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A NOTE ON HAUSDORFF MEASURES OF SIERPINSKI CARPETS

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Abstract

In this paper, regular Sierpinski carpet as a new concept is given. The exact value of Hausdorff measure of the regular Sierpinski carpet and the range of Hausdorff measures for all forms of generalized Sierpinski carpets is also obtained. For any one of the generalized Sierpinski carpets we show that there exists a regular carpet such that they have the same Hausdorff measures.

1 Introduction

To calculate and estimate the Hausdorff measure is a very difficult problem in fractal geometry. But in recent years, some progresses about the Hausdorff measures of Sierpinski carpets are maden. The exact value of Hausdorff measure of a Sierpinski carpet was calculated by Zhou and Wu (See[1]). On the base of [1], in [2] and [3] the exact values of Hausdorff measures of some generalized Sierpinski carpets are obtained. In this paper, we shall continue the research on the Housdorff measures of Sierpinski carpets.

Let S_0 be the unit square in R^2 and m be a real number satisfying $m \geqslant 4$. Retaining 4 smaller squares in S_0 arbitrarily such that their edges parallel the edges of S_0 respectively, and the length of their edge is $\frac{1}{m}$, and their interiors are disjoint. At the same time, the

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interior of the other part be cut out. Let S_1 be the union of the 4 smaller squares. In one of the 4 squares in S_1, we repeat this process such that the locations of the retained squares are the same as the last time. We obtain 4^2 smaller squares with edge to be $\frac{1}{m^2}$ and the union of them be denoted by S_2, see Fig1.

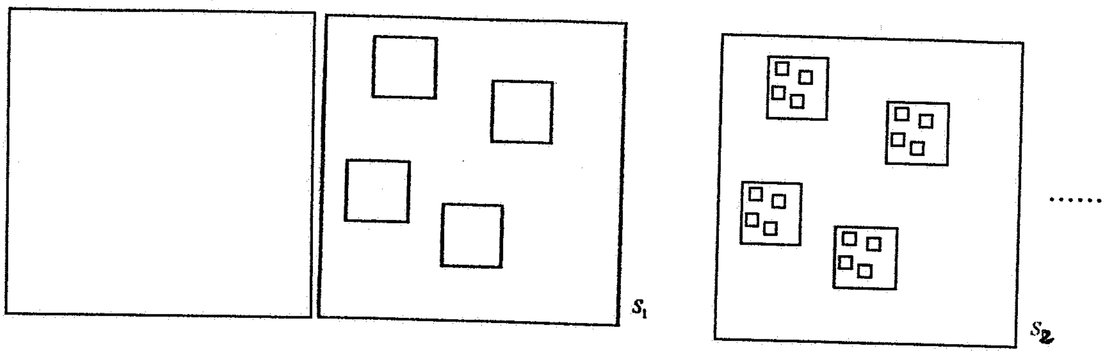

Fig 1

We do the above process infinitly, and obtain

$$S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_k \supset \cdots,$$

the nonempty set $S=\bigcap_{k=0}^{\infty} S_k$ is called a generalized Sierpinski carpet. It was studied in the case of the 4 squares of S_1 be located at the 4 angels of S_0 in [2]. Starting from S_0, by retaining different locations of the 4 smaller squares, different generalized Sierpinski carpets be obtained. This class of fractals this class are self-similar sets and satisfy the open set condition. Let s be the Hausdorff dimension of S, then $4(\frac{1}{m})^s=1$(see [4]). It imples $0<s=\log_m 4\leqslant 1$. For $k\in Z^+$, then S_k is composed of 4^k smaller squares and the length of their edge is $\frac{1}{m^k}$. One of these squares is called elementary square of S_k and denoted by $\square_k$.

Naturally we ask the following question: For generalized Sierpinski carpet S in Figure 1, what is the range of its Hausdorff measure? Are there two generalized Sierpinski carpets with the same Hausdorff measure? We shall give the answer in this section. The notations in this paper are the same as [1] and [3]. We give the following concept:

Defination. The following generalized Sierpinski carpet is called a regular Sierpinski carpet, see Figure 2.

The 4 smaller squares in S_1 are on the 4 angles of the square with the length of the edge p, where $\frac{2}{m}\leqslant p\leqslant 1$.

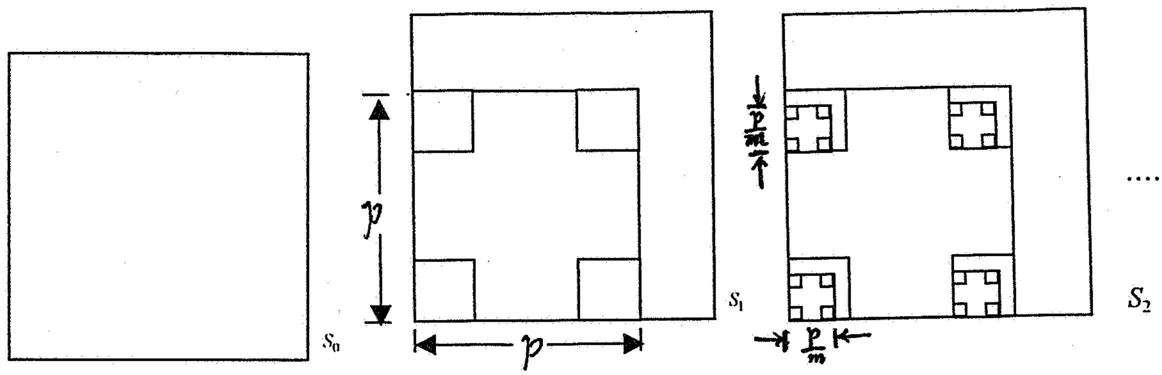

Fig 2

For regular Sierpinski carpet S, the following three theorems are well known.

Theorem A[1]. *If $m=4$, $p=1$, then $H(S)=\sqrt{2}$, where $s=1$.*

Theorem B[2]. *If $m\geqslant 4$, $p=1$, then $H^s(S)=\sqrt{2}^s$, where $s=\log_m 4$.*

Theorem C[3]. *If $m=4$, $p=\frac{2}{m}$, then $H(S)=\frac{\sqrt{2}}{3}$, where $s=1$.*

In this note, we shall prove the following theorem:

Theorem. *For all forms of generalized Sierpinski carpets in Figure 1, the following holds*

$$\left(\frac{1}{m-1}\right)^s \sqrt{2}^s \leqslant H^s(S) \leqslant \sqrt{2}^s;$$

and for any t, if $(\frac{1}{m-1})^s \sqrt{2}^s \leqslant t \leqslant \sqrt{2}^s$, then there exists a regular Sierpinski carpet S, such that $H^s(S)=t$. Hence $H^s(S)$ is continious.

Corollary. For any form of generalized Sierpinski carpets in Figure 1, there exists a regular Sierpinski carpet such that they have the same Hausdorff measure.

2 Proof of the Theorem

$H^s(S)\leqslant \sqrt{2}^s$ is trivial. Next we shall show $(\frac{1}{m-1})^s \sqrt{2}^s \leqslant H^s(S)$. For regular carpet, if $m=4$, $p=\frac{2}{m}$, then by Theorem C, we have $s=1$, $H(S)=\frac{\sqrt{2}}{3}$. If $m\geqslant 4$, $\frac{2}{m}\leqslant p\leqslant 1$, by putting S_0 in Figure 2 on a rectangular coordinates system, then the convex shell of S is a square with vertexes coordinates are $Z_1=(0,0)$, $Z_2=(\frac{mp-1}{m-1},0)$, $Z_3=(\frac{mp-1}{m-1},$

$\frac{mp-1}{m-1}$), $Z_4=(0,\frac{mp-1}{m-1})$, where $\frac{mp-1}{m-1}$ is obtained by calculating $1-(1-p)\sum_{n=0}^{\infty}\frac{1}{m^n}$. By using the same method in the proof of Theorem C[3] and combining with Theorem B, we obtain the Hausdorff measure of this regular Sierpinski carpet

$$H^s(S)=\left(\frac{pm-1}{m-1}\right)^s\sqrt{2}^s. \qquad (1)$$

Especialy, if $p=\frac{2}{m}$, then $H^s(S)=(\frac{1}{m-1})^s\sqrt{2}^s$. Therefore, we only need to show the Hausdorff measure of any form of generalized Sierpinski carpets in Figure 1 is greater than or equal to the Hausdorff measure of regular Sierpinski carpet with $p=\frac{2}{m}$.

Define a finite measure μ(mass distribution) on S_0 in Figure 1 satisfying

$$\mu(S_0)=\left(\frac{1}{m-1}\right)^s\sqrt{2}^s,$$

$$\mu(\square_n)=\frac{1}{4^n}\left(\frac{1}{m-1}\right)^s\sqrt{2}^s,$$

$$\mu(S_0-S)=0.$$

For regular Sierpinski carpet S' with $p=\frac{2}{m}$, if we define a finite measure μ' on S_0' to be the same as μ, then for any measurable set V in S_0', we have $\mu'(V)\leqslant|V|^s$(see [2]).

The following assersion is explicitly: For elementary squares $\square_i$ of S_i in Figure 1, some of them are chosen arbitrarily and denoted by $\square_i^1,\cdots,\square_i^l(1\leqslant l\leqslant 4^i)$. If the diameter of the union of them is denoted by d, then there exist l elementary squares of S_i' such that $d\geqslant d'$, where d' denotes the diameter of the union of the l elementary squares of S_i'. (For $i=1,2,\cdots$, we test this result by using the method of induction easily).

For any measurable set V in S_0, without loss of generality, we may choose V to be a simply connected region. Suppose there exist l_i elementary squares $\square_i$ in $V(i=1,2,\cdots,)$ then we have $l_i\mu(\square_i)\to\mu(V)(i\to\infty)$. Let U_i denote the union of these $\square_i$ contained in V and d_i denote the diameter of U_i. For these l_i elementary squares $\square_i$ in V, by using the above assersion, there exist l_i elementary squares in S_i' such that $d_i'\leqslant d_i$, $(i=1,2,\cdots)$, where d_i' denotes the diameter of U_i', and U_i' is the union of the l_i elementary squares in $S_i'(i=1,2,\cdots)$. By the definitions of μ and μ', we have

$$\mu'(U_i')=\mu(U_i)=l_i\mu(\square_i)\to\mu(V)(i\to\infty),$$

hence $\mu'(U_i')\leqslant d'^s_i\leqslant d_i^s\leqslant|V|^s$, and hence $\mu(V)\leqslant|V|^s$. By the mass distribution principle[4], we have

$$H^s(S)\geqslant\mu(S)=\left(\frac{1}{m-1}\right)^s\sqrt{2}^s.$$

For $\frac{2}{m} \leqslant p \leqslant 1$, by (1) and the intermediate value theorem, we have $H^s(S)$ is continuous relative p and for any t satisfying $(\frac{1}{m-1})^s \sqrt{2}^s \leqslant t \leqslant \sqrt{2}^s$, there exists a regular carpet S such that $H^s(S)=t$.

Proof or the Corollary. This is the direct result of the theorem.

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References

[1] Zhou Zuoling, Wu Min, The Hausdorff Measure of a Sierpinski Carpet, Science in China (Series A), 29(2), (1999), 138—144.

[2] Zhu Yican, Luo Jin, The Hausdorff Measure of Generalized Sierpinski Carpets, Approx. Theory and its Appl., 16:2, (2000), 13—18.

[3] Wu Min, Zhou Zuoling, About the Hausdorff Measures of Sierpinski Carpets, Progress in Science, 11(1999), 1032—1035.

[4] Falconer, K., Fractal Geometry-Mathematical Foundations and Applications. John Wiley and Sons. New York. 1990.

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