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On Periodic Perturbations of Uniform Motion of Maxwell's Planetary Ring

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We examine the case when equally sized small moons arrange themselves on the vertices of a regular n-gon for $n \geqslant 7$. For $n \geqslant 4$, there are at least 3 pure imaginary characteristic exponents, each of which has multiplicity $= 1$, a surprising result that makes it possible to apply the Lyapunov center theorem to verify the existence of some periodic perturbations. For sufficiently large n, when the regular n-gon is the unique central configuration, the number of families of periodic perturbations is at least equal to $2n - \lfloor (n+1)/4 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

KEY WORDS: Lyapunov center theorem; Maxwell's planetary ring, periodic perturbations; relative equilibrium.

1. INTRODUCTION

In the essay "On the Stability of the Motion of Saturn's Rings," which James Clerk Maxwell submitted for the Adams Prize in 1855, he discussed the linear stability of a singular planetary ring system made up of moons of the same mass surrounding a central mass. He formulated the necessary condition that the central mass be sufficiently larger than the total mass of the moons (Maxwell, 1983). His analysis of the system, which was a regular polygonal configuration, also yielded the following equation for the characteristic exponents of the linearized equations of motion:

$$\lambda^4 + a_1 \lambda^2 + a_2 = 0$$

where a_1 and a_2 are constants that depended on the central mass, the angular velocity, the mass of each moon, and the number of moons, n.

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Because of a minor error in his computations, allowing Maxwell to miscalculate one of the characteristic exponents, he missed discovering a necessary condition for the linear stability of the ring system, that $n \geqslant 7$, which Moeckel (1994) proved.

This ring system, which is sometimes referred to as Maxwell's Ring, is an example of what we call a *relative equilibrium* because, for an appropriately chosen uniformly rotating coordinate system, Maxwell's ring becomes a restpoint of Newton's equations of motion. Hall (1997) detailed a necessary condition for the existence of relative equilibria emanating from a limiting configuration where the mass of each moon has tended to 0. Dealing with the so-called $(1+n)$-body problem, Hall showed that the relative equilibria of the $(1+n)$-body problem must necessarily be a critical point of some potential function. He proved that, for sufficiently large values of n, the only relative equilibrium that can exist is Maxwell's ring. He also showed that there can be other relative equilibria when n is small. As an example, when $n=3$, there are two other relative equilibria besides Maxwell's ring.

The aim of this paper is to determine the existence of periodic perturbations to Maxwell's ring. The main result used is the Lyapunov center theorem, which requires certain information about the eigenvalues of the linearized equations. It is shown that, when n is sufficiently large, the eigenvalues turn out to be pure imaginary and distinct except for a pair of null eigenvalues. This is actually enough to show that there is at least 1 periodic perturbation. But it is shown that the actual number is at least $2n - \lfloor (n+1)/4 \rfloor$ where $\lfloor x \rfloor =$ the greatest integer less than or equal to x.

2. PRELIMINARY EQUATIONS

2.1. The Equations of Motion

For $i, j = 0, 1,..., n$, let $x_i, x_j \in \mathbf{R}^2$ and $r_{ij} = |x_i - x_j|$, the Euclidean distance between x_i and x_j. Consider the Newtonian potential function U with

$$U(x) = \sum_{i<j} \frac{m_i m_j}{r_{ij}} \tag{1}$$

where $m_i =$ the mass of the body at x_i. The equations of motion for the Newtonian $1+n$ body problem is given by

$$m_i \ddot{x}_i = \nabla_i U(x) \tag{2}$$

where ∇_i represents the partial gradient with respect to x_i.

Let $x_i = (x_{i1}, x_{i2}) = (r_i \cos \theta_i, r_i \sin \theta_i)$. Using the chain rule,

$$\begin{cases} \ddot{x}_{i1} = \ddot{r}_i \cos \theta_i - 2\dot{r}_i \dot{\theta}_i \sin \theta_i - r_i \dot{\theta}_i^2 \cos \theta_i - r_i \ddot{\theta}_i \sin \theta_i \\ \ddot{x}_{i2} = \ddot{r}_i \sin \theta_i + 2\dot{r}_i \dot{\theta}_i \cos \theta_i - r_i \dot{\theta}_i^2 \sin \theta_i + r_i \ddot{\theta}_i \cos \theta_i \end{cases} \tag{3}$$

Also,

$$(\partial U/\partial r_i, \partial U/\partial \theta_i) = (\partial U/\partial x_{i1}, \partial U/\partial x_{i2}) \begin{pmatrix} \cos \theta_i & -r_i \sin \theta_i \\ \sin \theta_i & r_i \cos \theta_i \end{pmatrix}$$

$$= (m_i \ddot{x}_{i1}, m_i \ddot{x}_{i2}) \begin{pmatrix} \cos \theta_i & -r_i \sin \theta_i \\ \sin \theta_i & r_i \cos \theta_i \end{pmatrix}$$

Define $\bar{F}_i(r, \theta) = (1/m_i)(\partial U/\partial r_i)$ and $\bar{G}_i(r, \theta) = (1/m_i r_i)(\partial U/\partial \theta_i)$. Hence, by Eq. (3),

$$\begin{cases} \bar{F}_i(r, \theta) = \ddot{r}_i - r_i \dot{\theta}_i^2 \\ \bar{G}_i(r, \theta) = r_i \ddot{\theta}_i + 2\dot{r}_i \dot{\theta}_i \end{cases} \tag{4}$$

which is a system of $2n + 2$ equations.

For a given relative equilibrium, we can choose $r_i(t) = \hat{r}_i$ and $\theta_i(t) = \omega t + \hat{\theta}_i$, where $\hat{r}_i$, $\hat{\theta}_i$, and ω are real and depend only on m_0, $m_1, ..., m_n$. Thus at a relative equilibrium, it follows from Eq. (4) that

$$\bar{F}_i(r, \theta) = -\omega^2 \hat{r}_i \qquad \text{and} \qquad \bar{G}_i(r, \theta) = 0$$

Let $m_0 = 1$ and $m_1 = m_2 = \cdots = m_n = \varepsilon$, where $0 < \varepsilon \ll 1$, and without loss of generality, set the center of mass at the origin, $\sum_{i=0}^{n} m_i x_i = 0$, so that

$$x_0 = -\varepsilon \sum_{i=1}^{n} x_i \tag{5}$$

For each $\varepsilon > 0$, denote a corresponding relative equilibrium x^ε and let x^0 be the limit of such equilibria as $\varepsilon \to 0$. Hall found that x^0 must necessarily be a critical point of the function

$$V(\theta) = [\tilde{U}(r, \theta) + \tilde{I}(r, \theta)]|_{r = (1, 1, 1, ..., 1)}$$

where

$$\tilde{U}(r, \theta) = \sum_{0 < j < k} \frac{1}{r_{jk}} \qquad \text{and} \qquad \tilde{I}(r, \theta) = \frac{1}{2} \sum_{0 < j < k} r_{jk}^2$$

By treating x^0 as a critical point of V, it is possible to prove the existence of certain families of relative equilibria for $\varepsilon > 0$ emanating from x^0 whenever the nullity of the Hessian matrix $(\partial^2 V/\partial\theta_i\,\partial\theta_k)$ at x^0 is at most 1. For example, the nullity of this matrix for Maxwell's ring is known to be exactly 1. Furthermore, we can discard the lone 0 eigenvalue by using a technique outlined by Siegel and Moser (1971). As a result, the families of relative equilibria turn out to be analytic in ε as well. By choosing an appropriate uniformly rotating coordinate system, we simply set

$$\left.\begin{aligned} \hat{r}_i &= 1 \\ \hat{\theta}_i &= \frac{2\pi i}{n} \end{aligned}\right\} \quad \text{for} \quad i = 1, 2, \ldots, n \quad \text{and} \quad \omega = 1 + \frac{1}{2}\varepsilon\,\omega_1 + O(\varepsilon^2) \tag{6}$$

for Maxwell's ring.

In order to examine the nonzero eigenvalues for the rest of paper, we need the following important theorem. Let $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

Theorem 1 (Lyapunov Center Theorem). *Consider a Hamiltonian system $\dot{w} = J\nabla H(w)$, where $H(w) = \frac{1}{2} w^T K w + O(|w|^2)$, a real power series in some neighborhood of $w = 0$ with JK having the $2n$ eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n, -\lambda_1, -\lambda_2, \ldots, -\lambda_n$. Let λ_1 be pure imaginary, with λ_j/λ_1 not an integer for $j > 1$. Then there exists a family of real periodic solutions to the Hamiltonian system which are analytic with respect to a real parameter $a \geqslant 0$ with period $\tau(a)$ which is also analytic in a and $\tau(0) = 2\pi/|\lambda_1|$.*

The reader should refer to Siegel and Moser (1971) or Chow and Hale (1982) for the details of the proof of the theorem, which we refer to as LCT.

2.2. The Main Results

We now present the main results that we wish to prove. Because of the symmetries of Maxwell's ring, it is not unreasonable to think that certain eigenvalues may be repeated, rendering the LCT useless and perhaps making the problem of deciding the validity of Theorem 1 extremely difficult if not impossible to resolve. Fortunately, this was not the case as indicated by the following surprising result.

Theorem 2. *The nonzero eigenvalues of Maxwell's ring for $n \geqslant 7$ are distinct. This theorem makes it possible for us to conclude that periodic perturbations to Maxwell's ring can actually exist.*

Definition 1. An open subset $\mathscr{I}$ of $\mathbf{R}$ is said to be a segmented interval if $\mathscr{I}=\bigcup_{k=1}^{\infty}(\varepsilon_k, \varepsilon_{k+1})$, where $\varepsilon_k \to 0$ as $k \to \infty$. If $\mathscr{I}$ is nonempty, then we say that $\mathscr{I}$ is a *n.s.i.* (nonempty segmented interval).

Theorem 3. *For Maxwell's ring and $n \geqslant 7$, there exists a n.s.i. $\mathscr{I}$ such that, whenever $\varepsilon \in \mathscr{I}$, there exists at least one family of periodic perturbations to uniform motion. These perturbations are analytic in ε.*

Finally, we have the following.

Theorem 4. *For Maxwell's ring and $n \geqslant 7$, there are at least $2n - \lfloor (n+1)/4 \rfloor$ families of analytic and periodic perturbations.*

The eigenvalues that we have to study are the eigenvalues of the linearization of (4) *after* we eliminate the 0th body from the equations. We accomplish this by using (5), leading to the following set of equations: for $i = 1$ to n,

$$\begin{cases} \bar{F}_i = \bar{F}_i(r_0(r_1,..., r_n, \theta_1,..., \theta_n), \quad r_1,..., r_n, \\ \qquad \theta_0(r_1,..., r_n, \theta_1,..., \theta_n), \quad \theta_1,..., \theta_n) \\ \bar{G}_i = \bar{G}_i(r_0(r_1,..., r_n, \theta_1,..., \theta_n), \quad r_1,..., r_n, \\ \qquad \theta_0(r_1,..., r_n, \theta_1,..., \theta_n), \quad \theta_1,..., \theta_n) \end{cases} \tag{7}$$

which is a reduced system of $2n$ equations. Let $\mu = \dot{r}$ and $v = \dot{\theta}$. Then the first-order system of differential equations corresponding to (4) is given by

$$(\dot{r}_i, \dot{\mu}_i, \dot{\theta}_i, \dot{v}_i) = \left(\mu_i, \bar{F}_i + r_i v_i^2, v_i, \frac{\bar{G} - 2\mu_i v_i}{r_i}\right) \quad \text{for} \quad i = 1, 2,..., n \tag{8}$$

Using Eq. (7), we define the following matrices:

$$F_r = \left(\frac{\partial \bar{F}_i}{\partial r_j} + \frac{\partial \bar{F}_i}{\partial r_0}\frac{\partial r_0}{\partial r_j} + \frac{\partial \bar{F}_i}{\partial \theta_0}\frac{\partial \theta_0}{\partial r_j}\right), \qquad F_\theta = \left(\frac{\partial \bar{F}_i}{\partial \theta_j} + \frac{\partial \bar{F}_i}{\partial r_0}\frac{\partial r_0}{\partial \theta_j} + \frac{\partial \bar{F}_i}{\partial \theta_0}\frac{\partial \theta_0}{\partial \theta_j}\right)$$

$$G_r = \left(\frac{\partial \bar{G}_i}{\partial r_j} + \frac{\partial \bar{G}_i}{\partial r_0}\frac{\partial r_0}{\partial r_j} + \frac{\partial \bar{G}_i}{\partial \theta_0}\frac{\partial \theta_0}{\partial r_j}\right), \qquad G_r = \left(\frac{\partial \bar{G}_i}{\partial \theta_j} + \frac{\partial \bar{G}_i}{\partial r_0}\frac{\partial r_0}{\partial \theta_j} + \frac{\partial \bar{G}_i}{\partial \theta_0}\frac{\partial \theta_0}{\partial \theta_j}\right)$$

It is important to realize that F_θ is not the same as G_r and that the matrices above are *not* submatrices of the Hessian of U.

If we linearize (8) about the relative equilibrium where $r_i = 1$, $\theta_i = (2\pi i/n)$, and $\dot{\theta} = \omega$, the resulting system is given by

$$\begin{pmatrix} \dot{r} \\ \dot{\mu} \\ \dot{\theta} \\ \dot{\nu} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & 0 \\ F_r + \omega^2 I & 0 & F_\theta & 2\omega I \\ 0 & 0 & 0 & I \\ G_r & -2\omega I & G_\theta & 0 \end{pmatrix} \begin{pmatrix} r \\ \mu \\ \theta \\ \nu \end{pmatrix}$$

The eigenvalues that we need will be precisely the eigenvalues of the coefficient matrix above evaluated at Maxwell's ring.

The equation for the eigenvalues λ is, then, given by

$$P(\lambda) = \det \begin{pmatrix} F_r + (\omega^2 - \lambda^2) I & F_\theta + 2\omega\lambda I \\ G_r - 2\omega\lambda I & G_\theta - \lambda^2 I \end{pmatrix} = 0 \tag{9}$$

$P(\lambda)$ is known to be an even polynomial (Siegel and Moser, 1971).

Near $\varepsilon = 0$, we find that

$$\begin{cases} F_r = 2I + \varepsilon A + O(\varepsilon^2), & G_\theta = \varepsilon B + O(\varepsilon^2) \\ F_\theta = \varepsilon C + O(\varepsilon^2), & G_r = \varepsilon D + O(\varepsilon^2) \end{cases} \tag{10}$$

where A, B, C, and D, are independent of ε. With $\omega = 1 + O(\varepsilon)$, it follows that

$$P(\lambda) = \det \begin{pmatrix} (3 - \lambda^2) I + O(\varepsilon), & 2\lambda I + O(\varepsilon) \\ -2\lambda I + O(\varepsilon), & -\lambda^2 I + O(\varepsilon) \end{pmatrix} = \lambda^{2n}(\lambda^2 + 1)^n + O(\varepsilon) \tag{11}$$

which leads us to believe that

$$\lambda^2 = -\varepsilon\zeta + O(\varepsilon^{3/2}) \qquad \text{or} \qquad \lambda^2 = -1 + \varepsilon\zeta + O(\varepsilon^2)$$

Each of the eigenvalues does have one of these forms, and we classify these eigenvalues as class (0) or class (-1), respectively. A few lemmas are now in order.

Lemma 1. *Let λ_k be a nonzero class (0) eigenvalue. Suppose that $\zeta_k = \zeta_j$ implies that either $j = k$ or $\bar{\lambda}_k = \lambda_j$. If $\zeta_k \in \mathbf{R}^+$, then λ_k is pure imaginary.*

Proof. Let $\zeta_k = \zeta_j$ with $j \neq k$. Then $\bar{\lambda}_k = \lambda_j$. Also, $\lambda_m = -\lambda_k$ is an eigenvalue since $P(\lambda)$ is an even polynomial. Because $\zeta_k = \zeta_m$, $\bar{\lambda}_k = \lambda_m = -\lambda_k$. This can happen only if λ_k is pure imaginary. □

A similar proof leads to the next lemma.

Lemma 2. *Let λ_k be class (-1). Furthermore, suppose that $\zeta_k = \zeta_j$ implies that either $j = k$ or $\bar{\lambda}_k = \lambda_j$. If $\zeta_k \in \mathbf{R}$, then λ_k is pure imaginary.*

Remark 1. Note that if λ_k is class (0) and λ_j is class (-1), we can see that $(\lambda_k^2/\lambda_j^2)$, cannot be an integer for small values of $\varepsilon > 0$.

Suppose

$$\frac{\lambda_j^2}{\lambda_k^2} = \frac{-1 + \varepsilon\zeta_j + O(\varepsilon^2)}{-\varepsilon\zeta_k + O(\varepsilon^{3/2})} = m$$

If ε is sufficiently small and $\zeta_k > 0$, then

$$\frac{1}{\varepsilon\zeta_k} + O(1) = m \tag{12}$$

For sufficiently small values of ε, the left side of Eq. (12) is monotonically increasing as $\varepsilon \to 0$. Thus, there exists $\varepsilon_{0j} > 0$ such that, given $m \in \mathbf{Z}$, there is at most one value of $\varepsilon \in (0, \varepsilon_{0j})$ that satisfies (12). Furthermore, these values of ε, as m ranges over all the integers, are isolated. Since, for each n, there is only a finite number of class (-1) eigenvalues, we can find $\varepsilon_0 = \min_j \{\varepsilon_{0j}\}$ that will work for all class (-1) λ_j. As such we have just proven the next lemma.

Lemma 3. *Let λ_j be class (-1) and $\lambda_k \neq 0$ be class (0). Then there exists a n.s.i. $\mathcal{J}$ such that, for every $\varepsilon \in \mathcal{J}$, (λ_j/λ_k) is neither an integer nor the reciprocal of an integer.*

We, also, have the following result.

Lemma 4. *Let λ_j and λ_k be two class (-1) eigenvalues with $\zeta_j \neq \zeta_k$. Then there exists a n.s.i. $\mathcal{J}$ such that, for every $\varepsilon \in \mathcal{J}$, (λ_j/λ_k) is neither an integer nor the reciprocal of an integer.*

Proof. Suppose $\zeta_k < \zeta_j$. If $\varepsilon > 0$ is sufficiently small so that $|\lambda_k^2 + 1|$, $|\lambda_j^2 + 1| < \frac{1}{2}$, then $(\lambda_j^2/\lambda_k^2) = (1 - \varepsilon\zeta_j + O(\varepsilon^2))/(1 - \varepsilon\zeta_k + O(\varepsilon^2)) = 1 + \varepsilon(\zeta_k - \zeta_j) + (\varepsilon^2)$. We can choose ε so that

$$\begin{cases} 0 < \varepsilon < \dfrac{1}{3} \\ 0 < \varepsilon(\zeta_k - \zeta_j) < \dfrac{1}{3} \\ \zeta_k - \zeta_j > \left| \dfrac{1}{\varepsilon} O(\varepsilon^2) \right| \end{cases} \tag{13}$$

in the last equation. Therefore, $1 < (\lambda_j^2/\lambda_k^2) < 1 + \frac{1}{3} + \frac{1}{3}$. So $(\lambda_j^2/\lambda_k^2)$ cannot be an integer. Obviously, the reciprocal is also not an integer. □

Eventually, we need to examine the higher-order terms of λ and, therefore, the matrices A, B, C and D in Eq. (10). Let

$$\begin{cases} \mathbf{C}_{ij} = \cos\theta_{ij}, & \mathbf{S}_{ij} = \sin\theta_{ij} \\ \mathbf{C}_j = \cos\left[\frac{\pi j}{n}\right], & \mathbf{S}_j = \sin\left[\frac{\pi j}{n}\right] \end{cases} \tag{14}$$

where $\theta_{ij} = \theta_j - \theta_i$. If $A = (a_{ij})$, etc., then for Maxwell's ring,

$$a_{ij} = \begin{cases} 2 - \sum_{k \neq i} \frac{1 - 3\mathbf{C}_{ik}}{16\,|\sin^3[(\pi/n)(k-i)]|} & \text{for } i = j \\ 2\mathbf{C}_{ij} + \frac{3 - \mathbf{C}_{ij}}{16\,|\sin^3[(\pi/n)(j-i)]|} & \text{for } i \neq j \end{cases} \tag{15a}$$

$$b_{ij} = \begin{cases} \sum_{k \neq i} \left\{ \mathbf{C}_{ik} + \frac{3 + \mathbf{C}_{ik}}{16\,|\sin^3[(\pi/n)(k-i)]|} \right\} & \text{for } i = j \\ -\mathbf{C}_{ij} - \frac{3 + \mathbf{C}_{ij}}{16\,|\sin^3[(\pi/n)(j-i)]|} & \text{for } i \neq j \end{cases} \tag{15b}$$

$$c_{ij} - d_{ij} = \begin{cases} 0 & \text{for } i = j \\ \frac{\mathbf{S}_{ij}}{8\,|\sin^3[(\pi/n)(j-i)]|} - \mathbf{S}_{ij} & \text{for } i \neq j \end{cases} \tag{15c}$$

These formulas were obtained with the help of Mathematica. Note that $C - D$ is skew-symmetric.

3. MAXWELL RING

3.1. Preliminaries

Maxwell had already shown that when $n \geqslant 7$ and if ε is small enough, then with two exceptions that he left out, the eigenvalues of the linearized system are 0 or pure imaginary. The two unaccounted eigenvalues turned out to be a conjugate pair of pure imaginary eigenvalues. We also remark that if $3 \leqslant n \leqslant 6$, exactly four of the eigenvalues have nonzero real parts (Moeckel, 1997). Let $r_j = 1$ and $\theta_j = (2\pi j/n)$ for $j = 1, 2, \ldots, n$.

Lemma 5. *For $n \geqslant 7$ and for sufficiently small $\varepsilon > 0$, except for a pair of zero eigenvalues, all the roots of $P(\lambda) = 0$ are distinct.*

The proof of the preceding lemma is spread throughout the rest of the paper. First, we provide a few preliminary definitions and lemmas. For $n \geqslant 4$ and $0 \leqslant k \leqslant n-1$, let

$$\sigma_k = \sum_{j=1}^{n-1} \frac{1}{S_j^2} \left\{ \sin \left[\frac{\pi j}{n} (2k-1) \right] \right\} \tag{16}$$

Note that $\sigma_0 = -\sigma_1$.

Lemma 6. *For $n \geqslant 7$,*

(a) If $1 \leqslant k \leqslant (n+1)/2$, then $\sigma_k \geqslant 0$. Equality holds only if $k = (n+1)/2$.

(b) If $1 < k < n$, then $\sigma_k = -\sigma_{n-k+1}$.

(c) If $1 \leqslant k \leqslant (n-1)/2$, then $\sigma_{k+1} - \sigma_k = \sum_{j=1}^{n-1} (2/S_j) \cos[(2\pi jk/n)]$.

(d) If $1 \leqslant k \leqslant (n-1)/2$, then $\sigma_k - \sigma_{k-1} > \sigma_{k+1} - \sigma_k$.

(e) $\sigma_2 > \sigma_1$ for $n \geqslant 7$.

(f) $\sigma_1 < (2n/\pi) \sum_{j=1}^{(n-1)/2} (1/j)$ for $n \geqslant 3$.

Proof. The proofs of parts (a), (b), and (c) of the lemma are given by Perko and Walter (1985). Also, the proof for part (d) was done for $2 \leqslant k \leqslant (n-1)/2$. For $k = 1$, using part (c) and the definition of σ_0, we have that

$$\sigma_1 - \sigma_0 = \sigma_1 + \sigma_1 = \sum_{j=1}^{n-1} \frac{2}{S_j} > \sum_{j=1}^{n-1} \frac{2}{S_j} \cos \left[\frac{2\pi j}{n} \right] = \sigma_2 - \sigma_1$$

which completes the proof of part (d).

For part (e),

$$\sigma_2 - \sigma_1 = \sum_{j=1}^{n-1} \frac{2}{S_j} \cos \left[\frac{2\pi j}{n} \right] = 2 \sum_{j=1}^{n=1} \left[\frac{1}{S_j} - 2S_j \right]$$

In order to estimate the last quantity, we apply the trapezoidal rule to obtain the following:

$$4 = 2 \int_0^{\pi} \sin x \, dx = \frac{\pi}{n} \left\{ \sin 0 + 2 \sum_{j=1}^{n-1} \sin \frac{\pi j}{n} + \sin \pi \right\} + \Delta, \qquad \text{where } |\Delta| \leqslant \frac{\pi^3}{6n^2}$$

Hence, $(2\pi/n) \sum_{j=1}^{n-1} S_j \leqslant 4 + (\pi^3/6n^2)$ so that $2 \sum_{j=1}^{n-1} S_j \leqslant (4n/\pi) + (\pi^2/6n) \leqslant (4n/\pi) + \frac{1}{2}$ if $n \geqslant 4$. Since $0 < (\pi j/2n) < \pi/2$ for $j = 1, 2, ..., n-1$,

$$
\begin{aligned}
2\sum_{j=1}^{n-1} \mathbf{S}_j^{-1} &\geqslant \sum_{j=1}^{n-1}\left\{\sin\frac{\pi j}{2n}\cos\frac{\pi j}{2n}\right\}^{-1} \\
&\geqslant \sum_{j=1}^{n-1}\left\{\sin\frac{\pi j}{2n}\right\}^{-1} \\
&\geqslant 2\sum_{j=1}^{(n-1/2)}\left\{\sin\frac{\pi j}{2n}\right\}^{-1} \\
&\geqslant \frac{4n}{\pi}\sum_{j=1}^{(n-1/2)} j^{-1} \qquad (17)
\end{aligned}
$$

for $n \geqslant 3$. The last quantity is bigger than $8.32n/\pi$ if $n \geqslant 9$. Therefore,

$$2\sum_{j=1}^{n-1}\left[\frac{1}{\mathbf{S}_j} - 2\mathbf{S}_j\right] \geqslant \frac{8.32n}{\pi} - 2\left(\frac{4n}{\pi} + \frac{1}{2}\right) > 0.101n - 1 > 0$$

if $n \geqslant 10$. We can directly verify that part (e) also holds for $n = 7$, 8, and 9.

Part (f) follows from (17). □

Definition 2. Let $y = f(x)$ be defined on an open interval (a, b). We say that f is *concave* if $f(x_0 + \eta(x_1 - x_0)) \geqslant f(x_0) + \eta(f(x_1) - f(x_0))$ for $x_0, x_1 \in (a, b)$ with $x_0 < x_1$ and $\eta \in (0, 1)$ (Fig. 1).

Corollary 1. *Let $0 \leqslant k \leqslant (n + 1/2)$. If the point (k, σ_k) is connected to the point $(k+1, \sigma_{k+1})$ by a line segment L_k, then $\bigcup_k L_k$ is a concave graph.*

Proof. Since $\bigcup_k L$ is a polygonal curve determined by the vertices (k, σ_k), we need only to show that $\sigma_k \geqslant (\sigma_{k+1} + \sigma_{k-1})/2$. This is immediate from Lemma 6(d). □

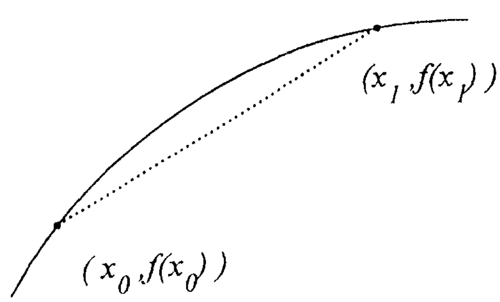


Fig. 1. The graph of a concave function.

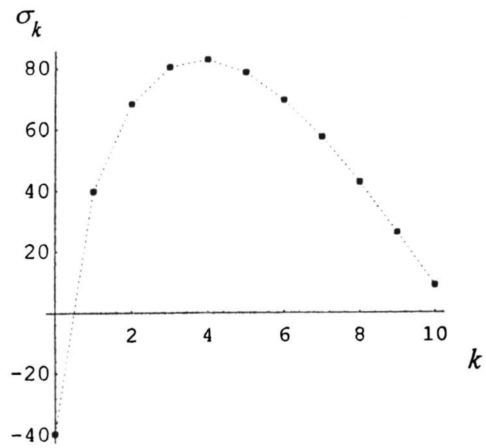


Fig. 2. The graph of $\bigcup_k L_k$ for $0 \leqslant k \leqslant 10 = (n/2)$ for $n = 20$.

The graph of $\bigcup_k L$ for $n = 20$ for $0 \leqslant k \leqslant 10 = (n/2)$ is given in Fig. 2.

Corollary 2. *Let* $1 \leqslant k \leqslant (n-1)/2$ *with* $k \neq (n/2) - 1$. *Then* $\sigma_{k+1} - \sigma_{k-1} > \sigma_{k+2} - \sigma_k$.

Proof. Using Lemma 6(d), we have $\sigma_k - \sigma_{k-1} > \sigma_{k+2} - \sigma_{k+1}$ for $1 \leqslant k \leqslant (n-3)/2$. By Lemma 6(a, b, d), this is also true if $k = (n-1)/2$. The corollary follows immediately. □

3.2. The Class (−1) Eigenvalues

Using Eq. (9),

$$P(\lambda) = \det\{(F_r + (\omega^2 - \lambda^2) I - (F_\theta + 2\omega\lambda I)(G_\theta - \lambda^2 I)^{-1} (G_r - 2\omega\lambda I)\}$$
$$\stackrel{\text{def}}{=} \det(E) = 0 \tag{18}$$

If $\lambda^2 \approx -1$, let $\lambda^2 = -1 + \varepsilon\hat{\zeta}$ where $\varepsilon\hat{\zeta} \to 0$ as $\varepsilon \to 0$. If $\varepsilon > 0$ is small enough, then $\lambda = \pm i[1 - \frac{1}{2}\varepsilon\hat{\zeta}]$. We can also choose ε to be small enough that $\lambda^2 < -\frac{1}{2}$ so that $(G_\theta - \lambda^2 I)^{-1} = (\varepsilon B - \lambda^2 I + O(\varepsilon^2))^{-1}$ exists and is given by

$$(G_\theta - \lambda^2 I)^{-1} = -\frac{1}{\lambda^2}\left\{I + \frac{\varepsilon}{\lambda^2} B\right\} + O(\varepsilon^2)$$

Using this equation and Eqs. (10) and (18),

$$\lambda^2 E = \varepsilon\{[\hat{\zeta} - 3\omega_1] I - [A + 4B \pm 2i(C - D)]\} + O(\varepsilon^2) \tag{19}$$

Hence, examining the $O(\varepsilon)$ part of Eq. (19), it must be that $\hat{\zeta} - 3\omega_1$ tends to an eigenvalue $\zeta - 3\omega_1$ of the matrix $A + 4B \pm 2i(C - D)$. It is enough to consider the eigenvalues of $A + 4B + 2i(C - D)$. In order to see this, we need to exploit the fact that the matrices A, B, C, and D are all *circulant*.

Definition 3. We say that an $n \times n$ matrix $Q = (q_{jk})$ is *circulant* if and only if $q_{(j+1, k+1)} = q_{jk}$, where we take $q_{(n+j, k)} = q_{(j, n+k)} = q_{jk}$.

The eigenvalues e_k and the eigenvectors $\vec{v}_k$ for $k = 0, 1, 2,..., n-1$ of such matrices are given by

$$\begin{cases} e_k = \sum_{j=0}^{n-1} q_{(1, j+1)} \rho^{jk} \\ \vec{v}_k = (1, \bar{\rho}, ..., (\bar{\rho})^{k(n-1)})^T \end{cases} \quad \text{where} \quad \rho = e^{2\pi/n} \quad \text{(Perko and Walter, 1985)} \tag{20}$$

Because they have the same eigenvectors, A, B, C, and D are simultaneously diagonalizable. Thus, $\zeta_k - 3\omega_1 = \alpha_k + 4\beta_k + 4\gamma_k$, where

$$\begin{aligned} \alpha_k &= \sum_{j=0}^{n-1} a_{(1, j+1)} \rho^{jk}, \\ \beta_k &= \sum_{j=0}^{n-1} b_{(1, j+1)} \rho^{jk} \\ \gamma_k &= \frac{i}{2} \sum_{j=0}^{n-1} [c_{(1, j+1)} - d_{(1, j+1)}] \rho^{jk} \end{aligned} \tag{21}$$

are the eigenvalues of A, B, and $\frac{1}{2}(C - D)$, respectively. Let

$$\alpha_n = \sum_{j=0}^{n-1} a_{(1, j+1)}$$

then $\alpha_n = \alpha_0$. Similarly, $\beta_n = \beta_0$ and $\gamma_n = \gamma_0$. We now have the following lemma.

***Lemma** 7. For $n \geqslant 3$ and for $k = 0, 1,..., n-1$, we have*

(a) $\alpha_k = \alpha_{n-k}$,

(b) $\beta_k = \beta_{n-k}$, *and*

(c) $\gamma_k = -\gamma_{n-k}$.

Proof. Using Eqs. (15) and (19),

$$\alpha_k = a_{11} + \sum_{j=1}^{n-1} \left[2\mathbf{C}_{(1,j+1)} + \frac{3-\mathbf{C}_{(1,j+1)}}{16\mathbf{S}_j^3} \right] \rho^{jk}$$
$$= a_{11} + \sum_{j=1}^{n-1} \left[2\mathbf{C}_{(1,j+1)} + \frac{3-\mathbf{C}_{(1,j+1)}}{16\mathbf{S}_j^3} \right] \cos\left[\frac{2\pi jk}{n}\right]$$

The imaginary part has dropped out from the last equation because of symmetries. Since $\cos[(2\pi jk/n)] = \cos[(2\pi j(n-k)/n)]$,

$$\alpha_k = a_{11} + \sum_{j=1}^{n-1} \left[2\mathbf{C}_{(1,j+1)} + \frac{3\mathbf{C}_{(1,j+1)}}{16\mathbf{S}_j^3} \right] \cos\left[\frac{2\pi j(n-k)}{n}\right] = \alpha_{n-k}$$

which proves part (a) of the lemma. A similar procedure can be used to show that parts (b) and (c) are also true. □

Corollary 3. $\gamma_0 = 0$.

Proof. From part (c) of the preceding lemma, $\gamma_0 = -\gamma_0$. □

Remark 2. Note that with part (c) of the preceding lemma, we need only to consider the eigenvalues of the matrix $A + 4B + 2i(C - D)$.

From (15) and (19), the eigenvalues of $A + 4B + 2i(C - D)$ are

$$\zeta_k - 3\omega_1 = a_{11} + 4b_{11} - \sum_{j=1}^{n-1} \left[2\mathbf{C}_{(1,j+1)} + \frac{9+5\mathbf{C}_{(1,j+1)}}{16\mathbf{S}_j^3} - \frac{i}{4}\frac{\mathbf{S}_{(1,j+1)}}{\mathbf{S}_j^3} + 2i\mathbf{S}_{(1,j+1)} \right] \rho^{jk}$$
$$= a_{11} + 4b_{11} - \sum_{j=1}^{n-1} \left[2(\mathbf{C}_{(1,j+1)} + i\mathbf{S}_{(1,j+1)}) + \frac{9+5\mathbf{C}_{(1,j+1)} - 4i\mathbf{S}_{(1,j+1)}}{16\mathbf{S}_j^3} \right] \rho^{jk}$$
$$= a_{11} + 4b_{11} - \sum_{j=1}^{n-1} \left[2\rho^j + \frac{18 + \rho^j + 9\rho^{-j}}{32\mathbf{S}_j^3} \right] \rho^{jk} \qquad (22)$$

Lemma 8. *If* $n \geqslant 7$, $\zeta_k - \zeta_{n-k+1}$ *for* $1 \leqslant k \leqslant n/2$.

Proof. By (22),

$$\zeta_k - \zeta_{n-k+1} = -\sum_{j=1}^{n-1} \left[2\rho^j + \frac{18+\rho^j+9\rho^{-j}}{32\mathbf{S}_j^3} \right] (\rho^{jk} - \rho^{-j(k-1)})$$

$$= \sum_{j=0}^{n-1} (-2\rho^{j(k+1)} + 2\rho^{-j(k-2)})$$

$$- \sum_{j=1}^{n-1} \frac{1}{32\mathbf{S}_j^3} (18\rho^{j/2} + \rho^{3j/2} + 9\rho^{-j/2})(\rho^{j(k-1/2)} - \rho^{-j(k-1/2)})$$

$$= 2n\delta_{(2,k)} - \sum_{j=1}^{n-1} \frac{i}{16\mathbf{S}_j^3} (18(\mathbf{C}_j + i\mathbf{S}_j) + (\mathbf{C}_j + i\mathbf{S}_j)^3$$

$$+ 9(\mathbf{C}_j - i\mathbf{S}_j)) \sin\left[\frac{(2k-1)\pi j}{n}\right]$$

$$= 2n\delta_{(2,k)} - \sum_{j=1}^{n-1} \frac{i}{16\mathbf{S}_j^3} (27\mathbf{C}_j + 9i\mathbf{S}_j$$

$$+ \mathbf{C}_j^3 + 3i\mathbf{C}_j^2\mathbf{S}_j - 3\mathbf{C}_j\mathbf{S}_j^2 - i\mathbf{S}_j^3) \sin\left[\frac{(2k-1)\pi j}{n}\right] \tag{23}$$

where

$$\delta_{(j,k)} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Only the real terms remain in (23), giving the following:

$$\zeta_k - \zeta_{n-k+1} = 2n\delta_{(2,k)} + \sum_{j=1}^{n-1} \frac{1}{16\mathbf{S}_j^2} (9 + 3\mathbf{C}_j^2 - \mathbf{S}_j^2) \sin\left[\frac{(2k-1)\pi j}{n}\right]$$

$$= 2n\delta_{(2,k)} + \sum_{j=1}^{n-1} \frac{1}{16\mathbf{S}_j^2} (10 + 2\mathbf{C}_j^2 - 2\mathbf{S}_j^2) \sin\left[\frac{(2k-1)\pi j}{n}\right]$$

$$= 2n\delta_{2,k)} + \sum_{j=1}^{n-1} \frac{1}{8\mathbf{S}_j^2} \left(5 + \cos\left[\frac{2\pi j}{n}\right]\right) \sin\left[\frac{(2k-1)\pi j}{n}\right]$$

$$= 2n\delta_{(2,k)} + \sum_{j=1}^{n-1} \frac{1}{8\mathbf{S}_j^2} \left(5\sin\left[\frac{(2k-1)\pi j}{n}\right]\right.$$

$$\left. + \frac{1}{2}\sin\left[\frac{(2k+1)\pi j}{n}\right] + \frac{1}{2}\sin\left[\frac{(2k-3)\pi j}{n}\right]\right) \tag{24}$$

Thus from (16),

$$\zeta_k - \zeta_{n-k+1} = 2n\delta_{(2,k)} + \tfrac{5}{8}\sigma_k + \tfrac{1}{16}\sigma_{k+1} + \tfrac{1}{16}\sigma_{k-1} \tag{25}$$

By Lemma 6(a), the right-hand side of (25) is positive if $1 < k < (n/2)$. Using Lemma 6(b) for $k = (n/2)$, $\zeta_{(n/2)} - \zeta_{(n/2)+1} = \frac{5}{8}\sigma_{(n/2)} + \frac{1}{16}\sigma_{(n/2)+1} + \frac{1}{16}\sigma_{(n/2)-1} = \frac{9}{16}\sigma_{(n/2)} + \frac{1}{16}\sigma_{(n/2)-1}$, which is also positive.

For $k = 1$, set $\zeta_n = \zeta_0$ and use Eq. (24) and Lemma 6(b) to get

$$\begin{aligned}\zeta_1 - \zeta_0 &= \tfrac{5}{8}\sigma_1 + \tfrac{1}{16}\sigma_2 + \tfrac{1}{16}\sigma_0 \\ &= \tfrac{9}{16}\sigma_1 + \tfrac{1}{16}\sigma_2 > 0\end{aligned}$$

□

Lemma 9. *If $1 \leqslant k \leqslant (n-1)/2$, then $\zeta_k - \zeta_{n-k} < 0$ for $n \geqslant 27$.*

Proof. Applying a procedure similar to that used for (22) and (23),

$$\begin{aligned}\zeta_k - \zeta_{n-k} &= 2n\delta_{(1,k)} - \frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{S_j^2}\left\{\sin\left[\frac{\pi}{n}(2k+1)\,j\right] + \sin\left[\frac{\pi}{n}(2k-1)\,j\right]\right\} \\ &= 2n\delta_{(1,k)} - \frac{1}{2}(\sigma_{k+1} + \sigma_k) < 0 \qquad \text{if} \quad 2 \leqslant k \leqslant \frac{n-1}{2}\end{aligned} \tag{26}$$

Also, $\zeta_1 - \zeta_{n-1} = 2n - \frac{1}{2}(\sigma_2 + \sigma_1)$, which is negative if $n \geqslant 27$ by Lemma 6(e, f).

□

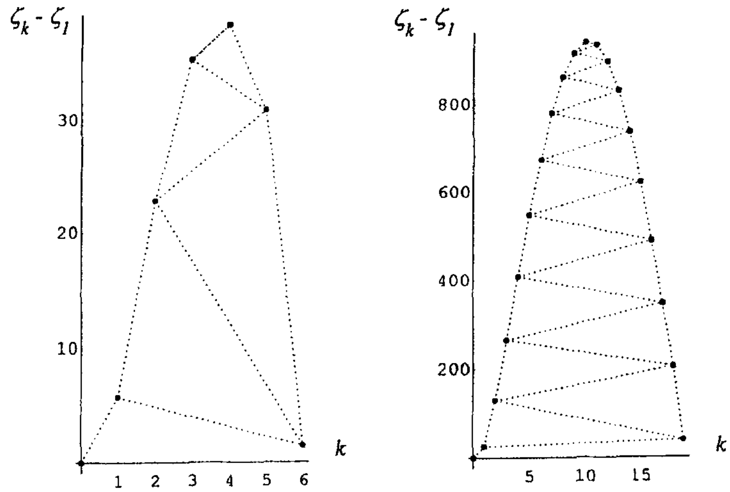
Remark 3. We can verify by direct computation that $\zeta_1 - \zeta_{n-1} > 0$ for $7 \leqslant n \leqslant 11$ and that $\zeta_1 - \zeta_{n-1} < 0$ when $12 \leqslant n \leqslant 26$ (Fig. 3).

Lemma 10. *$\zeta_0, \zeta_1, \ldots, \zeta_{n-1}$ are distinct for $n \geqslant 7$.*

Proof. For $n \geqslant 12$, we have that $\zeta_0 < \zeta_1$ by Lemma 8. First, suppose that n is even. Hence, $\zeta_2 > \zeta_{n-1}$, $\zeta_3 > \zeta_{n-2}, \ldots, \zeta_{n/2} > \zeta_{n/2+1}$. By Lemma 9 and Remark 3, $\zeta_1 < \zeta_{n-1}$, $\zeta_2 < \zeta_{n-2}, \ldots, \zeta_{n/2-1} < \zeta_{n/2+1}$. So that $\zeta_0, \zeta_1, \zeta_{n-1}, \zeta_2, \ldots, \zeta_k, \zeta_{n-k}, \ldots, \zeta_{n/2+1}, \zeta_{n/2}$ is strictly increasing.

If n is odd, then by Lemma 9, $\zeta_0, \zeta_1, \zeta_{n-1}, \zeta_2, \ldots, \zeta_k, \zeta_{n-k}, \ldots, \zeta_{n+1/2}$ is strictly increasing.

For $n = 7$, 8, 9, 10, and 11, we can directly verify that either $\zeta_0, \zeta_{n-1}, \zeta_1, \zeta_2, \ldots, \zeta_k, \zeta_{n-k}, \ldots, \zeta_{n/2+1}, \zeta_{n/2}$ or $\zeta_0, \zeta_{n-1}, \zeta_1, \zeta_2, \ldots, \zeta_k, \zeta_{n-k}, \ldots, \zeta_{n+1/2}$ is strictly increasing. □



a. $n = 7$ b. $n = 20$

Fig. 3. $\zeta_k - \zeta_0$ *vs* k for $n = 7$ and $n = 20$.

Remark 4.

(a) Because of the last lemma and Lemma 2, the class (-1) eigenvalues for Maxwell's ring must be all imaginary when $n \geqslant 7$. Also, the LCT is applicable to the n class (-1) eigenvalues, resulting in n families of periodic perturbations analytic in ε.

(b) By the implicit function theorem, one can show that each conjugate pair of class (-1) eigenvalues represents an analytic function of ε.

3.3. The Class (0) Eigenvalues

It is not enough simply to show that these eigenvalues are distinct because of the variation in the lead coefficient ζ_k at $O(\varepsilon)$, which happens to be the lowest order. Nonetheless, it is still possible to prove the existence of periodic perturbations for the class (0) eigenvalues. We show that the cardinality of this subset tends to $(3n/4)$ as $n \to \infty$. As usual, we proceed by examining the coefficients in the expansion of λ^2 in terms of ε;

Using (9),

$$P(\lambda) = \det\{(G_\theta - \lambda^2 I) - (G_r - 2\omega\lambda I)(F_r + (\omega^2 - \lambda^2) I)^{-1} (F_\theta + 2\omega\lambda I)\} \tag{27}$$

Using (10),

$$P(\lambda) = \det\{\varepsilon B - \lambda^2 I - (\varepsilon D - 2\omega\lambda I) \times [2I + \varepsilon A + (\omega^2 - \lambda^2) I + O(\varepsilon^2)]^{-1} (\varepsilon C + 2\omega\lambda I)\} \quad (28)$$

Let $\lambda_k^2 = -\varepsilon\hat{\zeta}_k$ for $k = 0, 2, ..., n-1$. With $\omega = 1 + \frac{1}{2}\varepsilon\omega_1 + O(\varepsilon^2)$ and if ε is small enough, then

$$[2I + \varepsilon A + (\omega^2 - \lambda^2) I + O(\varepsilon^2)]^{-1} = \tfrac{1}{3} I + O(\varepsilon)$$

Therefore,

$$3^n P(\lambda) = \varepsilon^n \det\{3B - \hat{\zeta} I + O(\varepsilon^{1/2})\}$$

Hence, $\hat{\zeta}$ tends to an eigenvalue of $3B$ as $\varepsilon \to 0$. We have just proven the following result.

Lemma 11. *For $n \geqslant 3$, every class (0) eigenvalue λ can be written as $\lambda^2 = \varepsilon(\zeta + \hat{\kappa})$, where $\hat{\kappa} \to 0$ as $\varepsilon \to 0$. The values of ζ are precisely the eigenvalues of $3B$.*

Lemma 12. *For Maxwell's ring, if $n \geqslant 7$, then*

(a) $\zeta_0, \zeta_1, ..., \zeta_{\lfloor n/2 \rfloor}$ is a strictly increasing sequence.

(b) The class (0) eigenvalues are at most pairwise equal.

Proof. (a) If $1 \leqslant k \leqslant \lfloor 1/2 \rfloor$, the next set of equations follows from Lemma 11, Eq. (15b) the definition of σ_k, and some trigonometric identities,

$$\begin{aligned}
\frac{1}{3}(\zeta_{k+1} - \zeta_k) &= -\sum_{j=1}^{n-1} \frac{3\mathbf{C}_{(1,j+1)}}{16\mathbf{S}_j^3} \left\{\cos\left[\frac{2\pi j(k+1)}{n}\right] - \cos\left[\frac{2\pi jk}{n}\right]\right\} + \frac{n}{2}\delta_{(1,k)} \\
&= \frac{1}{8}\sum_{j=1}^{n-1} \frac{1}{\mathbf{S}_j^2} \left\{3 \sin\left[\frac{\pi j(2k+1)}{n}\right]\right. \\
&\quad \left. + \mathbf{C}_{(1,j+1)} \sin\left[\frac{\pi j(2+1)}{n}\right]\right\} + \frac{n}{2}\delta_{(1,k)} \\
&= \frac{3}{8}\sigma_{k+1} + \frac{1}{16}\sum_{j=1}^{n-1} \frac{1}{\mathbf{S}_j^2} \left\{\sin\left[\frac{\pi j(2k+3)}{n}\right]\right. \\
&\quad \left. + \sin\left[\frac{\pi j(2k-1)}{n}\right]\right\} + \frac{n}{2}\delta_{(1,k)}
\end{aligned}$$

so that

$$\frac{1}{3}(\zeta_{k+1}-\zeta_k)=\frac{3}{8}\sigma_{k+1}+\frac{1}{16}\sigma_{k+2}+\frac{1}{16}\sigma_k+\frac{n}{2}\delta_{(1,k)} \tag{29}$$

which is positive if $1\leqslant k\leqslant(n-3)/2$.

Suppose that $k=(n/2)-1$. Applying Lemma 6(b) to (29),

$$\zeta_{(n/2)}-\zeta_{(n/2)-1}=\tfrac{5}{16}\sigma_{(n/2)}+\tfrac{1}{16}\sigma_{(n/2)-1}>0$$

Therefore, the sequence $\zeta_1,..., \zeta_{\lfloor n/2\rfloor}$ is strictly increasing.

We need to show only that $\zeta_0<\zeta_1$. Now

$$\frac{1}{3}(\zeta_1-\zeta_0)=\frac{3}{8}\sigma_1+\frac{1}{16}\sigma_2+\frac{1}{16}\sigma_0-\frac{n}{2}=\frac{5}{16}\sigma_1+\frac{1}{16}\sigma_2-\frac{n}{2}\geqslant\frac{3}{8}\sigma_1-\frac{n}{2}$$

since $\sigma_2>\sigma_1$ for $n\geqslant 7$. Applying Lemma 6(f), the last quantity above is positive for $n\geqslant 11$. We can directly verify that $\frac{5}{16}\sigma_1+\frac{1}{16}\sigma_2-(n/2)$ is positive for $n=7$, , 8, 9, and 10.

(b) That the class (0) eigenvalues are at most pairwise equal follows from part (a) and Lemma 7. □

Corollary *4. For $n\geqslant 7$, the following hold:*

(a) For $k=0,..., n-1$, $\zeta_k\geqslant 0$. Equality holds only if $k=0$.

(b) If n is even, then $\zeta_{(n/2)}>\zeta_k$ if $k\neq(n/2)$.

(c) If n is odd, then $\zeta_{(n-1)/2}=\zeta_{(n+1)/2}>\zeta_k$ if $k\neq(n-1)/2$ and $k\neq(n+1)/2$.

Proof. By Lemma 12, $\zeta_0, \zeta_1,..., \zeta_{\lfloor n/2\rfloor}$ is a strictly increasing sequence. Therefore, by Lemma 7, the sequence $\zeta_{n-\lfloor n/2\rfloor}, \zeta_{n-\lfloor n/2\rfloor+1},..., \zeta_{n-1}$ is strictly decreasing.

(a) Using the definition of ζ_k, Eqs. (15b) and (20), we get $\zeta_0=0$. The fact that $\zeta_{n-1}=\zeta_1>0=\zeta_0$ completes the proof of (a).

(b) If n is even, then $n\lfloor n/2\rfloor=n/2$, so that $\zeta_{n/2}$ is indeed the unique maximum.

(c) If n is odd, then $n-\lfloor n/2\rfloor=(n+1)/2$. Also, $\lfloor n/2\rfloor=(n-1)/2$. Therefore, by Lemmas 7 and 12, $\zeta_{(n-1)/2}$ and $\zeta_{(n+1)/2}$ represent the maximum value of ζ, completing the proof of the corollary (Fig. 4). □

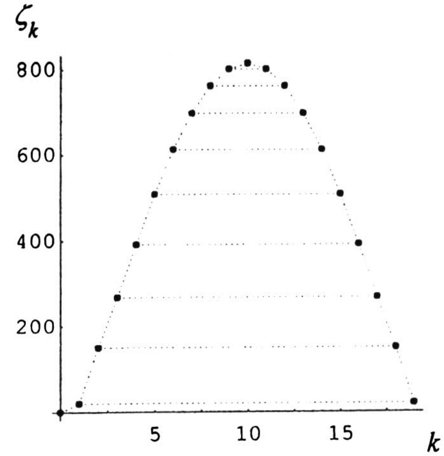


Fig. 4. ζ_k *vs* k for $n=20$.

Remark 5.

(a) We can see immediately that, if n is even, the nonresonance condition of the LCT holds for the index $n/2$.

(b) For $4 \leqslant n \leqslant 6$, we can verify directly that Lemma 12 is also true if we exclude the index 0. The fact is that only ζ_1 and ζ_{n-1} are negative.

For now, the most that we can say about the class (0) eigenvalues is that they are at most pairwise equal. We now proceed to examine the higher-order terms of λ_k^2. Let $\lambda^2 = -\varepsilon(\zeta + \hat{\kappa})$, where $\hat{\kappa} \to 0$ as $\varepsilon \to 0$. using (28),

$$P(\lambda) = \det\left\{\varepsilon B - \lambda^2 I - \frac{1}{3-\lambda^2}\left(-4\lambda^2 I + 2\varepsilon\lambda(C-D)\right) + O(\varepsilon^2)\right\}$$

$$= \frac{\varepsilon^n}{3^n} \det\left\{3B - (\zeta_k + \hat{\kappa}_k)\, I - 2\varepsilon^{1/2}\zeta_k^{1/2}\, i(C-D) + O(\varepsilon)\right\} \tag{30}$$

In obtaining the last matrix, we opted to use $\lambda_k = i\varepsilon^{1/2}\,\zeta_k^{1/2} + O(\varepsilon)$, which is made possible by Lemma 7.

Lemma 13. *Let $\lambda_k^2 = -\varepsilon(\zeta_k - \hat{\kappa}_k)$. Then as $\varepsilon \to 0$, $\varepsilon^{-1/2}\hat{\kappa}_k \to \kappa_k$, which is an eigenvalue of $2i\zeta_k^{1/2}(C-D)$. Furthermore,*

$$\kappa_k = 4\zeta_k^{1/2}\,\gamma_k \tag{31}$$

Proof. We can assume that the matrices in (30) are all diagonalized since they are all circulant. In particular, the entire matrix in (30) will also be in diagonal form. Because $(3-\lambda^2)^n P(\lambda)=0$, it must be that this matrix has some rows that have nothing but 0's as entries. Likewise, $3B-\zeta_k I$ must have some rows that are also 0 since the eigenvalues of $3B$ are precisely the ζ_k's by Lemma 11. Since these eigenvalues are at most pairwise equal, there must be exactly 1 or 2 of these rows that are equal to 0. By Lemma 7, if the kth row is one such row, then so is the $(n-k)$th row. Upon dividing the other terms in (30) by $\varepsilon^{1/2}$, the matrix $\varepsilon^{-1/2}\hat{\kappa}_k I-2\zeta_k^{1/2}\, i(C-D)+O(\varepsilon^{1/2})$ in (30) should still have either row k or row $n-k$ equaling 0. Allowing $\varepsilon\to 0$, we see that $\varepsilon^{-1/2}\hat{\kappa}_k$ tends to an eigenvalue κ_k of $2i\zeta_k^{1/2}(C-D)$. An application of the definition of γ_k completes the proof. □

Remark 6.

(a) It should be noted that $\kappa_k=-\kappa_{n-k}$ while $\zeta_k=\zeta_{n-k}$. Also, $\kappa_0=0$.

(b) If we can show that $\gamma_k\neq 0$, then the nonzero class (0) eigenvalues are distinct for sufficiently small values of ε. We state this result in the next lemma.

Lemma 14. *If $n\geqslant 27$, for $k=0, 1,..., \lfloor n/2\rfloor$, we have $\gamma_k\leqslant 0$. Equality holds only if $k=0$ or $n/2$.*

Proof. We already know that $\gamma_0=0$ by Corollary 3. For the rest, we first try to obtain an expression for the successive differences among the γ_k's and to get an inequality similar to the one given in Lemma 6(d). Then we show that $\gamma_{\lfloor n/2\rfloor}\leqslant 0$. We have that, for $0\leqslant k\leqslant (n-1)/2$,

$$\gamma_{k+1}-\gamma_k=-\frac{1}{16}\sum_{j=1}^{n-1}\frac{\mathrm{S}_{(1,\,j+1)}}{\mathrm{S}_j^3}\left\{\sin\left[\frac{2\pi j(k+1)}{n}\right]\right.$$
$$\left.-\sin\left[\frac{2\pi jk}{n}\right]\right\}+\frac{n}{4}\left(\delta_{1,\,k+1)}-\delta_{(1,\,k)}\right)$$
$$=-\frac{1}{16}\sum_{j=1}^{n-1}\frac{1}{\mathrm{S}_j^2}\left\{\sin\left[\frac{\pi j(2k+3)}{n}\right]-\sin\left[\frac{\pi j(2k-1)}{n}\right]\right\}$$
$$+\frac{n}{4}\left(\delta_{(1,\,k+1)}-\delta_{(1,\,k)}\right)$$

So that

$$\gamma_{k+1}-\gamma_k=-\frac{1}{16}(\sigma_{k+2}-\sigma_k)+\frac{n}{4}(\delta_{(1,k+1)}-\delta_{(1,k)}) \qquad (32)$$

By Corollary 2, for $1 \leqslant k \leqslant (n-1)/2$ with $k \neq (n/2)-1$, the quantity on the right increases as k increases. Therefore,

$$\gamma_{k+1}-\gamma_k<\gamma_{k+2}-\gamma_{k+1} \qquad (33)$$

completing the first part of the proof.

Now suppose that n is odd. Therefore, $\gamma_{\lfloor n/2 \rfloor}=\gamma_{(n-1)/2}=-\gamma_{(n+1)/2}$ by Lemma 7(c). By Eq. (32) and Lemma 6(b, c), $\gamma_{(n+1)/2}-\gamma_{(n-1)/2}=-\frac{1}{16}(\sigma_{(n+3)/2}-\sigma_{(n-1)/2})=\frac{1}{8}\sigma_{(n-1)/2}$. Also, by Lemma 6(a), we find that the right-hand quantity is positive. Since $\gamma_{(n-1)/2}=-\gamma_{(n+1)/2}$, it must be that $\gamma_{(n-1)}/2<0$.

Next, suppose that n is even. By Lemma 7(c), $\gamma_{n/2}=-\gamma_{n/2}$. Hence, $\gamma_{n/2}=0$. Using Lemma 6(a, b) and (33), $\gamma_{n/2}-\gamma_{(n/2)-1}=\frac{1}{16}(\sigma_{(n/2)+1}-\sigma_{n/2})=\frac{1}{8}\sigma_{n/2}>0$ that $\gamma_{(n/2)-1}<0$.

Next we show that $\gamma_1<0$.

$$\gamma_1-\gamma_0=-\frac{1}{16}(\sigma_2-\sigma_0)+\frac{n}{4}=-\frac{1}{16}(\sigma_2+\sigma_1)+\frac{n}{4}<0 \qquad \text{if} \quad n \geqslant 27$$

To complete the proof, suppose that $\gamma_k \geqslant 0$, for some k, with $1<k<(n/2)-1$. We can also suppose that $\gamma_{k+1}-\gamma_k>0$, which is made possible by the fact that $\gamma_1<0$ and that the successive differences form a strictly increasing sequence. Using Eq. (33), we have the following:

$$0<\gamma_{k+1}-\gamma_k<\dots<\gamma_{\lfloor (n-1)/2 \rfloor}-\gamma_{\lfloor (n-3)/2 \rfloor}$$

As a result, $0 \leqslant \gamma_k<\gamma_{k+1}<\dots<\gamma_{\lfloor (n-1)/2 \rfloor}$, which directly contradicts what we have found. Hence, we conclude that $\gamma_k \leqslant 0$, for $k=0, 1, \dots, \lfloor n/2 \rfloor$, and that equality holds only if $k=0$ or $(n/2)$ (Fig. 5). □

The following lemma is now immediate from Lemmas 12, 13, and 14.

Lemma 15. *For Maxwell's ring, if $n \geqslant 7$, then the nonzero class (0) eigenvalues are distinct.*

Proof. By Lemmas 7 and 12, equality between the nonzero class (0) eigenvalues can happen only when $\lambda_k=\lambda_{n-k}$. Lemmas 13 and 14 make this impossible. □

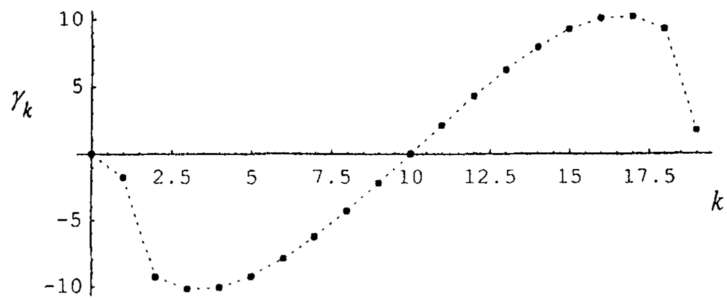


Fig. 5. γ_k *vs* k for $n = 20$.

Remark 7.

(a) Because the nonzero class (0) eigenvalues are distinct for $n \geqslant 7$, an extension of Lemma 1 to include κ_k leads to the fact that the nonzero eigenvalues are pure imaginary. Furthermore, each conjugate pair corresponds to an analytic function of $\varepsilon \geqslant 0$.

(b) Note that, for the case that n is odd, there is also a unique maximum absolute value for all class (0) eigenvalues and this corresponds to the index $k = (n-1)/2$.

(c) Applying the LCT, by the preceding remark and Remark 5, if n is even, there is at least 1 family of periodic perturbations corresponding to the index $(n/2)$. If n is odd, then there are at least 2 families of periodic perturbations corresponding to the indices $(n-1)/2$ and $(n+1)/2$.

We derive something more substantial than what the preceding remark states by examining the $O(\varepsilon^{3/2})$ a bit more closely.

Lemma 16. *If* $\lfloor (n+1)/4 \rfloor \leqslant k \leqslant n - \lfloor (n+1)/4 \rfloor$, *then* $\sigma_{k+1} < \sigma_k$.

Proof. Because of Lemma 6(a) and (b), we need to show only that this is true if $\lfloor (n+1)/4 \rfloor \leqslant k \leqslant \lfloor n/2 \rfloor$. There are four cases to consider: $n \equiv 0$, 1, 2, or 3 (*mod* 4). Because of Lemmas 6(a) and 6(d), we need to show only that $d = \sigma_{\lfloor (n+1)/4 \rfloor + 1} - \sigma_{\lfloor (n+1)/4 \rfloor} < 0$.

First, suppose that $n \equiv 0$ (*mod* 4). Using Lemma 6(c), we have that

$$d = \sigma_{n/4+1} - \sigma_{n/4} = 2 \sum_{j=1}^{n-1} \frac{1}{\mathbf{S}_j} \cos\left[\frac{2\pi j}{n} \cdot \frac{n}{4}\right] = 2 \sum_{j=1}^{n-1} \frac{1}{\mathbf{S}_j} \cos\left[\frac{\pi j}{2}\right]$$

Because $n \equiv 0$ $(mod\ 4)$, we have that $\cos[\pi j/n] = \cos[\pi(n-j)/2]$. Using this and the fact that $\mathbf{S}_j = \mathbf{S}_{n-j}$ and that $n \equiv 0$ $(mod\ 4)$,

$$d = 2\left[-\frac{2}{\mathbf{S}_2} + \frac{2}{\mathbf{S}_4} - \frac{2}{\mathbf{S}_6} + \cdots \pm \frac{1}{\mathbf{S}_{n/2}}\right]$$

This is an alternating sum whose terms are decreasing. Since the lead term is negative, $d < 0$.

Next, suppose that $n \equiv 1$ $(mod\ 4)$. Let $k = (n-1)/4$. Using Lemma 6(c),

$$d = \sigma_{(n+3)/4} - \sigma_{(n-1)/4} = 2 \sum_{j=1}^{n-1} \frac{1}{\mathbf{S}_j} \cos\left[\frac{2\pi j}{n} \cdot \frac{n-1}{4}\right]$$

$$= 2 \sum_{j=1}^{n-1} \frac{1}{\mathbf{S}_j} \left\{\cos\left[\frac{\pi j}{2}\right] \cos\left[\frac{\pi j}{2n}\right] + \sin\left[\frac{\pi j}{2}\right] \sin\left[\frac{\pi j}{2n}\right]\right\} \tag{34}$$

Using the definition of $\mathbf{S}_j$,

$$\sum_{j=1}^{n-1} \frac{1}{\mathbf{S}_j} \left\{\cos\left[\frac{\pi j}{2}\right] \cos\left[\frac{\pi j}{2n}\right]\right\} = \frac{1}{2} \sum_{j=1}^{n-1} \cos\left[\frac{\pi j}{2}\right] \left\{\sin\left[\frac{\pi j}{2n}\right]\right\}^{-1}$$

Since $0 < (\pi j/2n) < (\pi/2)$, this is a decreasing alternating sum whose lead term is negative. Therefore, this part of (34) is negative. The second part of the sum is

$$2 \sum_{j=1}^{n-1} \frac{1}{\mathbf{S}_j} \left\{\sin\left[\frac{\pi j}{2}\right] \sin\left[\frac{\pi j}{2n}\right]\right\} = \sum_{j=1}^{n-1} \sin\left[\frac{\pi j}{2}\right] \left\{\cos\left[\frac{\pi j}{2n}\right]\right\}^{-1}$$

Now this is an increasing alternating sum. The last nonzero term is equal to $\sin[(\pi/2)(n-2)]\{\cos[(\pi/2n)(n-2)]\}^{-1}$, which is negative since $\cos[(\pi/2n)(n-2)] > 0$ and since $\sin[(n/2)(n-2)] = -1$ because $n \equiv 1$ $(mod\ 4)$. Hence, the second part of the sum is negative. Therefore, $d < 0$.

The proofs for the two remaining cases, $n \equiv 2$ or 3 $(mod\ 4)$, are similar to the preceding cases and are omitted. □

Lemma 17. *For Maxwell's ring, if* $n \geqslant 27$ *and if* $\lfloor (n+1)/4 \rfloor \leqslant k < n - \lfloor (n-1)/4 \rfloor$, *then* $\gamma_{k+1} > \gamma_k$.

Proof. For $1 \leqslant k \leqslant (n-3)/2$, we need to show this only for $k = \lfloor (n+1)/4 \rfloor$ because of Lemma 7(c) and since $\gamma_{k+1} - \gamma_k$ is an increasing sequence. Using Eq. (32), we have

$$\gamma_{\lfloor (n+1)/4 \rfloor + 1} - \gamma_{\lfloor (n+1)/4 \rfloor} = -\tfrac{1}{16}\{\sigma_{\lfloor (n+1)/4 \rfloor + 2} - \sigma_{\lfloor (n+1)/4 \rfloor}\}$$

By the preceding lemma, the expression inside the brackets is negative. To complete the proof, we also have that $\gamma_{n/2-1} < 0 = \gamma_{n/2} < 0$ and $-\gamma_{(n+1)/2} = \gamma_{(n-1)/2} < 0$. □

Corollary 5. *If* $\gamma_m = \min_{0 \leqslant k \leqslant n-1} \{\gamma_k\}$, *then* $m \leqslant \lfloor (n+1)/4 \rfloor$ *for* $n \geqslant 27$.

Proof. According to Lemma 17, it must be that $m \leqslant \lfloor (n+1)/4 \rfloor$ or $m \geqslant n - \lfloor (n+1)/4 \rfloor$. Because of Lemma 14 and Lemma 7(c), we also know that $m \leqslant \lfloor n/2 \rfloor$. Thus, $m \leqslant \lfloor (n+1)/4 \rfloor$. □

Remark 8. Since γ_1, $\gamma_m < 0$ for $n \geqslant 27$, and since the $(k, -\gamma_k)$'s are the vertices of a concave polygonal graph when $1 \leqslant k \leqslant m$, it must be that $\gamma_1, \gamma_2, ..., \gamma_m$ is a strictly decreasing sequence.

Lemma 18. *If* $1 < j < k \leqslant m$, *then* $(\gamma_k - \gamma_1)/(\gamma_j - \gamma_1) < (k-1)/(j-1)$ *for* $n \geqslant 27$.

Proof. Since $(\gamma_k - \gamma_1)/(\gamma_j - \gamma_1) = (\gamma_k - \gamma_1)/(\gamma_{k-1} - \gamma_1)(\gamma_{k-1} - \gamma_1)/(\gamma_{k-2} - \gamma_1) \cdots (\gamma_{j+1} - \gamma_1)(\gamma_j - \gamma_1)$, it is enough to show that $(\gamma_{j+1} - \gamma_1)(\gamma_j - \gamma_1) < j/(j-1)$. We have seen that $\gamma_{j+1} - \gamma_j > \gamma_j - \gamma_{j-1} > \cdots > \gamma_2 - \gamma_1$. Then

$$\gamma_{j+1} - \gamma_j > \frac{1}{j-1} [\gamma_j - \gamma_{j-1} + \gamma_{j-1} - \gamma_{j-2} + \cdots + \gamma_2 - \gamma_1] = \frac{1}{j-1} [\gamma_j - \gamma_1]$$

As a result, $(j-1)[\gamma_{j+1} - \gamma_j + \gamma_j - \gamma_1] > j[\gamma_j - \gamma_1]$. And so, $(\gamma_{j+1} - \gamma_1)/(\gamma_j - \gamma_1) < j/(j-1)$. And the lemma follows. □

Lemma 19. *For* $n \geqslant 27$, *if* $(m+1)/2 \leqslant j \leqslant m$, *then* $0 < (\gamma_m/\gamma_j) < 2$.

Proof. By the preceding lemma, $(\gamma_m - \gamma_1)/(\gamma_j - \gamma_1) < (m-1)/(j-1)$ if $j < m$. Since $0 \geqslant \gamma_1 > \gamma_j \geqslant \gamma_m$ for $n \geqslant 27$, it must be that $(\gamma_m/\gamma_j) \leqslant (\gamma_m - \gamma_1)/(\gamma_j - \gamma_1) < (m-1)/(j-1)$. And $(m-1)/(j-1) \leqslant 2$ if $(m+1)/2 \leqslant j < m$. □

Lemma 20. *For* $n \geqslant 27$, *if* $n - m < j \leqslant n - (m+1)/2$. *Then* $0 < (\gamma_k/\gamma_j) < 2$.

Proof. The proof is similar to that of the preceding lemma. This time we make use of the fact that $\gamma_k = -\gamma_{(n-k)}$. □

Lemma 21. *If* $(m+1)/2 \leqslant j \leqslant n-(m+1)/2$, *then there exists a n.s.i* $\mathcal{J}$ *such that for* $k=1, 2, 3, ..., n$, *for every* $\varepsilon \in \mathcal{J}$, λ_k/λ_j *is not an integer for* $n \geqslant 27$.

Proof. Without loss of generality, suppose that $\lambda_k/\lambda_j = d$, a positive integer. Now, $d \geqslant 2$ since the class (0) eigenvalues are distinct. If this is the case for sufficiently small values of $\varepsilon > 0$, it must be that $\zeta_k/\zeta_j = (\lambda_k/\lambda_j)^2 \geqslant 4$. We consider four cases:

$$\text{(i)} \quad \frac{m+1}{2} \leqslant j < k < \frac{n}{2} \qquad \text{(iii)} \quad \frac{m+1}{2} \leqslant j < n-k \leqslant \frac{n}{2}$$

$$\text{(ii)} \quad n - \frac{m+1}{2} \geqslant j > k \geqslant \frac{n}{2} \qquad \text{(iv)} \quad n - \frac{m+1}{2} \geqslant j > n-k \geqslant \frac{n}{2}$$

Examining the $O(\varepsilon^{3/2})$ terms, using Eq. (31), it ought to be that $\gamma_k/\gamma_j = d$, which is at least 2. Now $\gamma_k/\gamma_j \leqslant 0$ for (iii) and (iv). Therefore, only (i) or (ii) can occur. If (i) is true, then we cannot have $j \leqslant m$, for this would contradict Lemma 19. On the other hand, if $j > m$, then $\gamma_j < \gamma_k \leqslant 0$ since these start to increase starting with the index m up to the index $n/2$. This is contrary to what we have of λ_k and λ_j and, thus, cannot happen either. The argument for case (ii) uses Lemma (7c) and is similar to that for case (i). □

The application of the LCT now makes it possible for us to make the following remarks.

Remark 9. (a) By the preceding lemma and Corollary 5, we now know that if

$$\frac{1}{2}\left(\left\lfloor \frac{n+1}{4} \right\rfloor + 1\right) \leqslant k \leqslant n - \frac{1}{2}\left(\left\lfloor \frac{n+1}{4} \right\rfloor + 1\right)$$

then at least one family of periodic perturbation exists, corresponding to λ_k. The number of indices satisfying the above inequality tends to $(3n/4)$ as $n \to \infty$. This, coupled with the fact that there are at least n families of class (-1) periodic perturbations, implies that total number of families of periodic perturbations is, at least, $2n - \lfloor (n+1)/4 \rfloor$. This includes only those that we have verified by the LCT. This also proves Theorem 4.

(b) We can show numerically that, if $4 \leqslant n \leqslant 6$, then there are $n-3$ periodic solutions. It is known that, for these values of n, four of the eigenvalues have nonzero real part.

(c) The question of the remaining approximately $n/4$ eigenvalues remains open. The higher-order coefficients [at $O(\varepsilon^2)$] have turned out to be extremely more complex than the expressions ζ_k and κ_k in that these involve not only sums but also products of the eigenvalues of A, B, C, and D, so that patterns, if any, are much more difficult to discern.

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