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QUANTUM ALGEBRA $U_q(gl(3))$ AND NONLINEAR OPTICS

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Abstract

Indecomposable representations are investigated for the $U_q(gl(3))$ quantum algebra. The matrix elements are explicitly determined for the elementary representations, and the extremal vectors which characterize invariant subspaces are given in explicit form. Quotient spaces are used to derive other representations from the elementary representations, including the finite-dimensional irreducible representations and infinite-dimensional representations which are bounded above. Applications to nonlinear-optical phenomena are discussed.

Keywords: nonlinear optics, quantum groups, indecomposable representations.

1. Introduction

The interaction of photons with atoms can depend on the intensity of electromagnetic fields. These effects can be described by Hamiltonians which are nonlinear functions of photon creation and annihilation operators. Nonlinearity can be associated with some algebraic structures. For example, the quantum group $SU_q(2)$ and q-deformed Heisenberg–Weyl group were used in [1–4] to discuss nonlinear effects in electrodynamics (see also [5, 6]). In order to use the formalism of quantum groups in nonlinear optics, one needs to investigate the properties of irreducible representations of quantum groups. Quantum groups can be naturally introduced using structures of the standard Lie groups and Lie algebras. The Lie groups and their irreducible representations give the possibility of describing different phenomena in quantum mechanics and quantum optics. A particular role is played by indecomposable representations of Lie groups.

Indecomposable representations (i.e., representations which are reducible but not completely) of Lie algebras have been known in physics for a long time. A well-known example of indecomposable representations encountered in physics is provided by (nontrivial) finite-dimensional indecomposable representations of the Euclidean groups [7]. Indecomposable representations of the de Sitter group $SO(3,2)$ have also found applications in physics (Dirac singletons [8–10]).

Indecomposable representations of the Lorenz group have been investigated in detail by Zhelobenko [11] and Gel'fand and Ponomarev [12]. Verma [13], on the other hand, studied indecomposable representations of semi-simple Lie algebras on certain spaces related to their universal enveloping algebra.

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Subsequently Bernstein, Gel'fand, and Gel'fand [14] added some new results to Verma's analysis. Gruber and Klimyk [15] analyzed the structure of indecomposable, as well as irreducible representations of semi-simple Lie algebras on Verma spaces in general. In particular, they analyzed the structure of nonmultiplicity-free indecomposable representations of the simple algebra $SU(1,1)$ [16].

In [17, 18], elementary representations d were analyzed for the case of the simple complex Lie algebra A_2. It should be noted that the concept of quasi-exactly solvable problems [19] is based on the theory of indecomposable representations.

Recently the indecomposable representation appeared to be topical in the representation theory of quantum algebras described by the parameter q equal to the root of unity.

In this connection, it seems interesting to extend the concept of indecomposable Verma modules on quantum algebras with generic q in the spirit of [17, 18]. In our paper, elementary representations d_f are considered for the quantum algebra $U_q(gl(3))$. This is done in a purely algebraic manner. The results of [17] concerning the invariant subspaces d_M are transferred to this quantum algebra. The matrix elements of the elementary representations d_f are obtained explicitly. The extremal vectors which define the invariant subspaces are given in explicit form. The representations induced by d_f on its quotient subspaces with respect to invariant subspaces are discussed. Attention is devoted to the finite-dimensional and infinite-dimensional representations with the highest weight which can be transformed into irreducible representations of the $U_q(U(3))$ and $U_q(U(2,1))$ algebras after unitarization. The concept of extremal vectors, developed for quantum algebras by Dobrev [20], is a main tool of our analysis.

2. Quantum Algebra $U_q(gl(3))$

Let us review the properties of generators of the quantum algebra $U_q(gl(3))$. A conventional choice of a basis of the quantum algebra $U_q(gl(3))$ is [21]

$$U_q(gl(3)) : \{A_{ii}, \quad i = 1,2,3; \quad A_{ik}, \quad i \neq k, \quad k = 1,2,3\}. \tag{2.1}$$

The permutation relations for the generators A_{ik} are of the form

$$\begin{aligned}
&[A_{ii}, A_{kk}] = 0, \quad [A_{12}, A_{32}] = 0, \quad [A_{21}, A_{23}] = 0,\\
&[A_{ii}, A_{ik}] = A_{ik}, [A_{ii}, A_{ki}] = -A_{ki}, \quad i \neq k,\\
&[A_{ik}, A_{ki}] = [A_{ii} - A_{kk}],\\
&[A_{12}, A_{23}]_q = A_{13}, \quad [A_{32}, A_{21}]_{q^{-1}} = A_{31},\\
&[A_{12}, A_{13}]_{q^{-1}} = 0, \quad [A_{23}, A_{13}]_q = 0,\\
&[A_{12}, A_{31}] = -q^{-A_{11}+A_{22}} A_{32},\\
&[A_{13}, A_{21}] = -A_{23} q^{A_{11}-A_{22}},\\
&[A_{13}, A_{32}] = q^{-A_{22}+A_{33}} A_{12},\\
&[A_{21}, A_{32}]_q = -qA_{31}, \quad [A_{23}, A_{31}] = A_{21} \quad q^{A_{22}-A_{33}},\\
&[A_{21}, A_{31}]_{q^{-1}} = 0, \quad [A_{32}, A_{31}]_q = 0,
\end{aligned} \tag{2.2}$$

where

$$[A,B]_x = AB - xBA = -x[B,A]_{x^{-1}}, \qquad [Y] = \frac{q^Y - q^{-Y}}{q - q^{-1}}. \tag{2.3}$$

It is easy to prove by induction that the following relations hold:

$$\begin{aligned}
&A_{ik}A_{ki}^n = A_{ki}^n A_{ik} + [n]A_{ki}^{n-1}[A_{ii} - A_{kk} - n + 1],\\
&A_{31}A_{21}^n = q^n A_{21}^n A_{31},\\
&A_{12}A_{31}^n = A_{31}^n A_{12} - [n]A_{31}^{n-1}A_{32}q^{-A_{11}+A_{22}+n-2},\\
&A_{23}A_{31}^n = A_{31}^n A_{23} + [n]A_{31}^{n-1}A_{21}q^{A_{22}-A_{33}-n+1},\\
&A_{13}A_{21}^n = A_{21}^n A_{13} - [n]A_{21}^{n-1}A_{23}q^{A_{11}-A_{22}-n+1},\\
&A_{13}A_{32}^n = A_{32}^n A_{13} + [n]A_{32}^{n-1}A_{12}q^{-A_{22}+A_{33}+n}.
\end{aligned} \tag{2.4}$$

Let us introduce the space Ω_- with basis

$$\Omega_- : \{\mathbf{1},\ A_{21}^p A_{31}^s A_{32}^t \mathbf{1},\quad t,s,p = 0,1,2,\dots\} \tag{2.5}$$

with t, s, and p not simultaneously equal to zero.

In our paper, those representations of $U_q(gl(3))$ are discussed for which one has

$$\rho(A_{12})\mathbf{1} = \rho(A_{23})\mathbf{1} = \rho(A_{13})\mathbf{1} = 0, \qquad \rho(A_{kk})\mathbf{1} = f_k\mathbf{1},\quad k = 1,2,3,\quad f_k \in C. \tag{2.6}$$

That is, the "raising" operators $\rho(A_{12})$, $\rho(A_{23})$, and $\rho(A_{13})$, which represent the elements A_{12}, A_{23}, and A_{13} as linear transformations on the space Ω_-, map the element $\mathbf{1} \in \Omega_-$ onto zero. At the same time, the element $\mathbf{1}$ is a simultaneous eigenvector of the commuting operators $\rho(A_{kk})$, which represent A_{kk} of the Cartan subalgebra of $U_q(gl(3))$ as a linear transformation on the space Ω_-. The spaces Ω_- defined by (2.5) and satisfying (2.6) are called Verma modules.

From Eqs. (2.4) one can obtain the following equations:

$$\begin{aligned}
\rho(A_{11})A_{21}^p A_{31}^s A_{32}^t\mathbf{1} &= (f_1 - s - p)A_{21}^p A_{31}^s A_{32}^t\mathbf{1},\\
\rho(A_{22})A_{21}^p A_{31}^s A_{32}^t\mathbf{1} &= (f_2 + p - t)A_{21}^p A_{31}^s A_{32}^t\mathbf{1},\\
\rho(A_{33})A_{21}^p A_{31}^s A_{32}^t\mathbf{1} &= (f_3 + s + t)A_{21}^p A_{31}^s A_{32}^t\mathbf{1},\\
\rho(A_{21})A_{21}^p A_{31}^s A_{32}^t\mathbf{1} &= A_{21}^{p+1} A_{31}^s A_{32}^t\mathbf{1},\\
\rho(A_{31})A_{21}^p A_{31}^s A_{32}^t\mathbf{1} &= q^p A_{21}^p A_{31}^{s+1} A_{32}^t\mathbf{1},\\
\rho(A_{32})A_{21}^p A_{31}^s A_{32}^t\mathbf{1} &= q^{-p+s}A_{21}^p A_{31}^s A_{32}^{t+1}\mathbf{1} + [p]A_{21}^{p-1}A_{31}^{s+1}A_{32}^t\mathbf{1},\\
\rho(A_{12})A_{21}^p A_{31}^s A_{32}^t\mathbf{1} &= [p][f_1 - f_2 - p - s + t + 1]A_{21}^{p-1}A_{31}^s A_{32}^t\mathbf{1}\\
&\quad -[s]q^{-f_1+f_2+s-t-2}\ A_{21}^p A_{31}^{s-1}A_{32}^{t+1}\mathbf{1},\\
\rho(A_{13})A_{21}^p A_{31}^s A_{32}^t\mathbf{1} &= -[p][t][f_2 - f_3 - t + 1]q^{f_1-f_2-p-s+t+1}A_{21}^{p-1}A_{31}^s A_{32}^{t-1}\mathbf{1}\\
&\quad +q^{-p}[s][f_1 - f_3 - p - s - t + 1]A_{21}^p A_{31}^{s-1}A_{32}^t\mathbf{1},\\
\rho(A_{23})A_{21}^p A_{31}^s A_{32}^t\mathbf{1} &= [s]q^{f_2-f_3-2t}A_{21}^{p+1}A_{31}^{s-1}A_{32}^t\mathbf{1}\\
&\quad +[t][f_2 - f_3 - t + 1]\ A_{21}^p A_{31}^s A_{32}^{t-1}\mathbf{1}.
\end{aligned} \tag{2.7}$$

Equations (2.7) define the representations under discussion. They are called elementary representations and will be denoted as d_f.

Representations of this kind are q-analogs of Verma modules [17]. These representations are either irreducible or reducible and indecomposable [10–12]. In what follows, our main interest will concern the reducible indecomposable representations. One can see that the representation operators satisfy nonlinear equations. Due to this, the Hamiltonians of nonlinear optical systems can be constructed on the basis of these operators. Examples of such constructions are available in [1–3].

3. Invariant Subspaces

In the following description, the invariant subspaces are determined for the elementary representations d_f, and the extremal vectors defining these invariant subspaces will be given in explicit form.

The invariant subspaces themselves are carrier spaces for elementary representations. Moreover, other types of representations can be obtained by making use of quotient spaces with respect to the invariant subspaces. In particular, all finite-dimensional representations of $U_q(u(3))$ can be obtained in this manner. Also the infinite-dimensional representations of $U_q(u(2,1))$ belonging to the negative discrete series can be found.

A vector $y \in \Omega_-$ is called extremal if

$$\rho(A_{kk})y = f_k y, \quad \rho(+)y = 0, \quad f_k \in C. \tag{3.1}$$

The symbol + denotes, collectively, the set of raising operators. Obviously the vector $\mathbf{1}$ is an extremal vector by definition. In order to find extremal vectors, the weight subspaces V_M, $M = (m_1, m_2, m_3)$, $m_s \in C$, have to be determined. Given the (highest) weight $f = (f_1, f_2, f_3)$ of the representation d_f, any other weight M is of the form

$$M = f + p\beta_{21} + s\beta_{31} + t\beta_{32}, \tag{3.2}$$

where $t, s, p = 0, 1, 2, \dots$ and

$$\beta_{ki} = e_k - e_i, \qquad k > i = 1, 2, 3,$$

are three negative roots of $sl(3)$.

Given f and M, each basis element

$$\xi = A_{32}^t A_{31}^s A_{21}^p \mathbf{1}$$

of Ω_-, with p, s, and t such that Eq. (3.2) is satisfied, belongs to V_M. That is, a basis for V_M is given by the set

$$V_M : \{A_{21}^p A_{31}^s A_{32}^t \mathbf{1} | p\beta_{21} + s\beta_{31} + t\beta_{32} = M - f\}, \tag{3.3}$$

where t, s, and p are nonnegative integers.

It follows from Eqs. (3.1) that, for an extremal vector

$$y = \sum_{pst} a^{pst} \quad A_{21}^p A_{31}^s A_{32}^t \mathbf{1}, \tag{3.4}$$

the following two equations must be satisfied:

$$\rho(A_{12})y = 0, \qquad \rho(A_{23})y = 0. \tag{3.5}$$

The equation

$$\rho(A_{13})y = 0$$

is satisfied automatically if Eqs. (3.5) are valid. We consider the case where $f_1 \geq f_2 \geq f_3$ and differences $f_i - f_k$ are integers, which provides the elementary representation d_f with the maximum number of extremal vectors.

Employing the same procedure as in [14], we can prove the following statement:

Statement 1. The space d_f with integers $f_1 \geq f_2 \geq f_3$ contains the following extremal vectors:

1) $M_1 = (f_1, f_2, f_3)$ (trivial case, holds by definition), $y_1 = \mathbf{1}$,

2) $M_2 = (f_2 - 1, f_1 + 1, f_3)$, $y_2 = A_{21}^{f_1 - f_2 + 1}\mathbf{1}$,

3) $M_3 = (f_1, f_3 - 1, f_2 + 1)$, $y_3 = A_{32}^{f_2 - f_3 + 1}\mathbf{1}$,

4) $M_4 = (f_3 - 2, f_1 + 1, f_2 + 1)$, $y_4 = A_{21}^{f_1 - f_3 + 2} A_{32}^{f_2 - f_3 + 1}\mathbf{1}$,

5) $M_5 = (f_2 - 1, f_3 - 1, f_1 + 2)$, $y_5 = A_{32}^{f_1 - f_3 + 2} A_{21}^{f_1 - f_2 + 1}\mathbf{1}$,

6) $M_6 = (f_3 - 2, f_2, f_1 + 2)$, $y_6 = A_{32}^{f_1 - f_2 + 1} A_{21}^{f_1 - f_3 + 2} A_{32}^{f_2 - f_3 + 1}\mathbf{1} = A_{21}^{f_2 - f_3 + 1} A_{32}^{f_1 - f_3 + 2} A_{21}^{f_1 - f_2 + 1}\mathbf{1}$.

It is easy to show by direct calculations that the vectors $y_i (i = 1, 2, \ldots, 6)$ satisfy Eqs. (3.5). Making use of the permutation relation [21]

$$A_{21}^a A_{31}^b = \sum_c (-1)^c \frac{[a]![b]!}{[c]![a-c]![b-c]!} A_{32}^{b-c} A_{31}^c A_{21}^{a-c} q^{c+(a-c)(b-c)}, \tag{3.6}$$

we can reduce the expressions for y_6 to the form (3.4):

$$\begin{aligned} y_6 \quad = \quad & \sum_c (-1)^c \frac{[f_1 - f_3 + 2]![f_2 - f_3 + 1]!}{[c]![f_1 - f_3 + 2 - c]![f_2 - f_3 + 1 - c]!} A_{32}^{f_1 - f_3 + 2 - c} A_{31}^c \\ & \times A_{21}^{f_1 - f_3 + 2 - c} \quad q^{c + (f_1 - f_3 + 2 - c)(f_2 - f_3 + 1 - c)}\mathbf{1}. \end{aligned} \tag{3.7}$$

The standard notation $[n]! = [n][n-1][n-2]\ldots[1]$ for q-factorials was used above.

Corollary 1. An elementary representation d_f with highest weight $f = (f_1, f_2, f_3)$, $f_i \in C$ has an invariant subspace if and only if

$$M = S(f + R) - R = f + m'\beta_{32} + n'\beta_{21} \tag{3.8}$$

is a weight of the representation d_f. Here S denotes an element of the Weyl group W of the simple Lie algebra $sl(3)$, and R denotes one-half of the sum over the positive roots. M is the weight of d_f if, for a

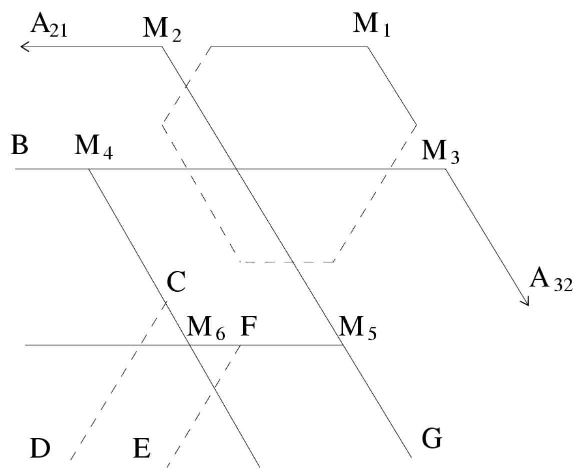

Fig. 1. Structure of the Verma module d_f at $f_1 \geq f_2 \geq f_3$.

given element S, Eq. (3.9) is satisfied with nonnegative integers m' and n'. For each $S \in W$ for which the above relation is satisfied for nonnegative m' and n', an invariant subspace of d_f is obtained.

Each subspace d_M is generated by the extremal vector y, i.e., a basis for d_M can be chosen as follows:

$$d_M : \{A_{21}^u A_{31}^v A_{32}^w y, \quad u, v, w = 0, 1, 2, \dots\}.$$

Obviously, each subspace d_M of Ω_- is the carrier space for a subrepresentation of d_f. This subrepresentation, induced by the representation d_f on the subspace d_M, will also be denoted by the symbol d_M. Each subrepresentation d_M is again an elementary representation with the matrix elements given by Eqs. (2.7) by replacing f with M. The invariant subspaces of d_M, their extremal vectors y, and the values for (t, s, p) are given by Corollary 1 by replacing f with M. The weights M are related to f through the elements S of the Weyl group W. One has

$$M = S(f + R) - R,$$

where S is some element of W and R is half of the sum over the positive roots of $sl(3)$ [22]. All extremal vectors (3.6) correspond to the same eigenvalues of Casimir operators C_2 and C_3 for the $U_q(sl(3))$ algebra.

The structure of the Verma module d_f is shown in Fig. 1. The points with integer coordinates at this weight diagram correspond to the basis vectors ξ. The extremal vectors are labeled by M_i. The boundaries of invariant subspaces d_M, generated by these extremal vectors, are shown by solid lines.

4. Representations on Quotient Spaces

Given an elementary representation d_f which has an invariant subspace d_M, new representations can be defined on the quotient spaces d_f/d_M and $d_f/(d_M + d_{M'})$, with d_M and $d_M + d_{M'}$ acting as ideals.

[The symbol + stands for the (not direct) sum of spaces d_M and $d_{M'}$.] In fact, all finite-dimensional irreducible representations of $U_q(sl(3))$ as well as some infinite-dimensional irreducible representations, which have the highest (or lowest) weight, can be obtained in this manner.

A. Below we consider the representations on the quotient space

$$\frac{d(f_1,f_2,f_3)}{d(f_2-1,f_1+1,f_3)+d(f_1,f_3-1,f_2+1)}\,.$$

It follows from Statement 1 given in Sec. 3 that, for the case $f_i - f_{i+1}$ = nonnegative integers, the weights f which satisfy the conditions $f_1 \geq f_2 \geq f_3$ correspond precisely to the highest weight of the finite-dimensional irreducible representations of $U_q(gl(3))$. In fact, the representation induced by the elementary representations

$$d_f \equiv d(f_1 f_2 f_3) \quad \text{on the quotient spaces} \qquad \frac{d(f_1,f_2,f_3)}{d(f_2-1,f_1+1,f_3)+d(f_1,f_3-1,f_2+1)}$$

are the finite-dimensional representations of $U_q(gl(3))$.

The ideal $d(f_2-1,f_1+1,f_3)+d(f_1,f_3-1,f_2+1)$ is generated by the elements

$$y_1 = A_{21}^{f_1-f_2+1}\mathbf{1} \qquad \text{and} \qquad y_2 = A_{32}^{f_2-f_3+1}\mathbf{1}.$$

Thus, modulo of the ideal, one has

$$A_{21}^{f_1-f_2+1}\mathbf{1}=0, \qquad A_{32}^{f_2-f_3+1}\mathbf{1}=0. \tag{4.1}$$

It should be noted that, because of these restrictions, not all vectors

$$\xi_{(M)}^{p_1} = A_{21}^{f_1-p_1}A_{31}^{p_1-m_1}A_{32}^{p_2-f_2}\mathbf{1}$$

with a given weight M are independent. In the Appendix, one of the possible sets of linearly independent vectors is discussed. These (main) vectors form a basis of the finite-dimensional representation d_f. All other (superfluous) vectors $\xi_{(M)}^{\tilde p_1}$ can be expressed in terms of this set of linearly independent (main) vectors as follows:

$$\xi_{(M)}^{\tilde p_1} = \sum_{p_1} D_{\tilde p_1 p_1}\xi_{(M)}^{p_1}. \tag{4.2}$$

(See Appendix for details.)

The representation

$$\frac{d(f_1,f_2,f_3)}{d(f_2-1,f_1+1,f_3)+d(f_1,f_3-1,f_2+1)} \qquad \text{of} \qquad U_q(gl(3))$$

is obtained from the elementary representation d_f given by Eqs. (2.7). It should be noted that, by the action of the generators $\rho(A_{ik})$ on the main vectors, some superfluous vectors could appear. In such a case, these superfluous vectors should be expressed in terms of the main ones, in view of relation (4.2). The area of the representation $\dfrac{d(f_1,f_2,f_3)}{d(f_2-1,f_1+1,f_3)+d(f_1,f_3-1,f_2+1)}$ is shown in Fig. 1 by a hexagon with a vertex M_1.

B. Now we consider the representations on the quotient space $\dfrac{d(f_2-1, f_3-1, f_1+2)}{d(f_3-2, f_2, f_1+2)}$.

If the difference $(f_1 - f_2)$ is a nonnegative integer, then the representation $d(f_2-1, f_3-1, f_1+2)$ is indecomposable.

If the difference $(f_1 - f_2)$ is not an integer, then this representation is irreducible.

In the following, we consider the indecomposable representation.

The representation induced by the elementary representation d_f on the quotient space $\dfrac{d(f_2-1, f_3-1, f_1+2)}{d(f_3-2, f_2, f_1+2)}$ is obtained from Eq. (2.7) by using (2.2) and the equivalence relation

$$A_{21}^{f_1-f_2+1}\mathbf{1} = 0.$$

The resulting infinite-dimensional representation is equivalent (after unitarization) to the irreducible representation of the $U_q(u(2,1))$ algebra belonging to negative discrete series [23]. This representation is shown in Fig. 1 by the broken line EFM_5G.

C. Let us consider the representations on the quotient space $\dfrac{d(f_3-2, f_1+1, f_2+1)}{d(f_3-2, f_2, f_1+2)}$.

If the difference $(f_2 - f_3)$ is a nonnegative integer, then the representation $d(f_3-2, f_1+1, f_2+1)$ is indecomposable.

If the difference $(f_2 - f_3)$ is not an integer, this representation is irreducible.

In the following, we consider the indecomposable representation.

The representation induced by the elementary representation d_f on the quotient space $\dfrac{d(f_3-2, f_1+1, f_2+1)}{d(f_3-2, f_2, f_1+2)}$ is obtained from Eqs. (2.7) by making use of Eqs. (2.2) and the relation

$$A_{32}^{f_2-f_3+1}\mathbf{1} = 0.$$

It is equivalent to the irreducible representation with the highest weight of the $U_q(u(1,2))$ algebra. This representation is shown in Fig. 1 by the broken line BM_4CD.

Appendix. The Structure of a Finite-Dimensional Irreducible Representation

We study here the properties of finite-dimensional representations.

The vectors $\xi = A_{21}^r A_{31}^s A_{32}^t \mathbf{1}$ have definite weight

$$M = (m_1, m_2, m_3), \qquad m_1 = f_1 - r - s, \qquad m_2 = f_2 + r - t, \qquad m_3 = f_3 + s + t. \tag{A.1}$$

Because of the condition

$$A_{32}^{f_2-f_3+1}\mathbf{1} = 0,$$

we obtain the restriction $0 \le t \le f_2 - f_3$. Therefore, instead of t a new parameter

$$t = p_2 - f_3 \qquad (f_3 \le p_2 \le f_2)$$

can be introduced.

The condition $A_{21}^{f_1-f_2+1}\mathbf{1}=0$ means that $r-t\le f_1-f_2$ or $r\le f_1-f_3$.

Let us take $r=f_1-p_1$; then $s=p_1-m_1$. As a result, the vectors ξ belonging to the finite-dimensional representation d_f could be reparametrized as follows:

$$\xi_{m_1}^{p_1p_2}\equiv\xi_{(M)}^{p_1}=A_{21}^{f_1-p_1}A_{31}^{p_1-m_1}A_{32}^{p_2-f_3}\mathbf{1}. \tag{A.2}$$

In this notation, we have

$$m_2=f_1+f_2+f_3-p_1-p_2, \qquad m_3=p_1+p_2-m_1$$

and the parameters m_1,p_1, and p_2 take only integer values satisfying the following conditions:

$$f_1\ge m_1\ge f_3, \qquad f_1\ge p_1\ge m_1, \qquad f_2\ge p_2\ge f_3. \tag{A.3}$$

From the set (A.2) we choose a subset of vectors $\xi_{m_1}^{p_1p_2}$ in which m_1,p_1, and p_2 obey the following requirements:

$$f_1\ge m_1\ge f_3, \qquad f_1\ge p_1\ge \max\,(m_1,f_2), \qquad \min\,(m_1,f_2)\ge p_2\ge f_3. \tag{A.4}$$

These vectors will be referred to as the main vectors while the remaining vectors of the set (A.2) will be called superfluous. It can readily be seen that all vectors $\xi_{m_1}^{p_1p_2}$ with $m_1\ge f_2$ are the main vectors. In the case $m_1<f_2$, both the main and superfluous vectors may occur.

The vectors (A.2) can be represented graphically (Fig. 2). To each vector $\xi_{m_1}^{p_1p_2}$ with fixed m_1 corresponds a point with integer coordinates (p_1,p_2) satisfying (A.3). Such points will be called the allowed points. Figure 2 illustrates the case $f_2>m_1\ge f_3$. The allowed points of the rectangle FHDT, including those lying on its perimeter, correspond to the main vectors $\xi_{m_1}^{p_1p_2}$, whereas the allowed points of the rectangle ABCD, excluding those of the rectangle FHDT, correspond to the superfluous vectors $\xi_{m_1}^{\tilde p_1\tilde p_2}$. The different vectors $\xi_{(M)}^{p_1}$ with fixed weight M are presented by the allowed points lying on a straight line of the type KN.

The following statement holds:

Statement 2. Any superfluous vector $\xi_{(M)}^{\tilde p_1}$ of weight M is either a zero vector, if there are no main vectors of weight M, or can be represented as a linear combination of the main vectors by the following formulas:

$$\xi_{(M)}^{\tilde p_1}=\sum_{p_1=b}^{g}D_{\tilde p_1p_1}^{(M)}\xi_{(M)}^{p_1}(a\le\tilde p_1\le b-1), \tag{A.5}$$

$$\begin{aligned} D_{\tilde p_1,p_1}^{(M)}&=(-1)^{b-\tilde p_1}q^{(p_1-\tilde p_1)(f_1-f_2-f_3+m_1+p_2-\tilde p_1+1)}\\ &\times\frac{[\tilde p_2-f_3]![f_1-\tilde p_1]![\tilde p_1-m_1]![p_1-a]![p_1-\tilde p_1-1]!}{[p_1-\tilde p_1]![b-\tilde p_1-1]![p_1-b]![p_2-f_3]![f_1-p_1]![p_1-m_1]![\tilde p_1-a]!}, \end{aligned} \tag{A.6}$$

where

$$g=m_1+m_3-f_3, \qquad a=\max\,(m_1,m_1+m_3-f_2),$$

$$b=a+f_2-m_1=\max\,(f_2,m_3), \qquad g=\max\,(f_2,m_1+m_3-f_3).$$

The proof of this statement is similar to the one given in [24] for the classical case $q=1$.

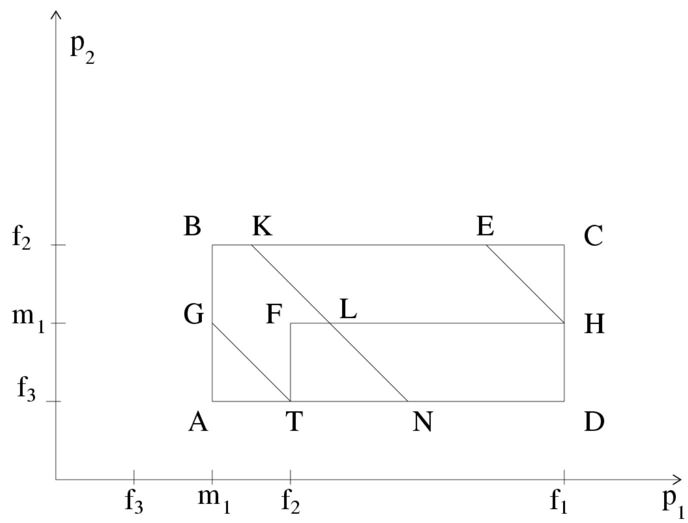

Fig. 2. Structure of the finite-dimensional representation of $U_q(gl(3))$.

Now let the vector $\xi^{\tilde p_1}_{(M)}$ have coordinates $(\tilde p_1, \tilde p_2)$ which satisfy the condition $f_2{+}f_3 < \tilde p_1{+}\tilde p_2 < f_1{+}m_1$. We assume for certainty that the point $(\tilde p_1, \tilde p_2)$ belongs to the straight line KN. The allowed points of this line correspond to all vectors $\xi^{p_1}_{(M)}$ with the same weight M. We find equations which involve all superfluous and main vectors corresponding to allowed points of the straight line KN.

It follows from conditions (4.1) that the vectors ξ satisfy the relations

$$A^x_{32}\mathbf{1} = 0, \quad x \geq f_2 - f_3 + 1, \quad A^y_{21}\mathbf{1} = 0, \quad y \geq f_1 - f_2 + 1. \tag{A.7}$$

This means that the vectors with $p_2 \leq f_2$ should be selected from the vectors $\xi^{p_1p_2}_{m_1}$. This restriction is taken into account in the form (A.2). From the second condition (A.7), we obtain a set of relations

$$A_{32}^{p_1+p_2+k-f_3}\; A_{31}^{f_2-k-m_1}\; A_{21}^{f_1-f_2+k}\mathbf{1} = 0, \qquad k = 1, 2, \ldots, f_2 - m_1. \tag{A.8}$$

Let us introduce instead of vectors (A.2) new vectors

$$\xi^{p_1p_2}_{m_1} = \frac{1}{[f_1 - p_1]![p_2 - f_3]!}\; \xi^{p_1p_2}_{m_1}\; q^{-(f_1-p_1)(p_2-f_2-f_3+m_1)}\,. \tag{A.9}$$

After permutations of operators A_{32} to the right and operators A_{21} to the left, in view of relation (3.7) and the relation

$$A^a_{32}A^b_{21} = \sum_c \frac{[a]![b]!}{[c]![a-c]![b-c]!}\; A^{a-c}_{21}A^c_{31}A^{b-c}_{32}q^{-(a-c)(b-c)}, \tag{A.10}$$

we obtain the following system of equations:

$$\sum_{p_1=a}^{g} B^{k,0}_{p_1}\; \xi^{p_1p_2}_{m_1} = 0, \quad k = 1, 2, \ldots, f_2 - m_1, \tag{A.11}$$

where

$$B_{p_1}^{k,0} = (-1)^k \frac{q^{p_1 k}}{[p_1 + k - f_2]!}. \tag{A.12}$$

[The lower and upper limits a and g in sum (A.11) (the abscissas of points N and K, respectively) will be explained below.]

The system of equations (A.11) contains all superfluous vectors with given M. It is important to point out that the number of equations is exactly equal to the number of superfluous vectors with given weight M.

We solve the system of equations (A.11) with respect to the superfluous vectors $\xi_{(M)}^{\tilde{p_1}}$. To do this, we multiply the kth equation of (A.11) by q^{a-f_2+k+1} and add to it the $(k+1)$th equation multiplied by $[a-f_2+k+1]q^{-f_2+k+1}$. The last equation ($k = f_2 - m$) remains as before. As a result, we obtain a new system of equations

$$\sum_{p_1=a}^{g} B_{p_1}^{k,1}\, \xi_{m_1}^{p_1 p_2} = 0, \tag{A.13}$$

with coefficients

$$B_{p_1}^{k,1} = (q^a B_{p_1}^{k,0} + [a - f_2 + k + 1] B_{p_1}^{k+1,0}) q^{-f_2+k+1} = (-1)^k \frac{[p_1 - a] q^{k p_1}}{[p_1 + k - f_2 + 1]!}. \tag{A.14}$$

Now let us perform similar transformations with the new system of equations (A.13) by multiplication of kth equation by q^{a-f_2+k+3} and the $(k+1)$th equation by $[a - f_2 + k + 3]q^{-f_2+k+2}$. The addition of these equations gives a system of equations of the form (A.11) and (A.13) with coefficients

$$B_{p_1}^{k,2} = (-1)^k \frac{q^{k p_1}[p_1 - a]!}{[p_1 - f_2 + k + 2]![p_1 - a - 2]!}. \tag{A.15}$$

Continuing with these transformations, we obtain after the ith iteration a system of equations of the type (A.11) with coefficients

$$B_{p_1}^{k,i} = (B_{p_1}^{k,i-1} q^a + [a - f_2 + k + 2i - 1[) q^{-f_2+k+i} = (-1)^k \frac{q^{k p_1}[p_1 - a]!}{[p_1 - f_2 + k + i]![p_1 - a - i]!}. \tag{A.16}$$

By collecting the equations from the different systems obtained, the final set of equations is constructed and it reads

$$\sum_{p_1=a}^{g} B_{p_1}^{k,f_2-m+k} \xi_{m_1}^{p_1 p_2} = 0, \qquad k = 1, 2, \ldots, f_2 - m_1. \tag{A.17}$$

Its explicit form is as follows:

$$\sum_{p_1=a}^{g} \frac{(-1)^k}{[p_1 + b + k]!} \xi_{m_1}^{p_1 p_2} = 0, \qquad k = 1, 2, \ldots, f_2 - m_1, \tag{A.18}$$

where $b = a + f_2 - m_1$. To solve these equations with respect to the vectors $\xi_{m_1}^{\tilde{p}_1,\tilde{p}_2}$ with $a \le \tilde{p}_1 \le b-1$, it is necessary to multiply (A.18) by $q^{-(\tilde{p}_1+1)(k-1)}/[b - \tilde{p}_1 - k]!$ and to carry out the summation over k. Making use of the known relations

$$\sum_{c=0} (-1)^c \frac{q^{(A-B-1)c}}{[A-c]![c-B]!} = (-1)^a q^{-A} \delta_{AB}$$

and

$$\sum_{c=0}(-1)^c \frac{q^{(A+D-1)c}}{[A-c]![c+D]!} = q^{-A}\frac{[A+D-1]!}{[A]![D-1]![A+D]!},$$

where A, B, D, and c are nonnegative integers, we obtain

$$\xi_{m_1}^{\tilde{p}_1\tilde{p}_2} = (-1)^{b-\tilde{p}_1}\sum_{p_1=b}^{g} q^{(p_1-\tilde{p}_1)}\frac{[p_1-a]![\tilde{p}_1-m_1]![p_1-\tilde{p}_1-1]!}{[p_1-m_1]![\tilde{p}_1-a]![b-\tilde{p}_1-1]![p_1-b]![p_1-\tilde{p}_1]!}\xi_{m_1}^{p_1p_2}. \tag{A.19}$$

Here $b(g)$ is the minimum (maximum) value of p_1 corresponding to the main vector with given M [abscissa of the point $L(N)$ in Fig. 2].

It follows from the comparison of (A.3) and (A.4) that

$$a = \max(m_1, m_1+m_3-f_2), \qquad b = \max(f_2, m_3), \qquad g = \max(f_2, m_1+m_3-f_3).$$

From formula (A.19) follows expression (A.6) for the vectors (A.2). Thus, Statement 2 is proved.

It is worth noting that the vectors $\xi_{(M)}^{\tilde{p}_1}$ with $\tilde{p}_1+\tilde{p}_2 > f_1+m_1$ or $\tilde{p}_1+\tilde{p}_2 < f_2+f_3$ are vanishing vectors because there are no main vectors in this case. In Fig. 2 these vanishing vectors are shown by allowed points of triangles AGT and ECH, excluding those lying on straight lines GT and EH.

It is easy to find the total number $N(f)$ of main vectors satisfying the conditions (A.4):

$$N(f) = \frac{1}{2}(f_1-f_2+1)(f_1-f_3+2)(f_2-f_3+1).$$

This expression is exactly equal to the dimension of the representation d_f of the quantum algebra $U_q(u(3))$. Thus, the main vectors form a complete basis of the finite-dimensional representation d_f of $U_q(gl(3))$. Generators of the quantum $SU_q(2)$-group can be constructed by means of q-oscillators. These oscillators correspond to nonlinear oscillators of the electromagnetic field at very high field intensities. As was shown in [4], the high field intensity can produce a blue-shift effect of the light frequency. This effect, if it exists, could be described within the framework of a nonlinear Hamiltonian based on the interaction of nonlinear q-oscillators. In view of the construction considered, the $U_g(gl(3))$ could also be interpreted in terms of nonlinear vibrations appropriate to describe nonlinear optical phenomena.

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