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Isoenergy Spectral Problem for Multidimensional Difference Operators*

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Abstract. In this paper, the direct and inverse isoenergy spectral problems are solved for a class of multidimensional periodic difference operators. It is proved that the inverse spectral problem is solvable in terms of theta functions of curves added to the spectral variety under compactification, and multidimensional analogs of the Veselov–Novikov relations are found.

Key words: multidimensional scattering problem, Bloch function, spectral data.

Introduction

The spectral problem for the multidimensional Schrödinger equation with spectral data taken from a single fixed energy level has a long history associated with the names of P. Newton, P. C. Sabatier, B. M. Levitan, and others. (For example, see the book by Chadan and Sabatier [1], containing a remarkable historical survey and an extensive bibliography.)

The algebro-geometric analysis of this problem was initiated in the well-known paper by Dubrovin, Krichever, and Novikov [2], in which a class of two-dimensional Schrödinger operators that are finite-gap with respect to a single energy level was introduced. The eigenfunction of such an operator is defined on an algebraic curve and is meromorphic outside some isolated points at which it has exponential singularities (a so-called Baker–Akhiezer function). In subsequent papers [3–5], Veselov and Novikov found sufficient conditions on the spectral curve and the divisor of poles ψ under which the related operator is purely potential.

The investigation of difference operators with similar properties was started by Krichever in [6, 7], where a construction was suggested for the inverse problem at a single energy level for operators on the two-dimensional lattice. The direct problem for operators on the two-dimensional lattice was investigated recently by Oblomkov in [8], where the answers to some questions posed in [7] were also given. We also note the paper [9], in which the inverse spectral problem was investigated for a class of two-dimensional difference operators with nonzero diagonal.

Kappeler and Bättig [10, 11] constructed the compactification of the spectral variety for the multidimensional Schrödinger difference operator at all energy levels and studied the question concerning operators with same spectral varieties was investigated. (The continuous version of these results was considered in [12, 13].)

In recent papers by Novikov and Dynnikov [14, 15], some examples of discrete operators integrable in the style of soliton theory on a multidimensional simplicial lattice were found. These papers stimulated further development of investigations in this direction [16], to which the present paper also belongs.

We investigate the direct and inverse spectral problems at a zero energy level for a class of operators L acting according to the following rule on the space V of functions defined on the lattice $\Gamma = \mathbb{Z}^N$:

$$(L\psi)(n) = \sum_{\zeta=(\pm 1,\dots,\pm 1)} a(n;\zeta)\psi(n+\zeta).$$

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It is assumed that the coefficients of the operator satisfy the selfadjointness and periodicity conditions $a(n;\zeta)=a(n+\zeta;-\zeta)$ and $a(n+T_ie_i;\zeta)=a(n;\zeta)$, $T_i\in\mathbb{Z}$, $i=1,\dots,N$.

We note that the diagonal elements of the operator L are zero. In the one-dimensional case, the algebro-geometric analysis of the spectral theory of "diagonal-free" operators $(L\psi)_n=c_n\psi_{n-1}+c_{n+1}\psi_{n+1}$ was carried out by Novikov (see [17]). In this situation, the spectral data were a curve with a symmetry and a divisor on this curve. In [18], the relationship between these operators and the Heisenberg chain was found and the distinction between the even and odd cases was revealed.

In the case we deal with, the spectral data are sets of hypersurfaces (spectral varieties) equipped with a set of divisors satisfying some conditions. A part of these conditions generalizes the Veselov–Novikov relations [3, 4], and the other part ensures the existence of multidimensional analogs of the Baker–Akhiezer function [6]. The spectral data possess a remarkable property, which we refer to as the existence of inductive structure. Namely, the components added under the compactification of the spectral variety together with the restrictions of the set of divisors to these components are the spectral data for the restriction of the original difference operator to some sublattice. This permits reducing the problem of reconstruction of an operator from the spectral data to the one-dimensional case and deriving formulas expressed in terms of theta functions. We shall describe these spectral data and the corresponding relations. The affine part of the spectral variety of an operator L is the hypersurface $Y=Y_L=\{\lambda\in(\mathbb{C}^*)^N\mid V(\lambda)\cap\ker L\neq 0\}$, where $V(\lambda)=\{\psi\in V\mid \psi(n+T_ie_i)=\lambda_i\psi(n)\}$ is the space of Bloch functions. For simplicity, we shall assume that all T_i's are odd. (This constraint is absent in the main part of this paper.) In this case, the hypersurface Y_L for the generic operator L is irreducible and invariant with respect to the action of the group $\mathbb{Z}_2^N$ that changes the signs of the components of the vector λ. We denote by $X=X_L$ the quotient of Y_L by this group and call it the *modified spectral variety*. The embedding $(\mathbb{C}^*)^N\subset(\mathbb{CP}^1)^N$ gives the natural compactification of X. The varieties added under the compactification have a similar structure; namely, $X^\gamma\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]=X_{L^\gamma\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]}$, where the operator $L^\gamma\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]$, $k\in\mathbb{Z}$, $1\leqslant i\leqslant N$, $\gamma=\pm1$, is the "reduction" of L to the $(N-1)$–dimensional sublattice $\Gamma\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]=\{\xi\in\Gamma\mid\xi_i=k\}$. (For precise definitions, see the main body of the text.) The involution $\sigma\colon\lambda\to\lambda^{-1}$ acts on X in a natural way. It interchanges the "infinities" $X^\gamma\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]$ in a special manner.

To the generic point $\lambda\in Y$, there corresponds a vector $\psi(\lambda,n)\in\ker L\cap V(\lambda)$, which is unique up to the multiplication by a constant. In this case, $\psi(\lambda,2n+\varepsilon)/\psi(\lambda,\varepsilon)$, $\varepsilon\in\{0,1\}^N$, is a well-defined function on X. Thus, the vector $\psi(\lambda,2n+\varepsilon)$, $n\in\Gamma$, normalized by the condition $\psi(\lambda,\varepsilon)\equiv1$ is well defined on X. To simplify the statements, we set $\varepsilon=0$. Then there is an effective divisor $\mathscr{D}_0$ without irreducible components at infinity (divisors with this property will be termed finite) that satisfies the conditions

$$\mathscr{D}_0+\sigma(\mathscr{D}_0)+\sum_{s=1}^{N}\sum_{p=0}^{(T_s-1)/2}\left(X^1\left[\begin{smallmatrix}s\\2p+1\end{smallmatrix}\right]-X^{-1}\left[\begin{smallmatrix}s\\2p+1\end{smallmatrix}\right]\right)-\sum_{s=1}^{N}X^{+1}\left[\begin{smallmatrix}s\\0\end{smallmatrix}\right]\sim\mathscr{K}_X, \tag{1}$$

$$\mathscr{D}_{2n}:=(\psi(\,\cdot\,,2n))-\mathscr{D}_0+Q\left[\begin{smallmatrix}0\\2n\end{smallmatrix}\right]-Q\left[\begin{smallmatrix}2n\\0\end{smallmatrix}\right]>0\qquad\forall n\in\Gamma, \tag{2}$$

$$X^{\pm1}\left[\begin{smallmatrix}r\\p+T_r\end{smallmatrix}\right]=X^{\pm1}\left[\begin{smallmatrix}r\\p\end{smallmatrix}\right],\quad Q\left[\begin{smallmatrix}2n+2T_re_r\\2n\end{smallmatrix}\right]\sim0,\qquad r=1,\dots,N. \tag{3}$$

Here $\mathscr{K}_X$ is the canonical divisor and $Q\left[\begin{smallmatrix}2n^1\\2n^0\end{smallmatrix}\right]=\sum_{s=1}^{N}\sum_{p=n_s^0}^{n_s^1}\left(X^1\left[\begin{smallmatrix}s\\2p+1\end{smallmatrix}\right]-X^{-1}\left[\begin{smallmatrix}s\\2p+1\end{smallmatrix}\right]\right)$, where the summation is assumed to be performed only if $n_s^0\leqslant n_s^1$. Relation (1) is the multidimensional discrete generalization of the Veselov–Novikov relation, formula (2) generalizes the standard requirements for the Baker–Akhiezer function (the presence of a fixed singularity in the finite part and the existence of poles (zeros) at infinity with multiplicities depending on n), and condition (3) ensures the quasiperiodicity of the function ψ. For $N=2$, formula (2) was written out by Krichever in [6], and relation (1) was given in our paper [8].

The variety X, together with the divisor $\mathscr{D}_0$, is the spectral data for the problem. The fact that $\{X^\gamma\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right],\mathscr{D}\cap X^\gamma\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]\}$ is the set of spectral data in the direct problem for the operators $L^\gamma\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]$ is the key point here. It permits reducing the solution of the inverse spectral problem for an N-dimensional operator to the solution of spectral problems for two-dimensional operators. Finally, in the two-dimensional case, this problem was solved in [7] (also see [8]), and there are explicit θ-functional formulas in this case.

On the space of operators L, the gauge group acts according to the formula $a'(n;\zeta) = g(n)g(n+\zeta)a(n;\zeta)$, where g is an arbitrary periodic function on the lattice. It can be seen easily that this group preserves the spectral data corresponding to the zero energy level. The main result in this paper is the assertion that the spectral data $\{X_L, \mathscr{D}_0\}$ uniquely define the gauge class of the operator L. The operator L can be reconstructed in terms of the theta functions of some curves lying at "infinity" in X_L. For example, if $N = 3$, then the inverse spectral problem can be solved as follows. We find $\mathscr{D}_{2n}$ from $\mathscr{D}_0$ using formula (2) and then reconstruct the two-dimensional operator $L^\gamma\left[{i \atop 2n_i}\right]$ from the spectral data $\{X^\gamma\left[{i \atop 2n_i-\gamma}\right], \mathscr{D}_{2n} \cap X^\gamma\left[{i \atop 2n_i-\gamma}\right]\}$ following the scheme in [6]. The expressions $L^\gamma\left[{i \atop 2s}\right]$, $i, s \in \mathbb{Z}$, $\gamma = \pm 1$, give all coefficients of the operator L.

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1. Geometry of the N-Dimensional Cubic Lattice. Reduced Operators and their Spectral Varieties

Here and henceforth, we use the following convention: let $\mu = (\mu_1, \dots, \mu_N)$, $k = (k_1, \dots, k_N) \in \mathbb{Z}^N$, l$=(l_1, \dots, l_r) \in \mathbb{Z}^r$, and $l_i \leqslant N$; then $\mu_l = (\mu_{l_1}, \dots, \mu_{l_r})$ and $\mu^k = \mu_1^{k_1} \cdots \mu_N^{k_N}$.

To describe the algebraic structure of spectral data for an operator L, we need the related operators $L_j^\gamma\left[{i \atop k}\right]$ acting in the space V^M of functions on the lattice $\Gamma^M = \mathbb{Z}^N$, $M \leqslant N$. From now on, $i \in \mathbb{Z}^{N-M}$ and $j \in \mathbb{Z}^M$ define a decomposition of the set $\{1, \dots, N\}$ into two disjoint subsets by the formula $\{i_1, \dots, i_{N-M}\} \cup \{j_1, \dots, j_M\} = \{1, \dots, N\}$, $k \in \mathbb{Z}^{N-M}$, $\gamma \in \{+1, -1\}^{N-M}$. The action of $L_j^\gamma\left[{i \atop k}\right]$ is determined by the formula

$$(L_j^\gamma\left[{i \atop k}\right]\psi)(m) = \sum_{\zeta, \zeta_i = \gamma} a(n;\zeta)\psi\biggl(m + \sum_{r=1}^{M} \zeta_{j_r} e_r\biggr),$$

where $\zeta = (\pm 1, \dots, \pm 1)$, $m \in \Gamma^M = \mathbb{Z}^M$, $n_i = k$, and $n_j = m$. The vector j specifies the order of the variables, and the subscript j will be omitted whenever this order is unessential. Note that the operator L is the special case of the operator $L_j^\gamma\left[{i \atop k}\right]$ for $M = N$, $\gamma, i = \varnothing$, and $j = (1, \dots, N)$.

We elucidate the informal meaning of the operator $L^\gamma\left[{i \atop k}\right]$. Let $M = N-1$, $\gamma = \pm 1$, and $i = N$. Then $(L\psi)(n) = (L^{+1}\left[{N \atop n_N}\right]\varphi^{+1})(\tilde{n}) + (L^{-1}\left[{N \atop n_N}\right]\varphi^{-1})(\tilde{n})$, where $\tilde{n} = (n_1, \dots, n_{N-1})$ and $\varphi^\gamma(m) = \psi(m_1, \dots, m_{N-1}, n_N + \gamma)$. This means that the operator L is represented "locally" as the sum $L^{+1}\left[{N \atop n_N}\right] \oplus L^{-1}\left[{N \atop n_N}\right]$. The operator $L^\gamma\left[{i \atop k}\right]$ can be regarded as the reduction of L to the sublattice $\Gamma\left[{i \atop k}\right] = \{\xi \in \Gamma^N | \xi_i = k\}$ in the direction γ. The points $n + \zeta$, $\zeta = (\pm 1, \dots, \pm 1)$ are the vertices of the N-dimensional cube. Furthermore, $L^\gamma\left[{i \atop k}\right]$ can be interpreted as an operator acting on the space $\{\psi \in V^N \mid \operatorname{supp}\psi \subset \Gamma\left[{i \atop k+\gamma}\right]\} \cong V^M$. Following the ideology of [14], we shall assume that the edges joining each vertex n to the vertices $n + \zeta$ enter this vertex and are marked by the numbers $a(n;\zeta)$. To calculate $(L^\gamma\left[{i \atop k}\right]\psi)(n_j)$, it is necessary to take the corresponding face of the cube, i.e., the vertices $n + \zeta$ such that $n_i = k$ and $\zeta_i = \gamma$, and perform the summation of the values of ψ over the vertices of this face with the weights $a(n;\xi)$ assigned to the corresponding edges.

We say that $V^M(\lambda) = \{\psi \in V^M \mid \psi(n + T_r e_r) = \lambda_r \psi(n),\ r = 1, \dots, M\}$ is the space of Bloch functions with Floquet multipliers λ. Consider the restriction of $L^\gamma\left[{i \atop k}\right]$ to the subspace $V^M(\lambda)$ and denote it by the symbol $L^\gamma\left[{i \atop k}\right]_\lambda$. The set of points $Y^{\mathrm{aff},\gamma}\left[{i \atop k}\right] = \{\lambda \in (\mathbb{C}^*)^M \mid \ker L^\gamma\left[{i \atop k}\right]_\lambda \neq 0\}$ will be called the affine part of the spectral variety of the operator $L^\gamma\left[{i \atop k}\right]$. Choosing the compactification

$(\mathbb{C}^*)^M \subset (\mathbb{CP}^1)^M$, which is natural in this case, we obtain the compactification $Y^\gamma\left[{i \atop k}\right]$ of the variety $Y^{\mathrm{aff},\gamma}\left[{i \atop k}\right]$. The set $Y^\gamma\left[{i \atop k}\right] \setminus Y^{\mathrm{aff},\gamma}\left[{i \atop k}\right]$ will be denoted by the symbol $Y^{\infty,\gamma}\left[{i \atop k}\right]$.

Generally speaking, the resulting variety $Y^\gamma\left[{i \atop k}\right]$ is reducible, and to describe its irreducible components, we need the following construction. The space V^M naturally decomposes into the sum $V^M = \bigoplus_\varepsilon V_\varepsilon^M$, $V_\varepsilon^M = \{\psi \in V^M \mid \operatorname{supp}\psi \subset \Gamma_\varepsilon^M\}$, where $\varepsilon \in \mathbb{Z}_2^M$ and $\Gamma_\varepsilon^M = \{\xi \in \Gamma^M \mid \xi = \varepsilon \pmod 2\}$. This decomposition cannot be transferred to the space of Bloch functions, since the shift by a vector having at least one odd component mixes lattice sites of different types. But this complication can be overcome using the construction below. Let $T'_r = T_r/2$ if T_r is even let $T'_r = T_r$ otherwise. We introduce the space $W_\varepsilon^M(\mu) = \{\psi \in V_\varepsilon^M \mid \psi(m + 2T'_{j_r} e_r) = \mu_r \psi(m),\ r = 1, \dots, M\}$ and the maps $\Phi\colon \mathbb{C}^M \to \mathbb{C}^M$ and $\Phi(\lambda) = \mu$, where $\mu_r = \lambda_r^2$ if T_{j_r} is odd and $\mu_r = \lambda_r$ otherwise. The even and odd periods will be denoted by $l \in \mathbb{Z}^S$, $h \in \mathbb{Z}^{M-S}$, $S \leqslant M$, $\{l_1, \dots, l_S\} \cup \{h_1, \dots, h_{M-S}\} = \{1, \dots, M\}$. Let $r \in \{l_1, \dots, l_S\}$ if T_{j_r} is even and $r \in \{h_1, \dots, h_{M-S}\}$ otherwise. We define a map $F_\varepsilon^\lambda\colon W_\varepsilon^M(\mu) \to V^M(\lambda)$, where $\mu = \Phi(\lambda)$, in the following way: $F_\varepsilon^\lambda(\varphi) = \psi$, and if $m_l = \varepsilon_l \pmod 2$, then $\psi(m) = \varphi(m - \sum_{r=1}^M \rho_r T_r e_r)\mu^\rho$, where $\rho_r = 1$ if T_{j_r} is odd and $m_r \neq \varepsilon_r \pmod 2$ and $\rho_r = 0$ otherwise, and if $m_l \neq \varepsilon_l \pmod 2$, then $\psi(m) = 0$.

The symbol $L_\varepsilon^\gamma\left[{i \atop k}\right]_\mu$ will denote the restriction of the operator $L^\gamma\left[{i \atop k}\right]$ to the space $W_{1-\varepsilon}^M(\mu)$, $L_\varepsilon^\gamma\left[{i \atop k}\right]_\mu\colon W_{1-\varepsilon}^M(\mu) \to W_\varepsilon^M(\mu)$. It can be seen easily that F_ε^λ is a well-defined injective map. In this case, we have $F_{1-\varepsilon}^\lambda(\ker L_\varepsilon^\gamma\left[{i \atop k}\right]_\mu) \subset \ker L^\gamma\left[{i \atop k}\right]_\lambda$, $\mu = \Phi(\lambda)$. Let $X_\varepsilon^{\mathrm{aff},\gamma}\left[{i \atop k}\right] = \{\mu \in (\mathbb{C}^*)^M \mid \ker L_\varepsilon^\gamma\left[{i \atop k}\right]_\mu \neq 0\}$, $X_\varepsilon^\gamma\left[{i \atop k}\right] = \overline{X_\varepsilon^{\mathrm{aff},\gamma}\left[{i \atop k}\right]} \subset (\mathbb{CP}^1)^M$, and $Y_\varepsilon^\gamma\left[{i \atop k}\right] = \Phi^{-1}(X_\varepsilon^\gamma\left[{i \atop k}\right]) \subset Y^\gamma\left[{i \atop k}\right]$. The infinite part of $X_\varepsilon^\gamma\left[{i \atop k}\right]$ added under the compactification will be denoted by $X_\varepsilon^{\infty,\gamma}\left[{i \atop k}\right] = X_\varepsilon^\gamma\left[{i \atop k}\right] \setminus X_\varepsilon^{\mathrm{aff},\gamma}\left[{i \atop k}\right]$. By the symbol $V_\varepsilon^M(\lambda) \subset V^M(\lambda)$, we denote $F_\varepsilon^\lambda(W_\varepsilon^M(\mu))$. Since the shift by the vector $T_{j_{h_r}} e_{h_r}$, $r = 1, \dots, M-S$, transforms $W_{1-\varepsilon}(\mu)$ to the space $W_{1-\varepsilon'}(\mu)$, $\varepsilon' = \varepsilon + e_{h_r} \pmod 2$ and preserves the invariance of the operator $L^\gamma\left[{i \atop k}\right]$, we have $V_{1-\varepsilon}(\lambda) = V_{1-\varepsilon'}(\lambda)$ and $F_{1-\varepsilon}^\lambda(\ker L_\varepsilon^\gamma\left[{i \atop k}\right]_\mu) = F_{1-\varepsilon'}^\lambda(\ker L_{\varepsilon'}^\gamma\left[{i \atop k}\right]_\mu)$. Consequently, $Y_\varepsilon^\gamma\left[{i \atop k}\right] = Y_{\varepsilon'}^\gamma\left[{i \atop k}\right]$ if $\varepsilon_l = \varepsilon'_l$. Dimensional calculations permits showing that the decomposition $V(\lambda) = \bigoplus_{\varepsilon, \varepsilon_h = 0} V_\varepsilon(\lambda)$ holds. Since $F_{1-\varepsilon}^\lambda(\ker L_\varepsilon^\gamma\left[{i \atop k}\right]_\mu) = V_\varepsilon^M(\lambda) \cap \ker L^\gamma\left[{i \atop k}\right]$, we have $Y^\gamma\left[{i \atop k}\right] = \bigcup_{\varepsilon, \varepsilon_h = 0} Y_\varepsilon^\gamma\left[{i \atop k}\right]$. Let $X^\gamma\left[{i \atop k}\right] = \Phi(Y^\gamma\left[{i \atop k}\right]) = \bigcup_{\varepsilon, \varepsilon_h = 0} X_\varepsilon^\gamma\left[{i \atop k}\right]$. The variety $X^\gamma\left[{i \atop k}\right]$ ($X_\varepsilon^\gamma\left[{i \atop k}\right]$) will be called the *modified spectral variety of the operator* $L^\gamma\left[{i \atop k}\right]$ ($L_\varepsilon^\gamma\left[{i \atop k}\right]$).

The above construction has the simplest form if all T_{j_r} are even. In this situation, $\mu = \lambda$, $W_\varepsilon(\mu) = V_\varepsilon(\lambda)$, $V(\lambda) = \bigoplus_\varepsilon V_\varepsilon(\lambda)$, and $X_\varepsilon^\gamma\left[{i \atop k}\right] = Y_\varepsilon^\gamma\left[{i \atop k}\right]$, and the problem of finding $\ker L^\gamma\left[{i \atop k}\right]_\lambda$ is equivalent to finding $\ker L_\varepsilon^\gamma\left[{i \atop k}\right]_\lambda$ for all ε. As to the generic period T_j, in this case the above construction is the reduction to the even case. Namely, we regard $L^\gamma\left[{i \atop k}\right]$ as a $2T'_j$-periodic operator rather than a T_j-periodic operator. Accordingly, the Floquet multipliers λ are replaced by μ. However, one should bear in mind that the coefficients of the operator $L^\gamma\left[{i \atop k}\right]$ are in fact T_j-periodic, whence it follows that

$$X_\varepsilon^\gamma\left[{i \atop k}\right] = X_{\varepsilon'}^\gamma\left[{i \atop k}\right] = X_\varepsilon^\gamma\left[{i \atop k + T_{i_r} e_r}\right], \qquad r = 1, \dots, N - M, \tag{4}$$

where $\varepsilon_l = \varepsilon'_l$. The necessity of this reduction is motivated by the fact that $\det L^\gamma\left[{i \atop k}\right]_\lambda$ is a rational function of $\lambda_{h_r}^2$, $r = 1, \dots, M-S$, and the transition from $Y^\gamma\left[{i \atop k}\right]$ to $X^\gamma\left[{i \atop k}\right]$ corresponds to the factorization of the variety $Y^\gamma\left[{i \atop k}\right]$ with respect to the involutions $\lambda_{h_r} \to -\lambda_{h_r}$, $r = 1, \dots, M-S$.

The Zariski open set of operators L such that $X_\varepsilon^\gamma\left[{i \atop k}\right]$ are irreducible for all ε will be denoted by the symbol $U_{ir}^\gamma\left[{i \atop k}\right]$. The nonemptiness of this set will be proved in Lemma 4.

2. Spectral Data. Inductive Structure

The existence of the inductive structure means the possibility of reducing the study of one operator L acting in dimension N to the study of a large number of operators $L_j^\gamma\left[{i \atop k}\right]$, which however already act in dimension $M < N$.

Let us take a basis $\{e^M_{\varepsilon,j}(\mu)\}$ of $W^M_\varepsilon(\mu)$ in which the matrix of $L^\gamma\left[{i \atop k}\right]$ has a block structure reflecting the inductive structure of the operator $L^\gamma\left[{i \atop k}\right]$. Let $0 \leqslant m'_r < T'_{j_r}$ and $a = 1 + \sum_{b=1}^M m'_b \prod_{r=b+1}^M T'_{j_r}$. Then $e^M_{\varepsilon,j,a}(\mu)$ is uniquely determined by the conditions $e_{\varepsilon,j,a}(\mu) \in W^M_\varepsilon(\mu)$ and $e_{\varepsilon,j,a}(\mu, m) = \prod_{b=1}^M \delta^{m_b}_{\varepsilon_b+2m'_b}$ for $0 \leqslant m - \varepsilon < 2T'_j$. The matrix of $L^\gamma_\varepsilon\left[{i \atop k}\right]_\mu$ written in the bases $\{e_{1-\varepsilon,j}(\mu)\}$ and $\{e_{\varepsilon,j}(\mu)\}$ will be denoted by the symbol $\mathbf{M}^\gamma_{\varepsilon,j}\left[{i \atop k}\right](\mu)$. It has a block bidiagonal structure with blocks of the form $\mathbf{M}^{\gamma,\pm 1}_{\tilde\varepsilon,\tilde j}\left[{i,t \atop k,u}\right]$; namely,

$$\mathbf{M}^\gamma_{\varepsilon,j}\begin{bmatrix} i \\ k \end{bmatrix} = \begin{vmatrix} \mathbf{M}^{\gamma'}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,0}\right] & 0 & \dots & \mathbf{M}^{\gamma''}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,0}\right]\mu_1^{-1} \\ \mathbf{M}^{\gamma''}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,2}\right] & \mathbf{M}^{\gamma'}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,2}\right] & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{M}^{\gamma'}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,2(T'_{j_1}-1)}\right] \end{vmatrix}$$

for $\varepsilon_{j_1} = 1$ and

$$\mathbf{M}^\gamma_{\varepsilon,j}\begin{bmatrix} i \\ k \end{bmatrix} = \begin{vmatrix} \mathbf{M}^{\gamma''}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,1}\right] & \mathbf{M}^{\gamma'}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,1}\right] & \dots & 0 \\ 0 & \mathbf{M}^{\gamma''}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,3}\right] & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}^{\gamma'}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,2T'_{j_1}-1}\right]\mu_1 & 0 & \dots & \mathbf{M}^{\gamma''}_{\tilde\varepsilon,\tilde j}\left[{i,j_1 \atop k,2T'_{j_1}-1}\right] \end{vmatrix}$$

for $\varepsilon_{j_1} = 0$. Here the notation $\tilde j = (j_2, \dots, j_M)$, $\tilde\varepsilon = (\varepsilon_2, \dots, \varepsilon_M)$, $\gamma' = (\gamma, 1)$, and $\gamma'' = (\gamma, -1)$ has been used. Induction on M permits showing that

$$(\mathbf{M}^\gamma_{\varepsilon,j}\left[{i \atop k}\right](\mu))^t = \mathbf{M}^{-\gamma}_{1-\varepsilon,j}\left[{i \atop k+\gamma}\right](\mu^{-1}), \tag{5}$$

where $\mu^{-1} = (\mu_1^{-1}, \dots, \mu_M^{-1})$ and t symbolizes transposition.

The equation $R^\gamma_{\varepsilon,j}\left[{i \atop k}\right](\mu) = \det(M^\gamma_{\varepsilon,j}\left[{i \atop k}\right](\mu)) = 0$ defines the variety $X^\gamma_\varepsilon\left[{i \atop k}\right]$. It can be seen easily that $R^\gamma_{\varepsilon,j}\left[{i \atop k}\right](\mu) = CR^\gamma_{\varepsilon,\tau(j)}\left[{i \atop k}\right](\tau(\mu))$, where $C \neq 0$ is a constant, $\tau \in S_M$ is a permutation, $\tau(j) = (j_{\tau(1)}, \dots, j_{\tau(M)})$, and $\tau(\mu) = (\mu_{\tau(1)}, \dots, \mu_{\tau(M)})$. Therefore, we shall omit j in the notation $R^\gamma_{\varepsilon,j}\left[{i \atop k}\right]$. The form of the matrix $M^\gamma_{\varepsilon,j}\left[{i \atop k}\right]$ implies that $R^\gamma_{\varepsilon,j}\left[{i \atop k}\right]$ is a polynomial in $\mu_1^{(1-2\varepsilon_{j_1})}$ of degree $\prod_{t=2}^M T'_{j_t}$, and it also follows from the preceding argument that $R^\gamma_\varepsilon\left[{i \atop k}\right]$ is a polynomial in $\mu_r^{(1-2\varepsilon_{j_r})}$ of degree $(T'_{j_r})^{-1}\prod_{p=1}^M T'_{j_p}$, $r = 1, \dots, M$. Summarizing the above observations, we arrive at the lemma below.

Lemma 1. *The following assertions hold for the generic operator* L: $R^\gamma_\varepsilon\left[{i \atop k}\right](\mu)$ *is a polynomial in* $\mu_r^{(1-2\varepsilon_{j_r})}$ *and* $\deg_{\mu_r^{(1-2\varepsilon_{j_r})}} R^\gamma_\varepsilon\left[{i \atop k}\right] = (T'_{j_r})^{-1}\prod_{t=1}^M T'_{j_t}$.

$$R^\gamma_\varepsilon\left[{i \atop k}\right] = R^\gamma_{\varepsilon'}\left[{i \atop k+T'_{i_r}}\right],$$

where $r = 1, \dots, N - M$, $\varepsilon_l = \varepsilon'_l$.

$$R^\gamma_\varepsilon\left[{i \atop k}\right](\mu) = R^{-\gamma}_{1-\varepsilon}\left[{i \atop k+\gamma}\right](\mu^{-1}). \tag{6}$$

$$R^\gamma_\varepsilon\left[{i \atop k}\right](\mu)\mu_p^{T'_{j_p}\delta^w_{2\varepsilon_p-1}}\Big|_{\mu_p^{-w}=0} = \prod_{r=0}^{T'_{j_p}-1} R^{\gamma'}_{\tilde\varepsilon}\left[{i,j_r \atop k,1-\varepsilon_p+2r}\right](\tilde\mu), \tag{7}$$

where $p = 1, \dots, M$, $w = \pm 1$, $\gamma' = (\gamma, w)$, $\tilde\mu = (\mu_1, \dots, \mu_{p-1}, \mu_{p+1}, \dots, \mu_M)$, *and* $\tilde\varepsilon = (\varepsilon_1, \dots, \varepsilon_{p-1}, \varepsilon_{p+1}, \dots, \varepsilon_M)$.

Formula (7) is an immediate consequence of the block bidiagonal structure of the matrix $\mathbf{M}_\varepsilon^\gamma\left[{i \atop k}\right]$.

It follows from relation (6) that the natural involution σ on $(\mathbb{CP}^1)^N$, $\sigma(\mu)=\mu^{-1}$, acts on $X_\varepsilon^\gamma\left[{t \atop u}\right]$ according to the following rule:

$$\sigma(X_\varepsilon^\gamma\left[{i \atop k}\right]) = X_{1-\varepsilon}^{-\gamma}\left[{i \atop k+\gamma}\right]. \tag{8}$$

The existence of this involution is a consequence of the selfadjointness of the operator $L^\gamma\left[{i \atop k}\right]$.

The symbol $(\mathbb{CP}^1)^M_\mu$ will be used to denote the variety $(\mathbb{CP}^1)^M$ with a fixed set of coordinates μ in which μ_r is the coordinate on the rth variety $\mathbb{CP}^1$. We now state the definition of a set of varieties which is compatible with the group of periods T. The set $\{X_\varepsilon \subset (\mathbb{CP}^1)^N_\mu\}$, where $X_\varepsilon \subset (\mathbb{CP}^1)^N$ are irreducible smooth hypersurfaces, is called a *set of varieties compatible with the group of periods T* if

- X_ε is a variety of degree $((T_1')^{-1},\dots,(T_N')^{-1})\prod_{u=1}^N T_u'$,
- $X_\varepsilon \cap \{\mu_{i_1}^{-\gamma_1}=0,\dots,\mu_{i_{N-M}}^{-\gamma_{N-M}}=0\} = \bigcup_{0\leqslant k<T_i'} X_{\varepsilon_j}^\gamma\left[{i \atop 2k+\varepsilon_i}\right]$, where $X_{\varepsilon_j}^\gamma\left[{i \atop \cdot}\right]$ is an irreducible smooth variety of degree $((T_{j_1}')^{-1},\dots,(T_{j_M}')^{-1})\prod_{u=1}^{N-M} T_{j_u}'$,
- relations (4) and (8) hold.

We use the symbol $U_{sm}^\gamma\left[{i \atop k}\right]$ to denote the set of operators L such that $X_\varepsilon^\gamma\left[{i \atop k}\right]$ is nonsingular for all ε. This set is Zariski open and, as will be shown in Lemma 4, nonempty. Lemma 1 implies that, for $L\in\bigcap_{i,k,\gamma} U_{sm}^\gamma\left[{i \atop k}\right]\cap U_{ir}^\gamma\left[{i \atop k}\right]$, the set $\{X_\varepsilon\}$ is compatible with the set of periods T, where X_ε is the set of spectral varieties for the operator L, i.e., $X_\varepsilon^\varnothing\left[{\varnothing \atop \varnothing}\right]$.

We note that it follows from this definition that $\{X_{\varepsilon_j}^\gamma\left[{i \atop k}\right]\}$ is a variety compatible with the set of the periods T_j and that

$$X_{\varepsilon_j}^\gamma\left[{i \atop k}\right]\cap X_{\varepsilon_{j'}}^{\gamma'}\left[{i' \atop k'}\right] = X_{\varepsilon_{\tilde j}}^{\gamma,\gamma'}\left[{i,i' \atop k,k'}\right], \tag{9}$$

where $\{i_1,\dots,i_M\}\cap\{i_1',\dots,i_{M'}'\}=\varnothing$, $\tilde j=\{j_1,\dots,j_M\}\cap\{j_1',\dots,j_{M'}'\}$, $\varepsilon\in\mathbb{Z}_2^N$, $k=\varepsilon_i \pmod 2$, and $k'=\varepsilon_{i'} \pmod 2$.

We proceed to studying the properties of Bloch functions in the kernel of $L_\varepsilon^\gamma\left[{i \atop k}\right]_\mu$. In what follows, for the generic $\mu\in X_\varepsilon^\gamma\left[{i \atop k}\right]$, the symbol $\psi_\varepsilon^\gamma\left[{i \atop k}\right](\mu,m)$ will be used to denote a vector in $W_\varepsilon^M(\mu)$ such that the relation $\ker L_\varepsilon^\gamma\left[{i \atop k}\right]_\mu = \langle\psi_\varepsilon^\gamma\left[{i \atop k}\right](\mu,\cdot)\rangle$ holds and m in $\psi_\varepsilon^\gamma\left[{i \atop k}\right](m)$ always belongs to $\Gamma_{1-\varepsilon}^M$. Note that $\psi_\varepsilon^\gamma\left[{i \atop \gamma}\right]\in\ker L_\varepsilon^\gamma\left[{i \atop k}\right]_\mu$ is defined up to the multiplication by a constant, but the ratio $\psi_\varepsilon^\gamma\left[{i \atop k}\right](m)/\psi_\varepsilon^\gamma\left[{i \atop k}\right](m')$ is a well-defined meromorphic function on $X_\varepsilon^\gamma\left[{i \atop k}\right]$. We investigate the behavior of this function in a neighborhood of $X_\varepsilon^\gamma\left[{i,r \atop k,p}\right]$. For this, we need the following formula:

$$\frac{\partial R_\varepsilon^\gamma\left[{i \atop k}\right]}{\partial a(\zeta;n)}\bigg/\frac{\partial R_\varepsilon^\gamma\left[{i \atop k}\right]}{\partial a(\xi;n)} = \frac{\psi_\varepsilon^\gamma\left[{i \atop k}\right](n_j+\zeta_j)}{\psi_\varepsilon^\gamma\left[{i \atop k}\right](n_j+\xi_j)}, \tag{10}$$

where $n_i=k$, $\zeta_i=\xi_i=\gamma$, and $n_j\in\Gamma_\varepsilon^M$. This formula is a direct consequence of the structure of the matrix $\mathbf{M}_\varepsilon^\gamma\left[{i \atop k}\right]$. Let $W_R^\gamma\left[{i \atop k}\right]$ denote the set of operators L such that

$$\frac{\partial R_\varepsilon^{\gamma,\gamma'}\left[{i,r \atop k,p}\right]}{\partial a(\xi;n)}\bigg|_{X_\varepsilon^{\gamma,\gamma'}\left[{i,r \atop k,p}\right]} \not\equiv 0$$

for any $\varepsilon\in\mathbb{Z}_2^N$, $r,p,\gamma'\in\mathbb{Z}$, $\xi_i=\gamma$, $\xi_r=\gamma'$, $n_i=k$, and $n_j\in\Gamma_\varepsilon^M$, let $U_R^\gamma\left[{i \atop k}\right]$ denote the set of operators L such that

$$\frac{\partial R_\varepsilon^\gamma\left[{i \atop k}\right](j)}{\partial a(\xi;n)}\,\mu_q^{\bar\gamma+\delta_{1-2\varepsilon_r}^{\bar\gamma}T_{i,r}'}\bigg|_{X_\varepsilon^{\gamma,\bar\gamma}\left[{i,z \atop k,p}\right]} \not\equiv 0,$$

where $j_q=r$, $T_{i,r}'=(T_r')^{-1}\prod_{a=1}^M T_{j_a}'$, $\xi_r=-\bar\gamma$, $\xi_i=\gamma$, $n_i=k$, $n_j\in\Gamma_\varepsilon^M$ for any $\varepsilon\in\mathbb{Z}_2^N$, $z,r,p,\bar\gamma\in\mathbb{Z}$, such that either $z=r$ or $p\neq m_r$, and let $U_{tr}^\gamma\left[{i \atop k}\right]$ denote the set of operators L such that the following condition holds for any $\varepsilon\in\mathbb{Z}_2^N$ and $r\in\mathbb{Z}$: $X_\varepsilon^\gamma\left[{i,r \atop k,p}\right]=X_\varepsilon^\gamma\left[{i,r \atop k,p'}\right]$ if and only if $p = p' \pmod{2T'_r}$. The above sets $U_R^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$, $W_R^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$, and $U_{tr}^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$ are Zariski open, and their nonemptiness will be proved in Lemma 4.

Proposition 1. *The following relations hold for* $L \in U_{sm}^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] \cap U_R^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] \cap W_R^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] \cap U_{tr}^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$:

$$\left(\frac{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](m^1)}{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](m^0)}\right) = \mathscr{D}_{m^1}^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] - \mathscr{D}_{m^0}^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] - Q^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]\left[\begin{smallmatrix} m^1 \\ m^0 \end{smallmatrix}\right] + Q^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]\left[\begin{smallmatrix} m^0 \\ m^1 \end{smallmatrix}\right], \tag{11}$$

$$Q^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]\left[\begin{smallmatrix} m^1 \\ m^0 \end{smallmatrix}\right] = \sum_{r=1}^{M} \sum_{p=(m_r^0-\varepsilon'_r)/2}^{(m_r^1-\varepsilon'_r)/2-1} \left(X_{\tilde\varepsilon}^{\gamma,1}\left[\begin{smallmatrix} i,j_r \\ k,2p+\varepsilon_r \end{smallmatrix}\right] - X_{\tilde\varepsilon}^{\gamma,-1}\left[\begin{smallmatrix} i,j_r \\ k,2p+\varepsilon_r \end{smallmatrix}\right]\right), \tag{12}$$

where $m, m^0, m^1 \in \Gamma_{\varepsilon'}^M$,

$$\mathscr{D}_m^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] = \sum_{\mathscr{C}} \mathscr{C} \max_{0 \leqslant p < T'_j} \operatorname{ord}_{\mathscr{C}}\left(\frac{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](m)}{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](\varepsilon' + 2p)}\right),$$

the summation extends over all finite divisors $\mathscr{C} \subset X_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$, $\tilde\varepsilon = (\varepsilon_1, \dots, \varepsilon_{r-1}, \varepsilon_{r+1}, \dots, \varepsilon_M)$, *and* $\varepsilon' = 1 - \varepsilon$.

Recall that a divisor $\mathscr{C}$ is finite if and only if it has no irreducible components at infinity, i.e., $\dim X_{\infty,\varepsilon}^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] \cap \mathscr{C} < M - 1$.

Proof. It is now easy to see that the assertion of the proposition is equivalent to the validity of the formula

$$\operatorname{ord}_{X_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]} \frac{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](m + 2\bar\gamma e_d)}{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](m)} = -\delta_{j_d}^r \delta_{m_r+\bar\gamma}^p,$$

where $\tilde\varepsilon = (\varepsilon_1, \dots, \varepsilon_{s-1}, \varepsilon_{s+1}, \dots, \varepsilon_M)$ and $j_s = r$ for arbitrary $\bar\gamma, r, p, d \in \mathbb{Z}$ and $\gamma' = (\gamma, \bar\gamma)$.

Let $\zeta = \xi + 2\bar\gamma e_d$, $\xi_d = -\bar\gamma$, $n_i = k$, $\xi_i = \gamma$, and $n_j = m - \xi$.

We first consider the case $j_d \neq t$. Let us rewrite formula (10) using (7) with regard to the fact that $L \in U_{tr}^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$,

$$\left.\frac{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](m + 2\bar\gamma e_d)}{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](m)}\right|_{X_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]} = \frac{\partial R_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]}{\partial a(\zeta; n)} \Big/ \left.\frac{\partial R_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]}{\partial a(\xi; n)}\right|_{X_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]} = \frac{\partial R_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]}{\partial a(\zeta; n)} \Big/ \left.\frac{\partial R_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]}{\partial a(\xi; n)}\right|_{X_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]} \tag{13}$$

for $\xi_r = \zeta_r = \bar\gamma$. The desired assertion now obviously follows from the definition of $W_R^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$.

Next, we consider the case $r = j_d$. Let us prove the assertion for $\varepsilon_r = 1$, $\bar\gamma = -1$, and $j_1 = r$. (In the remaining cases, the proof is carried out in a similar way.) Since $L \in U_{tr}^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$, there is a $q \neq 1$ such that $\partial R_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] / \partial\mu_q \not\equiv 0$. Consequently, $\mu_1, \dots, \mu_{q-1}, \mu_{q+1}, \dots, \mu_M$ are coordinates on $X_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$ in a neighborhood of $X_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]$, and $X_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]$ is defined in these coordinates by the equation $\mu_1 = 0$. We write out the expansion in terms of these coordinates using formulas (10) and (7),

$$\begin{aligned} \frac{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](m + 2\bar\gamma e_d)}{\psi_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right](m)} &= \frac{\partial R_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]}{\partial a(\zeta; n)} \Big/ \frac{\partial R_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]}{\partial a(\xi; n)} \\ &= \left(\frac{\partial R_\varepsilon^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]}{\partial a(\xi; n)} \mu_1^{-1}\right)^{-1} \mu_1^{-1}\left(\frac{\partial R_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,n_r-1 \end{smallmatrix}\right]}{\partial a(\zeta; n)} \left(R_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,n_r-1 \end{smallmatrix}\right]\right)^{-1} \prod_{s=0}^{T'_r-1} R_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,s \end{smallmatrix}\right] + \mu_1 Q(\mu)\right), \end{aligned}$$

where

$$\operatorname{ord}_{X_{\tilde\varepsilon}^{\gamma'}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]} Q(\mu) = \operatorname{ord}_{X_{\tilde\varepsilon}^{\gamma}\left[\begin{smallmatrix} i,r \\ k,p \end{smallmatrix}\right]} \frac{\partial R_{\tilde\varepsilon}^\gamma\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]}{\partial a(\zeta; n)} \mu_1^{-1} = 0.$$

An elementary analysis of the resulting formula in a neighborhood of $\mu_1 = 0$ leads to the assertion of the proposition. $\square$

Note that formula (13) implies the relation

$$\frac{\psi^{\gamma}_{\varepsilon_j}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](n'_j)}{\psi^{\gamma}_{\varepsilon_j}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](n_j)} = \frac{\psi_{\varepsilon}(n')}{\psi_{\varepsilon}(n)}\bigg|_{X^{\gamma}_{\varepsilon_j}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]}, \tag{14}$$

and it follows from (12), (9), and (11) that

$$Q^{\gamma}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]\left[\begin{smallmatrix} n'_j\\ n_j\end{smallmatrix}\right] = Q\left[\begin{smallmatrix} n'\\ n\end{smallmatrix}\right] X^{\gamma}_{\varepsilon_j}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right], \tag{15}$$

$$\mathscr{D}^{\gamma}_{n_j}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] = \mathscr{D}_n X^{\gamma}_{\varepsilon_j}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right], \tag{16}$$

where $n_i = k+\gamma$ and $n, n' \in \Gamma^N_{1-\varepsilon}$. The existence of the inductive structure of spectral data consists in fact in the above three formulas relating the spectral data and the eigenfunction of the operator L to those of the operator $L^{\gamma}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$.

We introduce the additional notation $\Delta^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](m,m')$ for the algebraic adjunction of $\mathbf{M}^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]_{u,w}$, where $u = 1+\sum_{b=1}^{M} m_b \prod_{t=b+1}^{M} T'_{j_t}$ and $w = 1+\sum_{b=1}^{M} m'_b \prod_{t=b+1}^{M} T'_{j_t}$, $0 \leqslant m, m' < T'_j$. Let us define the following divisor on $X^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$:

$$\mathscr{E}^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] = \sum_{\mathscr{C}} \mathscr{C} \min_{0\leqslant m,m'<T'_j} (\mathrm{ord}_{\mathscr{C}}\, \Delta^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](m,m')),$$

where the summation extends over all finite divisors $\mathscr{C} \subset X^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$.

We now define the following important differential on $X^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$:

$$\Omega^{\gamma}_{\varepsilon;n;\zeta}\begin{bmatrix} i\\ k\end{bmatrix} = \left(\frac{\partial R^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]}{\partial \hat{\mu}_r}\right)^{-1} \frac{\partial R^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]}{\partial a(\zeta;n)} \frac{\bigwedge_{p\neq r} d\hat{\mu}_p}{\prod_{p=1}^{M} \hat{\mu}_p},$$

where $\gamma = \zeta_i$, $n_i = k$, $n_j \in \Gamma^M_{\varepsilon}$, and $\hat{\mu}_r = \mu_r^{1-2\varepsilon_{j_r}}$.

Proposition 2. *The relation*

$$(\Omega^{\gamma}_{\varepsilon;n;\zeta}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]) = \mathscr{E}^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] + \mathscr{D}^{\gamma}_{n_j+\zeta_j}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] + \sigma(\mathscr{D}^{-\gamma}_{n_j}\left[\begin{smallmatrix} i\\ k+\gamma\end{smallmatrix}\right]) - \sum_{r=1}^{M} X^{\gamma,\zeta_{j_r}}_{\tilde{\varepsilon}}\left[\begin{smallmatrix} i,j_r\\ k,n_{j_f}\end{smallmatrix}\right],$$

where $\tilde{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_{r-1},\varepsilon_{r+1},\dots,\varepsilon_M)$, *holds for any* $L \in U^{\gamma}_R\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] \cap W^{\gamma}_R\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] \cap U^{\gamma}_{ir}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] \cap U^{\gamma}_{sm}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] \cap U^{\gamma}_{tr}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$.

Proof. An elementary manipulation with Laurent series expansions in a neighborhood of $X^{\infty,\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$ (by analogy with Proposition 1) gives

$$(\Omega^{\gamma}_{\varepsilon;n;\zeta}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]) \sim -\sum_{r=1}^{M} X^{\gamma,\zeta_{j_r}}_{\tilde{\varepsilon}}\left[\begin{smallmatrix} i,j_r\\ k,n_{j_r}\end{smallmatrix}\right] + \widetilde{\mathscr{D}},$$

where $\widetilde{\mathscr{D}} > 0$ is the effective finite divisor.

Let us calculate $\mathrm{ord}_{\mathscr{C}}(\widetilde{\mathscr{D}} - \mathscr{D}^{\gamma}_{n_j+\zeta_j}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] - \sigma(\mathscr{D}^{-\gamma}_{n_j}\left[\begin{smallmatrix} i\\ k+\gamma\end{smallmatrix}\right]))$, where $\mathscr{C} \subset X^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$ is a finite divisor. We carry out the argument for $\varepsilon = 0$ and $\zeta = 1$. (The consideration is similar in the remaining cases.) In view of the T_j-periodicity, it can be assumed without loss of generality that $0 \leqslant n_j < 2T'_j$. The structure of the matrix $\mathbf{M}^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$ implies the relation $\partial R^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]/\partial a(\zeta;n) = \mu^{\kappa}\Delta(n_j/2, n_j/2)$, where $\kappa \in \mathbb{Z}^M$. Thus, by the holomorphy of the differential $(\partial R^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]/\partial\hat{\mu})^{-1}\prod_{p\neq r} d\hat{\mu}_p$ on $X^{\mathrm{aff},\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$ and formula (5), we have

$$\begin{aligned}\mathrm{ord}_{\mathscr{C}}\,\widetilde{\mathscr{D}} &= \min_{m',m''} \mathrm{ord}_{\mathscr{C}}\left(\frac{\psi^{\gamma}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](2m'+1)}{\psi^{\gamma}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](2m+1)}\,\frac{\psi^{\gamma}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](2m''+1)}{\psi^{\gamma}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](2m+1)}\,\Delta^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](m,m)\right)\\ &= \min_{m'',m'} \mathrm{ord}_{\mathscr{C}}\left(\frac{\Delta^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](m'',m')}{\Delta^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](m'',m)}\,\frac{\Delta^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](m'',m)}{\Delta^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](m,m)}\,\Delta^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right](m,m)\right) = \mathrm{ord}_{\mathscr{C}}\,\mathscr{E}^{\gamma}_{\varepsilon}\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right],\end{aligned}$$

where $2m = n_j$. ☐

We shall use the symbol $U_E^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$ to denote the Zariski open set of operators $L \in U_{sm}^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] \cap U_{ir}^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] \cap W_R^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$ satisfying the condition $\mathscr{E}_\varepsilon^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] = 0$ for all ε.

Remark. A formula similar to (14) holds for the above differential,

$$P.R._{X_{\tilde{\varepsilon}}^{\gamma,\zeta_{j_r}}\left[\begin{smallmatrix} i,j_r\\ k,n_{j_r}\end{smallmatrix}\right]} \Omega_{\varepsilon;n;\zeta}^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] = \Omega_{\tilde{\varepsilon};n;\zeta}^{\gamma,\zeta_{j_r}}\left[\begin{smallmatrix} i,j_r\\ k,n_{j_r}\end{smallmatrix}\right],$$

where $P.R.$ denotes the Poincaré residue. (See [19].) This formula reflects the presence of the inductive structure in the spectral problem under study. □

We state the definition of a set of data that are compatible with the set of periods T. Let $\{X_\varepsilon \subset (\mathbb{CP}^1)_\mu^N\}$ be a set of varieties compatible with the set of periods T and let there be a set of divisors $\mathscr{D}_n$, $n \in \mathbb{Z}^N$, on $\{X_\varepsilon \subset (\mathbb{CP}^1)_\mu^N\}$ satisfying the conditions

$$\mathscr{D}_n = \mathscr{D}_{n+T_r e_r}, \tag{17}$$

$$\mathscr{K}_{X_\varepsilon} \sim \mathscr{D}_{n+\zeta} + \sigma(\mathscr{D}_n) - \sum_{r=1}^N X_{\tilde{\varepsilon}}^{\zeta_r}\left[\begin{smallmatrix} r\\ n_r\end{smallmatrix}\right], \tag{18}$$

$$\mathscr{D}_{n^1} - \mathscr{D}_{n^0} - Q\left[\begin{smallmatrix} n^1\\ n^0\end{smallmatrix}\right] + Q\left[\begin{smallmatrix} n^0\\ n^1\end{smallmatrix}\right] \sim 0 \tag{19}$$

(where $\tilde{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{r-1}, \varepsilon_{r+1}, \dots, \varepsilon_N)$) for all $n^0, n^1 \in \Gamma_\varepsilon^N$, $n, \zeta, \varepsilon \in \mathbb{Z}^N$, and $r = 1, \dots, N$. (Here formula (12) with $j = (1, \dots, N)$ has been used.) This set $\{X_\varepsilon \subset (\mathbb{CP}^1)_\mu^N, \mathscr{D}_n\}$ will be called a *set of data compatible with the set of periods* T. Let $\mathfrak{M}_T$ denote the family of sets of data compatible with the set of periods T.

If the system of equations (17)–(19) is regarded as a set of relations between the divisors, then it is obvious that this system is redundant, since equation (19) can be eliminated from it and all $\mathscr{D}_n$'s can be expressed via the divisors $\mathscr{D}_\varepsilon$, $\varepsilon_h = 0$, which, in turn, satisfy the relations

$$\mathscr{K}_{X_\varepsilon} \sim \mathscr{D}_{\varepsilon'} + \sigma(\mathscr{D}_\varepsilon) - \sum_{r=1}^N X_{\tilde{\varepsilon}}^1\left[\begin{smallmatrix} r\\ \varepsilon_r\end{smallmatrix}\right] + \sum_{r=1}^{N-S} \sum_{p=0}^{(T_{h_r}-1)/2} (X_{\tilde{\varepsilon}'}^1\left[\begin{smallmatrix} h_r\\ 2p+1\end{smallmatrix}\right] - X_{\tilde{\varepsilon}'}^{-1}\left[\begin{smallmatrix} h_r\\ 2p+1\end{smallmatrix}\right]), \tag{20}$$

$$\mathscr{D}_\varepsilon - Q\left[\begin{smallmatrix} \varepsilon\\ \varepsilon+2n\end{smallmatrix}\right] + Q\left[\begin{smallmatrix} \varepsilon+2n\\ \varepsilon\end{smallmatrix}\right] > 0, \qquad 0 \leqslant n < T', \tag{21}$$

where $\varepsilon_l + \varepsilon_l' = 1$, $\varepsilon_h = \varepsilon_h' = 0$, and $\tilde{\varepsilon}' = 1 - \tilde{\varepsilon}$, and it is assumed that $j = (1, \dots, N)$. Consequently, if we wish to treat $\mathfrak{M}_T$ as the *space of spectral data for the direct spectral problem*, then the elements $\mathfrak{m} \in \mathfrak{M}_T$ should be interpreted as the sets $\{X_\varepsilon \subset (\mathbb{CP}^1)_\mu^N, \mathscr{D}_\varepsilon\}$, $\varepsilon_h = 0$, where $\{X_\varepsilon \subset (\mathbb{CP}^1)_\mu^N\}$ is compatible with the set of periods T, while $\mathscr{D}_\varepsilon$ satisfies relations (20) and (21). We note that the description of the space of spectral data for a multidimensional problem is similar to that of the spectral data for two-dimensional problems [6, 8]. The only distinction consists in condition (21), which holds automatically in the two-dimensional case in view of the Riemann–Roch theorem, but is a strong constraint on $\mathscr{D}_\varepsilon$ and on X_ε in the multidimensional case.

The proposition below formalizes the assertion about the existence of the inductive structure on the spectral data.

Proposition 3. *If* $\{X_\varepsilon \subset (\mathbb{CP}^1)_\mu^N, \mathscr{D}_\varepsilon\} \in \mathfrak{M}_T$, *then* $\{X_\varepsilon^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] \subset (\mathbb{CP}^1)_{\mu_j}^M, \mathscr{D}_{\varepsilon_j}^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]\} \in \mathfrak{M}_{T_j}$, *where* $\mathscr{D}_{n_j}^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$ *is given by formula* (16). *(This means that* $\{X_\varepsilon^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right] \subset (\mathbb{CP}^1)_{\mu_j}^M, \mathscr{D}_{\varepsilon_j}^\gamma\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]\}$ *are spectral data for the* T_j*-periodic spectral problem.)*

Proof. It suffices to show that $\{X_\varepsilon^{\bar{\gamma}}\left[\begin{smallmatrix} r\\ p\end{smallmatrix}\right] \subset (\mathbb{CP}^1)_{\bar{\mu}}^{N-1}, \mathscr{D}_{\varepsilon,m}^{\bar{\gamma}}\left[\begin{smallmatrix} r\\ p\end{smallmatrix}\right]\} \in \mathfrak{M}_{\bar{T}}$, $\bar{T} = (T_1, \dots, T_{r-1}, T_{r+1}, \dots, T_N)$. To prove this, we note that, by the adjunction formula (see [19]), we have $\mathscr{K}_{X_\varepsilon^{\bar{\gamma}}\left[\begin{smallmatrix} r\\ p\end{smallmatrix}\right]} \sim (\mathscr{K}_{X_\varepsilon} + X_\varepsilon^{\bar{\gamma}}\left[\begin{smallmatrix} r\\ p\end{smallmatrix}\right]) X_\varepsilon^{\bar{\gamma}}\left[\begin{smallmatrix} r\\ p\end{smallmatrix}\right]$. Substituting (18) corresponding to the T-periodic problem with $n_r = p + \bar{\gamma}$ and $\zeta_r = -\bar{\gamma}$ into this formula and using (16) and (9), we conclude that condition (18) holds for the T'-periodic problem. The validity of condition (19) is a trivial consequence of (15) and (9). □

There is a gauge group acting of the space of operators L. For any T-periodic function $g \in V(1)$ such that $g(n) \neq 0$ for all $n \in \mathbb{Z}^N$, the map $F_g \colon F_g(L) = L'$, $a'(n,\zeta) = g(n)g(n+\zeta)a(n,\zeta)$ is defined. The quotient space of the space of operators L by the action of the gauge group will be denoted by $\mathfrak{L}_T$. The images of the open sets U_{sm}, U_{ir}, W_R, U_R, U_E, and U_{tr} under this factorization will be denoted by the same symbols. Let $\mathfrak{U} = \bigcap_{i,k,\gamma}(U^\gamma_{sm}\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] \cap U^\gamma_{ir}\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] \cap W^\gamma_R\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] \cap U^\gamma_R\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] \cap U^\gamma_E\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right] \cap U^\gamma_{tr}\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right])$. Then, as follows from the above sequence of lemmas, the map $\mathfrak{D} \colon \mathfrak{L}_T \to \mathfrak{M}_T$ corresponding to the direct scattering problem is defined on the set $\mathfrak{U}$. In the next section, the density of the set $\mathfrak{U}$ in $\mathfrak{L}_T$ will be proved and the inverse map $\mathfrak{I}$ of $\mathfrak{D}$ will be found.

3. Inverse Problem

A solution method for the two-dimensional inverse scattering problem was suggested by Krichever in [6]. (In the case considered in the present paper, the solution of the inverse scattering problem is equivalent to the construction of the map $\mathfrak{I}$.) We shall use the results in [6] to show that the set $\mathfrak{U}$ is dense in $\mathfrak{L}_T$ and also to prove the invertibility of the map $\mathfrak{D}$.

Lemma 2. *Let $N = 2$ and let $h_\varepsilon\left[\begin{smallmatrix} n \\ n' \end{smallmatrix}\right] = H^0(X_\varepsilon, \mathscr{O}(Q\left[\begin{smallmatrix} n \\ n' \end{smallmatrix}\right] + \mathscr{D}_{n'}))$. Then $\mathfrak{M}_T \neq \varnothing$, and we have*

$$\dim h_\varepsilon\left[\begin{smallmatrix} n \\ n' \end{smallmatrix}\right] = 1 \tag{22}$$

for a generic representative $\mathfrak{m} \in \mathfrak{M}_T$.

Proof. Let us consider the situation in which T_1 and T_2 are even. In this case, the spectral data consist of the curves $X_{0,0}$, $X_{0,1}$, $\sigma(X_{0,0})$, $\sigma(X_{0,1})$ and the divisors $\mathscr{D}_\varepsilon$. Here $X_{0,0}$ and $X_{0,1}$ are curves of degree (T_1', T_2') that are defined by equations with generic coefficients, and therefore $X_{0,0}$ and $X_{0,1}$ are smooth curves of genus $(T_1'-1)(T_2'-1)$. It follows from (18) that $|\mathscr{D}_n| = (T_1'-1)(T_2'-1)$. We choose $m_e \in \Gamma^2_{1,1-e}$, $e = 0,1$. Then relations (18) and (19) permit expressing all $\mathscr{D}_n$'s via $\mathscr{D}_{m_e}$. Moreover, if two generic positive divisors $\mathscr{D}_{m_e}$ of degree $(T_1'-1)(T_2'-1)$ are taken, then, in view of the relation $|Q_\varepsilon\left[\begin{smallmatrix} n^1 \\ n^0 \end{smallmatrix}\right]| = |Q_\varepsilon\left[\begin{smallmatrix} n^1 \\ n^0 \end{smallmatrix}\right]| = 0$ and the Riemann–Roch theorem, we have $\dim h_{0,e}\left[\begin{smallmatrix} n \\ m_e \end{smallmatrix}\right] = 1$, and, by the Jacobi theorem (see [19]), for $\mathscr{D}_{m_e}$, there are unique positive divisors $\mathscr{D}_n$ for which conditions (18) and (19) hold. Since the chosen $\mathscr{D}_{m_e}$'s are generic, the Riemann–Roch theorem implies that relation (22) holds. The remaining cases are considered in a similar way. ☐

Theorem 1. *Let $N = 2$. Then the set $\mathfrak{U}$ is dense in $\mathfrak{L}_T$ and the map $\mathfrak{D} \colon \mathfrak{L}_T \to \mathfrak{M}_T$ is invertible on an everywhere dense set. The map $\mathfrak{I} = \mathfrak{D}^{-1}$ can be described using the theta functions of the curves X_ε.*

Proof. We begin with the case of even T_1 and T_2. Let us choose $m_e \in \Gamma_{1,1-e}$, $e = 0,1$, and take T-Bloch functions $\psi(n)$ such that (11) holds. Using the method presented in [6], we construct an operator $\widetilde{L}_{0,e}$ from the functions $\varphi_{0,e} = \psi_{0,e}(n)/\psi_{0,e}(m_e) \in h_{0,e}\left[\begin{smallmatrix} n \\ m_e \end{smallmatrix}\right]$ which acts on $V_{1,1-e}$, $\widetilde{L}_{0,e}\varphi_{0,e} = \sum_{(\zeta_1,\zeta_2)} \tilde{a}(n,\zeta)\varphi_{0,e}(n+\zeta)$, and satisfies the condition $\widetilde{L}_{0,e}\varphi_{0,e} = 0$. The operator $\widetilde{L}_{0,e}$ is not uniquely determined by the functions $\varphi_{0,e}$. Indeed, the operator $\widetilde{L}'_{0,e}$ with coefficients $\tilde{a}'(n,\zeta) = \tilde{a}(n,\zeta)C(n)$, where $C(n)$ are arbitrary constants, also satisfies the condition $\widetilde{L}'_{0,e}\varphi_{0,e} = 0$. If we now set $\tilde{a}(n,\zeta) = \tilde{a}(n+\zeta,-\zeta)$ for $n \in \Gamma^2_{1,1-e}$, $e = 0,1$, this results in an operator $\widetilde{L}$ acting on V. Let us fix $n_e \in \Gamma^2_{0,e}$, $e = 0,1$. By proposition 1 and Lemma 2, the eigenfunction $\varphi_{1,1-e}$, $e = 0,1$, $\widetilde{L}\varphi_{1,1-e} = 0$, satisfies the relation

$$\frac{\varphi_{1,1-e}(n)}{\varphi_{1,1-e}(n_e)} = \widehat{C}(n)\,\frac{\psi_{1,1-e}(n)}{\psi_{1,1-e}(n_e)},$$

where $\widehat{C}(n)$ are some constants. Setting $a(n,\zeta) = \tilde{a}(n,\zeta)\widehat{C}(n)$, $n \in \Gamma^2_{0,e}$, we obtain an operator L such that $L\psi = 0$. The arbitrariness in the choice of the functions $\psi(n)$ in constructing the operator L possessing the property $L\psi = 0$ corresponds to the action of the gauge group. We have thus obtained a well-defined map $\mathfrak{I} \colon \mathfrak{M}_T \to L_T$. In this case, the foregoing argument shows that there is a unique operator L which satisfies the normalization condition $a(m_0;1,1) = a(m_1;1,1) = 1$

and the relation $L\psi = 0$. Hence, $\mathfrak{I}\mathfrak{D} = \mathfrak{D}\mathfrak{I} = \mathrm{id}$. Since the generic $\mathfrak{m} \in \mathfrak{M}_T$ satisfies the relation $\mathfrak{I}(\mathfrak{m}) \in \mathfrak{U}$, we have $\mathfrak{U} \neq \varnothing$.

The construction of $\mathfrak{I}$ in the remaining cases is quite similar. The only distinction from the above situation is that if T_1 is odd and T_2 is even, then ψ satisfies relation (11) and we have $\psi_{0,\varepsilon}(n) = \psi_{1,\varepsilon}(n+T_1e_1)$, $\varepsilon = 0, 1$, and if both T_1 and T_2 are odd, then ψ satisfies (11) and we have $\pi_{0,0}(n) = \psi_{0,1}(n+T_2e_2) = \psi_{1,0}(n+T_1e_1) = \psi_{1,1}(n+T_1e_1+T_2e_2)$. □

We now prove that the set $\mathfrak{U}$ is dense in the case of an arbitrary dimension N. The proof will be carried out by induction on N. It can be assumed that the spectral varieties of all operators L lie in the same space $(\mathbb{CP}^1)^N$. In this case, if the spectral variety of L is denoted by the symbol $X_\varepsilon(L)$, then the lemma stated below is true.

Lemma 3. *The relation* $\bigcap_L X_\varepsilon(L) = \varnothing$ *holds for an arbitrary* $\varepsilon \in \mathbb{Z}^N$.

Proof. We shall use induction on N.

Let us fix some arbitrary numbers $1 \leqslant r \leqslant N$, $k \in \mathbb{Z}$, and $\kappa \in \mathbb{C}^*$. Note that if two operators L and L' such that $a(n;\zeta) = a'(n;\zeta)$ for $n_r \neq k \pmod{T_r}$ and $a(n;\zeta) = \kappa a(n;\zeta)$ for $n_r = k \pmod{T_r}$ are given, then

$$(\mu_1, \dots, \mu_r, \dots, \mu_N) \in X_\varepsilon^{aff}(L') \iff (\mu_1, \dots, \kappa\mu_r, \dots, \mu_N) \in X_\varepsilon^{aff}(L).$$

Therefore, if $\mu \in \bigcap_L X_\varepsilon(L)$, then $\tilde{\mu} = (\mu_1, \dots, \kappa\mu_r, \dots, \mu_N) \in \bigcap_L X_\varepsilon^{aff}(L)$ for arbitrary r and κ. Thus, we conclude that if $\mu \in \bigcap_L X_\varepsilon(L)$, then $\mu \in \bigcap_L X_\varepsilon^\infty(L)$. Furthermore, we can use the induction hypothesis, since $X_\varepsilon^\infty(L)$ is the union of the spectral varieties of the operators $L_\varepsilon^\gamma \left[\begin{smallmatrix} r \\ p \end{smallmatrix}\right]$. □

Lemma 4. *The set* $\mathfrak{U}$ *is dense in* $\mathfrak{L}_T$ *for any* N.

Proof. Since the set under consideration is the intersection of Zariski open sets, it suffices to show that the sets U_{sm}, U_{ir}, W_R, U_E, and U_{tr} are nonempty.

The fact that $U_{tr} \neq \varnothing$ immediately follows from Lemma 3.

Let us prove that $U_R \neq \varnothing$. The proof will be carried out by induction on N. Suppose that the desired assertion is not true, i.e., there are $r, \bar{\gamma} \in \mathbb{Z}$, $\varepsilon \in \mathbb{Z}_2^N$, and ζ such that

$$F = \frac{\partial R_\varepsilon}{\partial a(n;\zeta)}\, \mu_r^{\bar{\gamma}+\delta^{\bar{\gamma}}_{1-2\varepsilon_{j_r}} Q_{j_r}}, \qquad F|_{X_\varepsilon^\gamma \left[\begin{smallmatrix} r \\ n_r \end{smallmatrix}\right]} \equiv 0, \tag{23}$$

where $Q_{j_r} = (T'_{j_r})^{-1} \prod_{a=1}^N T'_a$. Let $s \neq r$. Then

$$F|_{X_{\tilde{\varepsilon}}^{\gamma'} \left[\begin{smallmatrix} s \\ n_s \end{smallmatrix}\right]} = \frac{\partial R_{\tilde{\varepsilon}}^{\gamma'} \left[\begin{smallmatrix} s \\ n_s \end{smallmatrix}\right]}{\partial a(n;\zeta)}\, \mu_r^{\bar{\gamma}+v} = \frac{\partial R_{\tilde{\varepsilon}}^{\gamma'} \left[\begin{smallmatrix} s \\ n_s \end{smallmatrix}\right]}{\partial a(n;\zeta)}\, \mu_r^{v'+\bar{\gamma}} \prod_{p \neq n_s} (R_{\tilde{\varepsilon}}^{\gamma'} \left[\begin{smallmatrix} s \\ p \end{smallmatrix}\right] \mu_r^{v'}), \tag{24}$$

where $\tilde{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{s-1}, \varepsilon_{s+1}, \dots, \varepsilon_N)$, $\gamma' = \zeta_s$, $v = \delta^{\bar{\gamma}}_{1-2\varepsilon_{j_r}} Q_{j_r}$, $v' = \delta^{\bar{\gamma}}_{1-2\varepsilon_{j_r}} Q_{j_r, j_s}$, and $Q_{j_r, j_s} = Q_{j_r}/T'_{j_s}$. Since the relation $L \in U_{tr}$ holds for the generic L, it follows from (23) and (24) that

$$\frac{\partial R_{\varepsilon'}^{\gamma'} \left[\begin{smallmatrix} s \\ n_s \end{smallmatrix}\right]}{\partial a(n;\zeta)}\, \mu_r^{v'+\bar{\gamma}}|_{X_{\varepsilon'}^{\gamma', \bar{\gamma}} \left[\begin{smallmatrix} s, r \\ n_s, n_r \end{smallmatrix}\right]} \equiv 0,$$

where $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_{r-1}, \varepsilon_{r+1}, \dots, \varepsilon_{s-1}, \varepsilon_{s+1}, \dots, \varepsilon_N)$, which contradicts the induction hypothesis. The fact that $W_R \neq \varnothing$ can be proved in a similar way.

Let us prove that $U_{sm} \neq \varnothing$. Indeed, the polynomials R_ε depend linearly on the coefficients $a(n,\zeta)$, i.e., the X_ε form a linear system of divisors on $(\mathbb{CP}^1)^N$. The Bertini theorem on linear systems of divisors asserts that the generic term of the linear system has no singular points outside the base set, i.e., if $z \in \operatorname{Sing} X_\varepsilon$ for the generic L, then $z \in X_\varepsilon(L)$ for all L. Thus, Lemma 3 completes the proof of the fact that $U_{sm} \neq \varnothing$.

We shall prove that $U_E \neq \varnothing$ by induction on N. Suppose that $U_E = \varnothing$. (This means that $\mathscr{E}_\varepsilon \neq 0$ for the generic operator L_ε.) Then it follows from elementary intersection theory that there are $\gamma, t, u \in \mathbb{Z}$ such that $\mathscr{E}_\varepsilon^\gamma\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] = \mathscr{E} \cap X_\varepsilon^\gamma\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] \neq 0$. Consequently, $U_E^\gamma\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] = \varnothing$, which contradicts the induction hypothesis.

If the variety $X_\varepsilon \subset (\mathbb{CP}^1)^N$ is reducible and nonsingular, then $X_\varepsilon = X'_\varepsilon \cup X''_\varepsilon$ and $X'_\varepsilon \cap X''_\varepsilon = \varnothing$, and hence X'_ε and X''_ε have degrees $(a'_1, \dots, a'_N)$ and $(a''_1, \dots, a''_N)$ such that $a'_r a''_r = 0$, $r = 1, \dots, N$. In exactly the same way as before, it can be proved by induction on the dimension that this decomposition is impossible for the components of the modified spectral variety of the generic operator L. $\square$

We have thus shown that $\mathfrak{D}$ is defined on an open subset in the space $\mathfrak{L}_T$. Let us introduce the notation $\mathfrak{D}(\mathfrak{L}_T) = \mathfrak{M}'_T \subset \mathfrak{M}_T$.

Theorem 2. *The map $\mathfrak{D}\colon \mathfrak{L}_T \to \mathfrak{M}'_T$ is invertible on some dense set. The map $\mathfrak{I} = \mathfrak{D}^{-1}$ can be described using the theta functions of the curves $X_\varepsilon\left[\begin{smallmatrix} i \\ k \end{smallmatrix}\right]$, $i, k \in \mathbb{Z}^{N-2}$.*

Proof. By Proposition 3, the map $\mathfrak{I}$ can be constructed by induction. So, we suppose that $\mathfrak{I}$ has already been constructed for dimensions lower than N and then construct $\mathfrak{I}$ in the dimension N.

We note that the eigenfunction ψ, $L\psi = 0$, satisfies the equation

$$a(n; -1, \gamma)\psi^\gamma\left[\begin{smallmatrix} 2,3,\dots,N \\ k_1,k_2,\dots,k_{N-1} \end{smallmatrix}\right](n_1 - 1) + a(n; 1, \gamma)\psi^\gamma\left[\begin{smallmatrix} 2,3,\dots,N \\ k_1,k_2,\dots,k_{N-1} \end{smallmatrix}\right](n_1 + 1) = 0.$$

The validity of this equation for all $n \in \mathbb{Z}^N$ is equivalent to the following relations for all $n \in \mathbb{Z}^N$:

$$\frac{a(n; 1, \gamma)}{a(n_1 + 2, \tilde{n}; 1, \gamma)} = \frac{\psi^{-\gamma}\left[\begin{smallmatrix} 2,\dots,N \\ k_1,\dots,k_{N-1} \end{smallmatrix}\right](n_1 + 2)}{\psi^{-\gamma}\left[\begin{smallmatrix} 2,\dots,N \\ k_1,\dots,k_{N-1} \end{smallmatrix}\right](n_1)} \frac{\psi^{\gamma}\left[\begin{smallmatrix} 2,\dots,N \\ k_1,\dots,k_{N-1} \end{smallmatrix}\right](n_1 + 3)}{\psi^{\gamma}\left[\begin{smallmatrix} 2,\dots,N \\ k_1,\dots,k_{N-1} \end{smallmatrix}\right](n_1 + 1)}, \tag{25}$$

$$\frac{a(n; -1, \gamma)}{a(n_1 + 2, \tilde{n}; -1, \gamma)} = \frac{\psi^{\gamma}\left[\begin{smallmatrix} 2,\dots,N \\ k_1,\dots,k_{N-1} \end{smallmatrix}\right](n_1 + 1)}{\psi^{\gamma}\left[\begin{smallmatrix} 2,\dots,N \\ k_1,\dots,k_{N-1} \end{smallmatrix}\right](n_1 - 1)} \frac{\psi^{-\gamma}\left[\begin{smallmatrix} 2,\dots,N \\ k_1,\dots,k_{N-1} \end{smallmatrix}\right](n_1 + 2)}{\psi^{-\gamma}\left[\begin{smallmatrix} 2,\dots,N \\ k_1,\dots,k_{N-1} \end{smallmatrix}\right](n_1)}, \tag{26}$$

where $\tilde{n} = (n_2, \dots, n_N)$.

We now choose meromorphic functions $\psi_\varepsilon(n)$ on X_ε such that relations (11) and $\psi_\varepsilon(n) = \psi_{\varepsilon+e_{h_r}}(n + T'_{h_r} e_r)$ hold, where $\varepsilon_h = 0$, $r = 1, \dots, N - S$, $n \in \Gamma^N_{1-\varepsilon}$, and $\psi(n)$ are T'-Bloch functions with Floquet multipliers μ. By the induction hypothesis, we can construct the operators $M_0\psi^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = 0$ and $M_1\psi^{-1}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = 0$ using the functions $\psi^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\psi^{-1}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ defined by formula (14). Let the coefficients of the operator M_e, $e = 0, 1$, be $a_e(k, \zeta)$, $\zeta, k \in \mathbb{Z}^{N-1}$, $\zeta_r = \pm 1$, $r = 1, \dots, N - 1$. Formulas (25) and (26) permit expressing the coefficients $a(2s + 1, \tilde{n}; \gamma)$ in terms of $a(1, \tilde{n}; \gamma)$. Thus, formulas (25) and (26) together with the relation $a(1, k; 1 - 2e, \tilde{\gamma}) = a_e(k; \tilde{\gamma})$, $e = 0, 1$, give the map $\mathfrak{I}$, and the arbitrariness in the choice of ψ corresponds to the action of the gauge group. Consequently, $\mathfrak{I}$ is a well-defined map from $\mathfrak{M}_T$ into$\mathfrak{L}_T$, and, by construction, we have $\mathfrak{I}\mathfrak{D} = \mathrm{id}$.

We note that the fractions involved in formulas (25) and (26) can be expressed via the theta functions of the curves $X_\varepsilon\left[\begin{smallmatrix} 3,4,\dots,N \\ k_1,k_2,\dots,k_{N-2} \end{smallmatrix}\right]$. (See [6].) $\square$

It should be noted that the proof of Theorem 2 provides an inductive method for constructing $\mathfrak{I}$. In particular, the procedure of solving the inverse spectral problem for $N = 3$ and $h = \varnothing$ (i.e., $S = 3$) is as follows. First, it is necessary to consider the intersection of $\mathscr{D}_\varepsilon$,d $\varepsilon_i = 0, 1$, $i = 1, 2, 3$, with the curves $X_{\tilde{\varepsilon}}^{-1}\left[\begin{smallmatrix} 3 \\ 2-\varepsilon_3+2t \end{smallmatrix}\right]$, $1 \leqslant t \leqslant T_i/2$. This results in the divisors $\mathscr{F}_{\tilde{\varepsilon}}(t) = X_{\tilde{\varepsilon}}^{-1}\left[\begin{smallmatrix} 3 \\ 2-\varepsilon_3+2t \end{smallmatrix}\right] \cap \mathscr{D}_\varepsilon$. In view of (16) and (15), the intersection of the left-hand side of (19) with $X_{\tilde{\varepsilon}}^{-1}\left[\begin{smallmatrix} 3 \\ 2-\varepsilon_3+2t \end{smallmatrix}\right]$ for $n^0 = 1 - \varepsilon + 2te_3$ and $n^1 = 1 - \varepsilon$ gives a relationship between $\mathscr{F}_{\tilde{\varepsilon}}^\gamma(t)$ and $\mathscr{D}_{1-\tilde{\varepsilon}}^{-1}\left[\begin{smallmatrix} 3 \\ 2-\varepsilon_3+2t \end{smallmatrix}\right]$, which has the form $\mathscr{D}_{1-\tilde{\varepsilon}}^{-1}\left[\begin{smallmatrix} 3 \\ 2-\varepsilon_3+2t \end{smallmatrix}\right] \sim \mathscr{F}_{\tilde{\varepsilon}}(t) - t(T_1 \sum_{i=1}^{T_2} X_{\varepsilon_1}^{-1,1}\left[\begin{smallmatrix} 3,2 \\ 2-\varepsilon_3+2t,\varepsilon_2+2i \end{smallmatrix}\right] + T_2 \sum_{i=1}^{T_1} X_{\varepsilon_2}^{-1,1}\left[\begin{smallmatrix} 3,1 \\ 2-\varepsilon-3+2t,\varepsilon_1+2i \end{smallmatrix}\right])$. The divisors $\mathscr{D}_{\tilde{\varepsilon}}^1\left[\begin{smallmatrix} 3 \\ 1-\varepsilon_3+2t \end{smallmatrix}\right]$ are constructed from $\mathscr{D}_{1-\tilde{\varepsilon}}^{-1}\left[\begin{smallmatrix} 3 \\ 2-\varepsilon_3+2t \end{smallmatrix}\right]$ using formula (16).

As a consequence of Lemma 2, the relation $\dim H^0(X_{\tilde{\varepsilon}}^\gamma\left[\begin{smallmatrix} 3 \\ \varepsilon_3+2t \end{smallmatrix}\right], \mathscr{O}(\mathscr{D}_{1-\varepsilon}^\gamma + Q^\gamma\left[\begin{smallmatrix} 3 \\ \varepsilon_3+2t \end{smallmatrix}\right]\left[\begin{smallmatrix} m^1 \\ m^0 \end{smallmatrix}\right])) = 1$, $m^0, m^1 \in \Gamma^2_{1-\tilde{\varepsilon}}$, holds. Let us take the functions $\psi_{\tilde{\varepsilon}}^\gamma\left[\begin{smallmatrix} 3 \\ \varepsilon_3+2t \end{smallmatrix}\right] \in H^0(X_{\tilde{\varepsilon}}^\gamma\left[\begin{smallmatrix} 3 \\ \varepsilon_3+2t \end{smallmatrix}\right], \mathscr{O}(\mathscr{D}_{1-\varepsilon}^\gamma + Q^\gamma\left[\begin{smallmatrix} 3 \\ \varepsilon_3+2t \end{smallmatrix}\right]\left[\begin{smallmatrix} m^1 \\ m^0 \end{smallmatrix}\right]))$. The arbitrariness in the choice of these functions is related to the action of the gauge group. By definition, $L^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]$ is an operator from $V^2_{1-\tilde\varepsilon}$ into $V^2_{\tilde\varepsilon}$, the operator $L^{-\gamma}_{1-\tilde\varepsilon}\left[{3 \atop \varepsilon_3+\gamma+2t}\right]$ acts from $V^2_{\tilde\varepsilon}$ into $V^2_{1-\tilde\varepsilon}$, and, by formula (5), these operators are mutually conjugate. These two operators should be regarded as two parts of one selfadjoint operator $\widehat{L}^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]$ acting on the space $V^2_{\tilde\varepsilon} \oplus V^2_{1-\tilde\varepsilon}$ according to the formula $(\widehat{L}^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]\psi)(m) = (L^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]\psi)(m)$ if $m \in \Gamma_{1-\tilde\varepsilon}$ and according to $(\widehat{L}^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]\psi)(m) = (L^{-\gamma}_{1-\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t+\gamma}\right]\psi)(m)$ if $m \in \Gamma_{\tilde\varepsilon}$. We also note that $\widehat{L}^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right] = \widehat{L}^{-\gamma}_{1-\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t+\gamma}\right]$ and that the involution σ identifies the curves $X^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]$ and $X^{-\gamma}_{1-\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t+\gamma}\right]$. It follows from the results in [6] that there is a unique operator $\widehat{L}^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]$ such that $\widehat{L}^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]\hat\psi^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right] = 0$, where $\hat\psi^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right](m) = \psi^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right](m)$ if $m \in \Gamma^2_{1-\tilde\varepsilon}$ and $\hat\psi^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right](m) = \psi^{-\gamma}_{1-\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t+\gamma}\right](m)$ if $m \in \Gamma^2_{\tilde\varepsilon}$. The coefficients of the operator $\widehat{L}^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]$ can be written down in terms of the theta functions of the curve $X^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]$. It now remains to remove the arbitrariness in the choice of the constant up to which $\widehat{L}^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]$ is defined. This can be done if we require that relations (25) and (26) hold. (For $N = 3$.) After that the operator L is uniquely determined by the functions $\psi^{\gamma}_{\tilde\varepsilon}\left[{3 \atop \varepsilon_3+2t}\right]$ up to the multiplication by a constant. Consequently, we have reconstructed the operator L from the spectral data $\{X_\varepsilon, \mathscr{D}_\varepsilon\}$.

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