\gamma = \mathcal{M} \circ I$.

b) We have $\nu(\frac{3}{2}) = 2$, i.e., $\bar{p} = \frac{3}{2}$ if $q = 2$. As $H^{1,2}(\Omega) \subset H^{1,\frac{3}{2}}(\Omega)$, we can set $p = 2$ in this case and the considerations of part a) remain without changes.

Another proof: As $L_4(\partial\Omega) \subset L_2(\partial\Omega)$ (algebraically and topologically), assertion b) is a special case of Theorem 10.9 for $p = 2$ in the case $N = 3$. □

Our approach enables us to prove the following trace theorem which does not follow from the theory developed in [Ne] and [KJF]. (For example, the theory from [Ne] and [KJF] does not allow us to consider domains with cusp points.)

10.10. Theorem (trace theorem). *Let $\bar{\Omega} \subset G$, where $\bar{\Omega} \subset \mathbb{R}^3$ is a domain with a piecewise smooth boundary and $G \subset \mathbb{R}^3$ a domain with a continuous boundary in the sense of Nečas. Let us consider functions $v \in H^{1,p}(\Omega)$ as restrictions of functions $v \in H^{1,p}(G)$ to Ω and functions $v \in C^\infty(\bar{\Omega})$ as restrictions of functions $v \in C^\infty(\overline{G})$.*

a) *Let $1 \leqslant p < 3$, $q = 2p/(3-p)$. Then there exists a uniquely determined continuous linear mapping $\gamma\colon H^{1,p}(\Omega) \to L_q(\partial\Omega)$ such that $\gamma u = u\big|_{\partial\Omega}$ for all $u \in C^\infty(\bar{\Omega})$.*

b) *Let $p \geqslant 3$. Then for any $q \geqslant 1$ there exists a unique continuous linear mapping $\gamma\colon H^{1,p}(\Omega) \to L_q(\partial\Omega)$ such that $\gamma u = u\big|_{\partial\Omega}$ for all $u \in C^\infty(\overline{\Omega})$.*

c) *Let $p = 2$. Then for $q = 2$ there exists a unique continuous linear mapping $\gamma\colon H^{1,2}(\Omega) \to L_2(\partial\Omega)$ such that $\gamma u = u\big|_{\partial\Omega}$ for all $u \in C^\infty(\overline{\Omega})$.*

The proof is almost identical with the proofs of Theorems 10.8 and 10.9 and thus we omit it.

11. Flow of a vector through a part of a surface. Surface integral of the second kind

11.1. In physics we often speak about the flow of a vector of magnetic induction through an oriented surface, about the flow of a vector of the intensity of an electric field through an oriented surface, about the flow of a vector of velocity of a liquid through an oriented surface, etc. All these considerations have a common base: We are given a vector field

$$\mathbf{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z)), \tag{11.1}$$

where P, Q, R are three continuous functions (components of a vector field in the directions of the axes x, y, z), which are defined in a three-dimensional domain Ω, and an oriented regular part $\bar{S}$ which lies in Ω, $\bar{S} \subset \Omega$. (The orientation of $\bar{S}$ is given by prescribing a unit normal vector $\mathbf{n}(x,y,z)$, $[x,y,z] \in \bar{S}$—see (1.10).) We are interested in whether we can integrate the function

$$\mathbf{F} \cdot \mathbf{n} \equiv P\cos\alpha + Q\cos\beta + R\cos\gamma$$

over the part $\bar{S}$, i.e., whether the surface integral

$$\iint_{\bar{S}} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma \tag{11.2}$$

exists. This integral is called the flow of the vector $\mathbf{F} = (P,Q,R)$ through the oriented part $\bar{S}$.

Let us explain the physical meaning of the integral (11.2). Let (P, Q, R) be, for example, the velocity field of a flowing liquid. If this field is constant (i.e., the components P, Q, R are constants) and the part $\bar{S}$ is a part of a plane, then during a unit of time the volume

$$(P\cos\alpha + Q\cos\beta + R\cos\gamma)|S|. \tag{11.3}$$

will pass through the part $\bar{S}$. This volume has a positive (negative) value, if the liquid flows out from the upper (lower) side of the part $\bar{S}$. In the case of a variable field (P, Q, R) and a curved part $\bar{S}$ we consider the expression

$$\sum_{k=1}^{r_n}\big[P(x_k^{(n)}, y_k^{(n)}, z_k^{(n)})\cos\alpha(x_k^{(n)}, y_k^{(n)}, z_k^{(n)}) + Q(x_k^{(n)}, y_k^{(n)}, z_k^{(n)})\cos\beta(x_k^{(n)}, y_k^{(n)}, z_k^{(n)})$$
$$+ R(x_k^{(n)}, y_k^{(n)}, z_k^{(n)})\cos\gamma(x_k^{(n)}, y_k^{(n)}, z_k^{(n)})\big] \cdot \left|\Delta\sigma_k^{(n)}\right|$$

as an approximate value of the volume flowing out during a unit of time (an analogue to (4.3)), which gives, according to Theorem 4.3, the integral (11.2) (for the time being, under the assumption that the part $\bar{S}$ is strongly regular—this restrictive assumption will be removed in this section).

11.2. Theorem. *Let functions $P\colon \bar{S} \to \mathbb{R}^1$, $Q\colon \bar{S} \to \mathbb{R}^1$, $R\colon \bar{S} \to \mathbb{R}^1$ be continuous on $\bar{S}$, where $\bar{S}$ is a strongly regular part of a surface.*

a) *If the part $\bar{S}$ is strongly regular with respect to (x, y), then*

$$\begin{aligned}\iint_{\bar{S}} &(P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma \\ &= \varepsilon_z \iint_{\bar{S}_{xy}} \{-P(x, y, f(x, y))f_x(x, y) \\ &\quad - Q(x, y, f(x, y))f_y(x, y) + R(x, y, f(x, y))\}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{11.4}$$

b) *If the part $\bar{S}$ is strongly regular with respect to (x, z), then*

$$\begin{aligned}\iint_{\bar{S}} &(P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma \\ &= \varepsilon_y \iint_{\bar{S}_{xz}} \{-P(x, g(x, z), z)g_x(x, z) \\ &\quad + Q(x, g(x, z), z) - R(x, g(x, z), z)g_z(x, z)\}\,\mathrm{d}x\,\mathrm{d}z.\end{aligned} \tag{11.5}$$

c) *If the part* $\bar{S}$ *is strongly regular with respect to* (y, z)*, then*

$$\begin{aligned}(11.6)\qquad &\iint_{\bar{S}} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma \\ &= \varepsilon_x \iint_{\bar{S}_{yz}} \{P(h(y,z), y, z) \\ &\quad - Q(h(y,z)y, z)h_y(y,z) - R(h(y,z), y, z)h_z(y,z)\}\,\mathrm{d}y\,\mathrm{d}z.\end{aligned}$$

P r o o f. All three assertions follow immediately from Definition 3.4, Notation 3.1 and Theorem 1.9. □

Assumptions of Theorem 11.2 are too strong and expressions on the right-hand sides of relations (11.4)–(11.6) complicated. Both these drawbacks will be removed in Theorem 11.4.

11.3. Lemma. a) *Let a part* $\bar{S}$ *be regular with respect to* (y, z) *and let* $P\colon \bar{S} \to \mathbb{R}^1$ *be a continuous function. Then we can integrate the function* $P\cos\alpha\colon \bar{S} \to \mathbb{R}^1$ *over* $\bar{S}$ *with respect to* y, z *and we have*

$$(11.7)\qquad \iint_{\bar{S}} P(x,y,z)\cos\alpha(x,y,z)\,\mathrm{d}\sigma = \varepsilon_x \iint_{\bar{S}_{yz}} P(h(y,z), y, z)\,\mathrm{d}y\,\mathrm{d}z.$$

b) *Let a part* $\bar{S}$ *be regular with respect to* (x, z) *and let* $Q\colon \bar{S} \to \mathbb{R}^1$ *be a continuous function. Then we can integrate the function* $Q\cos\beta\colon \bar{S} \to \mathbb{R}^1$ *over* $\bar{S}$ *with respect to* x, z *and we have*

$$(11.8)\qquad \iint_{\bar{S}} Q(x,y,z)\cos\beta(x,y,z)\,\mathrm{d}\sigma = \varepsilon_y \iint_{\bar{S}_{xz}} Q(x, g(x,z), z)\,\mathrm{d}x\,\mathrm{d}z.$$

c) *Let a part* $\bar{S}$ *be regular with respect to* (x, y) *and let* $R\colon \bar{S} \to \mathbb{R}^1$ *be a continuous function. Then we can integrate the function* $R\cos\gamma\colon \bar{S} \to \mathbb{R}^1$ *over* $\bar{S}$ *with respect to* x, y *and we have*

$$(11.9)\qquad \iint_{\bar{S}} R(x,y,z)\cos\gamma(x,y,z)\,\mathrm{d}\sigma = \varepsilon_z \iint_{\bar{S}_{xy}} R(x, y, f(x,y))\,\mathrm{d}x\,\mathrm{d}y.$$

P r o o f. a) As $P\colon S \to \mathbb{R}^1$ is a continuous function on $\bar{S}$ (and thus bounded on $\bar{S}$), the integral on the right-hand side of (11.7) exists. Let us choose $\varepsilon > 0$ arbitrarily and let us set

$$\delta = \varepsilon / \max_{\bar{S}} |P(x,y,z)| = \varepsilon / \max_{\bar{S}_{yz}} |P(h(y,z), y, z)|.$$

Then every part $\overline{D} \subset S$ for which

$$\operatorname{meas}_2(S_{yz} - D_{yz}) < \delta$$

satisfies, according to (1.13), (3.3) and (3.13), the relations

$$\begin{aligned}
&\left| \iint_{\overline{D}} P(x,y,z) \cos\alpha(x,y,z) \,\mathrm{d}\sigma - \varepsilon_x \iint_{\bar{S}_{yz}} P(h(y,z),y,z) \,\mathrm{d}y \,\mathrm{d}z \right| \\
&= \left| \iint_{\bar{S}_{yz} - \overline{D}_{yz}} P(h(y,z),y,z) \,\mathrm{d}y \,\mathrm{d}z \right| \leqslant \max_{\bar{S}_{yz}} |P(h(y,z),y,z)| \operatorname{meas}_2(\bar{S}_{yz} - \overline{D}_{yz}) \\
&< \max_{\bar{S}_{yz}} |P(h(y,z),y,z)| \delta < \varepsilon.
\end{aligned}$$

From here, according to Definitions 5.1 and 5.5, we obtain (11.7). □

11.4. Theorem. *Let a part $\bar{S}$ be regular with respect to all three coordinate planes and let $P\colon \bar{S} \to \mathbb{R}^1$, $Q\colon \bar{S} \to \mathbb{R}^1$, $R\colon \bar{S} \to \mathbb{R}^1$ be functions continuous on $\bar{S}$. Then the integral* (11.2) *exists and satisfies*

$$\begin{aligned}
(11.10)\quad &\iint_{\bar{S}} (P\cos\alpha + Q\cos\beta + R\cos\gamma) \,\mathrm{d}\sigma \\
&= \varepsilon_x \iint_{\bar{S}_{yz}} P(h(y,z),y,z) \,\mathrm{d}y \,\mathrm{d}z \\
&\quad + \varepsilon_y \iint_{\bar{S}_{xz}} Q(x,g(x,z),z) \,\mathrm{d}x \,\mathrm{d}z + \varepsilon_z \iint_{\bar{S}_{xy}} R(x,y,f(x,y)) \,\mathrm{d}x \,\mathrm{d}y.
\end{aligned}$$

P r o o f. By Lemma 11.3 the integrals

$$\iint_{\bar{S}} P\cos\alpha \,\mathrm{d}\sigma, \quad \iint_{\bar{S}} Q\cos\beta \,\mathrm{d}\sigma, \quad \iint_{\bar{S}} R\cos\gamma \,\mathrm{d}\sigma$$

exist. According to Theorem 6.6, the integral (11.2) also exists and is equal to the sum of these integrals. Relation (11.10) can be now obtained by summing up relations (17.7)–(17.9). □

R e m a r k. From the point of view of the Gauss-Ostrogradskij theorem, at which we are aimed, the assumption of Theorem 11.4 concerning the part $\bar{S}$ is not restrictive.

11.5. Definition. a) Let a part $\bar{S}$ be regular with respect to all three coordinate planes, let $\mathbf{n}(x,y,z)$ be its oriented unit normal and let P, Q, R be functions continuous on $\bar{S}$. We set

$$\iint_{\bar{S}} P(x,y,z)\,\mathrm{d}y\,\mathrm{d}z := \varepsilon_x \iint_{\bar{S}_{yz}} P(h(y,z),y,z)\,\mathrm{d}y\,\mathrm{d}z, \tag{11.11}$$

$$\iint_{\bar{S}} Q(x,y,z)\,\mathrm{d}x\,\mathrm{d}z := \varepsilon_y \iint_{\bar{S}_{xz}} Q(x,g(x,z),z)\,\mathrm{d}x\,\mathrm{d}z, \tag{11.12}$$

$$\iint_{\bar{S}} R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y := \varepsilon_z \iint_{\bar{S}_{xy}} R(x,y,f(x,y))\,\mathrm{d}x\,\mathrm{d}y. \tag{11.13}$$

Integrals (11.11)–(11.13) are called projective surface integrals or surface integrals of the second kind.

b) Under the same assumptions as in a) we denote

$$\begin{aligned} &\iint_{\bar{S}} (P(x,y,z)\,\mathrm{d}y\,\mathrm{d}z + Q(x,y,z)\,\mathrm{d}x\,\mathrm{d}z + R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y) \\ &:= \iint_{\bar{S}} P(x,y,z)\,\mathrm{d}y\,\mathrm{d}z + \iint_{\bar{S}} Q(x,y,z)\,\mathrm{d}x\,\mathrm{d}z + \iint_{\bar{S}} R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{11.14}$$

The left-hand side of (11.14) is very often written in the form

$$\iint_{\bar{S}} P(x,y,z)\,\mathrm{d}y\,\mathrm{d}z + Q(x,y,z)\,\mathrm{d}x\,\mathrm{d}z + R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y.$$

Theorem 11.4 and Definition 11.5 yield

11.6. Corollary. *Under the same assumptions as in Definition* 11.5 *we have*

$$\iint_{\bar{S}} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma = \iint_{\bar{S}} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y). \tag{11.15}$$

If the part $\bar{S}$ is regular with respect to all three coordinate planes then

$$\operatorname{meas}_2 S_{xy} > 0, \quad \operatorname{meas}_2 S_{xz} > 0, \quad \operatorname{meas}_2 S_{yz} > 0. \tag{11.16}$$

In the case that (11.16) does not hold we proceed according to the following definition.

11.7. Definition. Let $\bar{S}$ be a regular part of a surface (i.e., $\bar{S}$ is regular at least with respect to one coordinate plane), let $\mathbf{n}$ be its oriented unit normal, i.e.,

$$\mathbf{n}(x,y,z) = (\cos\alpha(x,y,z), \cos\beta(x,y,z), \cos\gamma(x,y,z)),$$

and let P, Q, R be functions continuous on $\bar{S}$.

a) If $\bar{S}$ is a part of the plane $z = z_0$ then

$$\cos\alpha(x,y,z) = \cos\beta(x,y,z) = 0 \quad \forall[x,y,z] \in S.$$

In this case we set

$$\iint_{\bar{S}} P(x,y,z)\,\mathrm{d}y\,\mathrm{d}z = 0, \quad \iint_{\bar{S}} Q(x,y,z)\,\mathrm{d}x\,\mathrm{d}z = 0.$$

b) If $\bar{S}$ is a part of the plane $y = y_0$ then

$$\cos\alpha(x,y,z) = \cos\gamma(x,y,z) = 0 \quad \forall[x,y,z] \in S.$$

In this case we set

$$\iint_{\bar{S}} P(x,y,z)\,\mathrm{d}y\,\mathrm{d}z = 0, \quad \iint_{\bar{S}} R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

c) If $\bar{S}$ is a part of the plane $x = x_0$ then

$$\cos\beta(x,y,z) = \cos\gamma(x,y,z) = 0 \quad \forall[x,y,z] \in S.$$

In this case we set

$$\iint_{\bar{S}} Q(x,y,z)\,\mathrm{d}x\,\mathrm{d}z = 0, \quad \iint_{\bar{S}} R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

d) If $\bar{S}$ is regular with respect to (x,y) and (x,z) and if $\bar{S}$ is a part of a surface formed by straight lines which are parallel to the x-axis, which means that it can be expressed in either of the forms $z = f(y)$, $[x,y] \in \bar{S}_{xy}$ and $y = g(z)$, $[x,z] \in \bar{S}_{xz}$, then

$$\cos\alpha(x,y,z) = 0 \quad \forall[x,y,z] \in S.$$

In this case we set

$$\iint_{\bar{S}} P(x,y,z)\,\mathrm{d}y\,\mathrm{d}z = 0.$$

e) If $\bar{S}$ is regular with respect to (x,y) and (y,z) and if $\bar{S}$ is a part of a surface formed by straight lines which are parallel to the y-axis, which means that it can be expressed in either of the forms $z = f(x)$, $[x,y] \in \bar{S}_{xy}$ and $x = h(z)$, $[y,z] \in \bar{S}_{yz}$, then

$$\cos\beta(x,y,z) = 0 \quad \forall[x,y,z] \in S.$$

In this case we set

$$\iint_{\bar{S}} Q(x,y,z)\,\mathrm{d}x\,\mathrm{d}z = 0.$$

f) If $\bar{S}$ is regular with respect to (x, z) and (y, z) and if $\bar{S}$ is a part of a surface formed by straight lines which are parallel to the z-axis, which means that it can be expressed in either of the forms $y = g(x)$, $[x, z] \in \bar{S}_{xz}$ and $x = h(y)$, $[y, z] \in \bar{S}_{yz}$, then

$$\cos\gamma(x, y, z) = 0 \quad \forall [x, y, z] \in S.$$

In this case we set

$$\iint_{\bar{S}} R(x, y, z)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

11.8. Definition. We say that a part $\bar{S}$ has property (R) if it satisfies one of the following three conditions:

a) the part $\bar{S}$ is regular with respect to all three coordinate planes;

b) the orthogonal projection of the part $\bar{S}$ onto one of the three coordinate planes has the two-dimensional measure equal to zero; the part $\bar{S}$ is regular with respect to the remaining two coordinate planes;

c) two components of the vector $\mathbf{n}(x, y, z)$ equal zero for all points $[x, y, z] \in \bar{S}$.

11.9. Theorem. *Let a part $\bar{S}$ have property (R), let $\mathbf{n}$ be its oriented unit normal and let P, Q, R be three functions continuous on $\bar{S}$. Then relation* (11.15) *is satisfied, i.e.,*

$$\iint_{\bar{S}} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma = \iint_{\bar{S}} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y).$$

P r o o f. Relation (11.15) follows from Theorem 11.4 and Definitions 11.5, 11.7 and 11.8. In more detail:

In the case 11.8a the assertion of the theorem coincides with Corollary 11.6.

In the case 11.8b let, for example, $\cos\alpha \equiv 0$. Then

$$\begin{aligned}
&\iint_{\bar{S}} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma \\
&= \iint_{\bar{S}} (Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma \\
&= \varepsilon_y \iint_{\bar{S}_{xz}} Q(x, g(x, z), z)\,\mathrm{d}x\,\mathrm{d}z + \varepsilon_z \iint_{\bar{S}_{xy}} R(x, y, f(x, y))\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\bar{S}} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y),
\end{aligned}$$

where the second equality follows from Theorem 11.4 and the third equality from (11.12), (11.13) and Definition 11.7d.

In the case 11.8c let, for example, $\cos\alpha \equiv 0$, $\cos\beta \equiv 0$. Then

$$\iint_{\bar S}(P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma = \iint_{\bar S} R\,\mathrm{d}\sigma$$
$$= \varepsilon_z \iint_{\bar S_{xy}} R(x,y,z_0)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\bar S}(P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y).$$

□

11.10. Theorem. *Let a surface $\bar S$ be the union of n parts with property (R) which have mutually disjoint interiors, let $\mathbf{n}$ be its oriented unit normal and let P, Q, R be three functions continuous on $\bar S$. Then we can set*

$$\iint_{\bar S}(P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y) := \sum_{i=1}^{n}\iint_{\bar S^i}(P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y).$$

Proof. The assertion of the theorem follows from the preceding results. □

At the end of this section we prove a basic theorem concerning the parametric representation of a surface integral of the second kind.

11.11. Theorem. *Let a part $\bar S$, which has a smooth parametric representation (7.1), be regular with respect to all three coordinate planes, let $\mathbf{n}$ be its oriented unit normal and let P, Q, R be three function continuous on $\bar S$. Then*

$$\iint_{\bar S} P(x,y,z)\,\mathrm{d}y\,\mathrm{d}z = \beta\iint_M P(X(u,v),Y(u,v),Z(u,v))A(u,v)\,\mathrm{d}u\,\mathrm{d}v, \tag{11.17}$$
$$\iint_{\bar S} Q(x,y,z)\,\mathrm{d}x\,\mathrm{d}z = \beta\iint_M Q(X(u,v),Y(u,v),Z(u,v))B(u,v)\,\mathrm{d}u\,\mathrm{d}v, \tag{11.18}$$
$$\iint_{\bar S} R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y = \beta\iint_M R(X(u,v),Y(u,v),Z(u,v))C(u,v)\,\mathrm{d}u\,\mathrm{d}v, \tag{11.19}$$

where $\beta = +1$ ($\beta = -1$) if the vector (A,B,C) is oriented similarly (not similarly) as the normal vector $\mathbf{n} = (\cos\alpha,\cos\beta,\cos\gamma)$. (For the definition of the components A, B, C see (7.12)–(7.14) and for the definition of the set M see Definition 7.1.)

Proof. The proof of all three expressions is the same. Thus we prove only (11.19). We start from expression (11.13), i.e., from

$$\iint_{\bar S} R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y = \varepsilon_z\iint_{\bar S_{xy}} R(x,y,f(x,y))\,\mathrm{d}x\,\mathrm{d}y. \tag{11.20}$$

By the theorem on transformation of a two-dimensional integral, Lemma 7.2a and the fact that $C(u,v) \neq 0$ for all points $[u,v] \in \overline{M}$ we can write

$$\iint_{\bar{S}_{xy}} R(x,y,f(x,y))\,\mathrm{d}x\,\mathrm{d}y = \iint_M R(X,Y,Z)|C|\,\mathrm{d}u\,\mathrm{d}v, \tag{11.21}$$

where for greater simplicity we omit the arguments u, v on the right-hand side.

A) Let $\varepsilon_z = 1$, which means that $\cos\gamma > 0$. Let the vector (A,B,C) be oriented similarly as the vector $\mathbf{n}$. As $\cos\gamma > 0$, we have, due to the same orientation of the vectors mentioned, $C > 0$ (the fact that $C \neq 0$ follows from the considerations of Section 8). Hence $|C| = C$ and relations (11.20), (11.21) yield relation (11.19) with $\beta = +1$.

Let now the orientation of the vectors (A,B,C) and $\mathbf{n}$ be opposite. In this case $C < 0$. Hence $|C| = -C$ and relations (11.20), (11.21) imply relation (11.19) with $\beta = -1$.

B) Let $\varepsilon_z = -1$, which means that $\cos\gamma < 0$. Let the vectors (A,B,C) and $\mathbf{n}$ be oriented similarly. As $\cos\gamma < 0$, we have $C < 0$ in this case; hence $|C| = -C$. Let us insert this relation into (11.21) and multiply the relation obtained by minus one. This yields

$$\varepsilon_z \iint_{\bar{S}_{xy}} R(x,y,f(x,y))\,\mathrm{d}x\,\mathrm{d}y = \iint_M R(X,Y,Z)C\,\mathrm{d}u\,\mathrm{d}v. \tag{11.22}$$

Relation (11.19) with $\beta = +1$ follows from (11.20) and (11.22).

Let now the vectors (A,B,C) and $\mathbf{n}$ be oriented in the opposite way. In this case $C > 0$; hence $|C| = C$. If we multiply (11.21) by the relation $\varepsilon_z = -1$ and use (11.20) we obtain (11.19) where $\beta = -1$. □

12. The elementary form of the Gauss-Ostrogradskij theorem

12.1. Definition. a) A bounded domain Ω is called *elementary with respect to the coordinate plane* (x,y) if every straight line parallel to the z-axis intersects the boundary $\partial\Omega$ at two points or has with $\partial\Omega$ a common segment which can degenerate into a point.

b) Similarly we define *domains elementary with respect to the plane* (x,z), or *with respect to the plane* (y,z).

c) A bounded domain Ω is called *elementary* if it is elementary with respect to all three coordinate planes.

12.2. Lemma. *Let a domain Ω be elementary with respect to (x, y) and let its boundary $\partial\Omega$ consist of a finite number of parts with property (R), which have mutually disjoint interiors. Then these parts can be divided into three groups with the following properties:*

a) *The union of parts belonging to the first group forms a part $\overline{D}^1$, whose points $[x, y, z]$ satisfy the equation*

$$z = z_1(x, y), \quad [x, y] \in \overline{D}^1_{xy}, \tag{12.1}$$

where z_1 is a continuous function.

b) *The union of parts belonging to the second group forms a part $\overline{D}^2$, whose points $[x, y, z]$ satisfy the equation*

$$z = z_2(x, y), \quad [x, y] \in \overline{D}^2_{xy}, \tag{12.2}$$

where z_2 is a continuous function. At the same time we have

$$\overline{D}^1_{xy} = \overline{D}^2_{xy},$$
$$z_1(x, y) \leqslant z_2(x, y) \quad \forall [x, y] \in \overline{D}^1_{xy}.$$

c) *The normal vector of the parts belonging to the third group satisfies*

$$\cos\gamma \equiv 0.$$

The set of the parts belonging to the third group can be empty.

P r o o f. The assertion is evident. □

12.3. Theorem. *Let the boundary $\partial\Omega$ of an elementary domain Ω be the union of a finite number of parts with property (R). Let functions P, Q, R be continuous on $\overline{\Omega}$ and let the derivatives $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$ be continuous and bounded in Ω. Let the positive direction of the unit normal $\mathbf{n}$ be the direction of the outer normal. Then*

$$\iiint_{\overline{\Omega}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iint_{\partial\Omega} (P \, \mathrm{d}y \, \mathrm{d}z + Q \, \mathrm{d}x \, \mathrm{d}z + R \, \mathrm{d}x \, \mathrm{d}y). \tag{12.3}$$

P r o o f. By Lemma 12.2 and the Fubini theorem

$$\begin{aligned} \iiint_{\overline{\Omega}} \frac{\partial R}{\partial z} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= \iint_{\overline{D}^1_{xy}} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z} \, \mathrm{d}z \right\} \mathrm{d}x \, \mathrm{d}y \\ &= \iint_{\overline{D}^2_{xy}} R(x, y, z_2(x, y)) \, \mathrm{d}x \, \mathrm{d}y - \iint_{\overline{D}^1_{xy}} R(x, y, z_1(x, y)) \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{12.4}$$

Owing to the orientation of the normal, we have $\cos\gamma < 0$ on D^1 and $\cos\gamma > 0$ on D^2. Thus relation (12.4) can be rewritten in the form

$$(12.5)\quad \iiint_{\overline{\Omega}} \frac{\partial R}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \varepsilon_z \iint_{\overline{D}^2_{xy}} R(x,y,z_2(x,y))\,\mathrm{d}x\,\mathrm{d}y + \varepsilon_z \iint_{\overline{D}^1_{xy}} R(x,y,z_1(x,y))\,\mathrm{d}x\,\mathrm{d}y.$$

As the boundary $\partial\Omega$ can be expressed as the union of the surfaces (12.1), (12.2) and the parts for which $\cos\gamma = 0$, the right-hand side of (12.5) is equal to the surface integral $\iint_{\partial\Omega} R\,\mathrm{d}x\,\mathrm{d}y$. Hence

$$(12.6)\qquad \iiint_{\overline{\Omega}} \frac{\partial R}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} R\,\mathrm{d}x\,\mathrm{d}y.$$

Similarly we obtain

$$(12.7)\qquad \iiint_{\overline{\Omega}} \frac{\partial P}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} P\,\mathrm{d}y\,\mathrm{d}z,$$

$$(12.8)\qquad \iiint_{\overline{\Omega}} \frac{\partial Q}{\partial y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} Q\,\mathrm{d}x\,\mathrm{d}z.$$

Summing (12.6)–(12.8), we obtain (12.3). □

12.4. Theorem. *Let a domain $\overline{\Omega}$ be the union of a finite number of elementary domains $\overline{\Omega}^1, \dots, \overline{\Omega}^n$, which have mutually disjoint interiors. Let the boundary $\partial\Omega^i$ of each domain Ω^i ($i = 1, \dots, n$) be the union of a finite number of parts with property (R). Let functions P, Q, R be continuous on $\overline{\Omega}$ and let the derivatives $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$ be continuous and bounded in Ω. Let the unit normal $\mathbf{n}$ of the boundary $\partial\Omega$ be oriented in the direction of the outer normal. Then*

$$(12.9)\quad \iiint_{\overline{\Omega}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y).$$

P r o o f. The assumption concerning the normal $\mathbf{n}$ enables us to orient the normal of each boundary $\partial\Omega^i$ in the direction of the outer normal of Ω^i; hence

$$\iiint_{\overline{\Omega}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sum_{i=1}^{n} \iiint_{\overline{\Omega}^i} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \sum_{i=1}^{n} \iint_{\partial\Omega^i} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y) = \iint_{\partial\Omega} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y),$$

because at every point $P \in \Omega$ which satisfies the relation $P \in \partial\Omega^j \cap \partial\Omega^k$ ($j \neq k$) two opposite normals meet—one belonging to $\partial\Omega^j$ and the other to $\partial\Omega^k$. □

R e m a r k. Every bounded convex domain is elementary.

13. A more general form of the Gauss-Ostrogradskij theorem

Verifying the assumptions of Theorem 12.4 concerning the domain Ω is in most cases very difficult: Let us consider, for example, a domain (the so called "cheese ball with many bubbles")

$$\bar{\Omega} = \overline{K}_0 - \bigcup_{i=1}^{n} K_i,$$

where $\overline{K}_0, \overline{K}_1, \ldots, \overline{K}_n$ are balls with properties

$$\overline{K}_i \subset K_0 \quad (i = 1, \ldots, n), \quad \overline{K}_i \cap \overline{K}_j = \emptyset \quad (i \neq j;\ i, j = 1, \ldots, n).$$

To make the Gauss-Ostrogradskij theorem applicable in general use we must substitute its assumption concerning the domain Ω by an assumption which would enable us to check only the properties of the boundary $\partial\Omega$. This is the aim of this section.

13.1. Definition. We say that a part $\bar{S}$ has property (R^*) (or property (R^{**})) if it satisfies conditions a)–c) (or conditions a)–d)) where

a) the part $\bar{S}$ has property (R);

b) if

$$z = f(x, y), \quad y = g(x, z), \quad x = h(y, z)$$

are functions appearing in the analytical expressions of the part $\bar{S}$ with respect to the coordinate planes then at least one of the three relations $f \in C^2(\bar{S}_{xy})$, $g \in C^2(\bar{S}_{xz})$, $h \in C^2(\bar{S}_{yz})$ holds;

c) if $\operatorname{meas}_2 S_{st} > 0$, then the boundary ∂S_{st} is piecewise of class C^2 and has no cusp-points;

d) at least one of the plane domains $\bar{S}_{xy}$, $\bar{S}_{xz}$, $\bar{S}_{yz}$ is starlike. (A domain $\overline{D}$ is starlike if there exists at least one point $Q \in D$ with the property that every half-line starting from this point intersects ∂D just at one point.)

E x a m p l e. Let us divide the sphere

$$\partial K = \{[x, y, z] \colon x^2 + y^2 + z^2 = 1\}$$

by the coordinate planes into eight parts $\bar{S}^1, \ldots, \bar{S}^8$, or alternatively by the coordinate planes and the planes $y = x$, $y = -x$, $z = \frac{1}{2}$, $z = -\frac{1}{2}$ into 32 parts $\overline{D}^1, \ldots, \overline{D}^{32}$. The parts $\overline{D}^1, \ldots, \overline{D}^{32}$ have property (R^{**}), whilst the parts $\bar{S}^1, \ldots, \bar{S}^8$ do not satisfy condition 13.1b.

13.2. Lemma. *Let $g(0) = \eta_1$, $g(l) = \eta_2$ and $|g''(s)| \leqslant K_2$ for $s \in (0, l)$. Then*

$$|g(s)| \leqslant \max|\eta_j| + \frac{1}{8}K_2 l^2 \quad \forall s \in \langle 0, l\rangle.$$

P r o o f. According to the Lagrange interpolation theorem, we have

$$g(s) = \eta_1\Big(1 - \frac{s}{l}\Big) + \eta_2\frac{s}{l} + \psi(s), \quad s \in \langle 0, l\rangle,$$

where the remainder $\psi(s)$ satisfies

$$|\psi(s)| \leqslant \frac{1}{2}K_2|s(s-l)|, \quad s \in \langle 0, l\rangle.$$

Hence

$$|g(s)| \leqslant \max|\eta_j| + \frac{1}{2}K_2 \max_{s\in\langle 0,l\rangle}|s(s-l)|, \quad s \in \langle 0, l\rangle.$$

From here the assertion of Lemma 13.2 follows, because $\max\limits_{s\in\langle 0,l\rangle}|s(s-l)| = \frac{1}{4}l^2$. □

13.3. Lemma. *Let $\bar{T}$ be a closed triangle in the plane (s,t) with points P_1, P_2, P_3 as vertices. Let $\varphi \in C^2(\bar{T})$ and let M_2 be a constant bounding the second partial derivatives of the function φ on $\bar{T}$. Then the linear polynomial $p(s,t)$, for which*

$$p(P_i) = \varphi(P_i) \quad (i = 1, 2, 3), \tag{13.1}$$

satisfies

$$|\varphi(s,t) - p(s,t)| \leqslant \frac{1}{2}M_2\delta_T^2, \quad [s,t] \in T, \tag{13.2}$$

where δ_T is the length of the largest side of the triangle $\bar{T}$.

P r o o f. Let us set

$$\chi(s,t) := \varphi(s,t) - p(s,t). \tag{13.3}$$

Then M_2 is a constant bounding the second partial derivative of χ on T and by (13.1) we have

$$\chi(P_i) = 0 \quad (i = 1, 2, 3).$$

Let us consider the function $g = \chi|_{P_2P_3}$. Then Lemma 13.2, where we set $K_2 = 2M_2$, implies

$$|\chi(s,t)| \leqslant \frac{1}{4}M_2\delta_T^2, \quad [s,t] \in P_2P_3.$$

Let $P \in T$ $(P \neq P_i)$ and let B be the cross-point of the segment P_2P_3 and the straight-line determined by the points P_1, P. Then by Lemma 13.2

$$|\chi(P)| \leqslant \frac{1}{4}M_2\delta_T^2 + \frac{1}{4}M_2\delta_T^2.$$

This estimate and (13.3) yield (13.2). □

13.4. Lemma. *If a function $u(s,t)$ belongs to the class $C^n(\overline{D})$, where $D \in \mathbb{R}^2$ is a bounded closed domain whose boundary ∂D is piecewise of class C^n, without cusp-points and with the outer normal existing at almost all points of ∂D, then this function can be extended with keeping its class onto the whole plane (s,t).*

P r o o f. The proof of this theorem is presented, for example, in [Fil, Appendix]. □

13.5. Theorem (Gauss-Ostrogradskij). *Let $\overline{\Omega}$ be a three-dimensional bounded closed domain whose boundary $\partial\Omega$ is the union of a finite number of parts with property (R^*), which have mutually disjoint interiors. Let functions*

$$P,\ Q,\ R,\ \partial P/\partial x,\ \partial Q/\partial y,\ \partial R/\partial z$$

be continuous and bounded in a bounded three-dimensional domain $\widetilde{\Omega}$ satisfying $\widetilde{\Omega} \supset \overline{\Omega}$. Let the unit normal $\mathbf{n}$ of the boundary $\partial\Omega$ be oriented in the direction of the outer normal of $\partial\Omega$, which exists at almost all points of $\partial\Omega$. Then

$$\iiint_{\overline{\Omega}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y). \tag{13.4}$$

P r o o f. First we prove the theorem in the case that the parts forming $\partial\Omega$ have property (R^{**}). At the end we show how to change the proof when these parts have only property (R^*).

A) Let us choose $\delta > 0$ arbitrarily but fixed ($\delta < 1$). In this part of the proof we show how we shall approximate a part with property (R^{**}) by a "panel-shaped" surface which consists of triangular panels whose longest side is less or equal to δ. This approximation will be constructed in such a way that if

$$\partial\Omega = \bigcup_{i=1}^{n} \bar{S}_i, \quad S_i \cap S_j = \emptyset \ (i \neq j) \tag{13.5}$$

is a decomposition of $\partial\Omega$ into parts with property (R^{**}) and $\bar{S}_i^\delta$ is a panel-shaped surface approximating $\bar{S}_i$, then

$$\partial\Omega^\delta := \bigcup_{i=1}^{n} \bar{S}_i^\delta \tag{13.6}$$

is a polyhedron (or a union of polyhedrons if the domain Ω is multiply connected) satisfying

$$S_i^\delta \cap S_j^\delta = \emptyset \quad (i \neq j;\ i,j = 1,\dots,n) \tag{13.7}$$

and with vertices lying on $\partial\Omega$. The closed bounded three-dimensional domain with the boundary $\partial\Omega^\delta$ will be denoted $\bar{\Omega}^\delta$.

Let us consider one part $\bar{S} := \bar{S}_i$ appearing in the decomposition (13.5) and let, for example, its orthogonal projection $\bar{S}_{xy}$ onto the plane (x, y) be a starlike plane domain. The part $\bar{S}$ can be analytically expressed in the form

$$z = f(x, y), \quad [x, y] \in \bar{S}_{xy}. \tag{13.8}$$

If $\operatorname{meas}_2 S_{xz} > 0$ then we can express $\bar{S}$ also in the form

$$y = g(x, z), \quad [x, z] \in \bar{S}_{xz}, \tag{13.9}$$

and if $\operatorname{meas}_2 S_{yz} > 0$, then also in the form

$$x = h(y, z), \quad [y, z] \in \bar{S}_{yz}. \tag{13.10}$$

Let us choose the nodal points on the boundary ∂S so dense that the distance of two adjacent nodal points is not greater than δ. (If ∂S has corners, i.e., if ∂S_{xy} is a piecewise smooth curve, then these corners are also nodal points. Also each point $P \in \partial S$ at which at least two "edges" of the decomposition (13.6) meet will be a nodal point; the expression "edge" is in quotation marks because the surface $\partial\Omega$ can be smooth—for example, if Ω is a sphere.) We obtain in such a way a closed curve ∂S^δ, which is the union of a finite number of segments of lengths not greater than δ. The orthogonal projection of the curve ∂S^δ onto the plane (x, y) will be denoted ∂S^δ_{xy}. It is the boundary of a simply connected polygonal domain $\bar{S}^\delta_{xy}$. As the domain $\bar{S}_{xy}$ is starlike, the domain $\bar{S}^\delta_{xy}$ is also starlike (with respect to the same point Q). Connecting all nodal points lying on ∂S^δ_{xy} (they are the orthogonal projections of the nodal points lying on ∂S^δ onto the plane (x, y)) with the point Q we obtain a rough triangulation of the domain $\bar{S}^\delta_{xy}$. Choosing a sufficiently great integer s, dividing each segment which has the point Q as one end point into 2^s equal parts and triangulating each triangle which has the point Q as one vertex in such a way that each quadrilateral, which has arisen by connecting two opposite points, will be divided by two diagonals into four triangles, we obtain a sufficiently fine triangulation $\mathcal{T}(\bar{S}^\delta_{xy})$ of the domain $\bar{S}^\delta_{xy}$. The nodal points on $\bar{S}$ which correspond, according to (13.8), to the nodal points of this triangulation, determine uniquely a panel-shaped surface $\bar{S}^\delta$. This panel-shaped surface can be analytically expressed in the form

$$z = f^\delta(x, y), \quad [x, y] \in \bar{S}^\delta_{xy}, \tag{13.11}$$

where $f^\delta\colon \bar{S}^\delta_{xy} \to \mathbb{R}^1$ is a continuous function which is linear on triangles of the triangulation $\mathcal{T}(\bar{S}^\delta_{xy})$.

If $\operatorname{meas}_2 S_{xz} > 0$ (or $\operatorname{meas}_2 S_{yz} > 0$) then the orthogonal projection of the surface $\bar{S}^\delta$ onto the plane (x,z) (or (y,z)) defines the triangulation $\mathcal{T}(\bar{S}^\delta_{xz})$ (or $\mathcal{T}(\bar{S}^\delta_{yz})$) of the polygonal domain $\bar{S}^\delta_{xz}$ (or $\bar{S}^\delta_{yz}$) and the surface $\bar{S}^\delta$ can be expressed analytically in the form

$$y = g^\delta(x,z), \quad [x,z] \in \bar{S}^\delta_{xz}, \tag{13.12}$$

or

$$x = h^\delta(y,z), \quad [y,z] \in \bar{S}^\delta_{yz}, \tag{13.13}$$

where $g^\delta\colon \bar{S}^\delta_{xz} \to \mathbb{R}^1$ (or $h^\delta\colon \bar{S}^\delta_{yz} \to \mathbb{R}^1$) is a continuous function which is linear on triangles of the corresponding triangulation.

B) As $\bar{\Omega}^\delta$ is a polyhedron, we can express $\bar{\Omega}^\delta$ by [Že, Lemma 4.2] in the form

$$\bar{\Omega}^\delta = \bigcup_{j=1}^{m} \bar{U}_j, \tag{13.14}$$

where $\bar{U}_1, \dots, \bar{U}_m$ are closed convex polyhedrons. (It was Křížek who inspired the very idea of the proof of this lemma—see [Kř].) Let us orientate the normal to ∂U_j as the outer normal of $\bar{U}_j$ $(j = 1, \dots, m)$. Relation (13.14) and Theorem 12.4 yield

$$\begin{aligned} &\iiint_{\bar{\Omega}^\delta} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \sum_{j=1}^{m} \iiint_{\bar{U}_j} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \sum_{j=1}^{m} \iint_{\partial U_j} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y) \\ &= \iint_{\partial\Omega^\delta} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y), \end{aligned} \tag{13.15}$$

because the surface integrals over $\partial U_j \cap \Omega^\delta$ altogether cancel.

C) Now we prove that

$$\lim_{\delta\to 0} \iiint_{\bar{\Omega}^\delta} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\bar{\Omega}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{13.16}$$

We have

$$
\begin{aligned}
(13.17)\quad & \left|\iiint_{\bar\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right.\\
&\left.-\iiint_{\bar\Omega^\delta}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\\
&=\left|\iiint_{\bar\Omega-\bar\Omega^\delta}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right.\\
&\left.-\iiint_{\bar\Omega^\delta-\bar\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\\
&\leqslant\left\{\max_{\tilde\Omega}\left(\left|\frac{\partial P}{\partial x}\right|+\left|\frac{\partial Q}{\partial y}\right|+\left|\frac{\partial R}{\partial z}\right|\right)\right\}\left\{\mathrm{meas}_3(\Omega-\Omega^\delta)+\mathrm{meas}_3(\Omega^\delta-\Omega)\right\}.
\end{aligned}
$$

By Lemma 13.3 we have

$$
(13.18)\qquad \mathrm{meas}_3(\Omega-\bar\Omega^\delta)\leqslant\sum_{i=1}^{n}\sum_{\bar T\subset\bar S^{i\delta}_{st}}\widetilde{M}_2\delta^2\,\mathrm{meas}_2\,\bar T+\Delta(\delta),
$$

where $\bar S^i\equiv\bar S_i$ and $\bar S^i_{st}$ is the one from the orthogonal projections $\bar S^i_{xy}$, $\bar S^i_{xz}$, $\bar S^i_{yz}$ which satisfies both the inequality $\mathrm{meas}_2\,\bar S^i_{st}>0$ and condition b) from Definition 13.1. (The first term on the right-hand side of (13.18) can be split into two or three terms with different meanings of s,t.) The constant $\widetilde{M}_2$ is given by the relation

$$
\widetilde{M}_2=\max_{i=1,\dots,n}\max_{\widetilde S^i_{st}}\left(\left|\frac{\partial^2\widetilde\varphi^i}{\partial s^2}\right|,\left|\frac{\partial^2\widetilde\varphi^i}{\partial s\partial t}\right|,\left|\frac{\partial^2\widetilde\varphi^i}{\partial t^2}\right|\right),
$$

where $\widetilde S^i_{st}\supset\bar S^i_{st}\cup\bar S^{i\delta}_{st}$ for all δ, $\widetilde\varphi^i$ is the extension of the function φ^i (see Lemma 13.4) and φ^i denotes that of the functions f^i, g^i, h^i which belongs to the variables s,t.

As it is not guaranteed that the first term on the right-hand side of (13.18) is the upper bound of $\mathrm{meas}_3(\Omega-\bar\Omega^\delta)$ (in general, this term does not cover neighbourhoods of edges and "edges" in which the parts $\bar S_i$ meet), we also have the term $\Delta(\delta)$ on the right-hand side of (13.18). A rough estimate of this term is

$$
\Delta(\delta)\leqslant|\partial\Omega|\delta;
$$

thus relation (13.18) implies

$$
(13.19)\qquad \mathrm{meas}_3(\Omega-\bar\Omega^\delta)\leqslant(1+2\widetilde{M}_2\delta)|\partial\Omega|\delta.
$$

Similarly

$$\text{(13.20)} \qquad \operatorname{meas}_3(\overline{\Omega}^\delta - \Omega) \leqslant (1 + 2\widetilde{M}_2\delta)|\partial\Omega|\delta.$$

Relations (13.17), (13.19) and (13.20) yield (13.16).

D) Now we show that

(13.21)

$$\lim_{\delta\to 0}\iint_{\partial\Omega^\delta}(P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y) = \iint_{\partial\Omega}(P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y).$$

We prove only

$$\text{(13.22)} \qquad \lim_{\delta\to 0}\iint_{\partial\Omega^\delta} R\,\mathrm{d}x\,\mathrm{d}y = \iint_{\partial\Omega} R\,\mathrm{d}x\,\mathrm{d}y$$

since the remaining two relations

$$\lim_{\delta\to 0}\iint_{\partial\Omega^\delta} P\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} P\,\mathrm{d}y\,\mathrm{d}z, \quad \lim_{\delta\to 0}\iint_{\partial\Omega^\delta} Q\,\mathrm{d}x\,\mathrm{d}z = \iint_{\partial\Omega} Q\,\mathrm{d}x\,\mathrm{d}z,$$

which together with (13.22) give (13.21), can be derived in the same way.

By (13.5)–(13.7) we have

$$\text{(13.23)} \qquad \iint_{\partial\Omega^\delta} R\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{n}\iint_{\overline{S}_i^\delta} R\,\mathrm{d}x\,\mathrm{d}y,$$

$$\text{(13.24)} \qquad \iint_{\partial\Omega} R\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{n}\iint_{\overline{S}_i} R\,\mathrm{d}x\,\mathrm{d}y.$$

Thus, it is sufficient to prove that

$$\text{(13.25)} \qquad \lim_{\delta\to 0}\iint_{\overline{S}_i^\delta} R\,\mathrm{d}x\,\mathrm{d}y = \iint_{\overline{S}_i} R\,\mathrm{d}x\,\mathrm{d}y.$$

In this part of the proof we restrict ourselves to the case $\operatorname{meas}_2 S_{xy}^i > 0$; the case $\operatorname{meas}_2 S_{xy}^i = 0$ will be analyzed in the next part of the proof.

Let $\operatorname{meas}_2 S_{xy} > 0$, where we set $\overline{S} := \overline{S}_i$. Then

$$\begin{aligned}
\text{(13.26)} \quad & \left|\iint_{\overline{S}} R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y - \iint_{\overline{S}^\delta} R(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&= \left|\varepsilon_z\iint_{\overline{S}_{xy}} R(x,y,f(x,y))\,\mathrm{d}x\,\mathrm{d}y - \varepsilon_z\iint_{\overline{S}_{xy}^\delta} R(x,y,f^\delta(x,y))\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\leqslant |I_1| + |I_2| + |I_3| := \iint_{\overline{S}_{xy}\cap\overline{S}_{xy}^\delta} |R(x,y,f(x,y)) - R(x,y,f^\delta(x,y))|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_{\overline{S}_{xy}-\overline{S}_{xy}^\delta} |R(x,y,f(x,y))|\,\mathrm{d}x\,\mathrm{d}y + \iint_{\overline{S}_{xy}^\delta-\overline{S}_{xy}} |R(x,y,f^\delta(x,y))|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

According to the Lagrange theorem on the increment, we have

$$(13.27)\qquad \begin{aligned} |I_1| &= \iint_{\bar S_{xy}\cap \bar S^\delta_{xy}} \left|\frac{\partial R}{\partial z}(x,y,f^\delta(x,y)+\vartheta(f(x,y)-f^\delta(x,y))\right| \\ &\quad \times |f(x,y)-f^\delta(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \max_{\widetilde\Omega}\left|\frac{\partial R}{\partial z}\right| \operatorname{meas}_2 \bar S_{xy}\cdot \max_{\bar S_{xy}\cap \bar S^\delta_{xy}} |f(x,y)-f^\delta(x,y)|, \end{aligned}$$

where $0<\vartheta<1$. Further,

$$(13.28)\qquad |I_2| \leqslant \max_{\widetilde\Omega} |R| \operatorname{meas}_2(S_{xy}-S^\delta_{xy}),$$

$$(13.29)\qquad |I_3| \leqslant \max_{\widetilde\Omega} |R| \operatorname{meas}_2(S^\delta_{xy}-S_{xy}).$$

Let $[x,y]\in \bar S_{xy}\cap \bar S^\delta_{xy}$ be arbitrary but fixed. Let the line perpendicular to the plane (x,y) which passes through the point $[x,y]$ intersect the part $\bar S$ (or $\bar S^\delta$) at the point P_1 (or P_2). If $f\in C^2(\bar S_{xy})$ then

$$(13.30)\qquad \operatorname{dist}(P_1,P_2)\leqslant \frac12 \widetilde{M}_2\delta^2.$$

If $f\notin C^2(\bar S_{xy})$ then either $g\in C^2(\bar S_{xz})$, or $h\in C^2(\bar S_{yz})$. Let, for example, $g\in C^2(\bar S_{xz})$. Then the line perpendicular to the plane (x,z) which passes through the point P_1 will intersect the part $\bar S^\delta$ at a point, which will be denoted by P_3, and we have, according to Lemma 13.3,

$$\operatorname{dist}(P_1,P_3)\leqslant \frac12 \widetilde{M}^*_2\delta^2.$$

For a small δ the distance between $\bar S$ and $\bar S^\delta$ is very small and the point P_3 belongs either to the same triangle $\overline{T}\subset \bar S^\delta$ as P_2, or to a very near triangle. We deduce from here that $\operatorname{dist}(P_2,P_3)\leqslant C\delta$ (where $C\leqslant 5$); hence

$$(13.31)\qquad \operatorname{dist}(P_1,P_2)\leqslant C\delta,$$

because P_1, P_2, P_3 are the vertices of the right triangle with the hypotenuse P_2P_3. Thus (13.30) and (13.31) yield

$$(13.32)\qquad \max_{\bar S_{xy}\cap \bar S^\delta_{xy}} |f(x,y)-f^\delta(x,y)|\leqslant C\delta.$$

Assumption 13.1c, Lemma 13.2 (with $\eta_1=\eta_2=0$) and the fact that the number of segments which form ∂S^δ_{xy} is $O(\delta^{-1})$ imply

$$(13.33)\qquad \operatorname{meas}_2(\bar S_{xy}-\bar S^\delta_{xy})\leqslant C\delta,\quad \operatorname{meas}_2(\bar S^\delta_{xy}-\bar S_{xy})\leqslant C\delta.$$

Relations (13.27)–(13.29), (13.32) and (13.33) give

$$|I_1| + |I_2| + |I_3| \leqslant C\delta. \tag{13.34}$$

Using (13.26) and (13.34), we obtain (13.25) provided $\mathrm{meas}_2\, S_{xy} > 0$.

E) Let now $\mathrm{meas}_2\, S_{xy} = 0$. In this case

$$\iint_{\bar{S}} R \,\mathrm{d}x\,\mathrm{d}y = 0$$

and we have to prove

$$\lim_{\delta\to 0} \iint_{\bar{S}^\delta} R \,\mathrm{d}x\,\mathrm{d}y = 0. \tag{13.35}$$

In this part of the proof we shall need the following property of triangles $\overline{T} \subset \bar{S}^\delta$ ($0 < \delta < 1$ is arbitrary): There exist

$$0 < \vartheta_0 < \gamma_0 < \pi \tag{13.36}$$

such that for every $0 < \delta < 1$ we can find in every triangle $\overline{T} \subset \bar{S}^\delta$ an angle ϑ_T satisfying

$$\vartheta_0 \leqslant \vartheta_T \leqslant \gamma_0. \tag{13.37}$$

This requirement can be easily achieved: Let us consider a panel-shaped part $\bar{S}^{\delta^*}$. If a triangle $\overline{T} \subset \bar{S}^{\delta^*}$ has two angles smaller than ϑ_0 and the remaining angle greater than γ_0 then we refine locally the triangulation $\mathcal{T}(\bar{S}^{\delta^*}_{st})$ in such a way that the orthogonal projection $\overline{T}_{st}$ of $\overline{T}$ is divided into two right angled triangles. Thus we can assume that all triangles $\overline{T} \subset \bar{S}^{\delta^*}$ satisfy (13.37). Panel-shaped parts $\bar{S}^{\delta_i}$ with $\delta_i < \delta^*$ ($i = 1, 2, \ldots$) can be obtained if we refine $\mathcal{T}(\bar{S}^{\delta^*}_{st})$ by the bisection process.

We have

$$\begin{aligned} \left|\iint_{\bar{S}^\delta} R \,\mathrm{d}x\,\mathrm{d}y\right| &= \left|\sum_{\overline{T}\subset\bar{S}^\delta} \iint_{\overline{T}} R \,\mathrm{d}x\,\mathrm{d}y\right| = \left|\sum_{\overline{T}\subset\bar{S}^\delta} \iint_{\overline{T}} R \cos\gamma_T \,\mathrm{d}\sigma\right| \\ &\leqslant \max_{\widetilde{\Omega}} |R| \sum_{\overline{T}\subset\bar{S}^\delta} \mathrm{meas}_2\, T\, |\cos\gamma_T| \\ &\leqslant 2|\bar{S}| \max_{\widetilde{\Omega}} |R| \max_{\overline{T}\subset\bar{S}^\delta} |\cos\gamma_T|. \end{aligned} \tag{13.38}$$

If we show that

$$\max_{\overline{T}\subset\bar{S}^\delta} |\cos\gamma_T| \to 0 \quad \text{for} \quad \delta \to 0, \tag{13.39}$$

then (13.35) will follow from (13.38).

As we assume that the part $\bar{S}$ has property (R^*), Definitions 13.1, 11.8 and 11.7 guarantee that at least one of the following inequalities holds:

$$|g'(x)| \leqslant K_1, \quad x \in \langle a_1, a_2 \rangle, \tag{13.40}$$

$$|h'(y)| \leqslant K_2, \quad y \in \langle b_1, b_2 \rangle. \tag{13.41}$$

Let, for example, (13.40) hold. Let us consider an arbitrary triangle $\overline{T} \subset \bar{S}^\delta$ and let $P_i(x_i, g(x_i), z_i)$ $(i = 1,2,3)$ be its vertices, P_1 being the vertex at ϑ_T. The z-coordinate τ of the vector product $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is of the form

$$\begin{aligned}\tau &= (g(x_3) - g(x_1))\bar{x}_{21} - (g(x_2) - g(x_1))\bar{x}_{31} \\ &= \bar{x}_{21}\bar{x}_{31}(g'(x_1 + \vartheta_1\bar{x}_{31}) - g'(x_1 + \vartheta_2\bar{x}_{21})),\end{aligned}$$

where $0 < \vartheta_1 < 1$, $0 < \vartheta_2 < 1$ and $\bar{x}_{jk} = x_j - x_k$. Hence

$$\begin{aligned}|\cos\gamma_T| &= \frac{|\tau|}{2\operatorname{meas}_2 T} = \frac{|\tau|}{|\overrightarrow{P_1P_2}| \cdot |\overrightarrow{P_1P_3}| \sin\vartheta_T} \\ &\leqslant |g'(x_1 + \vartheta_1\bar{x}_{31}) - g'(x_1 + \vartheta_2\bar{x}_{21})| \sin^{-1}\vartheta_0.\end{aligned}$$

From here and from the uniform continuity of $g'(x)$ on $\langle a_1, a_2 \rangle$ relation (13.39) follows.

F) Passing to the limit in (13.15) for $\delta \to 0$ and taking into acccount (13.16) and (13.21), we obtain (13.4).

G) As we have not had any requirements as far as the triangulation $\mathcal{T}(\bar{S}_{st})$ is concerned, we have used the assumption of the starlikeness of $\bar{S}_{st}$ only for the sake of greater simplicity of part A). Now we show that it is sufficient to assume property (R^*) in the case of parts of surfaces forming $\partial\Omega$.

On edges and "edges" of the boundary $\partial\Omega$ we choose nodal points so dense that the distance of two adjacent nodal points is less than δ. We triangulate now the simply connected and bounded polygonal domain $\bar{S}^\delta_{st}$ as follows: The vertices of the domain $\bar{S}^\delta_{st}$ will be denoted by $A_1, \ldots, A_m$. As the number of vertices A_i is finite it is easy to see that *if $m > 3$ then there exist at least two vertices A_i, A_j such that the interior of the segment A_iA_j lies in the interior S^δ_{st} of $\bar{S}^\delta_{st}$*. The segment A_iA_j divides the closed domain $\bar{S}^\delta_{st}$ into two closed domains $\bar{S}^{\delta 1}_{st}$ and $\bar{S}^{\delta 2}_{st}$. If $\bar{S}^{\delta i}_{st}$ is not a triangle we can divide it again into two polygonal domains without dividing the sides. After a finite number of steps the domain $\bar{S}^\delta_{st}$ is divided into triangles in such a way that each side A_iA_j of $\bar{S}^\delta_{st}$ is a side of some triangle.

Every triangle is a starlike domain. Hence, we can triangulate it in the same way as in part A). Thus the polygonal domain $\bar{S}^\delta_{st}$ is triangulated in such a way that the corresponding panel-shaped surface $\bar{S}^\delta$ has edges of a length which is less than δ. The proof of Theorem 13.5 is complete. □

E x a m p l e. In the case of a cheese ball with many bubbles the assumption of Theorem 13.5 concerning $\partial\Omega$ can be easily verified.

Similarly as in the case of Green's theorem we shall reduce the assumption of Theorem 13.5 concerning the functions P, Q, R and their derivatives to the assumption that they are continuous on a closed bounded three-dimensional domain $\overline{\Omega}$. This will be done in Subsections 13.6–13.9.

13.6. Definition. a) We say that a domain Ω has an R-continuous boundary if Ω has a continuous boundary $\partial\Omega$ in the sense of Nečas (see [Ne, pp. 14–15]) and if $\partial\Omega$ is the union of a finite number of parts with property (R^*).

b) We say that a domain Ω has an R-Lipschitz continuous boundary if Ω has a Lipschitz continuous boundary $\partial\Omega$ in the sense of Nečas and Hlaváček (see [NH, p. 17]) and if $\partial\Omega$ is the union of a finite number of parts with property (R^*).

13.7. R e m a r k. If a domain Ω has an R-Lipschitz continuous boundary (or an R-continuous boundary) then it has an S-Lipschitz continuous boundary (or an S-continuous boundary).

According to [Fil, p. 676], the following theorem holds:

13.8. Theorem. *Let a domain $\Omega \subset \mathbb{R}^3$ have an R-Lipschitz continuous boundary $\partial\Omega$ and let a function $f\colon \overline{\Omega} \to \mathbb{R}^1$ belong to $C^1(\overline{\Omega})$. Then the function $f(x,y,z)$ can be continuously extended to the whole space $\mathbb{R}^3$ with keeping its class.*

The following theorem is a consequence of Theorems 13.5 and 13.8:

13.9. Theorem (Gauss-Ostrogradskij). *Let a (simply or multiply connected) domain Ω have an R-Lipschitz continuous boundary $\partial\Omega$. Let the unit normal $\mathbf{n}$ of the boundary $\partial\Omega$ be oriented in the direction of the outer normal of Ω, which exists at almost all points of $\partial\Omega$. Finally, let $P, Q, R \in C^1(\Omega)$. Then*

$$\iiint_{\overline{\Omega}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} (P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}x\,\mathrm{d}z + R\,\mathrm{d}x\,\mathrm{d}y).$$

Using Theorem 11.4 and denoting

$$\mathbf{n} = (n_1, n_2, n_3) = (\cos\alpha, \cos\beta, \cos\gamma), \tag{13.42}$$

we can reformulate Theorem 13.9 as follows:

13.10. Theorem. *Let the assumptions of Theorem* 13.9 *be satisfied. Then*

$$\iiint_{\overline{\Omega}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}\sigma. \tag{13.43}$$

Setting one of the functions P, Q, R equal to uv, where $u, v \in C^1(\bar{\Omega})$, and the other two functions equal to zero, denoting $x_1 := x$, $x_2 := y$, $x_3 := z$ and using (13.42), we obtain from Theorem 13.10:

13.11. Theorem (Gauss-Ostrogradskij formula). *Let a (simply or multiply connected) domain Ω have an R-Lipschitz continuous boundary. Then for all functions $u \in C^1(\bar{\Omega})$, $v \in C^1(\bar{\Omega})$ we have*

$$\iiint_{\bar{\Omega}} \frac{\partial u}{\partial x_i} v \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \iint_{\partial\Omega} uvn_i \, \mathrm{d}\sigma - \iiint_{\bar{\Omega}} u \frac{\partial v}{\partial x_i} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3, \tag{13.44}$$

where (n_1, n_2, n_3) is the outer unit normal vector.

14. The Gauss-Ostrogradskij theorem in Sobolev spaces

14.1. Theorem (Gauss-Ostrogradskij formula in $\mathbf{H}^1$). *Let a (simply or multiply connected) domain Ω have an R-Lipschitz continuous boundary. Then for all functions $u \in H^1(\Omega)$, $v \in H^1(\Omega)$ we have*

$$\iiint_{\bar{\Omega}} \frac{\partial u}{\partial x_i} v \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \iint_{\partial\Omega} uvn_i \, \mathrm{d}\sigma - \iiint_{\bar{\Omega}} u \frac{\partial v}{\partial x_i} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3, \tag{14.1}$$

where (n_1, n_2, n_3) is the outer unit normal vector.

Proof. Let $\{u_n\} \subset C^\infty(\bar{\Omega})$, $\{v_n\} \subset C^\infty(\bar{\Omega})$ be such sequences that

$$u_n \to u, \quad v_n \to v \quad \text{in} \quad H^1(\Omega). \tag{14.2}$$

It should be noted that these sequences exist, by virtue of Lemma 10.2. By Theorem 13.11,

$$\iiint_{\bar{\Omega}} \frac{\partial u_n}{\partial x_i} v_n \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \iint_{\partial\Omega} u_n v_n n_i \, \mathrm{d}\sigma - \iiint_{\bar{\Omega}} u_n \frac{\partial v_n}{\partial x_i} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3. \tag{14.3}$$

We have

$$\begin{aligned} \iint_{\partial\Omega} u_n v_n n_i \, \mathrm{d}\sigma &= \iint_{\partial\Omega} (u_n - u + u)(v_n - v + v) n_i \, \mathrm{d}\sigma \\ &= \iint_{\partial\Omega} (u_n - u)(v_n - v) n_i \, \mathrm{d}\sigma + \iint_{\partial\Omega} u(v_n - v) n_i \, \mathrm{d}\sigma \\ &\quad + \iint_{\partial\Omega} (u_n - u) v n_i \, \mathrm{d}\sigma + \iint_{\partial\Omega} uvn_i \, \mathrm{d}\sigma. \end{aligned} \tag{14.4}$$

As by Theorem 10.9b

$$\|u_n - u\|_{L_2(\partial\Omega)} \leqslant C\|u_n - u\|_{H^1(\Omega)} \to 0,$$
$$\|v_n - v\|_{L_2(\partial\Omega)} \leqslant C\|v_n - v\|_{H^1(\Omega)} \to 0,$$

the first three terms on the right-hand side of (14.4) tend to zero with $n \to \infty$. Hence

$$\iint_{\partial\Omega} u_n v_n n_i \,\mathrm{d}\sigma \to \iint_{\partial\Omega} uvn_i \,\mathrm{d}\sigma \quad \text{for} \quad n \to \infty. \tag{14.5}$$

Similarly we find

$$\iiint_{\bar\Omega} \frac{\partial u_n}{\partial x_i} v_n \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}x_3 \to \iiint_{\bar\Omega} \frac{\partial u}{\partial x_i} v \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}x_3, \tag{14.6}$$

$$\iiint_{\bar\Omega} u_n \frac{\partial v_n}{\partial x_i} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}x_3 \to \iiint_{\bar\Omega} u \frac{\partial v}{\partial x_i} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}x_3. \tag{14.7}$$

Passing to the limit in (14.3) with $n \to \infty$ we obtain, by virtue of (14.5)–(14.7), relation (14.1). □

14.2. Theorem (divergence form of the Gauss-Ostrogradskij theorem). *Let a (simply or multiply connected) domain Ω have an R-Lipschitz continuous boundary. Then for all functions $P_i \in H^1(\Omega)$ $(i = 1, 2, 3)$ we have*

$$\iiint_{\bar\Omega} \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} + \frac{\partial P_3}{\partial z}\right) \mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \iint_{\partial\Omega} (P_1 n_1 + P_2 n_2 + P_3 n_3) \,\mathrm{d}\sigma, \tag{14.8}$$

where (n_1, n_2, n_3) is the unit vector of the outer normal to the boundary $\partial\Omega$.

P r o o f. Let us set $u := P_i$, $v \equiv 1$ in (14.1). Summing up the result from $i = 1$ to $i = 3$, we obtain relation (14.8). □

The most general form of the Gauss-Ostrogradskij formula is introduced in the following theorem:

14.3. Theorem (the general Gauss-Ostrogradskij formula). *Let a (simply or multiply connected) domain Ω have an R-Lipschitz continuous boundary. Let $u \in H^{1,p}(\Omega)$, $v \in H^{1,q}(\Omega)$ with one of the following possibilities:*

a) $\frac{1}{p} + \frac{1}{q} \leqslant 1 + \frac{1}{N}$, *where* $1 \leqslant p < N$, $1 \leqslant q < N$ *with* $N = 3$;
b) $p > 1$, $q \geqslant N$ $(N = 3)$;
c) $p \geqslant N$, $q > 1$ $(N = 3)$.

Then $(\gamma u)(\gamma v) \in L_1(\partial\Omega)$ *and we have*

$$\iiint_{\bar\Omega} \frac{\partial u}{\partial x_i} v \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \iint_{\partial\Omega} uvn_i \,\mathrm{d}\sigma - \iiint_{\bar\Omega} u \frac{\partial v}{\partial x_i} \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3, \tag{14.9}$$

where (n_1, n_2, n_3) *is the outer unit normal vector.*

Proof. The proof is long and complicated; thus we divide it into several parts. (We use the symbol N instead of 3 because the assertion and its proof hold also in the case of $N = 2$.)

A) First we prove (14.9) in the case

$$\frac{1}{p} + \frac{1}{q} \leqslant 1, \quad 1 \leqslant p < N,\ 1 \leqslant q < N\ (N = 3). \tag{14.10}$$

Let $\{u_n\} \subset C^\infty(\bar\Omega)$, $\{v_n\} \subset C^\infty(\bar\Omega)$ be such sequences that

$$u_n \to u \quad \text{in } H^{1,p}(\Omega), \quad v_n \to v \quad \text{in } H^{1,q}(\Omega). \tag{14.11}$$

Then by Theorem 13.11

$$\iiint_{\bar\Omega} \frac{\partial u_n}{\partial x_i} v_n \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \iint_{\partial\Omega} u_n v_n n_i \,\mathrm{d}\sigma - \iiint_{\bar\Omega} u_n \frac{\partial v_n}{\partial x_i} \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3. \tag{14.12}$$

We have

$$\begin{aligned}
&\iiint_{\bar\Omega} \frac{\partial u_n}{\partial x_i} v_n \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&= \iiint_{\bar\Omega} \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)(v_n - v) \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&\quad + \iiint_{\bar\Omega} \frac{\partial u}{\partial x_i}(v_n - v) \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 + \iiint_{\bar\Omega} \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) v \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&\quad + \iiint_{\bar\Omega} \frac{\partial u}{\partial x_i} v \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3.
\end{aligned} \tag{14.13}$$

A generalized Hölder inequality (see [KJF, p. 67]) can be written in the following form: Let $f_i \in H^{1,p_i}(\Omega)$, where $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then $f_1 f_2 \ldots f_n \in L_1(\Omega)$ and

$$\iiint_{\bar\Omega} |f_1 f_2 \ldots f_n| \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \leqslant \mathcal{N}_{p_1}(f_1)\mathcal{N}_{p_2}(f_2)\ldots\mathcal{N}_{p_n}(f_n)$$

with

$$\mathcal{N}_{p_i}(f_i) := \left(\iiint_{\bar\Omega} |f_i|^{p_i} \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right)^{1/p_i}.$$

Setting here $n = 3$, $f_3 \equiv 1$, $p_1 = p$, $p_2 = q$, where $\frac{1}{p} + \frac{1}{q} < 1$, and $\frac{1}{p_3} = 1 - \frac{1}{p} - \frac{1}{q}$ we obtain

$$\iiint_\Omega |f_1 f_2| \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \leqslant (\mathrm{meas}_3 \, \Omega)^{1/p_3} \|f_1\|_{L_p(\Omega)} \|f_2\|_{L_q(\Omega)}. \tag{14.14}$$

If $\frac{1}{p} + \frac{1}{q} = 1$ then we can use the standard Hölder inequality

$$\iiint_\Omega |f_1 f_2| \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \leqslant \|f_1\|_{L_p(\Omega)} \|f_2\|_{L_q(\Omega)}. \tag{14.15}$$

Relations (14.11), (14.14), (14.15) enable us to prove that the first three terms on the right-hand side of (14.13) tend to zero with $n \to \infty$. Hence

$$\iiint_\Omega \frac{\partial u_n}{\partial x_i} v_n \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \to \iiint_\Omega \frac{\partial u}{\partial x_i} v \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \quad \text{for } n \to \infty. \tag{14.16}$$

Similarly

$$\iiint_\Omega u_n \frac{\partial v_n}{\partial x_i} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \to \iiint_\Omega u \frac{\partial v}{\partial x_i} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \quad \text{for } n \to \infty. \tag{14.17}$$

Now we use the restrictions $1 \leqslant p < N$, $1 \leqslant q < N$ (with $N = 3$) which enable us to set

$$\frac{1}{p^*} = \frac{1}{p} - \frac{p-1}{(N-1)p} = \frac{N-p}{(N-1)p}, \quad \frac{1}{q^*} = \frac{1}{q} - \frac{q-1}{(N-1)q} = \frac{N-q}{(N-1)q}. \tag{14.18}$$

Then, according to Theorem 10.8, we have

$$\gamma u \in L_{p^*}(\partial\Omega) \text{ if } u \in H^{1,p}(\Omega), \quad \gamma v \in L_{q^*}(\partial\Omega) \text{ if } v \in H^{1,q}(\Omega). \tag{14.19}$$

By (14.18)

$$\frac{1}{p^*} + \frac{1}{q^*} = \frac{1}{p} + \frac{1}{q} - \frac{1}{N-1}\left(1 - \frac{1}{p} + 1 - \frac{1}{q}\right) = \left(1 + \frac{1}{N-1}\right)\left(\frac{1}{p} + \frac{1}{q}\right) - \frac{2}{N-1} < 1$$

because we assume that $\frac{1}{p}+\frac{1}{q} \leqslant 1$. Hence by an analogy to (14.14) $(\gamma u)(\gamma v) \in L_1(\partial\Omega)$ and

$$\iint_{\partial\Omega} \gamma u \gamma v \, \mathrm{d}\sigma \leqslant |\partial\Omega|^{1-1/p^*-1/q^*} \|\gamma u\|_{L_{p^*}(\partial\Omega)} \|\gamma v\|_{L_{q^*}(\partial\Omega)}.$$

We can write

$$\begin{aligned} \iint_{\partial\Omega} u_n v_n n_i \, \mathrm{d}\sigma = & \iint_{\partial\Omega} (u_n - u)(v_n - v) n_i \, \mathrm{d}\sigma + \iint_{\partial\Omega} u(v_n - v) n_i \, \mathrm{d}\sigma \\ & \iint_{\partial\Omega} (u_n - u) v n_i \, \mathrm{d}\sigma + \iint_{\partial\Omega} u v n_i \, \mathrm{d}\sigma. \end{aligned} \tag{14.20}$$

As $\partial\Omega$ is piecewise smooth we have

$$\left|\iint_{\partial\Omega}(u_n - u)(v_n - v)n_i \, d\sigma\right| \leqslant C\left|\iint_{\partial\Omega}(u_n - u)(v_n - v)\, d\sigma\right|.$$

Hence, according to the generalized Hölder inequality (14.14), Theorem 10.8 (which guarantees the trace inequalities) and (14.11), we obtain

$$\begin{aligned}(14.21)\qquad &\left|\iint_{\partial\Omega}(u_n - u)(v_n - v)n_i \, d\sigma\right| \\ &\leqslant C\|u_n - u\|_{L_{p^*}(\partial\Omega)}\|v_n - v\|_{L_{q^*}(\partial\Omega)} \\ &\leqslant C\|u_n - u\|_{H^{1,p}(\partial\Omega)}\|v_n - v\|_{H^{1,q}(\partial\Omega)} \to 0 \quad \text{for } n \to \infty.\end{aligned}$$

Similarly

$$(14.22)\qquad \left|\iint_{\partial\Omega} u(v_n - v)n_i \, d\sigma\right| \to 0, \quad \left|\iint_{\partial\Omega}(u_n - u)vn_i \, d\sigma\right| \to 0 \quad \text{for } n \to \infty.$$

Hence by (14.20)–(14.22)

$$(14.23)\qquad \iint_{\partial\Omega} u_n v_n n_i \, d\sigma \to \iint_{\partial\Omega} uvn_i \, d\sigma.$$

Passing to the limit for $n \to \infty$ in relation (14.12) and using (14.16), (14.17), (14.23), we obtain (14.9) under the assumption (14.10).

B) Now we shall prove (14.9) under the assumption

$$(14.24)\qquad \frac{1}{p} + \frac{1}{q} \leqslant 1 + \frac{1}{N}, \quad 1 \leqslant p < N, \ 1 \leqslant q < N \ (N = 3).$$

The proof is a modification of part A). If

$$(14.25)\qquad q < N, \quad \frac{1}{q^*} = \frac{1}{q} - \frac{1}{N}$$

then (see Lemma 10.4)

$$(14.26)\qquad H^{1,q}(\Omega) \subset L_{q^*}(\Omega) \quad \text{algebraically and topologically;}$$

this means that

$$\|v_n - v\|_{L_{q^*}(\Omega)} \leqslant C\|v_n - v\|_{H^{1,q}(\Omega)}.$$

The inequality $\frac{1}{q^*} + \frac{1}{p} \leqslant 1$ (i.e., the inequality $\frac{1}{q} + \frac{1}{p} \leqslant 1 + \frac{1}{N}$) implies, by virtue of (14.14), (14.15) and (14.11),

$$\begin{aligned}(14.27)\qquad &\left|\iiint_{\bar\Omega} \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)(v_n - v)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right| \\ &\leqslant C\left\|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right\|_{L_p(\Omega)} \|v_n - v\|_{L_{q^*}(\Omega)} \\ &\leqslant C\|u_n - u\|_{H^{1,p}(\Omega)}\|v_n - v\|_{H^{1,q}(\Omega)} \to 0 \quad \text{for } n \to \infty.\end{aligned}$$

Similarly we prove that for $n \to \infty$

$$(14.28)\qquad \begin{aligned}&\left|\iiint_{\bar\Omega} \frac{\partial u}{\partial x_i}(v_n - v)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right| \to 0,\\ &\left|\iiint_{\bar\Omega} \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) v\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right| \to 0.\end{aligned}$$

Hence relation (14.16) follows; relation (14.17) can be proved similarly.

The proof of (14.23) in the case $q < N$, $p < N$ remains without changes. Passing to the limit for $n \to \infty$ in relation (14.12) and using (14.16), (14.17), (14.23) we obtain (14.9) under the assumption (14.24).

C) In the remaining parts we prove Theorem 14.3 for possibility b). In the case of possibility c) the proof is similar. First we prove (14.16) and (14.17) in the case $q = N$.

By [KJF, Th. 5.7.7(ii)], for any $1 \leqslant q^* < \infty$ we have

$$(14.29)\qquad H^{1,N}(\Omega) \subset L_{q^*}(\Omega) \quad \text{algebraically and topologically.}$$

Let us choose $q^* = \frac{p}{p-1}$; hence

$$H^{1,N}(\Omega) \subset L_{p/(p-1)}(\Omega) \quad \text{algebraically and topologically;}$$

this means (setting $q = N$ in (14.11))

$$\|v_n - v\|_{L_{p/(p-1)}(\Omega)} \leqslant C\|v_n - v\|_{H^{1,N}(\Omega)} \to 0.$$

As

$$\left\|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right\|_{L_p(\Omega)} \leqslant \|u_n - u\|_{H^{1,p}(\Omega)} \to 0$$

we have by (14.15)

$$\begin{aligned}&\left|\iiint_{\bar\Omega} \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)(v_n - v)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right| \\ &\leqslant C\left\|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right\|_{L_p(\Omega)} \|v_n - v\|_{L_{p/(p-1)}(\Omega)} \to 0 \quad \text{for } n \to \infty,\end{aligned}$$

which results in (14.27). Similarly we prove convergences (14.28). Hence relation (14.16) follows.

To prove (14.17) let us write

$$(14.30)\quad \iiint_{\bar\Omega} u_n \frac{\partial v_n}{\partial x_i}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \iiint_{\bar\Omega}(u_n-u)\Big(\frac{\partial v_n}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 + \iiint_{\bar\Omega} u\Big(\frac{\partial v_n}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 + \iiint_{\bar\Omega}(u_n-u)\frac{\partial v}{\partial x_i}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 + \iiint_{\bar\Omega} u\frac{\partial v}{\partial x_i}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3.$$

If $p \geqslant N$ then $\frac{1}{p}+\frac{1}{N}<1$. Hence by (14.14)

$$\Big|\iiint_{\bar\Omega}(u_n-u)\Big(\frac{\partial v_n}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\Big| \leqslant C\|u_n-u\|_{L_p(\Omega)}\Big\|\frac{\partial v_n}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big\|_{L_N(\Omega)} \leqslant C\|u_n-u\|_{H^{1,p}(\Omega)}\Big\|\frac{\partial v_n}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big\|_{H^{1,N}(\Omega)} \to 0 \quad \text{for } n\to\infty.$$

Similarly we prove that for $n \to \infty$

$$\Big|\iiint_{\bar\Omega} u\Big(\frac{\partial v_n}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\Big| \to 0, \quad \Big|\iiint_{\bar\Omega}(u_n-u)\frac{\partial v}{\partial x_i}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\Big| \to 0.$$

Thus (14.30) implies (14.17) in the case $p \geqslant N$.

Let now $p<N$. Similarly as in (14.25) we set

$$\frac{1}{p^*}=\frac{1}{p}-\frac{1}{N};$$

then we have by (14.26)

$$H^{1,p}(\Omega)\subset L_{p^*}(\Omega) \quad \text{algebraically and topologically.}$$

This means that

$$\|u_n-u\|_{L_{p^*}(\Omega)} \leqslant C\|u_n-u\|_{H^{1,p}(\Omega)}.$$

The inequality $\frac{1}{p^*}+\frac{1}{N}=\frac{1}{p}-\frac{1}{N}+\frac{1}{N}<1$ implies, by virtue of (14.14),

$$\Big|\iiint_{\bar\Omega}(u_n-u)\Big(\frac{\partial v_n}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\Big| \leqslant C\|u_n-u\|_{L_{p^*}(\Omega)}\Big\|\frac{\partial v_n}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big\|_{L_N(\Omega)} \leqslant C\|u_n-u\|_{H^{1,p}(\Omega)}\|v_n-v\|_{H^{1,N}(\Omega)} \to 0 \quad \text{for } n\to\infty.$$

Similarly we prove that for $n \to \infty$

$$\left| \iiint_{\bar\Omega} u\left(\frac{\partial v_n}{\partial x_i} - \frac{\partial v}{\partial x_i}\right) \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \right| \to 0, \quad \left| \iiint_{\bar\Omega} (u_n - u)\frac{\partial v}{\partial x_i} \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \right| \to 0.$$

Thus (14.30) implies (14.17) in the case $p < N$.

D) The proof of (14.16) and (14.17) in the case $q > N$ follows from part C) because

$$H^{1,q}(\Omega) \subset H^{1,N}(\Omega) \quad \text{algebraically and topologically.} \tag{14.31}$$

The proof of (14.31) is a consequence of Hölder's inequality (14.15):

$$\begin{aligned} \|v\|_{L_N(\Omega)} &= \left(\iiint_{\bar\Omega} |v|^N \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right)^{\frac{1}{N}} \\ &\leqslant \left(\left(\iiint_{\bar\Omega} \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right)^{1-\frac{N}{q}} \left(\iiint_{\bar\Omega} (|v|^N)^{\frac{q}{N}} \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right)^{\frac{N}{q}}\right)^{\frac{1}{N}} \\ &= (\operatorname{meas}_N \Omega)^{\frac{1}{N}-\frac{1}{q}} \|v\|_{L_q(\Omega)} \quad \forall v \in L_q(\Omega). \end{aligned}$$

E) It remains to prove (14.23) in the case $p > 1$ and $q \geqslant N$ $(N = 3)$. If $q \geqslant N$ then, according to Theorem 10.9, for any $1 \leqslant q^* < \infty$ we have

$$\|\gamma v\|_{L_{q^*}(\partial\Omega)} \leqslant C\|v\|_{H^{1,q}(\Omega)} \quad \forall v \in H^{1,q}(\Omega). \tag{14.32}$$

Let us choose $p > 1$ arbitrary but fixed and let us distinguish between two cases:

a) The case $1 < p < N$. Using Theorem 10.8 we set $p^* = \frac{Np-p}{N-p}$; then

$$\|\gamma u\|_{L_{p^*}(\partial\Omega)} \leqslant C\|u\|_{H^{1,p}(\Omega)} \quad \forall u \in H^{1,p}(\Omega). \tag{14.33}$$

Let us choose q^* in this case such that

$$\frac{1}{q^*} + \frac{1}{p^*} = 1. \tag{14.34}$$

This means that

$$\frac{1}{q^*} = 1 - \frac{N-p}{Np-p} = \frac{Np-N}{Np-p}.$$

Hence $q^* > 1$ and by (14.34), (14.15), (14.32), (14.33) and (14.11) we have

$$\begin{aligned} \left| \iint_{\partial\Omega} (u_n - u)(v_n - v)n_i \right| \mathrm{d}\sigma &\leqslant C \left| \iint_{\partial\Omega} (u_n - u)(v_n - v)\,\mathrm{d}\sigma \right| \\ &\leqslant C\|u_n - u\|_{L_{p^*}(\partial\Omega)} \|v_n - v\|_{L_{q^*}(\partial\Omega)} \\ &\leqslant C\|u_n - u\|_{H^{1,p}(\Omega)} \|v_n - v\|_{H^{1,q}(\Omega)} \to 0 \end{aligned} \tag{14.35}$$

which yields (14.21). Relations (14.22) can be derived similarly. Hence (14.23) follows in the case $1 < p < N$.

b) The case $p \geqslant N$. Using Theorem 10.9, for any $1 \leqslant p^* < \infty$ we have

$$\|\gamma u\|_{L_{p^*}(\partial\Omega)} \leqslant C\|u\|_{H^{1,p}(\Omega)} \quad \forall u \in H^{1,p}(\Omega). \tag{14.36}$$

By (14.32) and (14.36) we can choose $p^* = q^* = 2$ and derive again (14.35) and then (14.23).

F) Using (14.16), (14.17) and (14.23) proved in parts C)–E), we obtain (14.9) in the case $p > 1$, $q \geqslant N$. □

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