Journal of Dynamics and Differential Equations, Vol. 10, No. 1, 1998

Stability and Instability of Fourth-Order Solitary Waves

Steven Levandosky[1]

Received August 15, 1995

We study ground-state traveling wave solutions of a fourth-order wave equation. We find conditions on the speed of the waves which imply stability and instability of the solitary waves. The analysis depends on the variational characterization of the ground states rather than information about the linearized operator.

KEY WORDS: Fourth-order solitary waves; stability; instability.

1. INTRODUCTION

This paper is an analysis of the stability of traveling wave solutions of the equation

$$u_{tt} + \Delta^2 u + u = f(u) \tag{1.1}$$

where $f(u) = |u|^{p-1} u$. We always assume $p > 1$. If $n \geqslant 5$, then we also assume $p < (n+4)/(n-4)$ and we define $2^* = 2n/(n-4)$. We show that there exist solitary wave solutions of (1.1) and prove criteria for their stability and instability. Our results parallel those for the analogous second-order Klein–Gordon equations (see [6], [17], [19]). We show that the solitary wave of speed c is stable when the action function $d_1(c)$, defined by (3.14), is convex and is unstable when $d_1(c)$ is concave.

The interesting feature of the problem is that the solitary wave satisfies a fourth-order elliptic equation. The stability of solitary waves of second-order equations has been studied in many papers, including [16–18] for the Klein–Gordon equations and [3], [22], and [23] for the Schrödinger

[1] Brown University, Providence, Rhode Island 02912.

1040-7294/98/0100-0151$15.00/0

equation. Also, higher-order equations such as the KdV equation [2, 20] and generalized Boussinesq equations [11, 15] have been examined. In each case, however, the solitary wave satisfies a second-order ODE. Using a nodal analysis of the ground state, it is possible in some cases to obtain information about the spectrum of the linearized operator. However, the solitary waves of (1.1) satisfy a genuine fourth-order PDE (2.1), for which there is no maximum principle available. Thus the ground states may not necessarily be positive and, in fact, may be oscillatory. So we cannot easily obtain this spectral information, and therefore some of the standard techniques for analyzing stability are no longer applicable. Instead we rely entirely on the variational characterization of the solitary wave.

In one dimension an equation similar to (1.1), with a different nonlinear term, has been studied as a model for the suspension bridge [12]. Numerical evidence in the case of an exponential nonlinearity suggests that traveling waves are unstable for small c and exhibit soliton-like behavior for c near the critical value $\sqrt{2}$ [13].

In Section 2 we prove the existence of a solitary wave. Solutions are obtained by using the method of concentrated compactness developed by Lions [9] to solve a constrained minimization problem. We use the scaling property of the pure power nonlinearity to verify the subadditivity conditions (2.5) and to scale away the Lagrange multiplier. In second-order and pure fourth-order problems we can verify these conditions and eliminate any multipliers by dilating in the independent variable [1]. The presence of both fourth- and second-order terms in (2.1), however, prohibits such an approach, and therefore we restrict our attention to homogeneous nonlinearities. We also note that the scale invariance allows us to solve the minimization without any restrictions on the dimension n.

The results in this section apply to a more general class of homogeneous nonlinear terms. For instance, take $F \in C^1(\mathbb{R}^{n+1})$ such that $\nabla F(y, z) \cdot (y, z) = (p+1) F(y, z)$ for all $(y, z) \in \mathbb{R}^{n+1}$. If $F(u, \nabla u) \in L^1(\mathbb{R}^n)$ for every $u \in H^2(\mathbb{R}^n)$ and there exists some $u \in H^2(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} F(u, \nabla u)\, dx > 0$, then nonlinearities of the form

$$f(u, \nabla u) = \nabla_y F(u, \nabla u) - div_x(\nabla_z F(u, \nabla u))$$

may be treated as well. In particular, the nonlinearities $f(u) = \pm |u|^p$ and $-3u^2 + (u_x)^2 - 2(uu_x)_x$ are of this form. The latter arises in the study of fifth-order KdV equations [4] and will be the subject of another paper [8].

In Section 3 we show that the evolution equation admits solutions in the space $X = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ which exist locally in time for given initial data in X, provided $p < 2^*/2$.

We discuss in Section 4 the properties of $d_1(c)$ we shall need for the stability analysis. Once again, we use the scaling property of our non-linearity to write $d_1(c)$ in terms of the minimum values of the functionals used to obtain the solitary wave. We also establish bounds on dl which imply concavity for small c and convexity for c near $\sqrt{2}$.

In Section 5 we show (Theorem 5.4) that the set of ground states is stable whenever $d_1(c)$ is strictly convex. The proof consists of a compactness argument based on the ideas of Shatah [17] and Cazenave and Lions [3]. We use the variational properties of the ground states, along with a convexity lemma of Shatah (Lemma 5.1), to establish the key inequality (5.4).

In Section 6 we use a Lyapunov functional construction due to Grillakis *et al.* [6] to show (Theorem 6.2) that a given ground state is orbitally unstable when $d_1''(c) < 0$. We need to make the additional assumption that there is a C^1 map $c \mapsto \varphi_c$, where φ_c is a ground state with speed c, in order to apply the implicit function theorem.

Finally, in Section 7 we consider standing wave solutions of (1.1). The results of Sections 5 and 6 extend quite easily to this case, and the scaling properties of the solitary wave equation (7.3) make it possible to determine explicitly the intervals of stability and instability.

2. EXISTENCE OF MINIMIZERS

In this section we prove the existence of traveling wave solutions. In the process, an essential result concerning the compactness of minimizing sequences is established. Let $u(x, t) = \varphi(x - \vec{c}t)$ for $\vec{c} \in \mathbb{R}^n$ be a solution of (1.1). Then φ must solve

$$\Delta^2\varphi + \sum_{i,j=1}^{n} c_i c_j \varphi_{x_i x_j} + \varphi = |\varphi|^{p-1}\varphi \tag{2.1}$$

For $|\vec{c}|^2 < 2$ we can obtain solutions of (2.1) by considering the following constrained minimization problem. Let

$$\begin{aligned} I_{\vec{c}}(w) &= \int_{\mathbb{R}^n} |\Delta w|^2 - |\vec{c} \cdot \nabla w|^2 + |w|^2\, dx \\ K(w) &= \int_{\mathbb{R}^n} |w|^{p+1}\, dx \end{aligned} \tag{2.2}$$

and define

$$I_\lambda = \inf\{I_{\vec{c}}(w) : w \in H^2(\mathbb{R}^n),\, K(w) = \lambda\}$$

for $0<\lambda\leqslant 1$. We say a sequence $\{w_k\}_{k=1}^{\infty}$ in $H^2(\mathbb{R}^n)$ is a minimizing sequence if

$$\lim_{k\to\infty} I_{\vec{c}}(w_k)=I_1$$

$$\lim_{k\to\infty} K(w_k)=1$$

Our main result in this section is the following.

Theorem 2.1. *Let $\{w_k\}_{k=1}^{\infty}$ be a minimizing sequence. Then there exists a subsequence $\{w_{k_j}\}$, $y_j\in\mathbb{R}^n$ and $w\in H^2(\mathbb{R}^n)$ such that $w_{k_j}(\cdot-y_j)\to w$ in $H^2(\mathbb{R}^n)$. The function w is a minimizer of $I_{\vec{c}}$ subject to the constraint $K(w)=1$ and is therefore a weak solution of the Euler–Lagrange equation,*

$$\Delta^2 w+\sum_{i,j=1}^{n} c_i c_j w_{x_i x_j}+w=\mu\,|w|^{p-1}\,w \tag{2.3}$$

Hence $\varphi=\mu^{1/(p-1)}w$ is the desired solution of (2.1). Solutions obtained in this manner are referred to as *ground states.*

We establish Theorem 2.1 by applying the method of concentrated compactness. By scaling it is easily seen that

$$I_\lambda=\lambda^{2/(p+1)}I_1 \tag{2.4}$$

and therefore the strict subadditivity condition

$$I_\lambda+I_{1-\lambda}>I_1, \qquad \lambda\in(0,1) \tag{2.5}$$

holds. Let $\{w_k\}_{k=1}^{\infty}$ be a minimizing sequence and define a sequence of measures on $\mathbb{R}^n$ by

$$\rho_k=|\Delta w_k|^2+|w_k|^2 \tag{2.6}$$

Since

$$I_{\vec{c}}(w)\geqslant(1-|\vec{c}|^2/2)\int_{\mathbb{R}^n}|\Delta w|^2+|w|^2\,dx=(1-|\vec{c}|^2/2)\,\|w\|^2_{H^2(\mathbb{R}^n)} \tag{2.7}$$

for any $w\in H^2(\mathbb{R}^n)$, $I_{\vec{c}}$ is coercive for $|\vec{c}|^2<2$, and therefore $\{w_k\}_{k=1}^{\infty}$ is bounded in $H^2(\mathbb{R}^n)$. So, upon passing to a subsequence if necessary, we may assume that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\rho_k(x)\,dx=L>I_1$$

and after normalizing appropriately we may suppose

$$\int_{\mathbb{R}^n} \rho_k(x)\, dx = L$$

for all k. By the concentration-compactness lemma of Lions [9], there is a subsequence (renamed ρ_k) satisfying one of the following three conditions.

1. *Tightness.* There exist $y_k \in \mathbb{R}^n$ such that for any $\varepsilon > 0$, there exists $R(\varepsilon)$ so that for all k

$$\int_{B(y_k,\, R(\varepsilon))} \rho_k\, dx \geqslant \int_{\mathbb{R}^n} \rho_k\, dx - \varepsilon \tag{2.8}$$

2. *Vanishing.* For every $R > 0$,

$$\lim_{k \to \infty} \sup_{y \in \mathbb{R}^n} \int_{B(y,\, R)} \rho_k\, dx = 0 \tag{2.9}$$

3. *Dichotomy.* There exists $\alpha \in (0, L)$ such that for any $\varepsilon > 0$ there exist R, $R_k \to \infty$, $y_k \in \mathbb{R}^n$, and k_0 so that

$$\left| \int_{B(y_k,\, R)} \rho_k\, dx - \alpha \right| < \varepsilon \qquad \text{and} \qquad \left| \int_{R < |x - y_k| < R_k} \rho_k\, dx \right| < \varepsilon \tag{2.10}$$

for $k \geqslant k_0$.

Lemma 2.2. *The sequence* $\{\rho_k\}_{k=1}^{\infty}$ *is tight modulo the sequence of translations* $\{y_k\}_{k=1}^{\infty}$ *in* $\mathbb{R}^n$.

Proof. The proof follows from arguments given in [9] and [10] which we present here.

First, suppose vanishing occurs. For R_0 fixed we have

$$\int_{B(y,\, R_0)} |w_k|^{p+1}\, dx \leqslant C(R_0) \left(\int_{B(y,\, R_0)} |\Delta w_k|^2 + |w_k|^2\, dx \right)^{(p+1)/2} \tag{2.11}$$

for all k and any $y \in \mathbb{R}^n$, since $p + 1 < 2^*$ and $\{w_k\}_{k=1}^{\infty}$ is bounded in $H^2(\mathbb{R}^n)$. By (2.9) we can choose $k(\varepsilon)$ so that $k > k(\varepsilon)$ implies

$$\sup_{y \in \mathbb{R}^n} \int_{B(y,\, R_0)} \rho_k\, dx < \varepsilon$$

so that by (2.11), we then have

$$\int_{B(y, R_0)} |w_k|^{p+1}\, dx \leqslant C\varepsilon^{(p-1)/2} \left(\int_{B(y, R_0)} |\Delta w_k|^2 + |w_k|^2\, dx \right) \tag{2.12}$$

for $k > k(\varepsilon)$ and any $y \in \mathbb{R}^n$. Now we cover $\mathbb{R}^n$ with balls of radius R_0 in such a way that for some integer m each point of $\mathbb{R}^n$ is contained in at most m balls. Then we sum (2.12) over these balls to get

$$\|w_k\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \leqslant mC\varepsilon^{(p-1)/2}\, \|w_k\|_{H^2(\mathbb{R}^n)}^2 \leqslant C\varepsilon^{(p-1)/2}$$

This implies that $\lim_{k \to \infty} K(w_k) = 0$ and therefore the constraint is lost. Hence vanishing cannot occur.

Next suppose dichotomy occurs and choose $\xi_1, \xi_2 \in C^\infty(\mathbb{R})$ that $0 \leqslant \xi_1,\ \xi_2 \leqslant 1$, and

$$\begin{aligned} \xi_1(x) &= 1 \quad \text{for } |x| \leqslant 1, & \xi_2(x) &= 1 \quad \text{for } |x| \geqslant 1 \\ \xi_1(x) &= 0 \quad \text{for } |x| \leqslant 2, & \xi_2(x) &= 0 \quad \text{for } |x| \geqslant \tfrac{1}{2} \end{aligned}$$

Then define

$$w_{k,1}(x) = \xi_1 \left(\frac{|x - y_k|}{R} \right) w_k(x)$$

and

$$w_{k,2}(x) = \xi_2 \left(\frac{|x - y_k|}{R_k} \right) w_k(x)$$

It is then easy to verify that

$$\begin{aligned} I_{\bar{c}}(w_k) &= I_{\bar{c}}(w_{k,1}) + I_{\bar{c}}(w_{k,2}) + O(\varepsilon) \\ K(w_k) &= K(w_{k,1}) + K(w_{k,2}) + O(\varepsilon) \end{aligned} \tag{2.13}$$

for $k \geqslant k_0$. By passing to a further subsequence we can define

$$\begin{aligned} \lambda_1(\varepsilon) &= \lim_{k \to \infty} K(w_{k,1}) \\ \lambda_2(\varepsilon) &= \lim_{k \to \infty} K(w_{k,2}) \end{aligned} \tag{2.14}$$

We clearly have $\lambda_1(\varepsilon), \lambda_2(\varepsilon) \in [0, 1]$ and $|\lambda_1(\varepsilon) + \lambda_2(\varepsilon) - 1| = O(\varepsilon)$ and we can therefore choose a sequence $\varepsilon_j \to 0$ such that $\lambda_1 = \lim_{j \to \infty} \lambda_1(\varepsilon_j)$ exists.

Then $\lambda_2 = \lim_{j\to\infty} \lambda_2(\varepsilon_j) = 1 - \lambda_1$, and there are two possibilities. If $\lambda_1 \in (0, 1)$ it follows from (2.13) and (2.4) that

$$\begin{aligned} I_{\vec{c}}(w_k) &\geqslant I_{K(w_{k,1})} + I_{K(w_{k,2})} + O(\varepsilon_j) \\ &= [K(w_{k,1})^{2/(p+1)} + K(w_{k,2})^{2/(p+1)}]\, I_1 + O(\varepsilon_j) \end{aligned}$$

Since $\{w_k\}_{k=1}^{\infty}$ is a minimizing sequence, we may send k to infinity and use (2.14) to obtain

$$I_1 \geqslant [\lambda_1(\varepsilon_j)^{2/(p+1)} + \lambda_2(\varepsilon_j)^2/(p+1)]\, I_1 + O(\varepsilon_j)$$

Letting $j \to \infty$, we arrive at the contradiction

$$I_1 \geqslant [(\lambda_1)^{2/(p+1)} + (\lambda_2)^{2/(p+1)}]\, I_1 > I_1$$

If $\lambda_1 = 0$ (and similarly if $\lambda_1 = 1$), we have, since $w_{k,1}$ is supported in $B(y_k, R)$,

$$\begin{aligned} I(w_{k,1}) &\geqslant (1 - |\vec{c}|^2/2)\, \|w_{k,1}\|^2_{H^2(\mathbb{R}^n)} \\ &= (1 - |\vec{c}|^2/2)\, \|w_{k,1}\|_{H_0^2(B(y_k, 2R))} \qquad \text{by (2.7)} \\ &\geqslant (1 - |\vec{c}|^2/2) \int_{B(y_k, 2R)} |\Delta w_{k,1}|^2 + |w_{k,1}|^2\, dx \\ &= (1 - |\vec{c}|^2/2)(\alpha + O(\varepsilon_j)) \qquad \text{by (2.10)} \end{aligned}$$

Thus, using (2.13) and (2.4) again,

$$\begin{aligned} I_{\vec{c}}(w_k) &\geqslant (1 - |\vec{c}|^2/2)(\alpha + O(\varepsilon_j)) + I_{K(w_{k,2})} + O(\varepsilon_j) \\ &= (1 - |\vec{c}|^2/2)\, \alpha + K(w_{k,2})^{2/(p+1)}\, I_1 + O(\varepsilon_j) \end{aligned}$$

and sending k to infinity gives

$$I_1 \geqslant (1 - |\vec{c}|^2/2)\, \alpha + \lambda_2(\varepsilon_j)^{2/(p+1)}\, I_1 + O(\varepsilon_j)$$

We let $j \to \infty$ to get the contradiction

$$I_1 \geqslant \tfrac{2}{3}(1 - |\vec{c}|^2/2)\, \alpha + I_1 > I_1$$

Hence dichotomy does not occur and the lemma is proved. □

Proof of Theorem 2.1. By Lemma 2.2 there exist $y_k \in \mathbb{R}^n$ such that $\rho_k(\cdot + y_k)$ is tight. Since $K(w_k) \to 1$, this implies that $|w_k(\cdot + y_k)|^{p+1}$ is also tight. Now, since $\{w_k\}_{k=1}^{\infty}$ is bounded in $H^2(\mathbb{R}^n)$, there is a subsequence $\{w_{k_j}\}_{j=1}^{\infty}$ and $w \in H^2(\mathbb{R}^n)$ such that

$$w_{k_j}(\cdot + y_{k_j}) \rightharpoonup w \in H^2(\mathbb{R}^n)$$
$$w_{k_j}(\cdot + y_{k_j}) \to w \in L^2_{loc}(\mathbb{R}^n)$$

Since $\{w_k\}$ is bounded in $L^{2^*}(\mathbb{R}^n)$ and $p+1 < 2^*$, it follows by interpolation that $w_{k_j}(\cdot + y_j) \to w$ in $L^{p+1}_{loc}(\mathbb{R}^n)$. We now claim that $w_{k_j}(\cdot + y_j) \to w$ strongly in $L^{p+1}(\mathbb{R}^n)$. Indeed, let $\varepsilon > 0$ and choose R_0 such that

$$\int_{|x| \geqslant R_0} |w(x)|^{p+1}\, dx < \varepsilon$$

By (2.8) there exists $R(\varepsilon) > R_0$ and $j_1(\varepsilon)$ so that $j \geqslant j(\varepsilon)$ implies

$$\int_{|x| \geqslant R(\varepsilon)} |w_{k_j}(x + y_j)|^{p+1}\, dx < \varepsilon$$

By the convergence in $L^{p+1}_{loc}(\mathbb{R}^n)$ we can find $j_2(\varepsilon) > j_1(\varepsilon)$, so that for $j > j_2(\varepsilon)$ we have

$$\|w_{k_j}(\cdot + y_j) - w\|^{p+1}_{L^{p+1}(B(0,\, R(\varepsilon)))} < \varepsilon$$

Thus

$$\int_{\mathbb{R}^n} |w_{k_j}(x + y_j) - w(x)|^{p+1}\, dx \leqslant \varepsilon + 2^{p+1}\varepsilon$$

and the claim is proved. Hence $K(w) = 1$. Since the weak convergence in $H^2(\mathbb{R}^n)$ implies $I(w) \leqslant I_1$ we therefore have $I(w) = I_1$. The lemma then follows since $\|\Delta w_{k_j}\|_{L^2(\mathbb{R}^n)} \to \|\Delta w\|_{L^2(\mathbb{R}^n)}$ and $\Delta w_{k_j} \rightharpoonup \Delta w$ in $L^2(\mathbb{R}^n)$. □

Lemma 2.3. *Suppose $1 < p < 2^* - 1$ and let $\varphi \in H^2(\mathbb{R}^n)$ be a weak solution of (2.1). Then $\varphi \in H^5(\mathbb{R}^n)$.*

Proof. We use the following bootstrap procedure. Suppose that $\varphi \in H^k(\mathbb{R}^n)$. We can assume $k < n/2$, since otherwise $\varphi \in L^\infty(\mathbb{R}^n)$ and therefore $|\varphi|^{p-1}\varphi \in H^1(\mathbb{R}^n)$, which implies $\varphi \in H^5(\mathbb{R}^n)$. So we have $\varphi \in L^s(\mathbb{R}^n)$, where $s = 2n/(n-2k)$ and $\varphi_{x_i} \in L^r(\mathbb{R}^n)$, where $r = 2n/(n-2k+2)$. It then follows that $|\varphi|^{p-1}\varphi_{x_i} \in L^q(\mathbb{R}^n)$ for

$$q = \frac{rs}{(p-1)r + s} = \frac{2n}{p(n-2k)+2}$$

If $q > 2$ then $|\varphi|^{p-1}\varphi_{x_i} \in L^2(\mathbb{R}^n)$ and we conclude as before that $\varphi \in H^5(\mathbb{R}^n)$. Otherwise, $|\varphi|^{p-1}\varphi_{x_i} \in H^l(\mathbb{R}^n)$ where

$$l = \frac{n - p(n-2k) - 2}{2}$$

Thus $\varphi \in H^{k'}(\mathbb{R}^n)$ for

$$k' = l + 5 = pk + 4 - \frac{n}{2}(p-1)$$

So we have after j iterations that $\varphi \in H^{k_j}(\mathbb{R}^n)$, where $k_j = h^{(j)}(2)$ and $h(x) = px + 4 - (n/2)(p-1)$. The only fixed point of h is $x_0 = (n/2) - [4/(p-1)]$, and since $p < 2^* - 1$, $x_0 < 2$. Since $p > 1$, it then follows that $h^{(j)}(2) \to \infty$. Thus $\varphi \in H^5(\mathbb{R}^n)$. □

Remark 2.4. The restriction on p for $n \geqslant 5$ permits the variational characterization of solutions of (2.1). It also allows us to solve the solitary wave equation for small c. If $p > 2^* - 1$, then, by a Pohozaev-type identity, the solitary wave equation cannot have solutions in $H^2(\mathbb{R}^n)$ for all c in some interval around zero.

3. LOCAL EXISTENCE

We can write the evolution equation (1.1) as a system of two equations,

$$\begin{aligned} u_t &= v \\ v_t &= -\Delta^2 u - u + |u|^{p-1} u \end{aligned} \tag{3.1}$$

which may be written in the form

$$\mathbf{u}_t = B\mathbf{u} + f(\mathbf{u}) \tag{3.2}$$

where

$$B = \begin{pmatrix} 0 & I \\ -\Delta^2 - I & 0 \end{pmatrix}$$

$\mathbf{u} = [u, v]$, and $f(\mathbf{u}) = [0, |u|^{p-1} u]$. We consider B as an operator on the space $X = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with domain $D(B) = H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$. The following functionals are formal invariants of the evolution equation and are essential to the stability analysis. Let

$$\begin{aligned} E(\mathbf{u}) &= \int_{\mathbb{R}^n} \frac{1}{2}|\Delta u|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|u|^2 - \frac{1}{p+1}|u|^{p+1}\,dx \\ Q_i(\mathbf{u}) &= \int_{\mathbb{R}^n} v u_{x_i}\,dx, \qquad i = 1,\dots, n \end{aligned} \tag{3.3}$$

and define $\vec{Q}: X \to \mathbb{R}^n$ by $\vec{Q}(\mathbf{u}) = (Q_1(\mathbf{u}),\dots, Q_n(\mathbf{u}))$. The evolution equation may be written in terms of E as

$$\frac{d\mathbf{u}}{dt} = JE'(\mathbf{u}) \tag{3.4}$$

where $J: X^* \to X$ has domain $D(J) = L^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ and is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

By a *solution* of (1.1) on the interval $[0, t_0)$ we mean a function $\mathbf{u} \in C([0, t_0); X)$ such that

$$\frac{d}{dt}\langle \mathbf{v}, \mathbf{u}(t)\rangle = \langle E'(\mathbf{u}(t)), -J\mathbf{v}\rangle \tag{3.5}$$

holds in the sense of distributions on $[0, t_0)$ for all $\mathbf{v} \in D(J)$, where $\langle \cdot, \cdot \rangle$ denotes the pairing of X^* with X. We assume the following throughout.

Assumption 3.1. *Given initial data $\mathbf{v} \in X$, there exists $t_0 > 0$ which depends only on $\|v\|_X$ and a unique solution $\mathbf{u}$ of (1.1) such that $\mathbf{u}(0) = \mathbf{v}$, $E(\mathbf{u}(t)) = E(\mathbf{v})$, and $\vec{Q}(\mathbf{u}(t)) = \vec{Q}(\mathbf{v})$ for all $t \in [0, t_0)$.*

The following result shows that the assumption holds in dimension $n < 5$ with no restrictions on p and in dimension $n \geqslant 5$ if $p < 2^*/2$.

Theorem 3.2. *Suppose $1 < p < 2^*/2$. Then for every $\mathbf{u}_0 \in X$, there exists $t_0 > 0$ such that Eq. (3.2) has a unique integral solution $\mathbf{u}(t) \in C([0, t_0); X)$ with $\mathbf{u}(0) = \mathbf{u}_0$, and if $t_0 < \infty$,*

$$\lim_{t \to t_0^-} \|\mathbf{u}(t)\|_X = \infty$$

The theorem follows from a theorem of Segal [16] once we show that B is the infinitesimal generator of a C_0-semigroup of bounded linear operators on X and that f is locally Lipschitz on X [14]. This is the content of the following lemmas.

Lemma 3.3. *The operator B is the infinitesimal generator of a C_0-semigroup of unitary operators on X.*

Proof. Define an inner product on X by

$$([u_1, v_1], [u_2, v_2]) = \int_{\mathbb{R}^n} (\Delta u_1 \, \Delta u_2 + u_1 u_2 + v_1 v_2) \, dx$$

Then for $\mathbf{u} \in D(B)$,

$$\begin{aligned}(B\mathbf{u}, \mathbf{u}) &= ([v, -\Delta^2 u - u], [u, v]) \\ &= \int_{\mathbb{R}^n} (\Delta v \, \Delta u + vu - v \, \Delta^2 u - vu) \, dx = 0\end{aligned}$$

and therefore B is skew adjoint. The lemma is thus a direct consequence of Stone's theorem. □

Lemma 3.4. *The map $f\colon X \to X$ given by $f(\mathbf{u}) = [0, |u|^{p-1} u]$ is locally Lipschitz.*

Proof. Let $\mathbf{u}_1, \mathbf{u}_2 \in X$ and compute

$$\begin{aligned}\|f(\mathbf{u}_1) - f(\mathbf{u}_2)\|_X^2 &= \int_{\mathbb{R}^n} ||u_1|^{p-1} u_1 - |u_2|^{p-1} u_2|^2 \, dx \\ &= \int_{\mathbb{R}^n} |p(\lambda(x) u_1 + (1 - \lambda(x)) u_2)^{p-1} (u_1 - u_2)|^2 \, dx \\ &\leqslant p^2 \, \|u_1 - u_2\|_{L^{2^*}(\mathbb{R}^n)}^2 \, \| |u_1| + |u_2| \|_{L^{n(p-1)/2}(\mathbb{R}^n)}^{2(p-1)}\end{aligned}$$

Since $p < 2^*/2$ this shows that

$$\|f(\mathbf{u}_1) - f(\mathbf{u}_2)\|_X \leqslant C(\|\mathbf{u}_1\|_X + \|\mathbf{u}_2\|_X)^{p-1} \, \|\mathbf{u}_1 - \mathbf{u}_2\|_X$$

and therefore f is locally Lipschitz. □

Next let φ be a ground state with velocity $\vec{c}$ and denote $\boldsymbol{\varphi} = [\varphi, \vec{c} \cdot \nabla\varphi]$. Then $\boldsymbol{\varphi}$ satisfies

$$E'(\boldsymbol{\varphi}) - \vec{c} \cdot \vec{Q}'(\boldsymbol{\varphi}) = 0 \tag{3.6}$$

We define, for $|\vec{c}|^2 < 2$,

$$d(\vec{c}) = E(\varphi) - \vec{c} \cdot \vec{Q}(\varphi) \tag{3.7}$$

The stability or instability of the ground state φ will be determined by whether or not d is convex in $\vec{c}$. The relation between E, $\vec{Q}$ and the functionals used to find the traveling wave is given by

$$E(\mathbf{u}) - \vec{c} \cdot \vec{Q}(\mathbf{u}) = \frac{1}{2} I_{\vec{c}}(u) - \frac{1}{p+1} K(u) + \frac{1}{2} \int_{\mathbb{R}^n} |v - \vec{c} \cdot \nabla u|^2 \, dx \tag{3.8}$$

Note that this shows that $d(\vec{c})$ is well defined and that, in fact,

$$d(\vec{c}) = \frac{p-1}{2(p+1)} K(\varphi) = \frac{p-1}{2(p+1)} I_{\vec{c}}(\varphi) \tag{3.9}$$

for any ground state φ.

Since Eq. (1.1) is invariant under orthogonal transformations, we may choose $\vec{c} = (c, 0,..., 0) = c\vec{e}_1$ for $c \in (-\sqrt{2}, \sqrt{2})$, so that (2.1) becomes

$$\Delta^2 \varphi + c^2 \varphi_{x_1 x_1} + \varphi - |\varphi|^{p-1} \varphi = 0 \tag{3.10}$$

If φ is a solution of (3.10) obtained as in Section 1, we say φ is a ground state with speed c. Then for $\varphi = [\varphi, c\varphi_{x_1}]$, we have

$$E'(\varphi) - cQ_1'(\varphi) = 0 \tag{3.11}$$

If we define

$$I_c(u) = \int_{\mathbb{R}^n} |\Delta u|^2 - c^2 |u_{x_1}|^2 + |u|^2 \, dx \tag{3.12}$$

we may then write (3.8) as

$$E(\mathbf{u}) - cQ_1(\mathbf{u}) = \frac{1}{2} I_c(u) - \frac{1}{p+1} K(u) + \frac{1}{2} \int_{\mathbb{R}^n} |v - cu_{x_1}|^2 \, dx \tag{3.13}$$

Let

$$d_1(c) = E(\varphi) - cQ_1(\varphi) \tag{3.14}$$

so that, as above, we have

$$d_1(c) = \frac{p-1}{2(p+1)} K(\varphi) = \frac{p-1}{2(p+1)} I_c(\varphi) \tag{3.15}$$

By the invariance of the laplacian under orthogonal transformations, it is easily shown that

$$d(\vec{c}) = d_1(|\vec{c}|) \tag{3.16}$$

and therefore d is radial.

By stability we mean the following. A set $S \subset X$ is *stable* with respect to (1.1) if, given $\varepsilon > 0$, there exists $\delta > 0$ so that for any $\mathbf{v} \in X$ with

$$\inf_{\mathbf{w} \in S} \|\mathbf{v} - \mathbf{w}\|_X < \delta$$

the solution $\mathbf{u}(t)$ of (1.1) with data $\mathbf{u}(0) = \mathbf{v}$ can be extended to a solution in $C([0, \infty); X)$ and

$$\sup_{0 \leqslant t < \infty} \inf_{\mathbf{w} \in S} \|\mathbf{u}(t) - \mathbf{w}\|_X < \varepsilon$$

Otherwise we say S is *unstable.*

4. PROPERTIES OF $d(\vec{c})$

In this section we examine the dependence of d on the parameter $\vec{c}$. We prove the following.

Lemma 4.1. *The function $d_1(c)$ is continuous, decreasing in $|c|$, and differentiable at all but countably many points of $(-\sqrt{2}, \sqrt{2})$.*

First define

$$M(c) = \inf_{u \in H^2(\mathbb{R}^n)} \frac{I_c(u)}{K(u)^{2/(p+1)}} \tag{4.1}$$

It is clear that this infimum is attained at any ground state, so that by (3.15) we have

$$d_1(c) = \frac{p-1}{2(p+1)} (M(c))^{(p+1)/(p-1)} \tag{4.2}$$

Define the set of all ground states with speed c as

$$S_c = \{\psi \in H^2(\mathbb{R}^n) \colon K(\psi) = I_c(\psi) = (M(c))^{(p+1)/(p-1)}\}$$

and define the tubular neighborhood about S_c by

$$U_{c,\varepsilon} = \{\mathbf{u} \in X : \inf\{\|\mathbf{u} - \boldsymbol{\psi}\|_X : \psi \in S_c\} < \varepsilon\}$$

where we denote $\boldsymbol{\psi} = [\psi, c\psi_{x_1}]$ for $\psi \in S_c$. We now prove Lemma 4.1 by investigating the behavior of $M(c)$.

Lemma 4.2. *$M(c)$ is continuous and strictly decreasing in $|c|$ on $(-\sqrt{2}, \sqrt{2})$.*

Proof. Suppose $0 \leqslant c_1 < c_2 < \sqrt{2}$. It is clear that $M(c_1) > M(c_2) > 0$. Let u be a minimizer with speed c_2 and compute

$$\begin{aligned} M(c_1) &\leqslant \frac{I_{c_1}(u)}{K(u)^{2/(p+1)}} = \frac{I_{c_2}(u)}{K(u)^{2/(p+1)}} + \frac{c_2^2 - c_1^2}{K(u)^{2/(p+1)}} \int_{\mathbb{R}^n} |u_{x_1}|^2 \, dx \\ &= M(c_2) + \frac{c_2^2 - c_1^2}{K(u)^{2/(p+1)}} \int_{\mathbb{R}^n} |u_{x_1}|^2 \, dx \end{aligned} \tag{4.3}$$

So

$$\begin{aligned} |M(c_2) - M(c_1)| &\leqslant \frac{3(c_2 - c_1)}{K(u)^{2/(p+1)}} \int_{\mathbb{R}^n} |u_{x_1}|^2 \, dx \leqslant \frac{3(c_2 - c_1) \, I_{c_2}(u)}{(2 - c_2^2) \, K(u)^{2/(p+1)}} \\ &= 3(c_2 - c_1) \frac{M(c_2)}{(2 - c_2^2)} \leqslant 3(c_2 - c_1) \frac{M(0)}{(2 - c_2^2)} \end{aligned} \tag{4.4}$$

and thus M is locally Lipschitz on $c^2 < 2$. □

Motivated by the bounds obtained in Lemma 4.2 we set

$$\alpha(c) = \inf \left\{ \int_{\mathbb{R}^n} |\psi_{x_1}|^2 \, dx : \psi \in S_c \right\}$$

$$\beta(c) = \sup \left\{ \int_{\mathbb{R}^n} |\psi_{x_1}|^2 \, dx : \psi \in S_c \right\}$$

Lemma 4.3. *M is differentiable at c_1 if and only if $\alpha(c_1) = \beta(c_1)$.*

Proof. From (4.3) we see that, for $c_1 < c_2$,

$$\frac{-(c_1 + c_2) \, \beta(c_1)}{(M(c_1))^{2/(p-1)}} \geqslant \frac{M(c_2) - M(c_1)}{c_2 - c_1} \geqslant \frac{-(c_1 + c_2) \, \alpha(c_2)}{(M(c_2))^{2/(p-1)}} \tag{4.5}$$

We now claim that

$$\limsup_{c \to c_1} \alpha(c) \leqslant \beta(c_1)$$

Let $\{c_k\}$ be any sequence so that $c_k \to c_1$ and let $v_k \in S_{c_k}$. Then by the continuity of M,

$$I_{c_1}(v_k) \to (M(c_1))^{(p+1)/(p-1)}$$

and

$$K(v_k) \to (M(c_1))^{(p+1)/(p-1)}$$

Thus $\{v_k\}_{k=1}^{\infty}$ is a minimizing sequence for I_{c_1} and, by Theorem 2.1, has a strongly convergent subsequence v_{k_j} (modulo translations) to some $v \in S_{c_1}$. Hence

$$\int_{\mathbb{R}^n} |(v_{k_j})_{x_1}|^2 \, dx \to \int_{\mathbb{R}^n} |v_{x_1}|^2 \, dx$$

and thus

$$\limsup_{j \to \infty} \alpha(c_{k_j}) \leqslant \beta(c_1)$$

which proves the claim. Applying the claim to (4.5) shows that

$$\lim_{c \to c_1^-} \frac{M(c) - M(c_1)}{c - c_1} = \frac{-c_1 \beta(c_1)}{(M(c_1))^{2/(p-1)}}$$

Similary we see that

$$\lim_{c \to c_1^-} \frac{M(c) - M(c_1)}{c - c_1} = \frac{-c_1 \alpha(c_1)}{(M(c_1))^{2/(p-1)}}$$

Thus the left and right derivatives of M exist everywhere and are equal whenever $\alpha(c_1) = \beta(c_1)$. □

Lemma 4.4. $\alpha(c_1) = \beta(c_1) \Leftrightarrow \alpha$ *is right continuous at* $c_1 \Leftrightarrow \beta$ *is left continuous at* c_1.

Proof. By the inequalities in (4.5),

$$\frac{\beta(c_2)}{(M(c_2))^{2/(p-1)}} \geqslant \frac{\alpha(c_2)}{(M(c_2))^{2/(p-1)}} \geqslant \frac{\beta(c_1)}{(M(c_1))^{2/(p-1)}} \geqslant \frac{\alpha(c_1)}{(M(c_1))^{2/(p-1)}} \quad (4.6)$$

for $c_1 < c_2$. Thus $\alpha(c)/(M(c))^{2/(p-1)}$ and $\beta(c)/(M(c))^{2/(p-1)}$ are increasing functions of c. Applying Theorem 2.1 once shows that $\alpha(c)$ is lower semicontinuous and $\beta(c)$ is upper semicontinuous. Hence $\alpha(c)$ is left continuous and $\beta(c)$ is right continuous. By (4.6) it follows that

$$\lim_{c \to c_1^+} \alpha(c) = \beta(c_1)$$

and

$$\lim_{c \to c_1^-} \beta(c) = \alpha(c_1)$$

and the lemma is proved. □

To prove Lemma 4.1, note first that (4.2) and Lemma 4.2 imply that $d_1(c)$ is continuous and strictly decreasing in $|c|$ on $(-\sqrt{2}, \sqrt{2})$. Next, the monotonicity of $\alpha/M^{2/(p-1)}$ and $\beta/M^{2/(p-1)}$, together with the continuity of M, shows that $\alpha(c)$ and $\beta(c)$ are continuous at all but countably many points of $(-\sqrt{2}, \sqrt{2})$. Hence by Lemma 4.2 and Lemma 4.3, it follows that $d_1(c)$ is differentiable at all but countably many points of $(-\sqrt{2}, \sqrt{2})$. □

We now wish to obtain bounds on $d_1(c)$ in order to determine regions of convexity and concavity. We first find an upper bound on $d_1(c)$ for c near $\sqrt{2}$.

Lemma 4.5. *Suppose* $1 < p < 2^* - 1$. *Then*

$$d_1(c) \leqslant C(2 - c^2)^{\gamma} \tag{4.7}$$

where $\gamma = (2n - (n-2)(p+1))/(2(p-1))$.

Proof. We consider the case $n = 1$ only, as the result for $n > 1$ follows similarly. First let $\zeta_1 = (\sqrt{2 - c^2})/2$, $\zeta_2 = (\sqrt{2 + c^2})/2$. Then $\zeta = \zeta_1 + \zeta_2 i$ solves $\zeta^4 + c^2\zeta^2 + 1 = 0$ and therefore $e^{\pm\zeta x}$, $e^{\pm\bar{\zeta} x}$ are solutions of the linear equation $\varphi_{xxxx} + c^2\varphi_{xx} + \varphi = 0$. Define $g_c \in H^2(\mathbb{R})$ by

$$g_c(x) = e^{-\zeta_1 |x|}\left(\cos \zeta_2 |x| + \frac{\zeta_1}{\zeta_2} \sin \zeta_2 |x|\right)$$

Integrating by parts we see that

$$I_c(g_c) = 2\sqrt{2 - c^2}$$

Also, for c near $\sqrt{2}$ we have for some constant C that

$$K(g_c) \geqslant \frac{C}{\sqrt{2-c^2}}$$

and therefore

$$M(c) \leqslant \frac{I_c(g_c)}{K(g_c)^{2/(p+1)}} \leqslant C(2-c^2)^{(p+3)/(2(p+1))}$$

Finally (4.2) implies

$$d_1(c) \leqslant C(2-c^2)^{(p+3)/(2(p-1))}$$

which proves the lemma. □

If $p < 1 + 4/n$ then $\gamma > 1$ and hence $(2-c^2)^\gamma$ vanishes to first order at $c = \sqrt{2}$. The positivity and monotonicity of d_1 then imply the existence of intervals of convexity arbitrarily close to $\sqrt{2}$. We next establish a lower bound on d_1 under the assumption that d_1 is differentiable.

Lemma 4.6. *Suppose that $d_1(c)$ is differentiable on $(-\sqrt{2}, \sqrt{2})$. Then*

$$d_1(c) \geqslant d_1(0)\left(1-\frac{c^2}{2}\right)^{(p+1)/(p-1)} \tag{4.8}$$

Proof. First, by Lemma 4.3 we have

$$d_1'(c) = -c\int_{\mathbb{R}^n} |\varphi_{x_1}|^2\, dx$$

for any ground state φ with speed c. By (3.14) and (2.7),

$$\begin{aligned} d_1(c) &= \frac{p-1}{2(p+1)} I_c(\varphi) \geqslant \frac{p-1}{2(p+1)}\left(1-\frac{c^2}{2}\right)\|\varphi\|^2_{H^2(\mathbb{R}^n)} \\ &= \frac{p-1}{2(p+1)}\left(1-\frac{c^2}{2}\right)\left(I_c(\varphi) + c^2\int_{\mathbb{R}^n}|\varphi_{x_1}|^2\,dx\right) \\ &= \left(1-\frac{c^2}{2}\right)\left(d_1(c) - \frac{p-1}{2(p+1)}cd_1'(c)\right) \end{aligned}$$

Thus

$$\frac{d_1'(c)}{d_1(c)} \geqslant \left(\frac{p+1}{p-1}\right) \frac{-2c}{2-c^2}$$

and the lemma follows. □

Hence there are intervals of concavity of d_1 arbitrarily close to $c=0$. Further, if there is a C^1 map $c \mapsto \varphi_c$ (as in Section 6), we have the following:

$$\begin{aligned} d_1''(c) &= -\int_{\mathbb{R}^n} |(\varphi_c)_{x_1}|^2 \, dx - 2c \int_{\mathbb{R}^n} (\varphi_c)_{x_1} \frac{\partial(\varphi_c)_{x_1}}{\partial c} \, dx \\ &\leqslant -(1-|c|) \int_{\mathbb{R}^n} |(\varphi_c)_{x_1}|^2 \, dx + |c| \int_{\mathbb{R}^n} \left| \frac{\partial(\varphi_c)_{x_1}}{\partial c} \right|^2 dx \end{aligned} \tag{4.9}$$

So $d_1''(c) < 0$ in some interval about zero. We need the following lemmas in order to relate the properties of $d(\vec{c})$ and $d_1(c)$.

Lemma 4.7. *Let $f: \mathbb{R}^n \to \mathbb{R}$ be C^2 and radial, with $f(x) = g(|x|)$. Then for $x \neq 0$, $D^2 f(x)$ is singular if and only if $g''(|x|) = 0$ or $g'(|x|) = 0$. ($D^2 f(0) = g''(0)\, I$).*

Proof. Since

$$f_{x_1 x_j} = \left(\frac{g''(|x|)}{|x|^2} - \frac{g'(|x|)}{|x|^3} \right) x_i x_j + \frac{g'(|x|)}{|x|} \delta_{ij}$$

we have

$$D^2 f(x) = M(|x|)(x \otimes x) + N(|x|)\, I$$

where

$$M(r) = \left(\frac{g''(r)}{r^2} - \frac{g'(r)}{r^3} \right) \qquad N(r) = \frac{g'(r)}{r}$$

Now $D^2 f(x)$ is singular if and only if $-N(|x|)/M(|x|)$ is an eigenvalue of $x \otimes x$. Since the eigenvalues of $x \otimes x$ are zero and $|x|^2$, we have either $N(|x|) = 0$, which implies $g'(|x|) = 0$, or $-N(|x|) = |x|^2 M(|x|)$, which implies $g''(|x|) = 0$.

Lemma 4.8. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be C^2 with $f(x) = g(|x|)$ and suppose $g''(|x|) \leqslant 0$ and $g'(|x|) \leqslant 0$. Then for $x \neq 0$, $D^2 f(x)$ is negative-semidefinite. Further, for $x \neq 0$, $D^2 f(x)$ is negative-definite if and only if $g''(|x|) < 0$ and $g'(|x|) < 0$.*

Proof. Since $D^2 f(x)$ is symmetric, it suffices to show that it has only nonpositive (negative) eigenvalues. If λ is an eigenvalue of $D^2 f$, then

$$\lambda = N(|x|) \qquad \text{or} \qquad \lambda = N(|x|) + |x|^2 M(|x|)$$

In the first case $\lambda = g'(|x|)/|x|$, and in the second case $\lambda = g''(|x|)$ and the lemma is proved. □

So if d_1 is differentiable and $d_1''(c) < 0$ we have by (3.16), Lemma 4.3 and Lemma 4.8 that $D^2 d(\vec{c})$ is negative definite for every $\vec{c}$ with $|\vec{c}| = c$.

5. STABILITY

Here we show that the set of ground states S_c is stable whenever d is strictly convex in a neighborhood of c. The variational nature of the ground states is used to show that sequences of later time data are minimizing sequences, provided the initial data are chosen close enough to S_c. First we state without proof a lemma due to Shatah [17] concerning strictly convex functions.

Lemma 5.1. *Let h be any function which is strictly convex in an interval I about c. Then given $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ so that for $c_1 \in I$, $|c_1 - c| \geqslant \varepsilon$ we have*

(1) $c_1 < c < c_0$, $|c_0 - c| < \varepsilon/2$, $c_0 \in I \Rightarrow$

$$\frac{h(c_1) - h(c_0)}{c_1 - c_0} \leqslant \frac{h(c) - h(c_0)}{c - c_0} - \frac{1}{N(\varepsilon)}$$

(2) $c_0 < c < c_1$, $|c_0 - c| < \varepsilon/2$, $c_0 \in I \Rightarrow$

$$\frac{h(c_1) - h(c_0)}{c_1 - c_0} \geqslant \frac{h(c) - h(c_0)}{c - c_0} + \frac{1}{N(\varepsilon)}$$

Lemma 5.2. *Suppose that d_1 is strictly convex in an interval I around c. Then for every $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ so that for $c_1 \in I$ with $|c_1 - c| \geqslant \varepsilon$, have*

$$d_1(c_1) \geqslant d_1(c) - c\beta(c)(c_1 - c) + \frac{1}{N(\varepsilon)}(c - c_1) \tag{5.1}$$

for $c_1 < c$ *and*

$$d_1(c_1) \geqslant d_1(c) - c\alpha(c)(c_1 - c) + \frac{1}{N(\varepsilon)}(c_1 - c) \tag{5.2}$$

for $c_1 > c$.

Proof. This follows by taking limits in (1) and (2) of Lemma 4.1 as $c_0 \to c$ and using the inequalities in (4.5). □

The next lemma uses the variational characterization of ground states to establish the key inequality in the proof of stability. First we use the fact that d is continuous and strictly monotone on $[0, \sqrt{2})$ to define, for $\mathbf{u}$ near φ (φ in S_c),

$$c(\mathbf{u}) = d_1^{-1}\left(\frac{p-1}{2(p+1)} K(u)\right) \tag{5.3}$$

Lemma 5.3. *Suppose that* d_1 *is strictly convex in an interval* I *about* c. *Then there exists* $\varepsilon > 0$ *so that for all* $\mathbf{u} \in U_{c,\varepsilon}$ *and any* $\psi_c \in S_c$,

$$E(\mathbf{u}) - E(\psi_c) - c(\mathbf{u})(Q_1(\mathbf{u}) - Q_1(\psi_c)) \geqslant \frac{1}{N(\varepsilon)} |c(\mathbf{u}) - c| \tag{5.4}$$

Proof. Since $c(\mathbf{u})$ is a continuous function of $\mathbf{u}$, we may choose ε small enough that $c(U_{c,\varepsilon})$ is within the interval I. Then by Lemma 5.2,

$$\begin{aligned}
E(\mathbf{u}) - c(\mathbf{u})\, Q_1(\mathbf{u}) &= \frac{1}{2} I_{c(\mathbf{u})}(u) - \frac{1}{p+1} K(u) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} |v - c(\mathbf{u})\, u_{x_1}|^2\, dx \qquad \text{by (3.13)} \\
&\geqslant \frac{1}{2} I_{c(\mathbf{u})}(u) - \frac{1}{p+1} K(u) \\
&\geqslant \frac{1}{2} I_{c(\mathbf{u})}(\psi_{c(\mathbf{u})}) - \frac{1}{p+1} K(\psi_{c(\mathbf{u})}) = d_1(c(\mathbf{u})) \qquad \text{by (3.15)} \\
&\geqslant d_1(c) - Q_1(\psi_c)(c(\mathbf{u}) - c) + \frac{1}{N(\varepsilon)} |c(\mathbf{u}) - c| \\
&= E(\psi_c) - c(\mathbf{u})\, Q_1(\psi_c) + \frac{1}{N(\varepsilon)} |c(\mathbf{u}) - c| \qquad \text{by (3.14)}
\end{aligned}$$

The second inequality follows since $(2(p+1))/(p-1)\, d_1(c(\mathbf{u})) = K(u) = K(\psi_{c(\mathbf{u})})$ and $\psi_{c(\mathbf{u})}$ minimizes $I_{c(\mathbf{u})}$ subject to this constraint.

Theorem 5.4. *Suppose that Assumption 3.1 holds and that* $1 < p < 2^* - 1$. *If* d_1 *is strictly convex in an interval around* c, *then the set of ground states* S_c *is stable.*

Proof. Assume that S_c is unstable and choose initial data $\mathbf{u}_k(0) \in U_{c,1/k}$. Then since $\mathbf{u}_k(t)$ is continuous in t, we can find t_k such that

$$\inf_{\psi \in S_c} \|\mathbf{u}_k(t_k) - \psi\|_X = \delta \tag{5.5}$$

Since $U_{c,1/k}$ is bounded for each k and since E and Q_1 are invariants of the equation, we can find $\psi_k \in S_c$ and a constant C such that

$$|E(\mathbf{u}_k(t_k)) - E(\psi_k)| < \frac{C}{k}$$

$$|Q_1(\mathbf{u}_k(t_k)) - Q_1(\psi_k)| < \frac{C}{k}$$

Now choose δ small enough so that Lemma 5.3 applies. Then

$$E(\mathbf{u}_k(t_k)) - E(\psi_k) - c(\mathbf{u}_k(t_k))(Q_1(\mathbf{u}_k(t_k)) - Q_1(\psi_k)) \geqslant \frac{1}{N(\varepsilon)} |c(\mathbf{u}_k(t_k)) - c|$$

So letting $k \to \infty$, it follows that $c(\mathbf{u}_k(t_k)) \to c$. By (5.3) and the continuity of d_1, we then have

$$\lim_{k \to \infty} K(u_k(t_k)) = \frac{2(p+1)}{p-1} d_1(c) \tag{5.6}$$

Since by (3.13),

$$\frac{1}{2} I_c(u_k(t_k)) = E(\mathbf{u}_k(t_k)) - cQ_1(\mathbf{u}_k(t_k)) + \frac{1}{p+1} K(u_k(t_k)) - \frac{1}{2} \int_{\mathbb{R}^n} |v_k(t_k) - c(u_k(t_k))_{x_1}|^2 \, dx \tag{5.7}$$

we also have

$$\limsup_{k \to \infty} I_c(u_k(t_k)) \leqslant 2d_1(c) + \frac{4}{p-1} d_1(c) = \frac{2(p+1)}{p-1} d_1(c)$$

By (5.6) we therefore have

$$\lim_{k \to \infty} I_c(u_k(t_k)) = \frac{2(p+1)}{p-1} d_1(c)$$

So $M(c)^{1/(1-p)} u_k(t_k)$ is a minimizing sequence, and by Theorem 2.1 there exist $\phi_k \in S_c$ such that

$$\lim_{k \to \infty} \|u_k(t_k) - \phi_k\|_{H^2(\mathbb{R}^n)} = 0 \tag{5.8}$$

Finally, by (5.7),

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} |v_k(t_k) - c(u_k(t_k))_{x_1}|^2 \, dx = 0$$

and by (5.8),

$$\lim_{k \to \infty} \|(u_k(t_k))_{x_1} - (\phi_k)_{x_1}\|_{L^2(\mathbb{R}^n)} = 0$$

Thus $\|v_k(t_k) - c(\phi_k)_{x_1}\|_{L^2(\mathbb{R}^n)} \to 0$ and, together with (5.8), this implies

$$\lim_{k \to \infty} \|\mathbf{u}_k(t_k) - \phi_k\|_X = 0$$

which contradicts (5.5). □

Remark 5.5. Together with the bound in Lemma 4.5, Theorem 5.4 implies the existence of stable traveling waves for some speeds near $\sqrt{2}$ when $p < 1 + (4/n)$. This differs from the second-order wave equation (in one dimension) for which all traveling waves are unstable [6].

6. INSTABILITY

We show in this section that, under more strict hypotheses, the concavity of d implies that the ground state is orbitally unstable. Using the concavity of d we show that there is an "unstable direction" which allows us to construct a Lyapunov functional. The main assumption is the following.

Assumption 6.1. *There exists a C^1 map*

$$c \mapsto \boldsymbol{\varphi}_c \equiv [\varphi_c, (\varphi_c)_{x_1}] \tag{6.1}$$

such that φ_c is a ground state with speed c.

This implies by (3.15) that d_1 is differentiable, and for $c \geqslant 0$ we have by Lemma 4.3,

$$d_1'(c) = -Q_1(\varphi_c) = -c \int_{\mathbb{R}^n} |(\varphi_c)_{x_1}|^2 \, dx \leqslant 0 \tag{6.2}$$

We also need to extend the map in (6.1) to a neighborhood of $c\vec{e}_1$ in $\mathbb{R}^n$. For any $\vec{c} \in \mathbb{R}^n$ we can choose $A_{\vec{c}} \in O(n)$ such that $A_{\vec{c}}\vec{e}_1 = \vec{c}/|\vec{c}|$. Then $\psi(x) = \varphi_{|\vec{c}|}(A_{\vec{c}}^t x)$ is a ground state with velocity $\vec{c}$. In a neighborhood of $c\vec{e}_1$ we can choose $A_{\vec{c}}$ continuously in $\vec{c}$. We denote by $\varphi_{\vec{c}}$ the ground state with velocity $\vec{c}$, while we still write φ_c for the ground state with speed c in the direction $\vec{e}_1$. It can be shown, as in Lemma 4.3, that

$$\nabla d(\vec{c}) = -Q(\varphi_{\vec{c}}) \tag{6.3}$$

As in Section 3, let $\langle \cdot, \cdot \rangle$ denote the pairing of X^* with X and let $\{T(\tau): \tau \in \mathbb{R}^n\}$ be the group of translations acting on X by

$$T(\tau)\, \mathbf{u}(x) = \mathbf{u}(x + \tau) \tag{6.4}$$

Define the orbit of φ_c under T by

$$\tilde{S}_c = \{ T(\tau)\, \varphi_c : \tau \in \mathbb{R}^n \}$$

and let

$$V_{c,\varepsilon} = \{ \mathbf{u} \in X : \inf_{\tau \in \mathbb{R}^n} \|\mathbf{u} - T(\tau)\, \varphi_c\|_X < \varepsilon \}$$

be the tubular neighborhood of radius ε about $\tilde{S}_c$. We prove the following.

Theorem 6.2. *Suppose that Assumption 3.1 and Assumption 6.1 hold and that* $1 < p < 2^* - 1$. *If* $d_1''(c) < 0$, *then* $\tilde{S}_c$ *is unstable.*

The proof of our first lemma is trivial.

Lemma 6.3. *For any* $\mathbf{u} \in X$, *if* $T(\tau_n)\, \mathbf{u} \to \mathbf{u}$ *in* X *then* $\tau_n \to 0$.

Lemma 6.4. *There exists* $\varepsilon > 0$ *and a* C^2 *map* $\sigma: V_{c,\varepsilon} \to \mathbb{R}^n$ *such that for any* $\mathbf{u} \in V_{c,\varepsilon}$ *and* $\tau \in \mathbb{R}^n$,

(1) $\|T(\sigma(\mathbf{u}))\, \mathbf{u} - \varphi_c\|_X \leqslant \|T(\tau)\, \mathbf{u} - \varphi_c\|_X$,

(2) $\langle T(\sigma(\mathbf{u}))\, \mathbf{u}, (\varphi_c)_{x_i} \rangle = 0$ *for* $i = 1, \ldots, n$, *and*

(3) $\sigma(T(\tau)\, \mathbf{u}) = \sigma(\mathbf{u}) - \tau$,

where we denote $\mathbf{u}_{x_i} = [u_{x_i}, v_{x_i}]$.

Proof. Define $\rho: X \times \mathbb{R}^n \to \mathbb{R}$ by

$$\rho(\mathbf{u}, \sigma) = \|T(\sigma)\,\mathbf{u} - \varphi_c\|_X^2 = \|T(\sigma)\,\mathbf{u}\|_X^2 + \|\varphi_c\|_X^2 - 2(\mathbf{u}, T(-\sigma)\,\varphi_c)$$

Since $\varphi \in H^5(\mathbb{R}^n)$, we may compute

$$\frac{\partial \rho}{\partial \sigma_i}(\mathbf{u}, \sigma) = 2(T(\sigma)\,\mathbf{u}, (\varphi_c)_{x_i}), \qquad i = 1,..., n$$

and

$$\frac{\partial^2 \rho}{\partial \sigma_i\, \partial \sigma_j}(\mathbf{u}, \sigma) = -2(T(\sigma)\,\mathbf{u}, (\varphi_c)_{x_i x_j}), \qquad i, j = 1,..., n$$

Since $\nabla_\sigma \rho(\varphi_c, 0) = 0$ and since $D_\sigma^2 \rho(\varphi_c, 0) = 2(((\varphi_c)_{x_i}, (\varphi_c)_{x_j}))_{i,j=1}^n$ is positive definite, the implicit function theorem implies the existence of a neighborhood U of φ_c in X, a ball $B(0, \varepsilon')$, and a C^2 map $\sigma: U \to B(0, \varepsilon')$ such that $\nabla_\sigma \rho(\mathbf{u}, \sigma(\mathbf{u})) = 0$ and $D_\rho^2 \sigma(\mathbf{u}, \sigma(\mathbf{u}))$ is positive definite for all $\mathbf{u} \in U$. Thus $\sigma(\mathbf{u})$ is the unique minimizer of $\rho(\mathbf{u}, \cdot)$ in $B(0, \varepsilon')$ for each $\mathbf{u} \in U$. By Lemma 6.3, there exists $\delta > 0$ such that $\|T(\tau)\,\varphi_c - \varphi_c\|_X < \delta$ implies $\tau \in B(0, \varepsilon')$. Choose $\varepsilon < \delta/4$ so that $V_\varepsilon = \{\mathbf{u}: \|\mathbf{u} - \varphi_c\|_X < \varepsilon\} \subset U$. Then (1) and (2) hold for $\mathbf{u} \in V_\varepsilon$. To show that (3) holds, compute

$$\begin{aligned}\|T(\sigma(\mathbf{u}) - \tau)\,T(\tau)\,\mathbf{u} - \varphi_c\|_X &\leqslant \|T(\sigma(T(\tau)\,\mathbf{u}) + T(\tau)\,\mathbf{u})\,\mathbf{u} - \varphi_c\|_X \\ &= \|T(\sigma(T(\tau)\,\mathbf{u}))\,T(\tau)\,\mathbf{u} - \varphi_c\|_X \\ &\leqslant \|T(\sigma(\mathbf{u}) - \tau)\,T(\tau)\,\mathbf{u} - \varphi_c\|_X\end{aligned}$$

Thus (3) follows if we can show that $\sigma(\mathbf{u}) - \tau \in (-\varepsilon', \varepsilon')$ when $\mathbf{u}$, $T(\tau)\,\mathbf{u} \in V_\varepsilon$. Since

$$\begin{aligned}\|T(\sigma(\mathbf{u}) - \tau)\,\varphi - \varphi_c\|_X &\leqslant \|T(\sigma(\mathbf{u}))(\varphi_c - \mathbf{u})\|_X + \|T(\sigma(\mathbf{u}))\,\mathbf{u} - \varphi_c\|_X \\ &\quad + \|T(\tau)\,\mathbf{u} - \varphi_c\|_X + \|T(\tau)(\mathbf{u} - \varphi_c)\|_X \leqslant 4\varepsilon < \delta\end{aligned}$$

it follows from our choice of δ that $\sigma(\mathbf{u}) - \tau \in B(0, \varepsilon')$. Now extend σ to $\mathbf{u} \in V_{c,\varepsilon}$ by first choosing τ such that $T(\tau)\,\mathbf{u} \in V_\varepsilon$. Then let $\sigma(\mathbf{u}) = \sigma(T(\tau)\,\mathbf{u}) - \tau$. Since (3) holds in V_ε, $\sigma(\mathbf{u})$ is independent of the choice of τ and properties (1)–(3) follow for $\mathbf{u} \in V_{c,\varepsilon}$. □

The next theorem proves the existence of an "unstable direction" and depends on the fact that there is an element of X for which the linearized operator

$$H_c = E''(\varphi_c) - cQ_1''(\varphi_c) \tag{6.5}$$

has negative expectation. In fact, evaluating H_c on φ_c yields

$$\langle H_c\varphi_c, \varphi_c\rangle = -(p-1)\,\|\varphi_c\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} < 0 \tag{6.6}$$

We need to modify φ_c in order to get a vector orthogonal to $\vec{Q}'(\varphi_c)$.

Theorem 6.5. *If $d_1''(c) < 0$, then there exists $\mathbf{y} \in Y \equiv H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ such that*

(1) $\langle H_c\mathbf{y}, \mathbf{y}\rangle < 0$, *and*

(2) $\langle \vec{Q}'(\varphi_c), \mathbf{y}\rangle = 0.$

Proof. Define $\vec{q}\colon B(0, \sqrt{2}) \times \mathbb{R} \to \mathbb{R}^n$ by

$$\vec{q}(\vec{h}, s) = \vec{Q}(\varphi_{\vec{h}} + s\varphi_c)$$

Then

$$\left.\frac{\partial\vec{q}}{\partial h_i}(\vec{h}, 0)\right|_{\vec{h}=c\vec{e}_1} = \left\langle \vec{Q}'(\varphi_c), \left.\frac{d\varphi_{\vec{h}}}{dh_i}\right|_{\vec{h}=c\vec{e}_1}\right\rangle = \left.\frac{\partial}{\partial c_i}(\vec{Q}'(\varphi_{\vec{c}}))\right|_{\vec{c}=c\vec{e}_1}$$

so that

$$\nabla_{\vec{h}}\vec{q}(c\vec{e}_1, 0) = -D^2 d(c\vec{e}_1) \qquad \text{by (6.3)}$$

By (6.2) and the assumption that $d''(c) < 0$, Lemma 4.6 implies that $D^2 d(c\vec{e}_1)$ is negative definite, and therefore the implicit function theorem implies that there exist $\varepsilon > 0$ and a C^2 function $\vec{h}\colon(-\varepsilon, \varepsilon) \to B(0, \sqrt{2})$ such that

$$\vec{Q}(\varphi_{\vec{h}}(s) + s\varphi_c) = \vec{Q}(\varphi_c) \tag{6.7}$$

for $s \in (-\varepsilon, \varepsilon)$. Now compute the Taylor expansions of E and $\vec{Q}$ about $\varphi_{\vec{h}}(s)$ to get

$$\begin{aligned} E(\varphi_{\vec{h}}(s) + s\varphi_c) = E(\varphi_{\vec{h}}(s)) &+ s\langle E'(\varphi_{\vec{h}}(s)), \varphi_c\rangle \\ &+ \tfrac{1}{2}s^2\langle E''(\varphi_{\vec{h}}(s))\,\varphi_c, \varphi_c\rangle + o(s^2) \end{aligned}$$

and

$$\begin{aligned} \vec{Q}(\varphi_c) &= \vec{Q}(\varphi_{\vec{h}}(s) + s\varphi_c) \\ &= \vec{Q}(\varphi_{\vec{h}}(s)) + s\langle \vec{Q}'(\varphi_{\vec{h}}(s)), \varphi_c\rangle + \tfrac{1}{2}s^2\langle \vec{Q}''(\varphi_{\vec{h}}(s))\,\varphi_c, \varphi_c\rangle + o(s^2) \end{aligned}$$

Thus, using (3.5) and (3.6) we have

$$E(\varphi_{\vec h}(s)+s\varphi_c)-\vec h(s)\cdot\vec Q(\varphi_c)=\tfrac{1}{2}s^2\langle[E''(\varphi_{\vec h}(s))-\vec h(s)\cdot\vec Q''(\varphi_{\vec h}(s))]\,\varphi_c,\varphi_c\rangle + d(\vec h(s))+o(s^2) \tag{6.8}$$

and by the concavity of d,

$$\begin{aligned} d(\vec h(s)) &\leqslant d(c\vec e_1)+(\vec h(s)-c\vec e_1)\cdot\nabla\, d(c\vec e_1)\\ &=E(\varphi_c)-\vec h(s)\cdot\vec Q(\varphi_c) \qquad \text{by (6.3)} \end{aligned} \tag{6.9}$$

Combining (6.8) and (6.9) gives

$$E(\varphi_{\vec h}(s)+s\varphi_c) \leqslant \tfrac{1}{2}s^2\langle[E''(\varphi_{\vec h}(s))-\vec h(s)\cdot\vec Q''(\varphi_{\vec h}(s))]\,\varphi_c,\varphi_c\rangle + E(\varphi_c)+o(s^2) \tag{6.10}$$

Since at $s=0$ the pairing on the right-hand side of (6.10) is precisely $\langle H_c\varphi_c,\varphi_c\rangle$, the continuity of E'', $\vec Q''$, $\vec h$ and φ_c implies that

$$E(\varphi_{\vec h}(s)+s\varphi_c)<E(\varphi_c)+\tfrac{1}{4}s^2\langle H_c\varphi_c,\varphi_c\rangle+o(s^2) \tag{6.11}$$

for s sufficiently small. Now define

$$\tilde{\mathbf y}=\frac{d}{ds}(\varphi_{\vec h}(s)+s\varphi_c)\Big|_{s=0}=\vec h'(0)\cdot\nabla_{\vec c}\,\varphi_{\vec c}\Big|_{\vec c}=c\vec e_1+\varphi_c \tag{6.12}$$

so that by (6.7),

$$\langle\vec Q(\varphi_c),\tilde{\mathbf y}\rangle=\frac{d\vec Q}{ds}(\varphi_{\vec h}(s)+s\varphi_c)\Big|_{s=0}=0$$

and thus (2) holds for $\tilde{\mathbf y}$. Using (6.7) again, it follows that

$$\frac{d^2E}{ds^2}(\varphi_{\vec h}(s)+s\varphi_c)\Big|_{s=0}=\langle H_c\tilde{\mathbf y},\tilde{\mathbf y}\rangle$$

Finally, since $(d/ds)\,E(\varphi_{\vec h}(s)+s\varphi_c)=0$ at $s=0$, the strict inequality in (6.11) implies that

$$\frac{d^2E}{ds^2}(\varphi_{\vec h}(s)+s\varphi_c)\Big|_{s=0}<\frac{1}{2}\langle H_c\varphi_c,\varphi_c\rangle<0$$

and hence

$$\langle H_c\tilde{\mathbf y},\tilde{\mathbf y}\rangle<0 \tag{6.13}$$

Since $H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ is dense in X, we can perturb $\tilde{\mathbf{y}}$ slightly to obtain a vector in Y satisfying (1) and (2). Without loss of generality, assume that the $Q_i'(\varphi_c)$ are linearly independent. We define by induction $\mathbf{w}_1,..., \mathbf{w}_n$ in Y with the property that

$$\langle Q_i'(\varphi_c), \mathbf{w}_j \rangle = \delta_{ij} \tag{6.14}$$

First, let $\mathbf{w}$ be any element of Y for which $\langle Q_1'(\varphi_c), \mathbf{w} \rangle \neq 0$ and let

$$\mathbf{w}_1 = \frac{\mathbf{w}}{\langle Q_1'(\varphi_c), \mathbf{w} \rangle}$$

Next, assume that we have constructed $\mathbf{w}_1,..., \mathbf{w}_k$ satisfying (6.14) and choose any $\mathbf{w}$ in Y such that

$$\langle Q_{k+1}'(\varphi_c), \mathbf{w} \rangle - \sum_{i=1}^{k} \langle Q_i'(\varphi_c), \mathbf{w} \rangle \langle Q_{k+1}'(\varphi_c), \mathbf{w}_i \rangle \neq 0 \tag{6.15}$$

This can be done by the assumption of independence. So define

$$\mathbf{w}_{k+1} = \mathbf{w} - \sum_{i=1}^{k} \langle Q_i'(\varphi_c), \mathbf{w} \rangle \, \mathbf{w}_i$$

Then by the induction hypothesis, $\langle Q_i'(\varphi_c), \mathbf{w}_{k+1} \rangle = 0$ for $i = 1,..., k$, and by (6.15), $\langle Q_{k+1}'(\varphi_c), \mathbf{w}_{k+1} \rangle \neq 0$. If we now subtract

$$\frac{\langle Q_{k+1}'(\varphi_c), \mathbf{w}_i \rangle}{\langle Q_{k+1}'(\varphi_c), \mathbf{w}_{k+1} \rangle} \mathbf{w}_{k+1}$$

from $\mathbf{w}_i$ for each $i = 1,..., k$ we obtain, upon normalizing $\mathbf{w}_{k+1}$, a collection $\mathbf{w}_1,..., \mathbf{w}_{k+1}$ satisfying (6.14). Finally, having chosen $\mathbf{w}_1,..., \mathbf{w}_n$ we let $\varepsilon > 0$ be given and choose $\mathbf{x}_\varepsilon \in Y$ with such that $\|\mathbf{x}_\varepsilon - \tilde{\mathbf{y}}\| < \varepsilon$. Define

$$\mathbf{y} = \mathbf{x}_\varepsilon - \sum_{i=1}^{n} \langle Q_i'(\varphi_c), \mathbf{x}_\varepsilon \rangle \, \mathbf{w}_i$$

Then $\mathbf{y} \in Y$ and, by (6.14), $\langle Q_i'(\varphi_c), \mathbf{y} \rangle = 0$ for $i = 1,..., n$. If ε is chosen small enough, it follows from (6.13) that $\langle H_c \mathbf{y}, \mathbf{y} \rangle < 0$. Thus (1) and (2) hold for $\mathbf{y}$. Later we will need that

$$\| y_1 - \tilde{y}_1 \|_{L^{p+1}(\mathbb{R}^n)} \leqslant \tfrac{1}{2} \, \|\varphi_c\|_{L^{p+1}(\mathbb{R}^n)} \tag{6.16}$$

which again follows by choosing ε small enough. □

We now define the Lyapunov functional $A\colon V_{c,\varepsilon} \to \mathbb{R}$ by

$$A(\mathbf{u}) = -\langle J^{-1}\mathbf{y}, T(\sigma(\mathbf{u}))\,\mathbf{u}\rangle$$

Lemma 6.6. *The functional A is C^1 on $V_{c,\varepsilon}$ and*

(1) $A(T(\tau)\,\mathbf{u}) = A(\mathbf{u})$ *for any* $\tau \in \mathbb{R}^n$,

(2) $JA'(\varphi_c) = -\mathbf{y}$, *and*

(3) $\langle Q'(\mathbf{u}), JA'(\mathbf{u})\rangle = 0$.

Proof. Part (1) follows from Lemma 6.4(3). For $\mathbf{u} \in V_{c,\varepsilon} \cap Y$ and $\mathbf{w} \in X$, compute

$$\langle A'(\mathbf{u}), \mathbf{w}\rangle = -\langle J^{-1}\mathbf{y}, T(\sigma(\mathbf{u}))\,\mathbf{w}\rangle - \langle J^{-1}\mathbf{y}, T(\sigma(\mathbf{u}))\nabla\mathbf{u}\rangle \cdot \langle \sigma'(\mathbf{u}), \mathbf{w}\rangle$$

where $\nabla\mathbf{u}$ is used to denote $([u_{x_1}, v_{x_1}],\ldots, [u_{x_n}, v_{x_n}]) \in X^n$. By Theorem 6.5, $\mathbf{y} \in X^n$, and therefore A' extends to all of $V_{c,\varepsilon}$. So A is C^1 and

$$\begin{aligned}\langle A'(\varphi_c), \mathbf{w}\rangle &= -\langle J^{-1}\mathbf{y}, \mathbf{w}\rangle - \langle J^{-1}\mathbf{y}, \nabla\varphi_c\rangle \cdot \langle \sigma'(\varphi_c), \mathbf{w}\rangle \\ &= -\langle j^{-1}\mathbf{y}, \mathbf{w}\rangle + \langle \vec{Q}'(\varphi_c), \mathbf{y}\rangle \cdot \langle \sigma'(\varphi_c), \mathbf{w}\rangle \\ &= -\langle J^{-1}\mathbf{y}, \mathbf{w}\rangle\end{aligned}$$

by Theorem 6.5(2). Differentiating (1) with respect to τ at $\tau = 0$ proves (3). □

We now wish to construct a curve in X through φ_c in the unstable direction $\mathbf{y}$, on which the functional Q_1 is constant and such that E is maximized at φ_c. First let $\mathbf{r}(\lambda, \mathbf{v})$ denote the solution of

$$\frac{d\mathbf{r}}{d\lambda} = -JA'(\mathbf{r})$$

with initial data $\mathbf{r}(0) = \mathbf{v} \in V_{c,\varepsilon}$ and let the components of $\mathbf{r}$ be given by $r_1(\lambda, \mathbf{v})$, $r_2(\lambda, \mathbf{v})$. By Lemma 6.6 (3),

$$\frac{\partial Q_1}{\partial\lambda}(\mathbf{r}(\lambda, \mathbf{v})) = 0$$

and thus Q_1 is constant in λ on $\mathbf{r}$. Also,

$$\left.\frac{\partial\mathbf{r}}{\partial\lambda}(\lambda, \varphi_c)\right|_{\lambda=0} = \mathbf{y} \tag{6.17}$$

by Lemma 6.6 (2). By Lemma 6.4 (3)

$$T(\tau)\,\mathbf{r}(\lambda, \mathbf{v}) = \mathbf{r}(\lambda, T(\tau)\,\mathbf{v}) \tag{6.18}$$

The next lemma shows that there is a point along the curve $\mathbf{r}(\lambda, \mathbf{v})$ at which the functional K attains the value $K(\varphi_c)$. This allows us once again to exploit the variational characterization of φ_c.

Lemma 6.7. *If $d_1''(c) < 0$, then there exists $\varepsilon > 0$ and a C^1 functional λ: $V_{c,\varepsilon} \to \mathbb{R}$ so that for $\mathbf{v} \in V_{c,\varepsilon}$*

$$K(r_1(\lambda(\mathbf{v}), \mathbf{v})) = K(\varphi_c) = \frac{2(p+1)}{p-1}\, d_1(c) \tag{6.19}$$

Proof. Since $r_1(0, \varphi_c) = \varphi_c$, the lemma follows from the implicit function theorem and (6.18) once it is shown that

$$\frac{\partial K}{\partial \lambda}(r_1(\lambda, \varphi_c))\bigg|_{\lambda=0} \neq 0$$

By (6.17) we have

$$\begin{aligned}\frac{\partial K}{\partial \lambda}(r_1(\lambda, \varphi_c))\bigg|_{\lambda=0} &= (p+1)\int_{\mathbb{R}^n} |\varphi_c|^{p-1}\,\varphi_c\, y_1\, dx \\ &= (p+1)\int_{\mathbb{R}^n} |\varphi_c|^{p-1}\,\varphi_c(y_1 - \tilde{y}_1)\, dx \\ &\quad + (p+1)\int_{\mathbb{R}^n} |\varphi_c|^{p-1}\,\varphi_c\, \tilde{y}_1\, dx \end{aligned} \tag{6.20}$$

By (6.16) we can bound the first integral in (6.20) by

$$\begin{aligned}\left|\int_{\mathbb{R}^n} |\varphi_c|^{p-1}\,\varphi_c(y_1 - \tilde{y}_1)\, dx\right| &\leq \|\varphi_c\|^{p}_{L^{p+1}(\mathbb{R}^n)}\, \|y_1 - \tilde{y}_1\|_{L^{p+1}(\mathbb{R}^n)} \\ &\leq \tfrac{1}{2}\, \|\varphi_c\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} \end{aligned} \tag{6.21}$$

Using (6.12) we can rewrite the second term as

$$\begin{aligned}(p+1)\int_{\mathbb{R}^n} |\varphi_c|^{p-1}\,\varphi_c\, \tilde{y}_1\, dx &= (p+1)\,\vec{h}'(0)\cdot\int_{\mathbb{R}^n} |\varphi_c|^{p-1}\,\varphi_c(\nabla_{\vec{c}}\,\varphi_{\vec{c}}|_{\vec{c}=c\vec{e}_1})\, dx \\ &\quad + (p+1)\, \|\varphi_c\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} \end{aligned} \tag{6.22}$$

But since $d(\vec{c})=[(p-1)/2(p+1)]\,K(\varphi_{\vec{c}})$, (6.2) implies

$$-\vec{Q}(\varphi_c)=\nabla_{\vec{c}}\,d(c\vec{e}_1)=\frac{p-1}{2}\int_{\mathbb{R}^n}|\varphi_c|^{p-1}\,\varphi_c(\nabla_{\vec{c}}\,\varphi_{\vec{c}}|_{\vec{c}=c\vec{e}_1})\,dx \tag{6.23}$$

and therefore

$$(p+1)\,\vec{h}'(0)\cdot\int_{\mathbb{R}^n}|\varphi_c|^{p-1}\,\varphi_c(\nabla_{\vec{c}}\,\varphi_{\vec{c}}|_{\vec{c}=c\vec{e}_1})\,dx=\frac{2(p+1)}{p-1}\,\vec{h}'(0)\cdot\vec{Q}(\varphi_c) \tag{6.24}$$

Using Theorem 6.5 we compute

$$\begin{aligned}0&=\langle\vec{Q}'(\varphi_c),\tilde{\mathbf{y}}\rangle=\langle\vec{Q}'(\varphi_c),,\vec{h}'(0)\cdot\nabla_{\vec{c}}\,\varphi_{\vec{c}}|_{\vec{c}=c\vec{e}_1}+\varphi_c\rangle \qquad \text{by (6.12)}\\ &=\vec{h}'(0)\cdot\nabla_{\vec{c}}(\vec{Q}(\varphi_c))+\langle\vec{Q}'(\varphi_c),\varphi_c\rangle\\ &=-h'(0)\cdot D^2\,d(c\vec{e}_1)+2\vec{Q}(\varphi_c) \qquad \text{by (6.1)}\end{aligned} \tag{6.25}$$

Thus by (6.25) and the negative definiteness of $D^2\,d(c\vec{e}_1)$,

$$-\frac{2(p+1)}{p-1}\,\vec{h}'(0)\cdot\vec{Q}(\varphi_c)=-\frac{(p+1)}{p-1}\,\vec{h}(0)\cdot D^2\,d(c\vec{e}_1)\cdot\vec{h}'(0)\geqslant 0 \tag{6.26}$$

In view of (6.22), (6.24), and (6.26), we therefore have

$$(p+1)\int_{\mathbb{R}^n}|\varphi_c|^{p-1}\,\varphi_c\tilde{u}_1\,dx\geqslant(p+1)\,\|\varphi_c\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}$$

which, together with (6.21), implies that

$$\left.\frac{\partial K}{\partial\lambda}(r_1(\lambda,\varphi_c))\right|_{\lambda=0}\geqslant\frac{1}{2}(p+1)\,\|\varphi_c\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}>0 \qquad (\Box)$$

Lemma 6.8. *Suppose $d_1''(c)<0$. Then there exists $\varepsilon>0$ and a C^1 functional λ: $V_{c,\varepsilon}\cap\{Q_1(\varphi_c)\}\to\mathbb{R}$ such that*

$$E(\mathbf{r}(\lambda(\mathbf{v}),\mathbf{v}))\geqslant E(\varphi_c) \tag{6.27}$$

Proof. Let $\mathbf{v}\in V_{c,\varepsilon}$ with $Q_1(\mathbf{v})=Q_1(\varphi_c)$ and let $\lambda(\mathbf{v})$ be given by Lemma 6.7. Then since φ_c minimizes I_c subject to the constraint $K(u)=K(\varphi_c)$, we have, using (3.8),

$$E(\mathbf{r}(\lambda(\mathbf{v}), \mathbf{v})) = \frac{1}{2} I_c(r_1(\lambda(\mathbf{v}), \mathbf{v})) + cQ_1(\mathbf{r}(\lambda(\mathbf{v}), \mathbf{v})) - \frac{1}{p+1} K(r_1(\lambda(\mathbf{v}), \mathbf{v}))$$

$$+ \frac{1}{2} \int_{\mathbb{R}^n} |(r_2 - c(r_1)_{x_1})(\lambda(\mathbf{v}), \mathbf{v})|^2 \, dx$$

$$\geqslant \frac{1}{2} I_c(r_1(\lambda(\mathbf{v}), \mathbf{v})) + cQ_1(\mathbf{r}(\lambda(\mathbf{v}), \mathbf{v})) - \frac{1}{p+1} K(r_1(\lambda(\mathbf{v}), \mathbf{v}))$$

$$= \frac{1}{2} I_c(r_1(\lambda(\mathbf{v}), \mathbf{v})) + cQ_1(\varphi_c) - \frac{1}{p+1} K(\varphi_c) \qquad \text{by (6.19)}$$

$$\geqslant \frac{1}{2} I_c(\varphi_c) + cQ_1(\varphi_c) - \frac{1}{p+1} K(\varphi_c) = E(\varphi_c)$$

which proves the lemma. □

Lemma 6.9. *Let* $\mathbf{v} \in V_{c,\varepsilon}$ *with* $Q_1(\mathbf{v}) = Q_1(\varphi_c)$ *and* $\mathbf{v} \notin \tilde{S}_c = \{T(\tau)\varphi_c \mid \tau \in \mathbb{R}^n\}$. *If* $d_1''(c) < 0$, *then*

$$E(\varphi_c) < E(\mathbf{v}) + \lambda(\mathbf{v}) P(\mathbf{v}) \tag{6.28}$$

where $P(\mathbf{v}) \equiv \langle E'(\mathbf{v}), -JA'(\mathbf{v}) \rangle$.

Proof. The lemma follows by computing the second-order Taylor expansion of $E(\mathbf{r}(\lambda, \mathbf{v}))$ at $\lambda = 0$.

$$\frac{\partial E}{\partial \lambda}(\mathbf{r}(\lambda, \mathbf{v}))\bigg|_{\lambda=0} = \left\langle E'(\mathbf{v}), \frac{\partial \mathbf{r}}{\partial \lambda}(\lambda, \mathbf{v})|_{\lambda=0} \right\rangle = P(\mathbf{v}) \tag{6.29}$$

and

$$\frac{\partial^2 E}{\partial \lambda^2}(\mathbf{r}(\lambda, \mathbf{v}))\bigg|_{\lambda=0} = \langle E''(\varphi_c)\, \mathbf{y}, \mathbf{y} \rangle + \left\langle E'(\varphi_c), \frac{\partial^2 \mathbf{r}}{\partial \lambda^2}(0, \varphi_c) \right\rangle \tag{6.30}$$

Since $Q_1(\mathbf{r}(\lambda, \mathbf{v})) = Q_1(\varphi_c)$,

$$\frac{\partial Q_1}{\partial \lambda}(\mathbf{r}(\lambda, \varphi_c))\bigg|_{\lambda=0} = \frac{\partial^2 Q_1}{\partial \lambda^2}(\mathbf{r}(\lambda, \varphi_c))\bigg|_{\lambda=0} = 0 \tag{6.31}$$

so that

$$0 = \langle Q_1''(\varphi_c)\, \mathbf{y}, \mathbf{y} \rangle + \left\langle Q_1'(\varphi_c), \frac{\partial^2 \mathbf{r}}{\partial \lambda^2}(0, \varphi_c) \right\rangle \tag{6.32}$$

Subtracting (6.32) from (6.30) and using (3.6) yields

$$\frac{\partial^2 E}{\partial \lambda^2}(\mathbf{r}(\lambda, \varphi_c))\bigg|_{\lambda=0} = \langle (E'' - cQ_1'')(\varphi_c)\, \mathbf{y}, \mathbf{y}\rangle < 0 \tag{6.33}$$

Thus, for λ near zero, $\lambda \neq 0$ and ε small enough, it follows that if $\mathbf{v} \in V_{c,\varepsilon} \cap (X - \tilde{S}_c)$, then

$$E(\mathbf{r}(\lambda, \mathbf{v})) < E(\mathbf{v}) + \lambda P(\mathbf{v})$$

So if $\mathbf{v}$ also satisfies $Q_1(\mathbf{v} = Q_1(\varphi_c)$, we have by Lemma 6.8

$$E(\varphi_c) \leqslant E(\mathbf{r}(\lambda(\mathbf{v}), \mathbf{v})) < E(\mathbf{v}) + \lambda(\mathbf{v})\, P(\mathbf{v})$$

and this completes the proof. □

Lemma 6.10. *There exists $\delta > 0$ and a C^2 curve $\psi: (-\delta, \delta) \to V_{c,\varepsilon}$ such that $\psi(0) = \varphi_c$, $\psi'(0) = \mathbf{y}$, $Q_1(\psi(s)) = Q_1(\varphi_c)$, $P(\psi(s))$ changes sign at $s = 0$ and $E(\psi(s))$ has a strict local maximum at $s = 0$.*

Proof. Since $\langle Q_1'(\varphi_c), \mathbf{y}\rangle = 0$, $\mathbf{y}$ is tangent to the manifold $N = \{\mathbf{v} \in X\colon Q_1(\mathbf{v}) = Q_1(\varphi)\}$, and thus there is a curve $\psi(s)$ in N with $\psi(0) = \varphi_c$ and $\psi'(0) = \mathbf{y}$. To show that $E(\psi(s))$ is maximized at $s = 0$, we differentiate in s to obtain

$$\begin{aligned} \frac{dE}{ds}(\psi(s))\bigg|_{s=0} &= \frac{d}{ds}(E(\psi(s)) - cQ_1(\psi(s)))\bigg|_{s=0} \\ &= \langle E'(\varphi_c) - cQ_1'(\varphi_c), \mathbf{y}\rangle = 0 \qquad \text{by (3.11)} \end{aligned}$$

Also

$$\begin{aligned} \frac{d^2E}{ds^2}(\psi(s)) &= \langle [E''(\psi(s)) - cQ_1''(\psi(s))]\, \psi'(s), \psi'(s)\rangle \\ &\quad + \langle E'(\psi(s)) - cQ_1'(\psi(s)), \psi''(s)\rangle \end{aligned}$$

and therefore

$$\frac{d^2E}{ds^2}(\psi(s))\bigg|_{s=0} = \langle H_c \mathbf{y}, \mathbf{y}\rangle < 0 \qquad \text{by Theorem 6.4(1)}$$

Thus $E(\psi(s))$ is locally maximized at $s = 0$, and we have by (6.28),

$$0 < E(\varphi_c) - E(\psi(s)) < \lambda(\psi(s))\, P(\psi(s))$$

It remains only to show that $\lambda(\psi(s))$ changes sign at $s=0$. Recall that λ is defined by (6.19) and satisfies

$$K(\varphi_c) = K(r_1(\lambda(\psi(s)), \psi(s))) = \int_{\mathbb{R}^n} |r_1(\lambda(\psi(s)), \psi(s))|^{p+1}\, dx$$

Differentiating at $s=0$ gives

$$0 = (p+1)\int_{\mathbb{R}^n} |\varphi_c|^{p-1}\, \varphi_c \left(\frac{\partial r_1}{\partial \lambda}\frac{\partial \lambda(\psi)}{\partial s} + \frac{\partial r_1}{\partial v_1}\frac{\partial \psi_1}{\partial s} + \frac{\partial r_1}{\partial v_2}\frac{\partial \psi_2}{\partial s}\right)\Bigg|_{s=0} dx \quad (6.34)$$

Since $\mathbf{r}(0, \mathbf{v}) = \mathbf{v} = (v_1, v_2) = (r_1(0, \mathbf{v}), r_2(0, \mathbf{v}))$,

$$\frac{\partial r_1}{\partial v_1}(0, \mathbf{v}) = Id \qquad \text{and} \qquad \frac{\partial r_1}{\partial v_2}(0, \mathbf{v}) = 0$$

Thus, since $(\partial \psi_1/\partial s)|_{s=0} = (\partial r_1/\partial \lambda)(\lambda, \varphi_c)|_{\lambda=0} = y_1$, (6.34) becomes

$$0 = (p+1)\left(\frac{\partial \lambda(\psi)}{\partial s}\bigg|_{s=0} + 1\right)\int_{\mathbb{R}^n} |\varphi_c|^{p-1}\, \varphi_c\, y_1\, dx$$

The integral is exactly

$$\frac{\partial K}{\partial \lambda}(r_1(\lambda, \varphi_c))\bigg|_{\lambda=0} > 0$$

as shown in Lemma 6.7. Thus

$$\frac{\partial \lambda(\psi(s))}{\partial s}\bigg|_{s=0} = -1$$

and since $\lambda(\varphi_c) = 0$, we have shown that $\lambda(\psi(s))$ changes sign at $s=0$.

□

Proof of Theorem 6.2. Fix $\varepsilon > 0$ small enough so that Lemma 6.9 applies. Choose s near zero so that $\lambda(\psi(s)) > 0$ and let $\mathbf{u}_0 = \psi(s)$. Then by Lemma 6.10, $Q_1(\mathbf{u}_0) = Q_1(\varphi_c)$, $E(\mathbf{u}_0) < E(\varphi_c)$, and we may assume that $P(\mathbf{u}_0) > 0$. From Section 3, there is an interval $[0, t_0)$ on which a solution $\mathbf{u}(t)$ exists and satisfies $\mathbf{u}(0) = \mathbf{u}_0$, $Q_1(\mathbf{u}(t)) = Q_1(\mathbf{u}_0)$, and $E(\mathbf{u}(t)) = E(\mathbf{u}_0)$. We may suppose that $t_0 = \infty$, because otherwise $\tilde{S}_c$ is unstable by definition. Now, by Lemma 6.9,

$$0 < E(\varphi_c) - E(\mathbf{u}_0) = E(\varphi_c) - E(\mathbf{u}(t)) < \lambda(\mathbf{u}(t))\, P(\mathbf{u}(t))$$

for all $t > 0$. Thus by the continuity of P, $P(\mathbf{u}(t)) > 0$ for all $t > 0$. We may assume that $\lambda(\mathbf{u}(t)) < 1$ so that

$$P(\mathbf{u}(t)) > E(\varphi_c) - E(\mathbf{u}_0) \equiv \varepsilon_0 > 0$$

Now let $W = D(J)$ with the graph norm $\|\mathbf{v}\|_W^2 = \|\mathbf{v}\|_{X^*}^2 + \|J\mathbf{v}\|_X^2$. Then $J: W \to X$ and $J^*: X^* \to W^*$ are continuous, and by definition (3.5) we have

$$\frac{d}{dt}\langle \mathbf{v}, \mathbf{u}(t)\rangle = \langle E'(\mathbf{u}(t)), -J\mathbf{v}\rangle = \langle -J^*E'(\mathbf{u}(t)), \mathbf{v}\rangle$$

where the last pairing is between W^* and W. Hence

$$\mathbf{u} \in C([0, t_0); X) \cap C^1([0, t_0); W^*)$$

and

$$\frac{d\mathbf{u}}{dt} = -J^*E'(\mathbf{u})$$

So (by Ref. 5, Lemma 4.6) we may compute

$$\begin{aligned}\frac{dA}{dt}(\mathbf{u}(t)) &= \left\langle \frac{d\mathbf{u}(t)}{dt}, A'(\mathbf{u}(t))\right\rangle = \langle -J^*E'(\mathbf{u}(t)), A'(\mathbf{u}(t))\rangle \qquad \text{by (3.4)}\\ &= \langle E'(\mathbf{u}(t)), -JA'(\mathbf{u}(t))\rangle = P(\mathbf{u}(t)) > \varepsilon_0\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing of W^* with W. But A is bounded on $V_{c,\varepsilon}$ and hence $\mathbf{u}(t)$ must leave $V_{c,\varepsilon}$ in finite time, and therefore $\tilde{S}_c$ is unstable. □

Remark 6.10. By (4.9) we see that $d''(c) < 0$ in some interval around zero. Thus at speeds traveling waves are unstable.

7. STANDING WAVES

In this section we extend our results to include the (easier) case of standing wave solutions of (1.1).

By a standing wave we mean a solution of (1.1) of the form

$$u(x, t) = e^{i\omega t}\varphi(x) \tag{7.1}$$

where the space $X = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ now consists of complex valued functions and has an inner product given by

$$\langle [u_1, v_1], [u_2, v_2] \rangle = Re \int_{\mathbb{R}^n} \Delta u_1 \, \overline{\Delta u_2} + u_1 \, \overline{u_2} + v_1 \, \overline{v_2} \, dx \qquad (7.2)$$

Substituting (7.1) into (1.1) shows that φ must satisfy

$$\Delta^2 \varphi + (1 - \omega^2) \, \varphi = |\varphi|^{p-1} \, \varphi \qquad (7.3)$$

We solve (7.3) for $\omega^2 < 1$ using the method of Section 1 to show that minimizing sequences for the pair

$$\begin{aligned} I_\omega(u) &= \int_{\mathbb{R}^n} |\Delta u|^2 + (1 - \omega^2) \, |u|^2 \, dx \\ K(u) &= \int_{\mathbb{R}^n} |u|^{p+1} \, dx \end{aligned} \qquad (7.4)$$

are relatively compact in $H^2(\mathbb{R}^n)$ up to translation. The absence of second-order terms in (7.3) allows us to use the scaling property of the nonlinearity to make a choice of the ground state which is smooth in ω. If φ_0 is a ground-state solution of (7.3) with $\omega = 0$ [i.e., φ_0 is a stationary state of (1.10)], then

$$\varphi_w(x) = (1 - \omega^2)^{1/(p-1)} \, \varphi_0((1 - \omega^2)^{1/4} \, x) \qquad (7.5)$$

is a ground state with frequency ω. Next we consider the invariants of (1.1) relevant to standing waves

$$\begin{aligned} E(\mathbf{u}) &= \int_{\mathbb{R}^n} \frac{1}{2} \, |\Delta u|^2 + \frac{1}{2} \, |v|^2 + \frac{1}{2} \, |u|^2 - \frac{1}{p+1} \, |u|^{p+1} \, dx \\ Q(\mathbf{u}) &= Im \int_{\mathbb{R}^n} \bar{u} v \, dx \end{aligned} \qquad (7.6)$$

If φ is any ground state with frequency ω, we define $\varphi = [\varphi, i\omega\varphi]$, and it follows that

$$E'(\varphi) - \omega Q'(\varphi) = 0 \qquad (7.7)$$

We define the action function $d(\omega)$ as before by

$$d(\omega) = E(\varphi) - \omega Q(\varphi) \qquad (7.8)$$

By the relation

$$E(\mathbf{u}) - \omega Q(\mathbf{u}) = \frac{1}{2} I_\omega(u) - \frac{1}{p+1} K(u) + \frac{1}{2} \int_{\mathbb{R}^n} |v - i\omega u|^2 \, dx \tag{7.9}$$

we see that $d(\omega)$ is well defined and

$$d(\omega) = \frac{p-1}{2(p+1)} I_\omega(\varphi) = \frac{p-1}{2(p+1)} K(\varphi) \tag{7.10}$$

which, by (7.5), yields the explicit formula

$$d(\omega) = \frac{p-1}{2(p+1)} K(\varphi_0)(1-\omega^2)^{\tilde{\gamma}}, \qquad \tilde{\gamma} = \frac{p+1}{p-1} - \frac{n}{4} \tag{7.11}$$

If we define the set of ground states with frequency ω to be

$$S_\omega = \left\{ \psi \in H^2(\mathbb{R}^n) \mid I_\omega(\psi) = K(\psi) = \frac{2(p+1)}{p-1} d(\omega) \right\} \tag{7.12}$$

then we have the following stability result.

Theorem 7.1. *Suppose that Assumption 3.1 holds and that* $1 < p < 2^* - 1$. *If* $d''(\omega) > 0$, *then* S_ω *is stable.*

Proof. We define

$$\omega(\mathbf{u}) = d^{-1}\left(\frac{p-1}{2(p+1)} K(u) \right) = \left(1 - \left(\frac{K(u)}{K(\varphi_0)} \right)^{1/\tilde{\gamma}} \right)^{1/2} \tag{7.13}$$

for $\mathbf{u}$ near S_ω. Under the assumption $d''(\omega) > 0$, we can improve the inequality (5.4) to

$$E(\mathbf{u}) - E(\psi) - \omega(\mathbf{u}) Q(\mathbf{u}) - Q(\psi)) \geqslant \tfrac{1}{4} d''(\omega) \, |\omega(\mathbf{u}) - \omega|^2 \tag{7.14}$$

for any $\psi \in S_\omega$, and $\mathbf{u}$ near S_ω. The rest of the proof is identical to the proof of Theorem 5.4. □

Solutions of (1.1) are invariant under the group action $T \colon \mathbb{R} \times X \to X$ given by

$$T(s) \, \mathbf{u} = e^{i\omega s} \mathbf{u} = [e^{i\omega s} u, e^{i\omega s} v] \tag{7.15}$$

Given a ground state φ with frequency ω, we define its orbit under T by

$$\tilde{S}_\omega = \{ T(s)\, \varphi \mid s \in \mathbb{R} \}$$

With these definitions we have the following.

***Theorem** 7.2. Suppose that Assumption 3.1 holds and* $1 < p < 2^* - 1$. *If* $d''(\omega) < 0$, *then* $\tilde{S}_\omega$ *is unstable.*

Proof. First, Lemma 6.3 and Lemma 6.4 are true for T as given above, σ, $\tau \in \mathbb{R}$, modulo 2π. Also, Theorem 6.5 follows more easily in this case since we no longer insist that the unstable direction $\mathbf{y}$ have any regularity properties. Thus we may define the Lyapunov functional by

$$A(\mathbf{u}) = -\langle J^{-1}\mathbf{y}, T(\sigma(\mathbf{u}))\, \mathbf{u} \rangle$$

The rest of the proof follows exactly as in Section 5 with Q in place of Q_1. □

Using expression (7.11) for $d(\omega)$, we may now explicitly determine the intervals in which ground states are stable and unstable. We compute

$$d''(\omega) = 2\tilde{\gamma}(1 - \omega^2)^{\tilde{\gamma}-2}(w^2(2\tilde{\gamma} - 1) - 1) \tag{7.16}$$

Thus if $\tilde{\gamma} \leqslant 1/2$, then $d''(\omega) < 0$ for all $\omega^2 < 1$. That is, when

$$p \geqslant 1 + \frac{8}{n-2}$$

all ground states are unstable. On the other hand, if

$$p < 1 + \frac{8}{n-2}$$

then ground states are stable in the interval $\omega^2 > 1/(2\tilde{\gamma} - 1)$ and unstable in the interval $\omega^2 < 1/(2\tilde{\gamma} - 1)$. Ground states at the critical value $\omega^2 = 1/(2\tilde{\gamma} - 1)$ are also unstable since, by the smooth choice of ground states, there are unstable states arbitrarily nearby.

ACKNOWLEDGMENT

The author wishes to thank Professor W. A. Strauss for his suggestions and guidance in the preparation of this paper.

REFERENCES

1. Berestycki, H., and Lions, P. L. (1983). Nonlinear scalar field equations, I. Existence of a ground state. *Arch. Rat. Mech Anal.* **82**, 313–375.
2. Bona, J. L., Souganidis, P. E., and Strauss, W. A. (1987). Stability and instability of solitary waves of Korteweg–de Vries type. *Proc Roy. Soc. London A* **411**, 395–412.
3. Cazenave, T., and Lions, P. L. (1982). Orbital stability of standing waves for some nonlinear Schrodinger equations. *Commun. Math. Phys.* **85**, 549–561.
4. Champneys, A. R. and Groves, M. D. (1996). A global investigation of solitary-wave solutions to a two-parameter model for water waves (preprint).
5. Evans, L. C. (1990). *Weak Convergence Methods for Nonlinear Partial Differential Equations.* CBMS Regional Conference Series, American Mathematics Society, pp. 34–37.
6. Grillakis, M., Shatah, J., and Strauss, W. (1987). Stability theory of solitary waves in the presence of symmetry, I. *J. Funct. Anal.* **74**, 160–197.
7. Grillakis, M., Shatah, J., and Strauss, W. (1990). Stability theory of solitary waves in the presence of symmetry, II. *J. Funct. Anal.* **94**, 308–348.
8. Levandosky, S., (1997). Stability of solitary-waves for fifth-order KdV equations (in preparation).
9. Lions, P. L. (1984). The concentration-compactness principle in the calculus of variations. The locally compact case. Part I and Part II. *Ann. Inst. Henri Poincaré Sect. A (N.S.)* **1**, 109–145, 223–283.
10. Lions, P. L. (1985). The concentration-compactness principle in the calculus of variations. The limit case, Part I and Part II. *Revista Mat. Iberoam.* **1**(1, 2), 145–201, 45–121.
11. Liu, Yue (1993). Instability of solitary waves for generalized Boussinesq equations. *J. Dynam. Diff. Eqs.* **5**, 537–558.
12. McKenna, P. J., and Walter, W. (1990). Traveling waves in a susp. bridge. *SIAM J. Appl. Math.* **50**, 703–715.
13. McKenna, P. J. (1995). Oral communication.
14. Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York.
15. Pego, R., and Weinstein, W. (1992). Eigenvalues and instability of solitary waves. *Phil. Trans. Roy. Soc. Lond. A* **340**, 47–94.
16. Segal, I. (1963). Non-linear semi-groups. *Ann. Math.* **78**, 339–364.
17. Shatah, J. (1983). Stable Standing waves of nonlinear Klein Gordon equations. *Commun. Math. Phys.* **91**, 313–327.
18. Shatah, J. (1985). Unstable ground states of nonlinear Klein Gordon equations. *Trans. AMS* **290**, 701–710.
19. Shatah, J., and Strauss, W. (1985). Instability of nonlinear bound states. *Commun. Math. Phys.* **100**, 173–190.
20. Souganidis, P. E., and Strauss, W. A. (1990). Instability of a class of dispersive solitary waves. *Proc. Roy. Soc. Edinburgh* **114A**, 195–212.
21. Strauss, W. (1977). Existence of solitary waves in higher dimensions. *Commun. Math. Phys.* **55**, 149–162.
22. Weinstein, M. (1986). Lyapunov stability of ground states of nonlinear dispersive wave equations. *CPAM* **39**, 51–68.
23. Weinstein, M. (1985). Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.* **16**, 472–491.