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The Best Extension Operators for Sobolev Spaces on the Half-Line*

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Abstract. We describe the construction of extension operators with minimal possible norm τ_m from the half-line to the entire real line for the spaces W_2^m and derive the asymptotic estimate $\ln \tau_m \approx K_0 m$ (as $m \to \infty$), where

$$K_0 := \frac{4}{\pi} \int_0^{\pi/4} \ln(\cot x)\, dx = 1.166243\ldots = \ln 3.209912\ldots.$$

The proof is based on the investigation of the maximum and minimum eigenvalues and the corresponding eigenvectors of some special matrices related to Vandermonde matrices and their inverses, which can be of interest in themselves.

Key words: extrapolations with minimal norms, Vandermonde matrices.

1. Notation and Statement of the Main Result

Let $W_2^m(I)$ be the Sobolev space of all functions $f(x)$ defined on the interval $I := (\alpha, \beta) \subset \mathbb{R}^1$, having absolutely continuous derivatives $f^{(m-1)}(x)$, and satisfying the inequality

$$\|f\|_{W_2^m(I)} := \left(\int_I (|f(x)|^2 + |f^{(m)}(x)|^2)\, dx \right)^{1/2} < \infty. \tag{1}$$

Certain extension operators $T_m \colon W_2^m(\mathbb{R}^1_-) \to W_2^m(\mathbb{R}^1)$ whose norms do not exceed 8^m were constructed in [1]. On the other hand, it was shown in [2] that $W_2^m(\mathbb{R}^1_-)$ contains a function $f_m(x)$ such that the norm in $W_2^m(\mathbb{R}^1)$ of any function $g(x)$ defined on the entire real line and coinciding with $f_m(x)$ for all $x < 0$ is greater than $0.08\, m^{-1/4} 2^m \|f_m\|_{W_2^m(\mathbb{R}^1_-)}$. Our aim is to bridge the gap between the upper and lower bounds and establish the asymptotic formula given below (which is sharp in the logarithmic scale) for the expression

$$\tau_m := \min \|T_m\|_{W_2^m(\mathbb{R}^1_-) \to W_2^m(\mathbb{R}^1)}. \tag{2}$$

By $\mathbf{G}$ we denote the *Catalan constant* (e.g., see [3], 865.03 and 48.32),

$$\mathbf{G} := \int_0^{\pi/4} \ln(\cot x)\, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.91596559\ldots. \tag{3}$$

Theorem. *One has* $\ln \tau_m \approx K_0 m$ *as* $m \to \infty$, *where*

$$K_0 := \frac{4}{\pi} \mathbf{G} = 1.166243\ldots = \ln 3.209912\ldots. \tag{4}$$

It will be established in the course of the proof that the minimum norm extension operator is linear and is closely related to the best extrapolation operator (i.e., extension from a "single point"), whose investigation was initiated by L. D. Kudryavtsev. It will also be shown that τ_m can be expressed explicitly in terms of the maximum and minimum eigenvalues of a matrix related to some special Vandermonde matrices.

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2. Preliminaries

Consider the quadratic functional

$$J_m[y] := \int_0^\infty (|y^{(m)}(x)|^2 + |y(x)|^2)\,dx = \|y(x)\|^2_{W_2^m(\mathbb{R}^1_+)}. \tag{5}$$

The minimum of the squared W_2^m-norm over all possible extrapolations with given initial data will be denoted by

$$\psi_m(a) := \min\{J_m[y] : y^{(s-1)}(0) = a_s,\ s \in \{1,\dots,m\}\}, \tag{6}$$

where $a := (a_1,\dots,a_m)$ is an arbitrary vector.

The extremals of $J_m[y]$ are solutions of the Euler equation

$$(-1)^m y^{(2m)} + y = 0 \tag{7}$$

tending to 0 as $x \to +\infty$ and hence representable as linear combinations of exponentials (the set of all these linear combinations will be denoted by Y_m)

$$y(x) = \sum_{k=1}^m b_k e^{\mu_k x}, \tag{8}$$

where $\mu_k = \mu_{k,m} := e^{i\pi(2k+m-1)/(2m)}$, $k \in \{1,\dots,m\}$, are the $2m$th roots of $(-1)^{m+1}$ lying in the left half-plane. We note that these roots satisfy the identities $\mu_k^m = i^{m-1}(-1)^k$ and $\mu_k^{-1} = \overline{\mu}_k = \mu_{m+1-k}$, which are important in what follows.

For an arbitrary function $f \in W_2^m(\mathbb{R}^1_-)$, by $y := y(f;x)$ we denote the function in $W_2^m(\mathbb{R}^1_+)$ having the same initial values as f and providing the minimum of the functional $J_m[y]$. Then (by virtue of the definitions) the extension of f to the positive half-line by $y(f;x)$ is just the desired extension operator of minimum norm. Since the map taking each function $f \in W_2^m(\mathbb{R}^1_-)$ to the set of its limit values $A_m := \{\{a_s\}\} = \{\{f^{(s-1)}(0-)\}\}$ and the map of the space A_m into the set Y_m of the corresponding extremals are linear (this is the case because the Euler equation (7) is linear), it follows that the best approximation operator described here is also linear.

The value $J_m[y]$ for the functions (8) can be rewritten as a quadratic form in the coefficients b_j:

$$\begin{aligned} J_m[y] &= \int_0^\infty \Biggl(\Biggl(\sum_{j=1}^m b_j e^{\mu_j x}\Biggr)\Biggl(\sum_{k=1}^m \overline{b_k e^{\mu_k x}}\Biggr) + \Biggl(\sum_{j=1}^m b_j \mu_j^m e^{\mu_j x}\Biggr)\Biggl(\sum_{k=1}^m \overline{\mu_k^m b_k e^{\mu_k x}}\Biggr)\Biggr)dx \\ &= \sum_{j=1}^m\sum_{k=1}^m q_{j,k}^{(m)} b_j \bar{b}_k, \quad q_{j,k}^{(m)} := -\frac{1+\mu_j^m\overline{\mu}_k^m}{\mu_j+\overline{\mu}_k} = -\frac{1+(-1)^{j+k}}{\mu_j+\overline{\mu}_k},\ j,k \in \{1,\dots,m\}. \end{aligned} \tag{9}$$

This expression can also be rewritten via the initial values $y^{(s-1)}(0) = a_s$, since the coefficients b_j are uniquely determined from the linear algebraic system

$$\sum_{k=1}^m \mu_k^{j-1} b_k = a_j, \qquad j \in \{1,\dots,m\}, \tag{10}$$

whose coefficient matrix is the classical Vandermonde matrix

$$V_m = \|v_{j,k}^{(m)}\| := \|\mu_k^{j-1}\|. \tag{11}$$

(The matrix (11) is nonsingular, since all numbers μ_k are distinct.) We denote the inverse matrix of V_m (explicit formulas for its entries can be found, say, in [4]) by $\Phi_m = \|\varphi_{j,k}^{(m)}\|$, $j,k \in \{1,\dots,m\}$; then $V_m b = a$ or $b = \Phi_m a$ in vector notation. Consequently,

$$\psi_m(a) = (Q_m b, b) = (Q_m\Phi_m a, \Phi_m a) = (G_m a, a), \qquad G_m := \Phi_m^* Q_m \Phi_m, \tag{12}$$

where Q_m is the matrix with entries $q_{j,k}^{(m)}$ defined in (9) and parentheses stand for the ordinary inner product in the m-dimensional unitary space of complex vectors.

On the other hand, integration by parts readily shows that two arbitrary functions $f(x)$ and $g(x)$ in $C^{2m}[0,+\infty)$ that tend to 0 together with their derivatives of order $\leqslant 2m-1$ as $x \to +\infty$ satisfy the identity

$$\int_0^\infty f^{(m)}(x)g^{(m)}(x)\,dx = -f^{(m)}(0)g^{(m-1)}(0) + f^{(m+1)}(0)g^{(m-2)}(0) - \cdots \\ + (-1)^m f^{(2m-1)}(0)g(0) + (-1)^m \int_0^\infty f^{(2m)}(x)g(x)\,dx. \quad (13)$$

Choosing an arbitrary function $y(x)$ of the form (8) as $f(x)$ and the complex conjugate function as $g(x)$ in (13) and taking account of the fact that, by (7), the last term on the right-hand side in (13) is equal to minus the integral of $|y(x)|^2$ over $[0,+\infty]$, we arrive at the relation

$$J_m[y] = \sum_{j=1}^m (-1)^j y^{(m+j-1)}(0)\overline{y^{(m-j)}(0)} = i^{m-1}\sum_{j=1}^m \left(\sum_{k=1}^m b_k(-1)^k \mu_k^{j-1}\right)(-1)^j\overline{y^{(m-j)}(0)}. \quad (14)$$

Now let a vector of initial data $a = (a_1,\dots,a_m)$ be given. Then, as was already noted, the coefficient vector $b = (b_1,\dots,b_m)$ is equal to a multiplied on the left by the matrix Φ_m, $b = \Phi_m a$, and the transition from the vector b to the vector $\tilde{b} := (-b_1, b_2, \dots, (-1)^m b_m)$ is carried out by multiplying b on the left by the matrix P_m whose main diagonal entries are $(P_m)_{kk} = (-1)^k$ and the offdiagonal entries are zero: $\tilde{b} = P_m b$. As to the transition from $\tilde{b}$ to the vector with coordinates equal to the sums with respect to k in the rightmost term in (14), it corresponds to the multiplication of $\tilde{b}$ by the matrix V_m. Finally, the vector with the coordinates $y^{(m-j)}(0)$ results from the multiplication of a on the left by the matrix S_m with units on the secondary diagonal and zeros in all other positions. We have thus derived an alternative formula for the quadratic form $\psi_m(a)$ and the corresponding matrix G_m (see (5), (6), and (14)),

$$\psi_m(a) = J_m[y] = (i^{m-1}V_m P_m \Phi_m a, P_m S_m a) = (G_m a, a), \\ G_m = i^{m-1} S_m P_m V_m P_m \Phi_m. \quad (15)$$

3. The Symmetry Properties of the Matrices G_m

Lemma 1. *The matrix G_m is real and symmetric.*

Proof. We first note that, by definition (see (12)), the matrix G_m is Hermitian and positive definite. Let us prove that it is real. Indeed, the set of solutions of equation (7) that tend to 0 as $x \to +\infty$ exactly coincides with the set of *all* solutions of the mth-order differential equation $(D-\mu_1)\cdots(D-\mu_m)y(x) = 0$, where $D := d/dx$. For even and odd m, it has the forms

$$\prod_{k=1}^{m/2}(D^2 - 2\operatorname{Re}\mu_k D + 1)y = 0, \qquad (D+1)\prod_{k=1}^{(m-1)/2}(D^2 - 2\operatorname{Re}\mu_k D + 1)y = 0, \quad (16)$$

respectively; i.e., $D^m y + B_1 D^{m-1} y + B_2 D^{m-2} y + \cdots + B_{m-1} D y + y = 0$, where all coefficients are real (and even positive). Therefore, if all numbers $y(0), y'(0), \dots, y^{(m-1)}(0)$ are real, then the function $y(x)$ is also real for all $x > 0$, and consequently, the numbers $y^{(m)}(0), y^{(m+1)}(0), \dots, y^{(2m-1)}(0)$ are also real, which, by the first relation in (14), implies that all entries of the matrix G_m are real; thus, it also follows that G_m is symmetric.

The proof of Lemma 1 is complete.

Lemma 2. *The matrix G_m is also symmetric with respect to the secondary diagonal: $G_m = S_m G_m S_m$.*

Proof. We start from the relation $G = i^{m-1} SPVP\Phi$ (see (15)), where the subscript m on the matrices is omitted for brevity. We introduce the diagonal matrix $H := \operatorname{diag}(\mu_k^{m-1})$. Recall that the multiplication of a square matrix by S on the *left* (*right*) is equivalent to the reflection with respect to the middle *horizontal* (*vertical*), and the multiplication on the *left* (*right*) by a diagonal matrix implies the multiplication of each *row* (*column*) by the corresponding diagonal entry. Therefore, the matrices VS and SV satisfy the relations (see (11) and the definition of μ_k after formula (8))

$$VS = \|\mu_{m+1-k}^{j-1}\| = \|\mu_k^{1-j}\|, \quad SV = \|\mu_k^{m-j}\| = VSH \iff \Phi S = H^{-1}S\Phi. \tag{17}$$

Furthermore, we have the chain of identities

$$\begin{aligned} SGS &= (-i)^{m-1}SP(SV)P(\Phi S) = (-i)^{m-1}SP(VSH)P(H^{-1}S\Phi) \\ &= (-i)^{m-1}SPV(SPS)\Phi = i^{m-1}SPVP\Phi = G, \end{aligned}$$

where we have used the fact that the diagonal matrices H and P commute and the simple matrix identities $SP = (-1)^{m-1}PS$ and $SS = PP = E := \mathrm{id}_m$.

The proof of Lemma 2 is complete.

However, the following property, containing, in particular, the assertion about the similarity of the matrix G_m and its inverse, is most unexpected and very essential.

Lemma 3. $G_m^{-1} = P_m G_m P_m$.

Proof. It readily follows from formula (15), Lemma 2, and the identity used in its proof that the desired chain of relations holds:

$$G^{-1} = (-i)^{m-1}(SPVP\Phi)^{-1} = (-i)^{m-1}VP\Phi PS = i^{m-1}PS(SPVP\Phi)SP = P(SGS)P = PGP.$$

The following assertions are an immediate consequence of related definitions and Lemma 3.

Corollary 1. *The spectra of the matrices G_m and G_m^{-1} coincide. In particular, the minimum and maximum eigenvalues of the matrix G_m satisfy the relation $\lambda_{\min}(G_m) = \lambda_{\max}^{-1}(G_m)$, and if $\xi = (\xi_1, \dots, \xi_m)$ is an eigenvector that corresponds to $\lambda_{\max}(G_m)$, then $P_m\xi = (-\xi_1, \xi_2, \dots, (-1)^m\xi_m)$ is also an eigenvector and corresponds to $\lambda_{\max}^{-1}(G_m)$.*

Corollary 2. *The maximum Ω_m and the minimum ω_m of the quadratic form $\psi_m(a)$ on the unit sphere $\sum |a_s|^2 = 1$ are related by the formula $\Omega_m = \omega_m^{-1} = \lambda_{\max}(G_m)$.*

Next, using the even extension from the left half-line to the right half-line and taking the definitions of the expression $\psi_m(a)$ (see (5) and (6)) and the matrix P_m into account, we conclude that the identity

$$\min\{\|y(x)\|^2_{W_2^m(\mathbb{R}^1_-)} : y^{(s-1)}(0) = a_s,\, s \in \{1, \dots, m\}\} = \psi_m(-P_m a), \tag{18}$$

holds, which, in turn, implies the following assertion in view of Corollary 1.

Corollary 3. *The norm of the best approximation operator $T_m\colon W_2^m(\mathbb{R}^1_-) \to W_2^m(\mathbb{R}^1)$ is given by the formula*

$$\tau_m = \sqrt{1 + \frac{\Omega_m}{\omega_m}} = \sqrt{1 + \lambda^2_{\max}(G_m)}\,. \tag{19}$$

Remark. Clearly, the expression $\Omega_m^{1/2}$ is the minimum possible norm of the extrapolation operators $A_m^{(2)} \to W_2^m(\mathbb{R}^1_+)$, where $A_m^{(2)}$ is the space of vectors of initial values $(f(0), f'(0), \dots, f^{(m-1)}(0))$ equipped with the Euclidean norm. Gabushin [5] (see also [6], 2.4.5) obtained explicit (but ineffective) formulas for the maximum possible values of the intermediate derivatives at zero, i.e., the numbers

$$\Gamma^+_{s,m} := \max\{|y^{(s)}(0)| : \|y\|_{W_2^m(\mathbb{R}^1_+)} = 1\}, \qquad s \in \{0, 1, \dots, m-1\}. \tag{20}$$

As can be shown with regard to Lemma 3, these constants can be expressed via the diagonal entries of the matrix G_m: $\Gamma^+_{s,m} = (g^{(m)}_{s+1,s+1})^{1/2}$. Taikov (see [7] and [6], 2.4.4) proved that the numbers differing from those in (20) in that the norm is taken in $W_2^m(\mathbb{R}^1)$ rather than $W_2^m(\mathbb{R}^1_+)$ can be expressed by the simple constructive formulas $\Gamma_{s,m} = (2m\sin((2s+1)\pi/(2m)))^{-1/2}$.

4. Upper Bounds for the Matrices $\boldsymbol{\Phi_m}$

Lemma 4. *The relations* $\ln\max|\varphi_{j,k}^{(m)}| = (K_0+\varepsilon_m)m$ *hold, where* K_0 *is the constant defined in* (4) *and* $\varepsilon_m \to 0$ *as* $m\to\infty$.

Proof. The matrix relation $\Phi V = E$ (the subscript m is omitted) can be rewritten in coordinates (see (11)) as follows:

$$\sum_{s=1}^{m}\varphi_{j,s}\mu_k^{s-1} = \delta_{j,k}, \quad \text{where } \delta_{j,k} := 0\ (j\neq k),\ \delta_{k,k} := 1, \tag{21}$$

which implies that the entries in each row $\{\varphi_{j,s}\}$ of the matrix Φ are the coefficients of the $(m-1)$st-order Lagrange algebraic interpolation polynomials

$$L_{m-1,j}(z) := \sum_{s=1}^{m}\varphi_{j,s}z^{s-1} \quad \text{such that } L_{m-1,j}(\mu_k) = 0,\ k\neq j,\ L_{m-1,j}(\mu_j) = 1, \tag{22}$$

and hence these polynomials can be represented as

$$L_{m-1,j}(z) = \prod_{k\neq j}\frac{z-\mu_k}{\mu_j-\mu_k} = \left(\prod_{k\neq j}\frac{1}{\mu_j-\mu_k}\right)\prod_{k\neq j}(z-\mu_k). \tag{23}$$

As follows from geometrical considerations, the modulus of the first factor in (23), which depends only on m and j and does not depend on z, tends to its maximum value for a given m as μ_j maximally approaches the point -1. Consequently, since $\mu_{r+1} = -1$, the following inequality holds for odd $m = 2r+1$ and any $j\in\{1,\dots,m\}$:

$$A_{m,j} := \prod_{k\neq j}\left|\frac{1}{\mu_j-\mu_k}\right| \leqslant A_{2r+1,r+1} = \prod_{s=1}^{r}\frac{1}{|\mu_s+1|^2} = \prod_{s=1}^{r}\left(2\sin\frac{\pi s}{2m}\right)^{-2}. \tag{24}$$

Likewise, for even $m = 2r$ we have

$$A_{m,j} \leqslant A_{2r,r} = A_{2r,r+1} = \frac{1}{\sqrt{2}}\prod_{s=1}^{r-1}\left(2\sin\frac{\pi s}{2m}\right)^{-2} \tag{25}$$

for all $j\in\{1,\dots,m\}$.

Thus, the relation

$$A_m := \max_j A_{m,j} = A_{m,[m/2]+1} \leqslant \prod_{s=1}^{[m/2]}\left(2\sin\frac{\pi s}{2m}\right)^{-2} \tag{26}$$

has been derived for all m.

For the case in which $m = 2r+1$ and $k = r+1$, by grouping the pairs of parentheses corresponding to complex conjugate roots in the second factor in (23), we find

$$\prod_{k\neq r+1}(z-\mu_k) = \prod_{s=1}^{r}(z^2-(2\operatorname{Re}\mu_s)z+1) =: \widetilde{L}_{2r,r+1}(z) = \sum_{s=0}^{2r}\tilde{\varphi}_{r,s}z^s. \tag{27}$$

All coefficients of the polynomial (27) are positive, and hence their sum is equal to the value of the polynomial at $z = 1$, i.e.,

$$B_{2r+1,r+1} := \sum_{s=0}^{2r}|\tilde{\varphi}_{r,s}| = \sum_{s=0}^{2r}\tilde{\varphi}_{r,s} = \widetilde{L}_{2r,r+1}(1) = \prod_{k\neq r+1}(1-\mu_k) = \prod_{s=1}^{r}\left(2\cos\frac{\pi s}{2m}\right)^2. \tag{28}$$

Here we have again used geometric considerations related to the location of the roots and taken account of the fact that the vectors $1-\mu$ and $1+\mu$ are orthogonal for μ lying on the unit circle.

Furthermore, if m is odd, $m = 2r+1$, but $j < r+1$, then

$$\widetilde{L}_{2r,j}(z) = (z+1)\prod_{s=1}^{r}{}^{j'}(z^2-2(\operatorname{Re}\mu_s)z+1), \tag{29}$$

where the superscript j' implies that the factor with $s = j$ has been discarded, and consequently

$$B_{2r+1,j} = B_{2r+1,2r+1-j} \leqslant 2 \prod_{s=1}^{r}{}^{j'} \Big(2\cos\frac{\pi s}{2m}\Big)^2 \leqslant B_{2r+1,r+1} \tag{30}$$

since $\pi j/(2m) < \pi/4$ and hence $(2\cos(\pi j/(2m)))^2 > 2$.

For even $m = 2r$ and for $j \leqslant r$, in a similar way we obtain

$$B_{2r,j} = B_{2r,2r-1-j} \leqslant 2 \prod_{s=1}^{r}{}^{j'} \Big(2\cos\frac{\pi(2s-1)}{4m}\Big)^2 \leqslant 4 \prod_{s=1}^{r} \Big(2\cos\frac{\pi s}{2m}\Big)^2. \tag{31}$$

Combining (31), (30), and (26), we arrive at the inequality

$$B_m := \max_j B_{m,j} \leqslant 4 \prod_{s=1}^{[m/2]} \Big(2\cos\frac{\pi s}{2m}\Big)^2 \tag{32}$$

for all m, which, together with (23) and (26), gives an estimate for the sum of moduli of the entries in an arbitrary row of the matrix Φ_m:

$$\sum_{k=1}^{m} |\varphi_{j,k}^{(m)}| \leqslant A_m B_m \leqslant 4 \prod_{s=1}^{[m/2]} \Big(\cot\frac{\pi s}{2m}\Big)^2. \tag{33}$$

Passing to logarithms in (33), we obtain the inequality

$$\ln \max_{j,k} |\varphi_{j,k}^{(m)}| \leqslant \ln 4 + 2 \sum_{s=1}^{[m/2]} \ln\Big(\cot\frac{\pi s}{2m}\Big) \tag{34}$$

for all m and j.

Next, useing the monotone decrease of the function $\ln(\cot x)$ on the interval $(0,\pi/4)$, we write out inequalities for a Riemann sum and related integrals:

$$\int_{\pi/(2m)}^{\pi/4} \ln(\cot x)\,dx \leqslant \frac{\pi}{2m} \sum_{s=1}^{[m/2]} \ln\Big(\cot\frac{\pi s}{2m}\Big) \leqslant \int_0^{\pi/4} \ln(\cot x)\,dx. \tag{35}$$

Since the integral on the left-hand side tends to that on the right-hand side, we obtain the relation

$$\sum_{s=1}^{[m/2]} \ln\Big(\cot\frac{\pi s}{2m}\Big) = \Big(\frac{2m}{\pi} + o(m)\Big) \int_0^{\pi/4} \ln(\cot x)\,dx. \tag{36}$$

Finally, from (25) and (30) we obtain the lower bound

$$\ln \max_{j,k} |\varphi_{j,k}^{(m)}| \geqslant \ln \Big| \sum_{k=1}^{m} \varphi_{[m/2]+1,k}^{(m)} \Big| - \ln m \geqslant 2 \sum_{s=1}^{[m/2]} \ln\cot\frac{\pi s}{2m} + o(m), \tag{37}$$

which, in conjunction with (32), (36), and definition (4) of the number K_0, implies the assertion of Lemma 4.

5. Bounds for the Least Eigenvalue of the Matrix Q_m

The following assertion pertaining to the matrices defined by formula (9) is also of interest in itself.

Lemma 5. $\ln \lambda_{\min}(Q_m) \approx -K_0 m$, $m \to \infty$.

Proof. Let m_o and m_e be the numbers of odd and even positive integers,respectively, not exceeding m, i.e., $m_o := [(m+1)/2]$ and $m_e := [m/2]$, so that $m_o + m_e = m$. Since $q_{j,k}^{(m)} = 0$ if j and k are of opposite parities, we conclude that the matrix Q_m has a block diagonal form in the basis $(e_1, e_3, \dots, e_{2m_o-1}; e_2, e_4, \dots, e_{2m_e})$ obtained from the standard basis $(e_1, e_2, e_3, \dots, e_m)$ by a permutation:

$$Q_m = \begin{pmatrix} Q^{(o)} & 0 \\ 0 & Q^{(e)} \end{pmatrix}, \qquad q^{(o)}_{j,k} := \frac{-2}{\mu_{2j-1} + \overline{\mu}_{2k-1}}, \qquad q^{(e)}_{j,k} := \frac{-2}{\mu_{2j} + \overline{\mu}_{2k}}, \tag{38}$$

where the diagonal blocks $Q^{(o)}$ and $Q^{(e)}$ are $m_o \times m_o$ and $m_e \times m_e$ matrices, respectively.

The assertion of Lemma 5 is equivalent to the following relations for the maximum eigenvalues of the inverse ($m_o \times m_o$ and $m_e \times m_e$) matrices $U^{(o)}_{m_o}$ and $U^{(e)}_{m_e}$ of $Q^{(o)}$ and $Q^{(e)}$, respectively:

$$\ln \lambda_{\max}(U^{(o)}_{m_o}) \approx K_0 m, \quad \ln \lambda_{\max}(U^{(e)}_{m_e}) \approx K_0 m, \qquad m \to \infty. \tag{39}$$

Using the formulas for Cauchy determinants in [8], § 14, (see also [9], Sec. 2, where one should set $x_r = \overline{y}_r = \mu_r$), we obtain the following explicit expressions for the diagonal entries of the matrices $U^{(o)}_{m_o}$ and $U^{(e)}_{m_e}$:

$$u^{(o)}_{k,k} = \frac{|\mu_{2k-1} + \overline{\mu}_{2k-1}|}{2} \prod_{j \neq k} \left| \frac{\mu_{2j-1} + \overline{\mu}_{2k-1}}{\mu_{2j-1} - \overline{\mu}_{2k-1}} \right|^2, \qquad u^{(e)}_{k,k} = \frac{|\mu_{2k} + \overline{\mu}_{2k}|}{2} \prod_{j \neq k} \left| \frac{\mu_{2j} + \overline{\mu}_{2k}}{\mu_{2j} - \overline{\mu}_{2k}} \right|^2. \tag{40}$$

The products with respect to j in (40) can be estimated in just the same manner as those in the proof of Lemma 4. (See the derivation of the estimates (26), (32), and (33)). Introducing the notation $k_o := [m_o/2] + 1$ and $k_e := [m_e/2] + 1$, for the products over odd indices we obtain

$$w_k := \prod_{j \neq k} \left| \frac{\mu_{2j-1} + \overline{\mu}_{2k-1}}{\mu_{2j-1} - \overline{\mu}_{2k-1}} \right| \leqslant \prod_{j \neq k} \left| \frac{\mu_{2j-1} - 1}{\mu_{2j-1} + 1} \right| \leqslant \prod_{s=1}^{k_o} \left(2 \cot \frac{\pi s}{m} \right)^2 =: p^{(o)}, \tag{41}$$

and for $k = k_o$ one has $w_k \geqslant p^{(o)}/4$.

Likewise, for the products over even indices we have

$$\prod_{j \neq k} \left| \frac{\mu_{2j} + \overline{\mu}_{2k}}{\mu_{2j} - \overline{\mu}_{2k}} \right| \leqslant \prod_{s=1}^{k_e} \left(2 \cot \frac{\pi s}{m} \right)^2 \leqslant 4 \prod_{j \neq k_e} \left| \frac{\mu_{2j} + \overline{\mu}_{2k_e}}{\mu_{2j} - \overline{\mu}_{2k_e}} \right|. \tag{42}$$

In addition, with regard to the inequality $\max |\mu_k + \overline{\mu}_k| \leqslant 2$, whereas both numbers $|\mu_{2k_o-1} + \overline{\mu}_{2k_o-1}|$ and $|\mu_{2k_e} + \overline{\mu}_{2k_e}|$ are close to 2 for all sufficiently large m, we conclude that the maximum of the diagonal entries of the matrices $U^{(o)}_{m_o}$ and $U^{(e)}_{m_e}$ is of the order of

$$\max_k u^{(o)}_{k,k} \asymp \max_k u^{(e)}_{k,k} \asymp u^{(o)}_{[m/4],[m/4]} \asymp u^{(e)}_{[m/4],[m/4]} \asymp \prod_{s=1}^{[m/4]} \left(\cot \frac{\pi s}{m} \right)^4. \tag{43}$$

The argument in the foregoing section (see (35)–(37)) implies that

$$\ln \max_k u^{(o)}_{k,k} \approx \max_k u^{(e)}_{k,k} \approx 4 \sum_{s=1}^{[m/4]} \ln \left(\cot \frac{\pi s}{m} \right) \approx \frac{4}{\pi} \mathbf{G} m = K_0 m, \qquad m \to \infty, \tag{44}$$

and since the inequalities $\ln \max u_{k,k} \leqslant \ln \lambda_{\max}(U_n) \leqslant \ln \sum u_{k,k} \leqslant \ln n + \ln \max u_{k,k}$ hold for any positive definite $n \times n$ matrix U_n, we see that relations (44) imply (39).

The proof of Lemma 5 is complete.

Remark. We note that, according to (9), the identity $Q_m = \widetilde{Q}_m + \widehat{Q}_m$ holds, where $\widetilde{Q}_m$ is the matrix with entries $\tilde{q}_{j,k} := -2(\mu_j + \overline{\mu}_k)^{-1}$, and we have $\widehat{Q}_m := P_m \widetilde{Q}_m P_m$. The argument in the proof of Lemma 5 implies that

$$\ln \lambda_{\min}(\widetilde{Q}_m) = \ln \lambda_{\min}(\widehat{Q}_m) \approx -2K_0 m,$$

whereas the logarithm of the minimum eigenvalue of the sum of $\widetilde{Q}_m$ and $\widehat{Q}_m$ decreases twice as slowly: $\ln \lambda_{\min}(Q_m) \approx -K_0 m$.

6. End of Proof of the Theorem

The definition of the matrix G_m (see formula (15)) implies the upper bound

$$(G_m a, a) = (i^{m-1} V_m P_m \Phi_m a, P_m S_m a) \leqslant m^2 \Big(\max_{j,k} |\varphi_{j,k}^{(m)}| \Big) |a|^2, \tag{45}$$

whence, with regard to definition (19), Lemma 4 (see (36)), and formula (4), we obtain the inequality

$$\ln \tau_m \leqslant \ln \lambda_{\max}(G_m) + o(1) \leqslant \ln \max_{j,k} \ln |\varphi_{j,k}^{(m)}| + 2 \ln m + O(1) \leqslant K_0 m + o(m). \tag{46}$$

Now it remains to verify that the opposite inequality also holds.

Formula (12) defining the matrix G_m implies the chain of inequalities

$$(G_m a, a) = (\Phi_m^* Q_m \Phi_m a, a) = (Q_m \Phi_m a, \Phi_m a) \geqslant \lambda_{\min}(Q_m)(\Phi_m a, \Phi_m a). \tag{47}$$

Let us take the unit vector $\tilde{a}$, $\|\tilde{a}\| = 1$, with equal coordinates $\tilde{a}_j = 1/\sqrt{m}$ as a. By Lemma 4 (see (37)), it satisfies the inequality

$$\|\Phi_m \tilde{a}\| \geqslant m^{-1/2} \left| \sum_{k=1}^{m} \varphi_{[m/2]+1,k} \right| \geqslant e^{(K_0 - \varepsilon_m)m}. \tag{48}$$

Therefore (47), (49), and Lemma 5 imply that

$$\ln \lambda_{\max}(G_m) \geqslant \ln(G_m \tilde{a}, \tilde{a}) \geqslant \ln \lambda_{\min}(Q_m) + 2 \ln \|\Phi_m \tilde{a}\| \geqslant (K_0 - \tilde{\varepsilon}_m)m + o(m), \tag{49}$$

which completes the proof of the theorem.

In conclusion, we note that initial data vectors $\tilde{a}$ with *equal coordinates* ensure the asymptotic growth of the order of $K_0 m/2$ for the logarithm of the norm of the extrapolation in $W_2^m(\mathbb{R}_+^1)$. However, the extrapolation (for even values of m) of these initial data to the *negative* half-line has the norm in $W_2^m(\mathbb{R}_-^1)$ exactly equal to 1. Therefore, it remains unclear what sets of initial data give the difference between the logarithms of the norms of the extrapolations in $W_2^m(\mathbb{R}_+^1)$ and $W_2^m(\mathbb{R}_-^1)$ equivalent to $K_0 m$.

References

1. V. I. Burenkov and A. L. Gorbunov, "Sharp estimates of the norms of extension operators for Sobolev spaces," Izv. RAN, **61**, No. 1, 3–44 (1997).
2. V. I. Burenkov and G. A. Kalyabin, "Lower estimates of the norm of extension operators for Sobolev spaces on the halfline," Math. Nachr., **218**, 19–23 (2000).
3. H. B. Dwight, Tables of Integral and Other Mathematical Data [in Russian], Nauka, Moscow, 1978.
4. A. Klinger, "The Vandermonde matrix, Amer. Math. Monthly," **74**, 571–574 (1967).
5. V. N. Gabushin, "Inequalities for the norms of the function and its derivatives in L_p metrics," Mat. Zametki, **1**, No. 3, 291–298 (1968).
6. V. M. Tikhomorov, Some Questions in Approximation Theory [in Russian], Moscow University Press, Moscow, 1976.
7. L. V. Taikov, "Kolmogorov type inequalities and the best formulas for numerical differentiation," Mat. Zametki, **4**, No. 2, 233–238 (1968).
8. N. I. Akhiezer, Lectures in Approximation Theory [in Russian], Nauka, Moscow, 1965; English transl.: N. I. Achieser, Theory of approximation, Dover Publications, New York, 1992.
9. G. A. Kalyabin, "Asymptotic formulas for smallest eigenvalues of Hilbert type matrices," Funkts. Anal. Prilozhen., **35**, No. 1, 80–84 (2001).

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