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BRIEF COMMUNICATIONS

Description of the Real von Neumann Algebras with Abelian Skew-Symmetric Part

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Abstract. In this note we describe (up to isomorphism) the real von Neiman algebras R with Abelian skew-symmetric part $R_k = \{x \in R : x^* = -x\}$, i.e., such that $xy - yx = 0$ for any $x, y \in R_k$.

Key words: real von Neimann algebra, symmetric element, skew-symmetric element, JW-algebra, spin factor.

Consider the *-algebra $B(H)$ of all bounded linear operators on a complex Hilbert space H. Recall [1, 2] that a weakly closed real *-subalgebra R in $B(H)$ is called *a real von Neumann algebra* if it contains the identity operator $\mathbb{I}$ and satisfies the condition $R \cap iR = \{0\}$. There is a close relationship between the real von Neumann algebras and the involutive (i.e., period two) *-antiautomorphisms of (complex) von Neumann algebras (for details, see [1, 2]).

For a real von Neumann algebra R, the set $R_s = \{x \in R : x^* = x\}$ of all symmetric elements of R forms a weakly closed Jordan algebra of self-adjoint operators (a *JW-algebra* [3]) with respect to the symmetrized product $x \circ y = \frac{1}{2}(xy + yx)$. The set $R_k = \{x \in R : x^* = -x\}$ of all skew-symmetric elements in R is a Lie algebra with respect to the commutator $[x, y] = xy - yx$.

It follows from the results in [4] that if a real von Neumann algebra R is Abelian, then it is isomorphic to the direct sum of algebras of the form $L^\infty(\Omega, \mu, \mathbb{R})$ and $L^\infty(\Omega, \mu, \mathbb{C})$, i.e., algebras of essentially bounded measurable (real or complex) functions on a measure space (Ω, μ). This result was generalized in [5], where it was proved that if R is a real von Neumann algebra with Abelian symmetric part R_s, then, along with the above summands, R can have a direct summand of the form $L^\infty(\Omega, \mu, \mathbb{Q})$, where $\mathbb{Q}$ is the quaternion skew field.

In the present note we describe the real von Neumann algebras R with Abelian skew-symmetric part R_k, i.e., such that $[x, y] = 0$ for any $x, y \in R_k$. Namely, we prove the following result.

Theorem. *Let R be a real von Neumann algebra whose skew-symmetric part R_k is Abelian. Then R is isomorphic to the direct sum of algebras of the following types*:

(i) $L^\infty(\Omega, \mu, \mathbb{R})$;
(ii) $L^\infty(\Omega, \mu, \mathbb{C})$;
(iii) $L^\infty(\Omega, \mu, M_2(\mathbb{R})) = L^\infty(\Omega, \mu, \mathbb{R}) \otimes M_2(\mathbb{R})$,

where $M_2(\mathbb{R})$ denotes the algebra of 2×2 real matrices.

Before passing to the proof of the theorem we present several preliminary results.

Lemma 1 (Putnam [6]). *Let $a, x \in B(H)$, where a is normal, i.e., $a^*a = aa^*$. Then $[a, [a, x]] = 0$ implies $[a, x] = 0$.*

Lemma 2. *Under the assumptions of the theorem, for any $x, y \in R_k$, the product xy is a central element of R, i.e., it commutes with every element of R.*

Proof. Since R_k is Abelian, we have $(xy)^* = xy \in R_s$, and xy commutes with every element of R_k. Further, since $xy \in R_s$, it follows that $[a, xy] \in R_k$ for any $a \in R_s$, and therefore $[a, xy]$ commutes with x and with y, and thus with xy, i.e., $[[a, xy], xy] = 0$. Since the symmetric element xy is normal, it follows from Lemma 1 that $[a, xy] = 0$ for any $a \in R_s$. Therefore, xy commutes with any element in $R = R_s + R_k$.

Lemma 3. *Under the assumptions of the theorem, if the JW-algebra R_s is Abelian, then the real von Neumann algebra R is commutative.*

Proof. By assumption, $[x, y] = 0$ for any $x, y \in R_k$, and $[a, b] = 0$ for any $a, b \in R_s$. Therefore, it suffices to prove that $[a, x] = 0$ for $a \in R_s$ and $x \in R_k$. Indeed, $[a, x]^* = [a, x] \in R_s$, and thus $[a, [a, x]] = 0$. Since $a = a^*$ is normal, it follows from Lemma 1 that $[a, x] = 0$.

Lemma 4. *Under the assumptions of the theorem, any family of mutually orthogonal pairwise equivalent projections in the JW-algebra R_s has at most two elements.*

Proof. Suppose the converse, i.e., let there exist three mutually orthogonal and pairwise equivalent projections $e = e_1$, e_2, e_3. By [3, Proposition 10] they are conjugate by symmetries, i.e., $e_2 = s_2 e s_2$ and $e_3 = s_3 e s_3$ for some symmetries $s_2, s_3 \in R_s$ (recall that $s_2^2 = s_3^2 = \mathbb{1}$). Consider the element $x = s_2 e s_3 - s_3 e s_2 \in R_k$. Since the projections e, e_2, and e_3 are orthogonal, it can readily be calculated that $x^2 = -e_2 - e_3$. By Lemma 2, the element x^2 is central in R. On the other hand, consider the element $a = e s_2 + s_2 e \in R_s$. Using the orthogonality of e, e_2, and e_3 again and the relations between them via symmetries, we can readily see that

$$[a, x^2] = -[e s_2 + s_2 e, e_2 + e_3] = s_2 e - e s_2.$$

Therefore,

$$[a, x^2]^2 = (s_2 e - e s_2)^2 = -e_2 - e \neq 0,$$

which contradicts the condition that the element x^2 is central, i.e., $[a, x^2] = 0$ for any $a \in R$. A contradiction.

Proof of the theorem. By Lemma 4, there exists a central projection z in R_s such that zR_s is of type I_1 (i.e., an Abelian JW-algebra) and $(\mathbb{1} - z)R_s$ is a type I_2 JW-algebra. The central element z in R_s is automatically central in R. Indeed, for $x \in R_k$, the commutator $[z, x]$ is in R_s, and therefore $[z, [z, x]] = 0$, and Lemma 1 implies that $[z, x] = 0$, i.e., z commutes with each element of R_k as well. Thus, $R = zR \oplus (\mathbb{1} - z)R$, where the real von Neumann algebra zR has the Abelian symmetric part zR_s and the Abelian skew-symmetric part $(zR)_k = zR_k$. By Lemma 3, the real von Neumann algebra zR is Abelian, and, by [4], zR is isomorphic to the direct sum of algebras of the form (i) and (ii). Therefore, it remains to consider the case in which R_s is a *JW-algebra of type* I_2.

Since $a_1 \cdots a_n + a_n a_{n-1} \cdots a_1 \in R_s$ for any $a_1, \ldots, a_n \in R_s$, it follows that the JW-algebra R_s is a *reversible* JW-algebra of type I_2. As is known [7], any JW-algebra of type I_2 can be decomposed into a direct sum of JW-algebras of the form

$$\sum_{m=3}^{\infty} L^\infty(\Omega_m, \mu_m, V_m),$$

where Ω_m is a locally compact Hausdorff space, μ_m is a Radon measure on Ω_m, and V_m is a spin factor (i.e., a JW-factor of type I_2) of dimension $m \geqslant 3$. On the other hand, if the original JW-algebra is reversible, then all spin factors V_m are reversible as well, which is possible only if $m = 3$, 4, or 6 (see [8]), i.e., if $V_3 \cong M_2(\mathbb{R})_s$, $V_4 \cong M_2(\mathbb{C})_s$, or $V_6 \cong M_2(\mathbb{Q})_s$. As is known [9, Theorem 3], $M_k = [M_s, M_s]$ for any real factor M except for $\mathbb{Q}$, i.e., any skew-symmetric element of M is a finite sum of commutators of symmetric elements in M. In particular, the skew-symmetric matrices in $M_2(\mathbb{F})$ are algebraically generated by $M_2(\mathbb{F})_s$ (where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}$). Therefore, if the reversible JW-algebra R_s contains direct summands of the form $L^\infty(\Omega, \mu, M_2(\mathbb{F})_s)$, then R contains direct summands of the form $L^\infty(\Omega, \mu, M_2(\mathbb{F}))$. However, for $\mathbb{F} = \mathbb{C}$ or $\mathbb{Q}$, the algebra $M_2(\mathbb{F})$ has skew-symmetric matrices which do not commute, and this contradicts the assumption of the theorem. Therefore, the only possible case is $R_s = L^\infty(\Omega, \mu, M_2(\mathbb{R})_s)$.

As noted above, in this case the algebra R contains the algebra $R^0 = L^\infty(\Omega, \mu, M_2(\mathbb{R}))$. We claim that $R = R^0$. Indeed, $R_s = R^0_s$. Let $x \in R_k$. Consider an arbitrary invertible skew-symmetric element $u \in R^0_k$, for instance, the function on Ω identically equal to the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $M_2(\mathbb{R})$.

Then xu is a symmetric element of $R_s = R_s^0$, and therefore $x = (xu)u^{-1} \in R_s^0 R_k^0 \subset R^0$. Thus, $R_k = R_k^0$ and $R = R_s + R_k = R_s^0 + R_k^0 = R^0$, which proves that if the JW-algebra R_s is of type I_2, then $R = L^\infty(\Omega, \mu, M_2(\mathbb{R}))$, i.e., case (iii) takes place. This completes the proof of the theorem.

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