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On the Branching Points of the Spectrum of an Algebraic Extension of a Uniform Algebra

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Abstract. The branching points of the spectrum of an algebraic extension of a uniform algebra A are algebraically described, and the so-called A-index of branching of such points is introduced and studied. The description of the character of branching in terms of local point derivations on the algebraic extension is obtained.

Key words: Banach algebra, algebraic extension, A-index of branching, point derivation.

Let $A[t]$ be the algebra of all polynomials over a semisimple commutative Banach algebra A with identity element over the field $\mathbb{C}$ of complex numbers. Denote by $A_U[t]$ the set of unital polynomials. Let $f \in A_U[t]$ and let

$$f = t^n + a_1 t^{n-1} + \dots + a_n, \qquad \deg f = n > 1.$$

Denote by B the algebraic extension of A generated by the polynomial f. In other words, $B = A[t]/(f)$, where (f) is the principal ideal in $A[t]$ generated by f.

As is well known, B is a free A-module of rank n, and B can naturally be endowed with the structure of a Banach algebra in such a way that the natural homomorphism $\rho\colon A \to B$ is an isometrically isomorphic embedding of A in B (for details, see [1, 2]). If X is the spectrum (the maximal ideal space) of the algebra A, then the spectrum of B is realized in the form

$$Y = \{(x, \lambda) \in X \times \mathbb{C} \mid \lambda^n + a_1(x)\lambda^{n-1} + \dots + a_n(x) = 0\}.$$

Moreover, the dual mapping $\pi\colon Y \to X$ of the embedding $A \subset B$ is a (generally ramified) covering. Recall that a point $y \in Y$ is said to be *branching* if there is no neighborhood V of this point for which $\pi|V$ is injective.

In this note we give an algebraic description of branching points and introduce and study the so-called A-index of branching of the covering π. In the case of uniform algebras, we describe the character of branching in terms of point derivations, and it turns out to be of importance whether or not the derivation is local (in general, a uniform Banach algebra can have nonlocal continuous derivations, namely, a function vanishing in a neighborhood of a given point can have a nonzero "derivative"). A nonlocal point derivation occurred first in [3] (see also [4, pp. 74 and 310 of the Russian translation]). In [5], a simple way to construct uniform algebras with nonlocal derivations was suggested.

Although the problems treated in the note seem to be natural, the author knows no other publication on this topic especially devoted to *Banach* algebras.

Denote by $\widehat{B} = B/\operatorname{Rad}(B)$ the Gel'fand representation of the algebra B. Note that the algebra B need not be semisimple, but its radical $\operatorname{Rad}(B)$ is nilpotent and thus coincides with the image $\rho(\operatorname{Rad}(f))$ of the radical of the ideal (f). In particular, the Gel'fand representation of the element $\rho(t)$ is a root of every polynomial in $\rho(\operatorname{Rad}(f))$.

By analogy with simple algebraic extensions of fields, we refer to the integer

$$m = \min\{\deg g \mid g \in \operatorname{Rad}(f) \cap A_U[t]\}$$

as the *degree* of the algebraic extension B, and every polynomial g on which this minimum is attained is called a *minimal polynomial*.

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Obviously, $m \leqslant n$. If the discriminant of the polynomial f is a nonzero element of A, then $m = n$. However, the inequality $m < n$ can hold even if f is irreducible over A, and hence the minimal polynomial g does not divide f in general. Recall that the (minimal) number of generators of a finite module is called the *corank* of the module.

Theorem 1. *The degree of an algebraic extension B of a semisimple commutative Banach algebra A is equal to the corank of the A-module $\widehat{B}$.*

In the proof, a well-known algebraic procedure is used that permits one to indicate a unital polynomial of degree k in $\mathrm{Rad}(f)$ for a given family of k generators of the module $\widehat{B}$.

By Theorem 1, A-isomorphic algebraic extensions have equal degrees. We also note that the relation $m = 1$ is equivalent to the condition that the mapping π is homeomorphic.

Let A be a uniform algebra, and let $y_0 = (x_0, \lambda_0)$ be an arbitrary point in Y. Consider a base $(\overline{U_i})$ of closed A-convex neighborhoods of the point x_0 and assign to any i the uniform algebra $A_i = \overline{A|U_i}$ and the algebraic extension $\mathbf{B}_i$ of A_i generated by the polynomial f. The spectrum $\pi^{-1}(\overline{U_i})$ of the algebra $\mathbf{B}_i$ is decomposed into the union of at most n disjoint closed subsets that project onto $\overline{U_i}$. Let $\overline{V_i}$ be one of these subsets, namely, the one containing y_0, let B_i be the corresponding direct summand in $\mathbf{B}_i$ (related to the Shilov idempotent theorem), and let m_i be the degree of the extension B_i. Since the inclusion $\overline{U_j} \subset \overline{U_i}$ implies the inequality $m_j \leqslant m_i$, it follows that (m_i) is a stabilizing sequence.

Definition. The number $m_0 = \min\{m_i\}$ is called the *A-index of branching* of the covering π at the point y_0.

Theorem 2. *Let B be an algebraic extension of a uniform algebra A. Then $y_0 = (x_0, \lambda_0)$ is a branching point for π if and only if $m_0 > 1$. Moreover, $s_0 \leqslant m_0 \leqslant \mu_0$, where*

$$s_0 = \min_i \{\max \operatorname{card}(\pi^{-1}(x) \cap \overline{V_i}) \mid x \in \overline{U_i}\}$$

is the topological index of branching of the covering π at the point y_0 [6] *and μ_0 is the multiplicity of the root λ_0 of the scalar equation*

$$\lambda^n + a_1(x_0)\lambda^{n-1} + \cdots + a_n(x_0) = 0;$$

finally, the index m_0 is invariant with respect to the A-automorphisms of the algebra B.

The proof is based on the reducibility of the polynomial f (for $\mu_0 < n$) over one of the algebras A_i and on Theorem 1.

Example. Let $A(\Delta)$ be the standard algebra of continuous functions on the unit disk Δ that are holomorphic inside Δ. Consider the uniform algebra $A = \{a(z) \in A(\Delta) \mid a'(0) = 0\}$ and its algebraic extension B generated by the following irreducible polynomial over A:

$$f = t^4 - 6z^2t^2 + 8z^3t - 3z^4 = (t - z)^3(t + 3z).$$

The point $y_0 = (0, 0)$ is the only branching point of the spectrum of the algebra B. It is clear that $s_0 = 2$ and $\mu_0 = 4$. The polynomial $g = t^3 - 7z^2t + 6z^3$ (or $g_1 = g + z^3t^2 + 2z^4t - 3z^5$) is minimal. Since g is a minimal polynomial for any algebra B_i, it follows that the A-index of the point y_0 is equal to 3. However, the $A(\Delta)$-index of the same point is equal to 2.

If $\overline{U_j} \subset \overline{U_i}$, then the restriction homomorphism $\sigma_{ij} \colon A_i \to A_j$ naturally induces homomorphisms $\tau_{ij} \colon B_i \to B_j$ and $\widehat{\tau}_{ij} \colon \widehat{B}_i \to \widehat{B}_j$.

Let A_0, B_0, and $\widehat{B}_0$ be the direct limits of the systems (A_i, σ_{ij}), (B_i, τ_{ij}), and $(\widehat{B}_i, \widehat{\tau}_{ij})$, respectively. It is clear that A_0 and $\widehat{B}_0$ are local algebras, and $\widehat{B}_0$ is a finite A_0-module. By Theorem 1, the index m_0 is equal to the corank of the module $\widehat{B}_0$. Therefore, it follows from the Nakayama lemma that $m_0 = \dim \widehat{B}_0 / M_0 \widehat{B}_0$, where M_0 is the maximal ideal of the algebra A_0.

The A-index of a branching point is also characterized by point derivations of special form. Let us define a functional D_i^k on $A_i[t]$ as follows: the number $D_i^k(R)$ is equal to the value of the formal kth derivative of the polynomial $R \in A_i[t]$ with respect to t at (x_0, λ_0).

It can be shown that, for $k \leqslant \mu_0 - 1$, the functional D_i^k is a point derivation of order k on B_i, and the kernel of this functional contains A_i. Since D_i^k are compatible with the homomorphisms τ_{ij}, it follows that the derivations $D^k \stackrel{\text{def}}{=} \lim D_i^k$ are defined on B_0.

Denote by N_0 the preimage in B_0 of the zero germ of the algebra $\widehat{B}_0$.

Theorem 3. *Let B be an algebraic extension of a uniform algebra A, and let $y_0 = (x_0, \lambda_0)$ be a point of the spectrum of B such that $\mu_0 > 1$. Then $m_0 = 1$ if $\ker D^1$ does not contain N_0 and*

$$m_0 = 1 + \max\{l \mid N_0 \subset \ker D^k,\ 1 \leqslant k \leq l\}$$

if $N_0 \subset \ker D^1$.

The main point of the proof is as follows: if $b \in N_0$, then there is an algebra $\widehat{B}_i$ in which $\hat{b} = \sum_{k=0}^{m_0-1} a_k \hat{t}^k$ and $a_k(x_0) = 0$ for any k.

The functional D^k defines a continuous point derivation of order k on the algebra B_i at the point y_0. Moreover, if the derivation D^1 is nonlocal on some of these algebras, then y_0 is a simple point; however, if there is a positive integer m_0 not exceeding $\mu_0 - 1$ such that the derivations $D^1, \dots, D^{m_0-1}$ are local on each of the algebras B_i and if the derivation D^{m_0} is nonlocal on some algebra B_j, then y_0 is a branching point of A-index m_0.

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