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A Remark on the Fourier Pairing and the Binomial Formula for the Macdonald Polynomials*

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Abstract. We concisely and directly prove that the interpolation Macdonald polynomials are orthogonal with respect to the Fourier pairing and briefly discuss immediate applications of this fact, in particular, to the symmetry of the Fourier pairing and to the binomial formula.

Key words: Macdonald polynomials, Fourier pairing, binomial formula.

1. Introduction

The Fourier pairing introduced by Cherednik [1] is a fundamental notion in the theory of Macdonald polynomials. In its simplest instance, it pairs the algebra Λ_n of symmetric polynomials in n variables with the algebra $\mathscr{D}_n$ of Macdonald commuting difference operators acting on Λ_n [6]. By definition,

$$\langle D, f\rangle = [D\cdot f](\widehat{0}), \qquad D\in\mathscr{D}_n,\ f\in\Lambda_n, \tag{1}$$

where $\widehat{0}$ is a distinguished point. There is a natural isomorphism $\mathscr{D}_n\cong\Lambda_n$, which makes (1) a quadratic form on Λ_n. The most important and useful property of this form is its symmetry (see [1, 6, 7]).

The main observation in this note is that there is a quite natural orthogonal basis for the form (1). Namely, this is the basis $\{I_\mu\}$ of the interpolation Macdonald polynomials which were intensively studied by Knop, Olshanski, Sahi, the author, and others (see, e.g., [3, 8, 11, 13] and the references therein). The polynomials I_μ are defined by very simple multivariate Newton-type interpolation conditions and have many remarkable applications.

The orthogonality of I_μ with respect to (1), stated in Theorem 1 below, readily follows from the definitions and needs no nontrivial properties of the polynomials I_μ. It directly implies that (1) is symmetric.

Moreover, the orthogonality of I_μ immediately implies an expansion of the simultaneous eigenfunctions of the operators in $\mathscr{D}_n$ with respect to the basis $\{I_\mu\}$, and these eigenfunctions are known as the symmetric Macdonald polynomials P_λ. This expansion, which is reproduced in Theorem 3 below, is the binomial formula for P_λ (see [9]). In fact, the orthogonality of I_μ is essentially equivalent to the binomial theorem, but it certainly appears to be a much more basic, natural, and appealing property.

The binomial theorem of [9] was extended [4, 10, 14] to a more general setting, including other classical root systems and the nonsymmetric Macdonald polynomials. We do not strive for the greatest possible generality in this paper. Our intention, rather, is to show how the basic idea works in the simplest nontrivial example of the ordinary symmetric Macdonald polynomials. We even first consider the (almost) trivial one-dimensional case to give a completely elementary illustration of what is going on.

It should be pointed out that there is another source of orthogonality relations for the polynomials I_μ. Namely, the polynomials I_μ can be obtained from the symmetric Macdonald polynomials of type BC_n (this can be seen explicitly by degenerating the binomial formula of [10] to the binomial formula for the polynomials I_μ [9]).

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2. Simplest Example

2.1. As a warm-up, let us first consider the one-dimensional case. Let the operator T act on polynomials in x by the formula

$$[Tf](x) = f(qx).$$

Obviously, the monomials x^n, $n = 0, 1, 2, \dots$, are the eigenfunctions of this operator with the eigenvalues q^n. Consider the following bilinear form:

$$\langle g, f \rangle = [g(T) \cdot f](1). \tag{2}$$

In the basis $\{x^n\}$, this form has the matrix

$$[q^{nm}]_{n,m=0,1,\dots} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & q & q^2 & q^3 & \dots \\ 1 & q^2 & q^4 & q^6 & \dots \\ 1 & q^3 & q^6 & q^9 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and it is clearly symmetric.

It is also clear that

$$x^* = T, \qquad T^* = x, \tag{3}$$

where x denotes the operator of multiplication by the independent variable and the asterisk indicates the adjoint operator with respect to (2). Clearly, (3) is an anti-automorphism of the q-Heisenberg algebra generated by T and x (subjected to the relation $Tx = qxT$) and deserves to be called the *Fourier transform*.

2.2. Now consider the polynomial

$$I_n = (x-1)(x-q)\cdots(x-q^{n-1}), \qquad n = 0, 1, \dots, \tag{4}$$

satisfying the following Newton interpolation conditions:

$$I_n \equiv x^n \mod \{x^m\}_{m<n}, \tag{5}$$

$$I_n(q^m) = 0, \qquad 0 \leqslant m < n. \tag{6}$$

We have the following assertion.

Proposition 1. *The polynomials I_n are orthogonal with respect to the form* (2)*, namely,*

$$\langle I_n, I_m \rangle = \delta_{n,m} I_n(q^n). \tag{7}$$

Proof. We advisedly avoid using the symmetry of (2) in our reasoning because we intend to obtain the analogous symmetry in the general case as a corollary.

It is clear from the definition (2) that

$$\langle x^n, f \rangle = f(q^n), \tag{8}$$

and since $T \cdot x^n = q^n x^n$, we also have

$$\langle g, x^n \rangle = g(q^n).$$

It now follows from (6) that

$$\langle x^m, I_n \rangle = \langle I_n, x^m \rangle = 0, \qquad m < n,$$

and

$$\langle x^n, I_n \rangle = \langle I_n, x^n \rangle = I_n(q^n).$$

Now property (5) completes the proof. □

The following expansion is immediate from (7) and (8):

$$x^n = \sum_m \frac{\langle x^n, I_m \rangle}{\langle I_m, I_m \rangle} I_m(x) = \sum_m \frac{I_m(q^n) I_m(x)}{I_m(q^m)} .$$

This is the Newton interpolation of the monomial x^n with nodes $1, q, q^2, \dots$ and also an instance of the q-binomial theorem.

3. Symmetric Macdonald Polynomials

3.1. We now turn to polynomials in n variables $x_1, \dots, x_n$. Write

$$[T_i f](x_1, \dots, x_n) = f(x_1, \dots, q x_i, \dots, x_n).$$

Let t be an additional parameter. Following Macdonald [6], we introduce the operators

$$D_k = t^{k(k-1)/2} \sum_{|S|=k} d_S(x) \prod_{i \in S} T_i,$$

where the summation is taken over the subsets $S \subset \{1, \dots, n\}$ of cardinality k and

$$d_S(x) = \prod_{i \in S,\, j \notin S} \frac{t x_i - x_j}{x_i - x_j} .$$

3.2. The operators D_k commute and take symmetric polynomials to symmetric polynomials. They act triangularly in the basis of monomial symmetric functions, namely,

$$D_k \cdot m_\lambda \equiv e_k(\widehat{\lambda}) m_\lambda \mod \{m_\mu\}_{\mu < \lambda},$$

where $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n)$ stands for a partition,

$$m_\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} + \text{permutations}$$

for the corresponding monomial symmetric function, $e_k = m_{(1^k)}$ for the kth elementary symmetric function, $\widehat{\lambda}$ for the point

$$\widehat{\lambda} = (q^{\lambda_1} t^{n-1}, q^{\lambda_2} t^{n-2}, \dots, q^{\lambda_{n-1}} t, q^{\lambda_n}),$$

and $\mu < \lambda$ for the dominance order on the partitions:

$$\mu \leqslant \lambda \iff \begin{pmatrix} \mu_1 \leqslant \lambda_1, \\ \mu_1 + \mu_2 \leqslant \lambda_1 + \lambda_2, \\ \dots \\ \mu_1 + \cdots + \mu_n = \lambda_1 + \cdots + \lambda_n \end{pmatrix} .$$

The simultaneous eigenfunctions P_λ of D_k,

$$D_k \cdot P_\lambda = e_k(\widehat{\lambda}) P_\lambda, \tag{9}$$

normalized by the condition

$$P_\lambda \equiv m_\lambda \mod \{m_\mu\}_{\mu < \lambda} \tag{10}$$

are known as the *Macdonald symmetric polynomials*.

3.3. Let Λ_n be the algebra of symmetric polynomials in n variables. It is clear from (9) that the mapping

$$D \colon \Lambda_n \ni e_k \mapsto D_k$$

extends to an algebra homomorphism such that

$$D(g) \cdot P_\lambda = g(\widehat{\lambda}) P_\lambda, \qquad g \in \Lambda_n. \tag{11}$$

Following Cherednik [1], we now define the *Fourier pairing*

$$\langle g, f \rangle = [D(g) \cdot f](\widehat{0}), \qquad f, g \in \Lambda_n. \tag{12}$$

This is an analog of (2). It is clear that

$$\langle hg, f\rangle = \langle g, D(h)f\rangle.$$

In other words, $D(h) = h^*$, where h is regarded as a multiplication operator and the asterisk indicates the *Fourier transform* of this operator, that is, the adjoint operator with respect to (12). It is also clear from (9) that the pairing (12) takes the normalized eigenfunction

$$N_\lambda = \frac{P_\lambda}{P_\lambda(\widehat{0})} \tag{13}$$

to the δ-function at $\widehat{\lambda}$, namely,

$$\langle g, N_\lambda\rangle = g(\widehat{\lambda}). \tag{14}$$

3.4. Our goal is now to produce an explicit orthogonal basis for the quadratic form (12). As in (4), this basis will consist of Newton interpolation polynomials.

Let $\lhd$ be any total ordering of the set of partitions λ that is compatible with both the ordering of partitions by their size $|\lambda|$ and the dominance ordering for partitions of the same number. Define the interpolation Macdonald polynomials I_μ by the following generalization of (5) and (6):

$$I_\mu \equiv m_\mu \mod \{m_\lambda\}_{\lambda\lhd\mu}, \tag{15}$$

$$I_\mu(\widehat{\lambda}) = 0, \qquad \lambda \lhd \mu. \tag{16}$$

For generic q and t, the existence and uniqueness of such polynomials are clear from their existence and uniqueness for $t = 1$, which is elementary.

3.5. It can be shown (see, e.g., [3, 8, 11, 13]) that the polynomials I_μ do not depend on the choice of the ordering $\lhd$ and satisfy the much stronger *extra vanishing* property

$$I_\mu(\widehat{\lambda}) = 0, \qquad \mu \not\subset \lambda. \tag{17}$$

By the binomial formula (22), this gives the following strengthening of (15):

$$I_\mu \equiv P_\mu \mod \{P_\lambda\}_{\lambda\subset\mu}. \tag{18}$$

However, we do not need the extra vanishing (17) below, and this makes our reasoning applicable in situations in which an analog of (17) is not available.

3.6. Our main result is the following assertion.

Theorem 1. *The polynomials I_μ are orthogonal with respect to the Fourier pairing* (12).

An immediate corollary of this theorem is the following central result of the theory of Macdonald polynomials:

Corollary 1 (Koornwinder [6]). *The Fourier pairing* (12) *is symmetric.*

Koornwinder actually proved an equivalent symmetry, namely, the following label-argument symmetry for the normalized polynomials (13):

$$N_\lambda(\widehat{\mu}) = N_\mu(\widehat{\lambda}).$$

Numerous application of this symmetry, for example, Pieri-type formulas for Macdonald polynomials, can be found in [1, 6, 7].

3.7. The proof of Theorem 1 goes in two steps. First, we claim that

$$\langle I_\mu, I_\lambda\rangle = 0, \qquad \mu \rhd \lambda.$$

Indeed, by (15), (10), and (14), this is equivalent to

$$\langle I_\mu, N_\lambda\rangle = I_\mu(\widehat{\lambda}) = 0, \qquad \mu \rhd \lambda,$$

which holds by (16).

3.8. Now we prove that

$$\langle I_\mu, I_\lambda\rangle = 0, \qquad \mu \lhd \lambda.$$

By (15), this is equivalent to proving that $\langle m_\mu, I_\lambda\rangle = 0$ if $\mu \lhd \lambda$. Since

$$e_\mu \stackrel{\text{def}}{=} e_{\mu_1} \cdots e_{\mu_n} \equiv m_\mu \mod \{m_\nu\}_{\nu<\mu},$$

it suffices to prove that

$$\langle e_\mu, I_\lambda\rangle = 0, \qquad \mu \lhd \lambda.$$

By the definition (12), this is equivalent to the relation

$$[D_\mu \cdot I_\lambda](\widehat{0}) = 0, \qquad D_\mu = D_{\mu_1} \cdots D_{\mu_n}, \tag{19}$$

which will now be established.

3.8. It is a crucial property of the operators D_k that

$$\begin{pmatrix} \lambda_i = \lambda_{i+1}, \\ i \notin S,\ i+1 \in S \end{pmatrix} \implies d_S(\widehat{\lambda}) = 0.$$

Hence,

$$[D_k \cdot f](\widehat{\lambda}) = \sum_{\nu/\lambda = \text{vertical } k\text{-strip}} d_{S(\nu,\lambda)}(\widehat{\lambda}) f(\widehat{\nu}),$$

where $S(\nu, \lambda) = \{i,\ \nu_i > \lambda_i\}$. It follows that

$$[D_\mu \cdot f](\widehat{0}) = \sum_{\nu<\mu} c_{\mu,\nu} f(\widehat{\nu}), \tag{20}$$

with some coefficients $c_{\mu,\nu}$. A similar property can be established in a more general context, e.g., for nonsymmetric Macdonald polynomials [2].

It is clear that (20) together with (16) imply (19), and this completes the proof of Theorem 1.

3.9. Theorem 1 can be sharpened as follows.

Theorem 2. *We have*

$$\langle I_\mu, I_\nu\rangle = \delta_{\mu,\nu} I_\mu(\widehat{\mu}) P_\mu(\widehat{0}) = \delta_{\mu,\nu} c_{\mu,\mu} I_\mu(\widehat{\mu}). \tag{21}$$

In particular, this shows that $P_\mu(\widehat{0}) = c_{\mu,\mu}$, which, after making the number $c_{\mu,\mu}$ explicit, can be seen to be equivalent to a known formula for $P_\mu(\widehat{0})$ (see [6]).

Proof. Arguing as in Sec. 3.7, we see that

$$\langle I_\mu, I_\mu\rangle = \langle I_\mu, P_\mu\rangle = I_\mu(\widehat{\mu}) P_\mu(\widehat{0}).$$

On the other hand, arguing as in Sec. 3.8, we obtain

$$\langle I_\mu, I_\mu\rangle = [D_\mu \cdot I_\mu](\widehat{0}) = c_{\mu,\mu} I_\mu(\widehat{\mu}).$$

3.10. Theorem 2 implies the following Newton interpolation formula:

$$f = \sum_\mu \frac{\langle I_\mu, f\rangle}{\langle I_\mu, I_\mu\rangle} I_\mu, \qquad f \in \Lambda_n.$$

In particular, applying this to N_λ and using (14), we obtain the following expansion (in which we explicitly keep the variables x to stress the label-argument symmetry).

Theorem 3 (Binomial theorem, [9]). *We have*

$$N_\lambda(x) = \sum_\mu \frac{I_\mu(\widehat{\lambda}) I_\mu(x)}{\langle I_\mu, I_\mu\rangle}\,. \tag{22}$$

It follows from (17) that a partition μ can occur in this expansion only if $\mu \subset \lambda$.

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