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Homogeneous Nondegenerate Hypersurfaces in $\mathbb{C}^3$ with Two-Dimensional Isotropy Groups*

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Abstract. We construct a complete list of nonspherical real hypersurfaces in $\mathbb{C}^3$ that are Levi nondegenerate and admit seven-dimensional transitive groups of local holomorphic transformations. The description splits into two cases corresponding to strictly pseudoconvex surfaces and surfaces with nondegenerate sign-indefinite Levi form.

Key words: homogeneous manifold, normal form of equation, vector field, isotropic group, Levi form.

The nondegenerate quadrics

$$\operatorname{Im} z_3 = |z_1|^2 \pm |z_2|^2 \tag{1}$$

are homogeneous with respect to holomorphic transformations of the space $\mathbb{C}^3$ with coordinates z_1, z_2, z_3 and have ten-dimensional isotropy groups (e.g., see [2]).

Furthermore, the isotropy group $\operatorname{Aut}_0(M)$ of any homogeneous hypersurface $M \in \mathbb{C}^3$ that is nonspherical (i.e., locally nonequivalent to any quadric (1)) and has a nondegenerate Levi form satisfies the inequality [10]

$$0 \leqslant \dim \operatorname{Aut}_0(M) \leqslant 3. \tag{2}$$

It was also shown in [10] that the relation $\dim \operatorname{Aut}_0(M) = 3$ holds only for the homogeneous Winkelmann surface (here and in the following, $u = \operatorname{Re} z_3$ and $v = \operatorname{Im} z_3$):

$$v = (z_1 \bar{z}_2 + z_2 \bar{z}_1) + |z_1|^4. \tag{3}$$

Below we give a complete list of homogeneous Levi nondegenerate hypersurfaces in $\mathbb{C}^3$ for which the isotropy group is two-dimensional and hence the group of holomorphic transformations is seven-dimensional.

Theorem 1. *Any homogeneous nonspherical hypersurface M in $\mathbb{C}^3$ with two-dimensional group $\operatorname{Aut}_0(M)$ and positive definite Levi form belongs to the following list of pairwise nonequivalent manifolds up to local holomorphic equivalence*:

$$v = \ln(1 + |z_1|^2) + b \ln(1 + |z_2|^2), \qquad b \in (0, 1], \tag{4}$$

$$v = \ln(1 + |z_1|^2) - b \ln(1 - |z_2|^2), \qquad b \in (0, 1) \cup (1, \infty), \tag{5}$$

$$v = \ln(1 - |z_1|^2) + b \ln(1 - |z_2|^2), \qquad b \in (0, 1], \tag{6}$$

$$v = |z_2|^2 + \varepsilon \ln(1 + \varepsilon |z_1|^2), \qquad \varepsilon = \pm 1. \tag{7}$$

Theorem 2. *The homogeneous nonspherical hypersurfaces in $\mathbb{C}^3$ with two-dimensional isotropy group and nondegenerate sign-indefinite Levi are defined up to local holomorphic equivalence by the*

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following list of pairwise nonequivalent manifolds:

$$v = (z_1\bar z_2 + z_2\bar z_1) + (1+\varepsilon|z_1|^2)\ln(1+\varepsilon|z_1|^2), \qquad \varepsilon = \pm 1, \tag{8}$$
$$v = e^{i\theta}\ln(1+z_1\bar z_2) + e^{-i\theta}\ln(1+z_2\bar z_1), \qquad \theta\in(-\pi/2,\pi/2), \tag{9}$$
$$v = \ln(1-|z_1|^2) - b\ln(1-|z_2|^2), \qquad 0<b<1, \tag{10}$$
$$v = \ln(1+|z_1|^2) + b\ln(1-|z_2|^2), \qquad 0<b<\infty, \tag{11}$$
$$v = \ln(1+|z_1|^2) - b\ln(1+|z_2|^2), \qquad 0<b<1, \tag{12}$$
$$v = |z_2|^2 + \varepsilon\ln(1-\varepsilon|z_1|^2), \qquad \varepsilon=\pm 1. \tag{13}$$

Let us outline the proof of these theorems, which is based on local normal forms for the equations of the homogeneous manifolds in question.

Let us define a real-analytic (not necessarily homogeneous) hypersurface M in $\mathbb{C}^3$ containing the origin and Levi nondegenerate at this point by an equation of the form

$$v = \langle z,z\rangle + \sum_{k+l+2m=s,\,\min(k,l)\geqslant 2} h_{klm}(z,\bar z)u^m. \tag{14}$$

Here $\langle z,z\rangle$ stands for a nondegenerate Hermitian form in two complex variables (the so-called Levi form of the surface M) and $h_{klm}(z,\bar z)$ is a homogeneous polynomial of bidegree $(k,\bar l)$ in $z=(z_1,z_2)$ and $\bar z = (\bar z_1,\bar z_2)$.

The possibility of this representation of the hypersurface M is ensured by the results in [2], where the additional conditions on the lower coefficients of the normal equation (14) are also described. The nonzero polynomial h_{220} in this normal equation was reduced to a more specific form in [9].

After this reduction, the problem on the group $G_1 \subset GL(2,\mathbb{C})$ of matrices preserving the pair of polynomials $\{\langle z,z\rangle, h_{220}\}$ up to a constant is substantially simplified. Since the isotropy group $\mathrm{Aut}_0(M)$ can be embedded in G_1 (see [3–6]), we obtain the following factorization of the estimate (2) given in [10].

Proposition 1. *If the dimension of the isotropy group of a Levi nondegenerate homogeneous real hypersurface M is not less than two, then the pair $\{\langle z,z\rangle, h_{220}\}$ in the normal equation* (14) *can be reduced to one of the following forms*:

1) $\{(z_1\bar z_2 + z_2\bar z_1), |z_1|^4\}$,
2) $\{(z_1\bar z_2 + z_2\bar z_1), i(z_1^2\bar z_1\bar z_2 - z_1z_2\bar z_1^1)\}$,
3) $\{(z_1\bar z_2 + z_2\bar z_1), (-z_1^2\bar z_2^2 + 4|z_1|^2|z_2|^2 - z_2^2\bar z_1^2)\}$,
4) $\{(|z_1|^2 - |z_2|^2), (|z_1|^4 + 4|z_1|^2|z_2|^2 + |z_2|^4)\}$,
5) $\{(|z_1|^2 + |z_2|^2), \pm(|z_1|^4 - 4|z_1|^2|z_2|^2 + |z_2|^4)\}$.

To continue our study of homogeneity, we use vector fields on the manifolds in question.

Briefly, the scheme of the proof is as follows (see also [11]). On a homogeneous hypersurface $M\subset\mathbb{C}^3$, we introduce the algebra of holomorphic vector fields of the form $Z=\sum_{k=1}^3 f_k(z_1,z_2,z_3)\,\partial/\partial z_k$.

The fact that such a field is tangent to a real hypersurface $M=\{\Phi(z,\bar z,u,v)=0\}$ means that the following identity holds:

$$\mathrm{Re}\{Z(\Phi)\}|_M \equiv 0. \tag{15}$$

For surfaces of the form (14), the left-hand side of this identity is an analytic function in the variables z, $\bar z$, and u. By equating all components of degrees k, $\bar l$, m with respect to the variables z, $\bar z$, u respectively, in this identity with zero, we obtain an infinite system of separate $(k,\bar l,m)$-equations. Since the above algebra of vector fields depends on seven real parameters, it follows that each of these equations is decomposed in turn into seven parts.

It remains to choose an appropriate subsystem of this infinite system of necessary homogeneity conditions to construct our classification. Here the use of normal equations describing M (rather than arbitrary ones) significantly simplifies technical details.

This scheme is realized, in dependence on the cases listed in Proposition 1, in the proof of the following two assertions.

Proposition 2. *There are no homogeneous real hypersurfaces corresponding to case* 2) *in Proposition* 1.

Proposition 3. *For the other cases in Proposition* 1, *the polynomial* h_{330} *in the normal equation* (14) *can be reduced to the following form*:

case 1): $h_{330} = \varepsilon|z_1|^6$, $\varepsilon \in \{-1, 0, 1\}$,

case 3): $h_{330} = it(z_1^3\bar{z}_2^3 - 3z_1^2 z_2 \bar{z}_1^2 \bar{z}_2 - 3z_1 z_2^2 \bar{z}_1^2 \bar{z}_2 + z_2^3 \bar{z}_1^3)$, $t \in \mathbb{R}$,

case 4): $h_{330} = t(|z_1|^6 - 3|z_1|^4|z_2|^2 - 3|z_1|^2|z_2|^4 + |z_2|^6)$, $t \in \mathbb{R}$,

case 5): $h_{330} = t(|z_1|^6 + 3|z_1|^4|z_2|^2 - 3|z_1|^2|z_2|^4 - |z_2|^6)$, $t \geqslant 0$.

Moreover, the entire normal equation is completely determined by the triple of polynomials $\{\langle z, z\rangle, h_{220}, h_{330}\}$.

Remark. In case 1) we obtain the homogeneous Winkelmann surface (3) with three-dimensional isotropy group for $\varepsilon = 0$, and the homogeneous surface (8) for $\varepsilon = \pm 1$. For these surfaces, we have $\dim \mathrm{Aut}_0(M) = 2$, as well as in the remaining cases 3), 4), 5).

The proof of Theorems 1 and 2 is completed by constructing normal equations for the homogeneous surfaces (4)–(7) and (8)–(13).

The normalization procedure itself is also very cumbersome and is not described here for this reason. We only note that, for equations (4)–(7), it leads to case 5) of Proposition 1. Moreover, for instance, for the surface (4) we obtain $h_{220} = -(|z_1|^4 - 4|z_1|^2|z_2|^2 + |z_2|^4)$, and the coefficient t of the polynomial h_{330} is equal to $3(1-b)/(1+b)$ and covers the half-open interval $[0, 3)$.

After the normalization, both surfaces (7) have the same coefficient $t = 3$, and their polynomials h_{220} differ only in the sign. The remaining possible pairs h_{220}, h_{330} correspond in this strictly pseudoconvex case to the surfaces (5) and (6).

The case of a sign-indefinite Levi form can be treated in a similar way. Here case 3) corresponds to the surfaces (9), and the normalization of equations (10)–(13) leads to case 4). The evaluation of the specific values of the coefficient t is similar to that in the pseudoconvex case.

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