Multibody System Dynamics **7:** 307–329, 2002.

On Derivation of Motion Equations for Systems with Non-Holonomic High-Order Program Constraints

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(Received: 7 January 2000; accepted in revised form: 8 February 2001)

Abstract. The paper develops and discusses the generalization of modeling methods for systems with non-holonomic constraints. The classification of constraints has been revisited and a concept of program constraints introduced. High-order non-holonomic constraints (HONC), as presented in examples, are the generalization of the constraint concept and may, as a constraint class, include many of motion requirements that are put upon mechanical systems. Generalized program motion equations (GPME) that have been derived in the paper can be applied to systems with HONC. Concepts of virtual displacements and a generalized variational principle for high-order constraints are presented. Classical modeling methods for non-holonomic systems based on Lagrange equations with multipliers, Maggi, Appell–Gibbs, Boltzman–Hamel, Chaplygin and others are peculiar cases of GPME. The theory has been illustrated with examples of high-order constraints. Motion equations have been derived for a system subjected to a constraint that programmed a trajectory curvature profile. Efficiency, advantages and disadvantages of GPME have been discussed.

Key words: non-holonomic high-order constraints, program constraints, generalized program motion equations.

1. Introduction

The concept of constraints in classical mechanics, i.e. mechanics based on Euler–Lagrange or Hamiltonian approaches and their modifications, is based on an assumption that constraints are given *a priori* and they are put upon a mechanical system through other bodies or physical systems. These constraints, position and kinematical ones are referred to as material constraints and are 'known' and 'given' by the nature. That understanding of the constraint concept and its nature is also reflected in a common assumption that there are two kinds of situations when non-holonomic constraints arise: when bodies are in contact with each other and roll without slipping or at angular momentum conservation in a multibody system, see,

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for example, [8, 24, 33]. Material constraints constitute a significant class of motion restrictions in engineering area but some problems may be formulated different way. For example, in synthesis or optimal synthesis problems, before a system is designed, we put requirement-constraints upon its performance. Constraints are formulated first and then we look for modeling methods to describe the system motion. Generally, these constraint sources are not in other bodies. Such non-material constraints then may arise as performance, designing, operating or safety requirements and be formulated analytically as algebraic or differential equations, or inequalities. For example, a constraint on a free robot link has been shown to be a second-order non-holonomic constraint and an underactuated manipulator is a typical example of a second-order non-holonomic system [37]. In navigation of wheeled mobile robots, to avoid the wheel slippage and mechanical shocks during motion, dynamic constraints such as acceleration limits have to be taken into account [25]. In path planning problems, for car-like robots, to secure motion smoothness two additional constraints are added. They are put upon a trajectory curvature and its time derivative [46] so constraints are of the second and third orders. Similarly, a vehicle operates within a limited workspace is subjected to a non-holonomic constraint – its path curvature cannot exceed some value [25, 46, 47]. Driving and task constraints are other examples of non-material constraints [33, 36].

Other research areas have also reported manifestation of high-order coordinate derivatives that are responsible for some dynamic phenomena, for example in bio-mechanical modeling, as reported in [16], a third time position derivative has been found to influence the smoothness of a limb motion. In what follows, maybe high-order constraints will have to be taken into account in advanced modeling and analysis or synthesis of physical systems.

These new 'constraint sources' that have been revealed were the motivation to call constraints any analytical formulations like

$$F_\beta(t, q_1, \ldots, q_\sigma, \dot{q}_1, \ldots, \dot{q}_\sigma) = 0, \quad \beta = 1, \ldots, b, \quad b \leq \sigma. \tag{1.1}$$

This extended concept of constraints is not completely new. First ideas were introduced by Mieszczerski at the beginning of the XXth century. At the same time a concept of servo-constraints was developed by Beghin. Appell [2] described these constraints as those 'that can be realized not through the direct contact'. Ideas proposed by these authors were limited to first-order constraints as expressed by Equation (1.1). In Section 3 we present an extension of the above constraint concept.

Concluding, motivations to revisit and investigate more intensively systems with non-holonomic non-material constraints are as follows:

1. Constraints that arise in analysis or synthesis of physical systems when some requirements are put upon their motion characteristics may be non-material ones and of the order higher than two.

2. For the first and the second-order non-holonomic constraints (these second-order constraints often come from differentiation of first-order ones) classical methods can be tedious and cumbersome or some of those methods serve for specific kinds of constrained systems only, see discussion in [41], so a unified approach to the modeling of non-holonomic systems would be welcome.

The purpose of the paper is to deliver a modeling analysis tool i.e. to derive generalized program motion equations (GPME) that allow the generation of motion equations of constrained systems to be treated in a new and uniform manner.

The paper is organized as follows: Section 2 provides classification of constraints and Section 3 describes and discusses a concept of program constraints. Examples of non-holonomic high-order constraints are given in Section 4. Section 5 is devoted to a short overview of modeling methods for non-holonomic systems. They serve as a background to the discussion of generalized program motion equations (GPME). Section 6 provides derivation of GPME, also includes an example of derivation of motion equations for a constrained system and discussion of GPME. Paper ends with final conclusions and a list of references.

Problems concerning integrability conditions and other mathematical aspects of non-holonomic systems are not discussed here. Readers interested in these problems can find some considerations in, for example, [33, 52, 53] and references provided there.

2. Classifications of Constraints

In this section classifications of constraints that exist in classical analytical mechanics are revisited. We define a system as being constrained when any restriction is put upon its motion conditions and it can be formulated in an algebraic or differential equation form, or expressed by an inequality. We follow then the more recent concept of the constraint definition and constraints can be material or non-material ones of arbitrary orders. One of the general classifications divides constraints into unilateral ones represented by inequalities, and bilateral ones represented by algebraic or differential equations. In the paper we deal with bilateral constraints. Constraints can be modeled as ideal or non-ideal ones. The paper addresses ideal constraints. Bilateral and ideal constraints are classified as follows [13, 27, 34, 42].

(a) POSITION OR GEOMETRICAL CONSTRAINTS

$$f_\alpha(t, q_1, \ldots, q_n) = 0, \quad \alpha = 1, \ldots, a, \quad a < n, \tag{2.1}$$

where $\{q_\sigma\}$, $\sigma = 1, \ldots, n$ are generalized coordinates, functions f_α, $\alpha = 1, \ldots, a$ are defined on $(n + 1)$ manifold and have continuous derivatives up to the second order at least. Position constraints also restrict velocities and accelerations of a system. Differentiating Equation (2.1) once and twice with respect to time, we get restrictions put upon velocities and accelerations.

(b) KINEMATICAL (VELOCITY) CONSTRAINTS

$$f_\beta(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) = 0, \quad \beta = 1, \ldots, b, \quad b < n. \tag{2.2}$$

(c) KINEMATICAL CONSTRAINTS LINEAR WITH RESPECT TO VELOCITIES

$$\sum_{\sigma=1}^{n} \alpha_{\beta_\sigma}(t, q_\sigma)\dot{q}_\sigma + a_{\beta_0}(t, q_\sigma) = 0, \quad \beta = 1, \ldots, b. \tag{2.2a}$$

We assume that $f_\beta, \beta = 1, \ldots, b$ are defined on $(2n+1)$ manifold and have continuous derivatives. Kinematical constraints (2.2) or (2.2a) also restrict accelerations. Constraints (2.1) and (2.2) are referred to as material constraints. Constraints (2.2) are in the form of first-order differential equations and we will refer to them as first-order constraints, for example, linear first-order constraints in the case of (2.2a). If Equation (2.2) can be integrated, constraints are called holonomic ones. In the other case they are called non-holonomic constraints. We will consider non-holonomic constraints only and systems with non-holonomic constraints will be referred to as non-holonomic systems. Many discussions were conducted about a problem of integrability conditions for kinematical constraints [33, 34]. We will not address that problem here. Constraints of the form (2.2a) written in differential forms are often referred to as Pfaffian constraints (see, for example, [31, 33, 41, 51]). The non-holonomic constraint definition is often formulated then, as that this is a Pfaffian constraint, which is not integrable. It should be emphasized that the above classifications are not the only ones listed in literature. For example, two kinds of constrained systems referred to as Chaplygin [13, 34] and non-Chaplygin [20] systems have been classified. Task or driving constraints [31, 34], mentioned in Section 1, are examples of other constraint classes.

3. Program Constraints

The overview of constraint kinds may be summarized as follows:

1. When synthesis problems are formulated constraints are put upon a system performance before it is designed and put into operation, and these constraints may be non-material ones.
2. When modeling or analysis is extended beyond purely mechanical systems, for example, to electromechanical or biomechanical ones, constraints different that material ones arise; see, for example, [11, 16] and references therein.
3. Constraint equations that represent motion requirements may be of the order higher than two.

All the above gave rise to introduce a program constraint concept.

DEFINITION 1. A program constraint is any demand or limitation put upon a physical system kinematic, dynamic or performance characteristics.

DEFINITION 2. A program motion is a system motion performed according to a program constraint.

Material constraints, if present in a system, accompany program constraints and have to be added to program ones. It is worth notifying that motion initial conditions do not have to satisfy program constraint equations. A separate problem is then how to bring a system to a program motion, i.e. how to design controllers. This topic is not considered in the present paper.

We introduce program constraint formulations as follows:

– geometrical program constraints:

$$f_p(t, q_1, \ldots, q_n) = 0, \quad p = 1, \ldots, a, \quad a < n; \tag{3.1}$$

– kinematical program constraints:

$$g_p(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) = 0, \quad p = 1, \ldots, b, \quad b < n. \tag{3.1a}$$

We can see that mathematical relations (3.1) and (3.1a) are the same as those for material constraints, i.e. (2.1) and (2.2) but their interpretation is absolutely different. These constraints are put upon a system in order to program its motion, i.e. to get its desired performance. They are not constraints coming for example from a rolling without sliding condition although a system motion can be realized on a plane. Finally, according to Definition 1 the general formulation of bilateral program constraints is:

$$G_r(t, q_\sigma, \dot{q}_\sigma, \ldots, q_\sigma^{(p)}) = 0, \quad \begin{cases} p = 1, 2, 3, \ldots, \\ r = 1, \ldots, b, \quad b < n, \\ \sigma = 1, \ldots, n. \end{cases} \tag{3.2}$$

Constraints (3.2) can be linear or non-linear with respect to $q_\sigma^{(p)}$.

Only very few works use the name ‘program constraints’, see [28, 57], and they discuss trajectory tracking and problems of motion stabilization and/or optimal trajectory seeking. However, according to Definitions 1 and 2, and (3.1), (3.2), driving and task constraints, performance goals or other requirements put upon a system motion to obtain its specified performance may be included into the ‘program constraint’ class. They can get the unified name as they play the same role: they program the motion.

Remark. Throughout the paper we will consider programs referred to as partly-specified ones, where a number of constraint equations is smaller than a number of independent coordinates describing a system performance, i.e. $b < n$ as indicated in (3.2).

4. Examples of Non-Holonomic Non-Material Constraint Equations

4.1. A CONDITION FOR A PSEUDO-REGULAR PRECESSION

According to the Grioli theorem [12], necessary and sufficient conditions for a rigid body to perform a pseudo-regular precession are:

$$(p\dot{q} - q\dot{p}) + r(p^2 + q^2) - \lambda(p^2 + q^2)^{3/2} = 0, \tag{4.1}$$

where $p = \omega_\xi$, $q = \omega_\eta$, $r = \omega_\zeta$ and $\lambda = \text{const}$ and

$$\begin{aligned} \omega_\xi &= \dot{\psi} \sin\vartheta \sin\varphi + \dot{\vartheta} \cos\varphi, \\ \omega_\eta &= \dot{\psi} \sin\vartheta \cos\varphi - \dot{\vartheta} \sin\varphi, \\ \omega_\zeta &= \dot{\psi} \cos\vartheta + \dot{\varphi}, \end{aligned} \tag{4.1a}$$

and φ, ψ, ϑ denote Euler angles. When $\dot{\varphi} = \text{const}$, $\dot{\psi} = \text{const}$, $\dot{\vartheta} = 0$ then we have a regular precession. In the pseudo-regular precession $\dot{\varphi}$ and $\dot{\psi}$ are arbitrary and $\dot{\vartheta} \neq \text{const}$. The proof of the Grioli theorem can be found in [12]. Inserting kinematical relations (4.1a) into the Grioli theorem equation, one gets the condition

$$\begin{aligned} &\ddot{\psi}\dot{\vartheta} \sin\vartheta - \ddot{\vartheta}\dot{\psi} \sin\vartheta + 2\dot{\psi}\dot{\vartheta}^2 \cos\vartheta \\ &\quad + \dot{\psi}^3 \sin^2\vartheta \cos\vartheta - \lambda(\dot{\varphi}^2 \sin^2\vartheta + \dot{\vartheta}^2)^{3/2} = 0. \end{aligned} \tag{4.2}$$

Equation (4.2) is the kinematical condition and it constitutes the non-holonomic program constraint equation of the second order. In other words, Equation (4.2) has to be satisfied to get the specific motion, i.e. the pseudo-regular precession.

4.2. A BODY MOTION ON A TRAJECTORY WITH PRESCRIBED CURVATURE PROFILES

In [25, 46, 47] for trajectory planning purposes, to preserve continuity of a trajectory curvature, constraints are put upon the curvature and its derivative. For a real car-like robot the orientation between directing wheels and its main axis is bounded, which implies that a turning radius is lower bounded or that the curvature is bounded $|\kappa| \leq \kappa_{\max}$. The orientation of the directing wheels can change with a limited speed and the derivative of the curvature has to be bounded $|\dot{\kappa}| \leq \dot{\kappa}_{\max}$. Within those bounds the curvature or its derivative may be prescribed functions. We may demand a trajectory $y(x)$ for which the curvature profile is a given function Φ. Also, we demand the curvature and its derivative to be bounded functions to make the resulting trajectory possible to perform by a real car-like robot. The curvature profile constraint, for a planar motion, according to the differential geometry formula, has the form:

$$\kappa^2 = \Phi^2(t) = \frac{\begin{vmatrix} \dot{x} & \dot{y} \\ \ddot{x} & \ddot{y} \end{vmatrix}^2}{(\dot{x}^2 + \dot{y}^2)^3}. \tag{4.3}$$

Equation (4.3) constitutes the non-linear non-holonomic constraint equation of the second order. To put a constraint on the change rate of the curvature profile, constraint (4.3) has to be differentiated and we get a linear non-holonomic constraint equation of the third order

$$\dot{\kappa} = F_0 + \frac{\dot{x}\dddot{y} - \dddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \tag{4.3a}$$

The function F_0 does not contain third-order coordinate derivatives.

Curvature profiles and curvature derivative functions are examined in synthesis of mechanisms so the example could have potential applications there; see [30] and references therein.

Also, in [51] some examples are presented where a particle has to move on a trajectory of a specified shape $f(x, y) = 0$. We can generalize those examples and demand not only a trajectory but also its other characteristics like a curvature to be prescribed. Classical modeling methods developed in [51] fail when constraints like (4.3a) are put upon motion.

5. Modeling Methods for Non-Holonomic Systems

5.1. SOME COMMENTS ON MOTION EQUATIONS FOR NON-HOLONOMIC SYSTEMS

Classical analytical mechanics provides modeling methods for systems with first-order non-holonomic constraints, i.e. with material constraints, see, for example, [1, 2, 5, 8, 9, 13–15, 17, 22–24, 27, 31, 34, 38–42, 51, 54]. Appell–Gibbs equations can also deal with second-order constraints. These methods are not adequate to systems constrained with high-order non-holonomic constraints. Also, at the beginning of the XXth century it has been observed (see [6]) that equations of motion for electro-mechanical systems do not have the form of Lagrange equations of the second kind. At that time a trend of 'leaving the Lagrange equations approach' began too. That trend has been accompanied by efforts to eliminate unknown reactions of constraints from motion equations.

Historically, the problem of elimination of unknown constraint reactions from motion equations was solved first by Chaplygin in 1897 [7]. In 1901 Voronetz [54] obtained equations for non-holonomic systems for a more general case in which kinetic and potential energies and coefficients in constraint equations may depend on all generalized coordinates as well as on time. In 1908 Volterra proposed equations of motion in so-called kinematical characteristics (later called quasi-velocities). In 1901 Maggi derived equations of motion in a form that yields Volterra's equations. Appell's equations were derived in 1899 on the basis of the principle of the least constraint. Boltzmann in 1902 [5] and Hamel in 1904 [14, 15] derived a form of equations in quasi-coordinates, which coincide for holonomic systems. Later versions of the equations were developed by Tzenoff [49], Vranceanu in 1926 and Schouten in 1929.

Between 1910–1930 dynamics of non-linear non-holonomic systems made a significant progress; Appell, Chetaev, Johnes and Hamel derived equations for systems subjected to non-linear velocity constraints.

Some later papers [43–45] deal with non-linear non-holonomic constraints but of the first order and considerations within equations of classical mechanics are presented there.

Some other efforts to leave the Lagrange equations approach resulted in equations obtained by Nielsen [35] and Tzenoff [50]. Nielsen equations can be derived from Lagrange equations or the Jourdain principle. They serve for systems with first-order constraints and have the form:

$$\frac{\partial \dot{T}}{\partial \dot{q}_\sigma} - 2\frac{\partial T}{\partial q_\sigma} = Q_\sigma, \quad \sigma = 1, \ldots, n. \tag{5.1}$$

Tzenoff derived new equations for systems with second-order constraints from the Gauss principle. They are referred to as Tzenoff equations of the second kind and have the form:

$$\frac{1}{2}\left(\frac{\partial \ddot{T}}{\partial \ddot{q}_\sigma} - 3\frac{\partial T}{\partial q_\sigma}\right) = Q_\sigma, \quad \sigma = 1, \ldots, n. \tag{5.2}$$

Other equations by Tzenoff that are referred to as Tzenoff equations of the third kind, were derived from the postulated variational principle in the form:

$$\sum_{v=1}^{N}(\mathbf{F}_v - m_v\ddot{\mathbf{r}}_v)\delta''\ddot{\mathbf{r}}_v = 0, \quad \delta t = 0, \quad \delta\mathbf{r}_v = \delta\dot{\mathbf{r}}_v = \delta\ddot{\mathbf{r}}_v = 0, \quad \delta''\ddot{\mathbf{r}}_v \neq 0, \tag{5.3}$$

and $\delta''\ddot{\mathbf{r}}_v$ are defined as:

$$\delta''\ddot{\mathbf{r}}_v = \sum_{\sigma=1}^{n} \frac{\partial \mathbf{r}_v}{\partial q_\sigma}\delta\dddot{q}_\sigma. \tag{5.4}$$

Tzenoff equations of the third kind have the form:

$$\frac{1}{3}\left(\frac{\partial \dddot{T}}{\partial \dddot{q}_\sigma} - 4\frac{\partial T}{\partial q_\sigma}\right) = Q_\sigma, \quad q\sigma = 1, \ldots, n. \tag{5.5}$$

They work for third-order constraints ($p = 3$) and variational principle (5.3) is a peculiar case of the generalized variatonal principle that is presented in a subsequent section.

Some papers [48, 49] deal with high-order constraints but what is suggested there is to 'treat non-holonomic constraints by changing them into formal holonomic systems' and apply the concept of 'derivative space' and 'correspondent kinetic energy' in a derivative space. Equations that have been derived there were in vector forms and no examples or applications to high-order constraints were presented.

5.2. VIRTUAL DISPLACEMENTS AND VARIATIONAL PRINCIPLES FOR NON-HOLONOMIC NON-LINEAR SYSTEMS

Virtual displacement concepts for non-holonomic systems have been extended, comparing to the classical one, to:

1. Virtual displacement in the Appell–Chetaev sense [13, 34] – it was introduced when non-holonomic non-linear first-order constraints $\varphi_\beta(t, q_\sigma, \dot{q}_\sigma) = 0$ were revealed. Appel and Chetaev postulated to define virtual displacement that satisfies the relation

$$\sum_{\sigma=1}^{n} \frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} \delta q_\sigma = 0, \quad \sigma = 1, \ldots, n, \quad \beta < b. \tag{5.6}$$

   The above definition includes the definition of virtual displacement for holonomic and first-order linear non-holonomic constraints. It holds for ideal constraints.
2. The generalized virtual displacement – this is an extension of virtual displacement in the Appell–Chetaev sense. For ideal, arbitrary order constraints

$$G_\beta(t, q_\sigma, \dot{q}_\sigma, \ldots, q_\sigma^{(p)}) = 0, \quad \beta = 1, \ldots, p, \quad p = 1, 2, 3, \ldots. \tag{5.7}$$

Mangeron and Deleanu [29] postulated the generalized virtual displacement definition as:

$$\sum_{\sigma=1}^{n} \frac{\partial G_\beta}{\partial q_\sigma^{(p)}} \delta q_\sigma = 0. \tag{5.8}$$

We take advantage of this definition in Section 6.

The existence of three variational principles in classical mechanics – d'Alembert, Jourdain and Gauss principles – gives rise to a question about their equivalence from the point of view of motion equations generation. It can be shown easily that for holonomic systems these three principles are equivalent. For non-holonomic systems, these principles are equivalent for constraints linear in velocities [13, 34]. When non-holonomic constraints are non-linear in velocities, the problem of equivalency of these principles should be examined separately. It means, for non-linear non-holonomic constraints it must be stated what it is virtual displacement for them and when they are ideal.

Appell and Chetaev postulated to define virtual displacement that would serve for non-holonomic non-linear systems and would be compatible with the classical definition for linear non-holonomic and holonomic systems. This way a certain class of constraints and virtual displacements have been selected and they are known as constraints and virtual displacements of the Appell–Chetaev type. D'Alembert, Jourdain and Gauss principles are equivalent for Appell–Chetaev systems. In general, for systems that are not Appell–Chetaev ones these principles are not equivalent. We deal with Appell–Chetaev systems in this paper. More details about the equivalency of variational principles can be found in [34]. The generalized variational principle for ideal constraints has been postulated by Mangeron and Deleanu [29]:

$$\sum_{v=1}^{N}(\mathbf{F} - m_v\ddot{\mathbf{r}}_v)\delta\mathbf{r}_v^{(p)} = 0, \tag{5.9}$$

$$\delta t = 0, \quad \delta\mathbf{r}_v = \delta\dot{\mathbf{r}}_v = \ldots = \delta\mathbf{r}_v^{(p-1)} = 0, \quad \delta\mathbf{r}_v^p \neq 0.$$

For Appell–Chetaev systems principle (5.9) for $p = 1, 2, 3$ coincides with classical principles and with the principle (5.3). The proof and comments to the principle can be found in [29].

Remark. In [48] principle (5.9) is cited following [49] and it is referred to as the universal d'Alembert principle. We refer to it as the generalized variational principle following its original name.

6. Generalized Program Motion Equations (GPME) for Systems with High-Order Non-Holonomic Constraints (HONC)

The purpose of this section is to present a complete and compact structure of analytical apparatus to derive motion equations for systems with high-order non-holonomic constraints (HONC). Equations that we derive here are referred to as generalized program motion equations (GPME). They can replace equations of classical mechanics that are their peculiar cases.

The background for development of motion equations for systems with HONC was established by Mangeron and Deleanu [29]. The generalized variational principle (5.9) was the basis for them to derive, in a vector form, motion equations for systems with HONC. This vector form of equations can be compared to one of 'principal forms' of motion equations in classical mechanics, see [42] for example. Mangeron and Deleanu have not derived motion equations for any system with HONC but applied the method they have proposed to cases of first-order constraints and investigated its coincidence with classical methods or examined some of its mathematical features. Also, a few other publications [9, 41] were their work was mentioned, did not deliver any modeling method for systems with HONC. In the paper GPME were derived a different way, i.e. not from principle (5.9). Final equations were developed in an analytical form. An extension of the work by Mangeron and Deleanu is presented.

6.1. DERIVATION OF GPME

Let us take the general form of bilateral non-holonomic constraints of the p-order:

$$G_\beta(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \ldots, q_1^{(p)}, \ldots, q_n^{(p)}) = 0,$$

$$\beta = 1, \ldots, b, \quad p = 1, 2, 3, \ldots.$$

These constraint equations can be non-linear with respect to some p-derivatives of coordinates. Differentiating constraint equations, with respect to time, we get constraints of the higher order but linear in at least one $(p+1)$ or $(p+a)$-order coordinate derivative. From now on, for simpler notation and without loss of generality, we assume that our p-order constraints are linear in, at least one p-order coordinate derivative. To develop GPME we formulate the following theorem:

THEOREM. *If the function $F = F(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ is regular enough, i.e. all derivatives up to the certain order p can be computed, then the following identity holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{q}_\sigma}\right) \equiv \frac{1}{p}\left[\frac{\partial F^{(p)}}{\partial q_\sigma^{(p)}} - \frac{\partial F}{\partial q_\sigma}\right], \quad \sigma = 1, \ldots, n, \quad p = 1, 2, 3, \ldots. \tag{6.1}$$

Proof. If we calculate total derivatives of order $p = 1, 2, 3, \ldots$ of the function $F = F(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$, then one can verify that $F^{(p)}$ has the form:

$$\begin{aligned} F^{(p)} &= p\left[\sum_{\sigma=1}^{n} \frac{\partial^2 F}{\partial \dot{q}_\sigma \partial t} q_\sigma^{(p)} + \sum_{\sigma=1}^{n}\sum_{\alpha=1}^{n} \frac{\partial^2 F}{\partial \dot{q}_\sigma \partial q_\alpha} \dot{q}_\alpha q_\sigma^{(p)} \right. \\ &\quad \left. + \sum_{\sigma=1}^{n}\sum_{\alpha=1}^{n} \frac{\partial^2 F}{\partial \dot{q}_\sigma \partial q_\alpha} \ddot{q}_\alpha q_\sigma^{(p)}\right] \\ &\quad + \sum_{\sigma=1}^{n} \frac{\partial F}{\partial q_\sigma} q_\sigma^{(p)} + \sum_{\sigma=1}^{n} \frac{\partial F}{\partial \dot{q}_\sigma} q_\sigma^{(p+1)} + \Gamma, \quad p > 2, \end{aligned} \tag{6.2}$$

where the function Γ does not contain derivatives $q_\sigma^{(p)}$ and $q_\sigma^{(p+1)}$, $\sigma = 1, \ldots, n$. Hence we have:

$$\frac{\partial F^{(p)}}{\partial q_\sigma^{(p)}} = p\left(\frac{\partial^2 F}{\partial \dot{q}_\sigma \partial t} + \sum_{\alpha=1}^{n} \frac{\partial^2 F}{\partial \dot{q}_\sigma \partial q_\alpha} \dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial^2 F}{\partial \dot{q}_\sigma \partial \dot{q}_\alpha} \ddot{q}_\alpha\right) + \frac{\partial F}{\partial q_\sigma}. \tag{6.3}$$

Relation (6.3) holds for $p = 1, 2, \ldots$ and $\sigma = 1, \ldots, n$. Indeed, for $p = 1$ we have:

$$\dot{F} = \frac{\partial F}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial F}{\partial q_\alpha} \dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial F}{\partial \dot{q}_\alpha} \ddot{q}_\alpha$$

and hence:

$$\begin{aligned} \frac{\partial \dot{F}}{\partial \dot{q}_\sigma} &= \frac{\partial^2 F}{\partial t \partial \dot{q}_\sigma} + \sum_{\alpha=1}^{n} \frac{\partial^2 F}{\partial q_\alpha \partial \dot{q}_\sigma} \dot{q}_\alpha \\ &\quad + \sum_{\alpha=1}^{n} \frac{\partial^2 F}{\partial \dot{q}_\alpha \partial \dot{q}_\sigma} \ddot{q}_\alpha + \frac{\partial F}{\partial q_\sigma}, \quad \sigma = 1, \ldots, n. \end{aligned} \tag{6.4}$$

The same way we show that (6.2) holds for $p = 2$. Next, we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{q}_\sigma}\right) = \frac{\partial^2 F}{\partial t \partial \dot{q}_\sigma} + \sum_{\alpha=1}^{w} \frac{\partial^2 F}{\partial q_\alpha \partial \dot{q}_\sigma}\dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial^2 F}{\partial \dot{q}_\alpha \partial \dot{q}_\sigma}\ddot{q}_\alpha, \quad \sigma = 1, \ldots, n. \tag{6.5}$$

From (6.3) and (6.5) we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{q}_\sigma}\right) = \frac{1}{p}\left(\frac{\partial F^{(p)}}{\partial q_\sigma^{(p)}} - \frac{\partial F}{\partial q_\sigma}\right), \quad \sigma = 1, \ldots, n, \quad p = 1, 2, \ldots. \tag{6.6}$$

Now if we replace F by $T = T(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ in (6.6) and insert it into Lagrange equations we get:

$$\frac{1}{p}\left[\frac{\partial T^{(p)}}{\partial q_\sigma^{(p)}} - (p+1)\frac{\partial T}{\partial q_\sigma}\right] = Q_\sigma, \quad p = 1, 2, \ldots, \quad \sigma = 1, \ldots, n. \tag{6.7}$$

Equations (6.7) are the generalized program motion equations (GPME). Their form to direct applications will be developed in Section 6.3. In the above derivation we have not assumed anything about kinetic energy, for example that it was a quadratic function of velocities. That particular form of kinetic energy has been assumed in [29]. Derivation of Equations (6.7) without that assumption is an important improvement because they can be applied to systems for which kinetic energy is any function of velocities [18, 19]. Equations (6.7) become Nielsen equations (5.1) for $p = 1$ and Tzenoff equations (5.2) for $p = 2$.

6.2. GPME AND EQUATIONS OF CLASSICAL ANALYTICAL MECHANICS

Now we will explain and discuss Equations (6.7) in more detail. First, we consider the case when $p = 1$. It gives more light to the structure of (6.7) and the case $p = 1$ is peculiar somehow. This is because for $p = 1$ a system kinetic energy depends on first derivatives of coordinates that are also present in non-holonomic first-order constraint equations

$$G_\beta(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) = 0, \quad \beta = 1, \ldots, b. \tag{6.8}$$

On the basis of the Nielsen equations (5.1), one can get motion equations for a system with constraints (6.8) applying Lagrange equations with multipliers in the form

$$\frac{\partial \dot{T}}{\partial \dot{q}_\sigma} - 2\frac{\partial T}{\partial q_\sigma} = Q_\sigma + \sum_{\beta=1}^{b} \lambda_\beta \frac{\partial G_\beta}{\partial \dot{q}_\sigma}, \quad \sigma = 1, \ldots, n. \tag{6.9}$$

These equations have been derived the same way as Lagrange equations with multipliers [18]. To eliminate multipliers from (6.9) we convert (6.8) to the form:

$$\dot{q}_\beta = g_\beta^{(1)}(t, q_1, \ldots, q_n, \dot{q}_{b+1}, \ldots, \dot{q}_n), \quad \beta = 1, \ldots, b. \tag{6.10}$$

Next, differentiating (6.10) we get:

$$\ddot{q}_\beta = \gamma_\beta^{(2)}(t, q_1, \ldots, q_n, \dot{q}_{b+1}, \ldots, \dot{q}_n, \ddot{q}_{b+1}, \ldots, \ddot{q}_n), \quad \beta = 1, \ldots, b. \tag{6.11}$$

Equations (6.7) for $p = 1$ can be written in the form:

$$\sum_{\beta=1}^{b}\left(\frac{\partial \dot{T}}{\partial \dot{q}_\beta} - 2\frac{\partial T}{\partial q_\beta} - Q_\beta\right)\delta q_\beta + \sum_{\mu=b+1}^{n}\left(\frac{\partial \dot{T}}{\partial \dot{q}_\mu} - 2\frac{\partial T}{\partial q_\mu} - Q_\mu\right)\delta q_\mu = 0. \tag{6.12}$$

Keeping in mind that we consider Appell–Chetaev systems (6.12) becomes:

$$\sum_{\beta=1}^{b}\left(\frac{\partial \dot{T}}{\partial \dot{q}_\beta} - 2\frac{\partial T}{\partial q_\beta} - Q_\beta\right)\sum_{\mu=1}^{n}\frac{\partial g_\beta^{(1)}}{\partial \dot{q}_\mu}\delta q_\mu + \sum_{\mu=b+1}^{n}\left(\frac{\partial \dot{T}}{\partial \dot{q}_\mu} - 2\frac{\partial T}{\partial q_\mu} - Q_\mu\right)\delta q_\mu = 0. \tag{6.13}$$

Variations ∂q_μ, $\mu = b+1, \ldots, n$, are independent, so we rewrite (6.13) in the form:

$$\frac{\partial \dot{T}}{\partial \dot{q}_\mu} - 2\frac{\partial T}{\partial q_\mu} - Q_\mu + \sum_{\beta=1}^{b}\left(\frac{\partial \dot{T}}{\partial \dot{q}_\beta} - 2\frac{\partial T}{\partial q_\beta} - Q_\beta\right)\frac{\partial g_\beta^{(1)}}{\partial \dot{q}_\mu} = 0, \quad \mu = b+1, \ldots, n. \tag{6.14}$$

We refer to Equations (6.14) as the Nielsen equations in the Maggi form for systems with constraints (6.10).

Now, consider the case when a system is constrained with HONC, i.e. when $p > 1$. In this case constraint equations have the form:

$$G_\beta(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \ldots, q_1^{(p)}, \ldots, q_n^{(p)}) = 0 \tag{6.15}$$

and the generalized definition of virtual displacements δq_σ, $\sigma = 1, \ldots, n$, is given by (5.8).

Assuming that first bp-order derivatives of generalized coordinates in (6.15) are dependent ones, constraints (6.15) are transformed to the form:

$$q_\beta^{(p)} = g_\beta^{(p)}(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \ldots, q_{b+1}^{(p)}, \ldots, q_n^{(p)}), \quad \beta = 1, \ldots, b. \tag{6.16}$$

Differentiating (6.16) with respect to time we get:

$$q_\beta^{(p+1)} = \gamma_\beta^{(p+1)}(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \ldots, q_{b+1}^{(p)}, \ldots, q_n^{(p)}, q_{b+1}^{(p+1)}, \ldots, q_n^{(p+1)}). \tag{6.17}$$

Now we rewrite (6.7) as

$$\sum_{\beta=1}^{b}\left[\frac{1}{p}\left(\frac{\partial T^{(p)}}{\partial q_{\beta}^{(p)}}-(p+1)\frac{\partial T}{\partial q_{\beta}}\right)-Q_{\beta}\right]\delta q_{\beta} + \sum_{\mu=b+1}^{n}\left[\frac{1}{p}\left(\frac{\partial T^{(p)}}{\partial q_{\mu}^{(p)}}-(p+1)\frac{\partial T}{\partial q_{\mu}}\right)-Q_{\mu}\right]\delta q_{\mu}=0. \tag{6.18}$$

Because of (5.8) and (6.16) we also have

$$\delta q_{\beta}=\sum_{\mu=b+1}^{n}\frac{\partial g_{\beta}^{(p)}}{\partial q_{\mu}^{(p)}}\delta q_{\mu}, \quad \beta=1,\ldots,b, \tag{6.19}$$

so (6.18) takes the form:

$$\sum_{\mu=b+1}^{n}\left\{\frac{1}{p}\left(\frac{\partial T^{(p)}}{\partial q_{\mu}^{(p)}}-(p+1)\frac{\partial T}{\partial q_{\mu}}\right)-Q_{\mu} + \sum_{\beta=1}^{n}\left[\frac{1}{p}\left(\frac{\partial T^{(p)}}{\partial q_{\beta}^{(p)}}-(p+1)\frac{\partial T}{\partial q_{\beta}}\right)-Q_{\beta}\right]\frac{\partial g_{\beta}^{(p)}}{\partial q_{\mu}^{(p)}}\right\}\delta q_{\mu}=0. \tag{6.20}$$

Because variations δq_{μ}, $\mu=b+1,\ldots,n$, are independent we get motion equations in the form:

$$\frac{1}{p}\left[\frac{\partial T^{(p)}}{\partial q_{\mu}^{(p)}}-(p+1)\frac{\partial T}{\partial q_{\mu}}\right]-Q_{\mu} + \sum_{\beta=1}^{b}\left\{\frac{1}{p}\left[\frac{\partial T^{(p)}}{\partial q_{\beta}^{(p)}}-(p+1)\frac{\partial T}{\partial q_{\beta}}\right]-Q_{\beta}\right\}\frac{\partial g_{\beta}^{(p)}}{\partial q_{\mu}^{(p)}}=0,$$

$$\mu=b+1,\ldots,n. \tag{6.21}$$

Equations (6.21) are motion equations for a system with HONC (6.15). They become the Nielsen equations in the Maggi form for $p=1$, so they hold for $p=1,2,3,\ldots$.

6.3. GPME GENERATION TO DIRECT APPLICATIONS

GPME forms (6.7) or (6.21) are not suitable to direct applications. To develop an algorithm to generate GPME easily we take advantage of considerations that resulted in Equations (6.14) and (6.21). Let us start from the first-order constraint equations

$$\sum_{\sigma=1}^{n}h_{\beta\sigma}\dot{q}_{\sigma}+h_{\beta}=0, \quad \beta=1,\ldots,b, \tag{6.22}$$

where $h_{\beta\sigma} = h_{\beta\sigma}(t, q_1, \dots, q_n)$, $h_\beta = h_\beta(t, q_1, \dots, q_n)$.

Partition of coordinates transforms (6.22) into:

$$\dot{q}_\rho = \sum_{\lambda=1}^{l} a_{\rho\lambda}\dot{q}_\lambda + a_\rho, \quad \rho = l+1, \dots, n. \tag{6.23}$$

Now, l denotes the number of dof, i.e. $l = (n - b)$.

For $p = 1$, we employ the Nielsen equations (5.1). First we construct the function P_1:

$$P_1 = \dot{T} - 2\dot{T}_0, \tag{6.24}$$

where for $p = 1$, $\dot{T}_0$ is computed according to the relation

$$T_0^{(p)} = \sum_{\sigma=1}^{n} \frac{\partial T}{\partial q_\sigma} q_\sigma^{(p)}. \tag{6.25}$$

Next we introduce the function R_1:

$$\begin{aligned} R_1 &= P_1 - \sum_{\sigma=1}^{n} \dot{q}_\sigma Q_\sigma = R_1(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, \ddot{q}_1, \dots, \ddot{q}_n) \\ &= R_1(t, q_\sigma, \dot{q}_\lambda, \dot{q}_\rho, \ddot{q}_\sigma). \end{aligned} \tag{6.26}$$

Replacing $\dot{q}_\rho$ in (6.26) by relations (6.23) we get:

$$R_1^* = R_1\left(t, q_\sigma, \dot{q}_\lambda, \sum_{\lambda=1}^{l} a_{\rho\lambda}\dot{q}_\lambda + a_\rho, \ddot{q}_\sigma\right) = R_1^*(t, q_\sigma, \dot{q}_\lambda, \ddot{q}_\sigma). \tag{6.27}$$

Assuming that $\partial Q_\sigma / \partial \dot{q}_\sigma = 0$, program motion equations have the form:

$$\frac{\partial R_1^*}{\partial \dot{q}_\lambda} = \frac{\partial R_1}{\partial \dot{q}_\lambda} + \sum_{\rho=l+1}^{n} \frac{\partial R_1}{\partial \dot{q}_\rho}\frac{\partial \dot{q}_\rho}{\partial \dot{q}_\lambda} = \frac{\partial R_1}{\partial \dot{q}_\lambda} + \sum_{\rho=l+1}^{n} \frac{\partial R_1}{\partial \dot{q}_\rho} a_{\rho\lambda} = 0. \tag{6.28}$$

Equations (6.28) and (6.23) constitute a set of $\lambda + \rho = l + (n - l) = n$ equations with n unknown q_σ's. In the general case of p-order constraints, as stated earlier, non-linear constraints can be transformed to linear ones by differentiation. We assume that p-order constraints are already in a linear form with respect to a highest derivative of at least one of coordinates. If so, we rewrite p-order constraint equations (6.16) in the form:

$$q_\rho^{(p)} = \sum_{\lambda=1}^{l} a_{\rho\lambda} q_\lambda^{(p)} + a_\rho, \tag{6.29}$$

where $a_{\rho\lambda} = a_{\rho\lambda}(t, q_\sigma, \dot{q}_\sigma, \dots, q_\sigma^{(p-1)})$, $a_\rho = a_\rho(t, q_\sigma, \dot{q}_\sigma, \dots, q_\sigma^{(p-1)})$.

Next we construct following functions:

$$P_p = \frac{1}{p}[T^{(p)} - (p+1)T_0^{(p)}] \tag{6.30}$$

and $T_0^{(p)}$ is defined by (6.25).

$$R_p = P_p - \sum_{\sigma=1}^{n} q_\sigma^{(p)} Q_\sigma = R_p(t, q_\sigma, \dot{q}_\sigma, \ldots, q_\lambda^{(p)}, q_\rho^{(p)}, q_\sigma^{(p+1)}), \tag{6.31}$$

$$\begin{aligned} R_p^* &= R_p^*\left(t, q_\sigma, \dot{q}_\sigma, \ldots, q_\lambda^{(p)}, \sum_{\lambda=1}^{l} a_{\rho\lambda} q_\lambda^{(p)} + a_\rho, q_\sigma^{(p+1)}\right) \\ &= R_p^*(t, q_\sigma, \dot{q}_\sigma, \ldots, q_\lambda^{(p)}, q_\sigma^{(p+1)}). \end{aligned} \tag{6.32}$$

Assuming that $\partial Q_\sigma / \partial q_\sigma^{(p)} = 0$, GPME for a system with p-order constraints (6.29) have the form:

$$\frac{\partial R_p^*}{\partial q_\lambda^{(p)}} = \frac{\partial R_p}{\partial q_\lambda^{(p)}} + \sum_{\rho=l+1}^{n} \frac{\partial R_p}{\partial q_\rho^{(p)}} \frac{\partial q_\rho^{(p)}}{\partial q_\lambda^{(p)}} = \frac{\partial R_p}{\partial q_\lambda^{(p)}} + \sum_{\rho=l+1}^{n} \frac{\partial R_p}{\partial q_\rho^{(p)}} a_{\rho\lambda} = 0. \tag{6.33}$$

Relations (6.30–6.33) constitute the algorithm that can be applied to generate motion equations for a system with arbitrary order constraints.

6.4. EXAMPLE

Consider a body motion according to constraints (4.3) or (4.3a). For simplicity we assume that a particle mass $m = 1$. Constraint equation (4.3) after time differentiation to get its linear form with respect to the highest coordinate derivative, for example $\dddot{x}$, has the form:

$$\dddot{x} = \frac{-\Phi_1(\dot{x}^2 + \dot{y}^2)^2[\dot{\Phi}_1(\dot{x}^2 + \dot{y}^2) + 3\Phi_1(\dot{x}\ddot{x} + \dot{y}\ddot{y})]}{\dot{y}(\dot{x}\ddot{y} - \ddot{x}\dot{y})} + \dddot{y}\frac{\dot{x}}{\dot{y}}$$

or

$$\dddot{x} = F_1 + \dddot{y}\frac{\dot{x}}{\dot{y}}.$$

The function F_1 does not contain coordinate derivatives of the third order.

Constraint equation (4.3a) is also of the third order. For illustrative purposes we select constraint (4.3) to simulation. The reason is that results may be verified easily with the radius of curvature ro_p calculated as

$$ro_p = \frac{(\dot{x}_p^2 + \dot{y}_p^2)^{3/2}}{|\dot{x}_p\ddot{y}_p - \dot{y}_p\ddot{x}_p|},$$

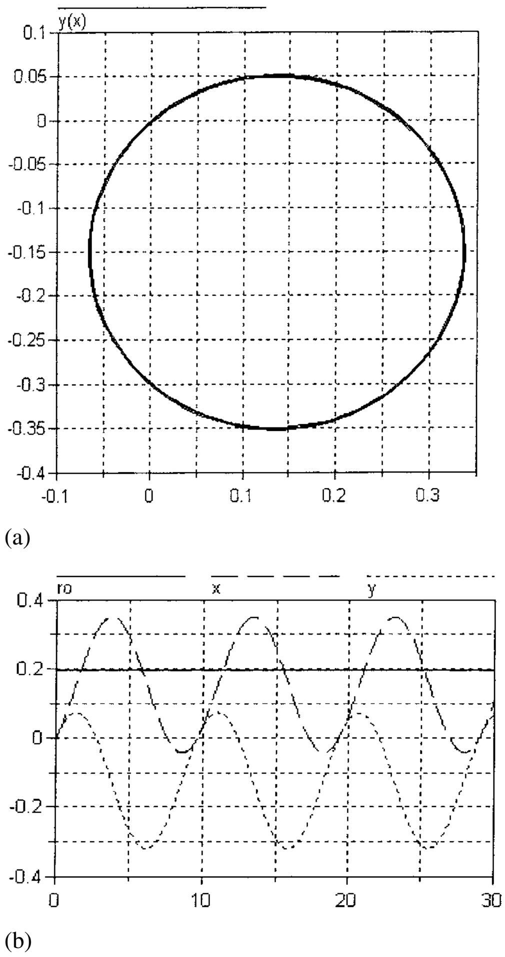
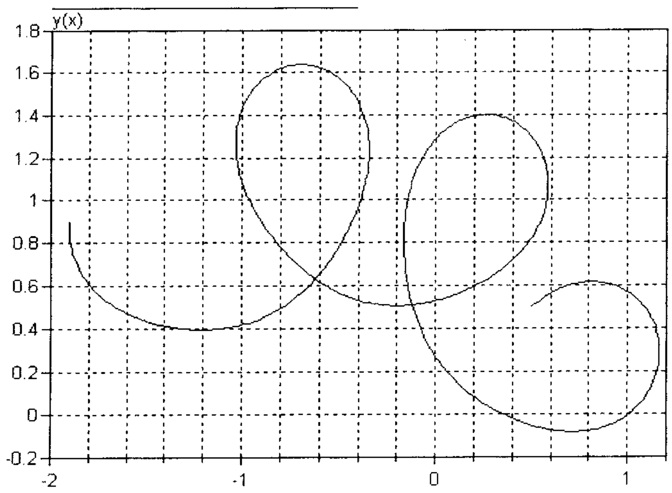


Figure 1. (a) Trajectory of the particle according to constant curvature Φ_1. (b) Radius of the curvature ro_p, x_p, y_p.

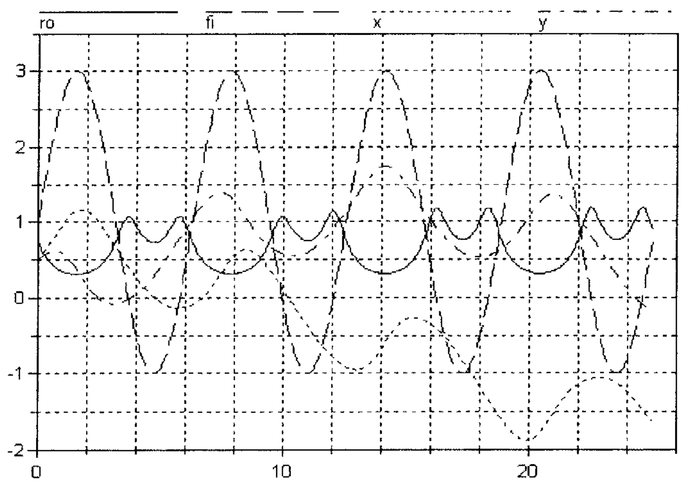
where the subscript p indicates that they are program coordinate functions. Assume that the generalized forces Q_x, Q_y, Q_z act upon the particle. On the basis of (6.30–6.33) we have:

$$P_3 = \frac{1}{3}(\dddot{T} - 4\dddot{T}_0) = \frac{1}{3}\dddot{T} = \frac{1}{3}(3\ddot{x}\dddot{x} + 3\ddot{y}\dddot{y} + \dot{y}y^{\mathrm{IV}} + \dot{x}x^{\mathrm{IV}}),$$

$$\begin{aligned} R_3 &= P_3 - \sum_{\sigma=1}^{2} Q_\sigma \dddot{q}_\sigma = \ddot{x}\dddot{x} + \ddot{y}\dddot{y} + F_2 - Q_x\dddot{x} - Q_y\dddot{y} \\ &= (\ddot{x} - Q_x)\dddot{x} + (\ddot{y} - Q_y)\dddot{y} + F_2, \end{aligned}$$



(a)



(b)

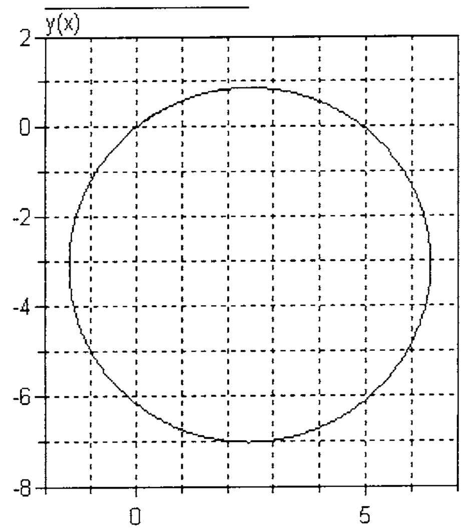
Figure 2. (a) Trajectory of the particle according to curvature changes Φ_2 (after 50 seconds). (b) Radius of the curvature ro_p, curvature Φ_2, x_p, y_p.

where F_2 does not contain terms with $\dddot{x}$, $\dddot{y}$. F_1 and F_2 will be omitted in what follows.

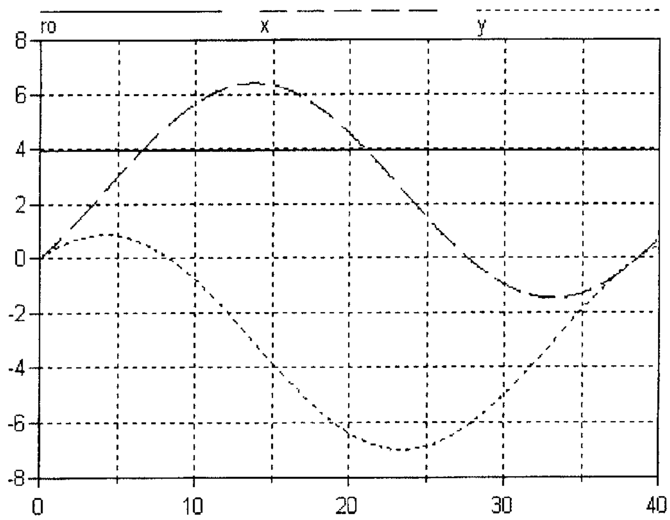
$$R_3^* = (\ddot{x} - Q_x)\dddot{y}\frac{\dot{x}}{\dot{y}} + \dddot{y}(\ddot{y} - Q_y).$$

The final program motion equation of the particle has the form:

$$\ddot{y} + (\ddot{x} - Q_x)\frac{\dot{x}}{\dot{y}} = Q_y, \tag{6.34a}$$



(a)



(b)

Figure 3. (a) Trajectory of the particle according to constant curvature Φ_1 (initial conditions do not satisfy constraints). (b) Radius of curvature ro_p, x_p, y_p.

and together with the constraint equation

$$\dddot{x} = \frac{-\Phi_1(\dot{x}^2 + \dot{y}^2)^2[\dot{\Phi}_1(\dot{x}^2 + \dot{y}^2) + 3\Phi_1(\dot{x}\ddot{x} + \dot{y}\ddot{y})\dot{y}\ddot{y}]}{\dot{y}(\dot{x}\ddot{y} - \ddot{x}\dot{y})} + \dddot{y}\,\frac{\dot{x}}{\dot{y}} \tag{6.34b}$$

they are the set of equations that describe the program motion.

For simulations we assume $Q_x = Q_y = Q_z = 0$ and select two functions: $\Phi_1 = 5$, $\Phi_2 = 2\sin t + 1$. Functions Φ are limited and have limited derivatives. Dymola-

Modelica [10] (DASSL solver) has been selected to simulate Equations (6.34) as it provides a number of advanced integration methods to simulate non-linear dynamic systems. Simulation results are shown in Figures 1, 2, and 3. Initial conditions have to include initial values of positions, velocities and accelerations. Initial conditions satisfy the constraint equation in cases shown in Figures 1 and 2. They do not satisfy it in the case in Figure 3 and then, program motion is still performed, i.e. the trajectory curvature is kept constant but its value is different than it was prescribed. Figure 3 presents the case of non-controlled motion, i.e. none force acts on the particle to bring it to the exact program trajectory.

The above example shows a potential power of modeling with GPME and possibilities of motion programming.

7. Conclusions and Discussion

The generalized program motion equations (GPME) have been derived in this paper. They can be applied to systems with arbitrary (finite) order non-holonomic constraints and this way allows generation of motion equations of constrained systems to be treated in a new and uniform manner. What follows, methods of classical analytical mechanics are peculiar cases of GPME. Additionally, GPME can be applied to systems for which kinetic energy is not a quadratic function of velocities. It makes possible to apply GPME to variable mass systems [18]. GPME result in motion equations referred to as program motion equations. A disadvantage of GPME is the necessity of computation of higher-order kinetic energy derivatives but it can be completed using most of commercial software. Constraint classifications have also been revisited and the class of HONC has been distinguished as program constraints. Concepts of the generalized variational displacement and the generalized variational principle for HONC have been introduced.

Problems of stability and stabilization of program motions, controller designs for systems with program constraints are topics of further research. Some work has already been done for simpler cases of non-holonomic constraints; see, for example, [21].

References

1. Appell, P., ‘Exemple de mouvement d’un point assujeti á une liason exprimée par une relation non linéaire entre les composantes de la vitesse’, *Comptes Rendus* **152**, 1911, 40–50.
2. Appell, P., *Traité de mécanique rationnelle*, Gauthier-Villars, Paris, Vol. 1, 1941, Vol. 2, 1953.
3. Arabyan, A. and Wu, F., ‘An improved formulation for constrained mechanical systems’, *Multibody System Dynamics* **2**(1), 1998, 49–69.
4. Beghin, H., *Course de mécanique*, 1947.
5. Boltzmann, L., ‘Über die Form der Lagrangerchen Gleichungen für nichtholonome generalisierte Koordinaten’, *Sitzungsberichte der Mathematisch-Naturwissenschaftliche Akademie der Wissenschaften zu Wien*, Bd. CXI, Ab. IIa, Heft 1–2, 1902, 1603–1614.

6. Carvallo, E., 'L'electricité de l'expérience et ramenée au principe des travaux virtuels', *Coll. Scienta* **19**, 1902.
7. Chaplygin, S.A., 'About a motion of a heavy body on a horizontal plane', in *Izbrannye Trudy Klassiki Nauki*, Nauka, Moscow, 1897, Chap. 3, 363–375 [in Russian].
8. de Jalon, J.G. and Bayo, E., *Kinematic and Dynamic Simulation of Multibody Systems*, Mechanical Engineering Series, Springer-Verlag, Berlin, 1994.
9. Dobronravov, V.V., *Foundations of Mechanics of Non-Holonomic Systems*, Mir, Moscow, 1970.
10. Elmqvist, H., Breuch, M., Mattsson, S.-E., Olsson, H. and Otter, M., *Dymola User's Manual*, Dynasi AB, Research Park Ideon, 223 70 Lund, Sweden, 2000, www.dynasim.se.
11. Gaponov, V., 'Nonholonomic systems of S.A. Chaplygin and the theory of commutative electrical machinery', *Doklady Akademii Nauk SSSR* **87**, 1952, 401 [in Russian].
12. Grioli, G., 'Particular solutions in stereodynamics', in *Stereodynamics*, Centro Internazionale Matematico Estivo, Edizioni Cremonese, Rome, 1972, 1–65 [in Italian].
13. Gutowski, R., *Analytical Mechanics*, PWN, Warsaw, 1972 [in Polish]. Also "Analytical mechanics", in *Foundation of Mechanics*, H. Zorski (ed.), Polish Scientific Publisher/Elsevier, 1985, 1–119 (original in Polish 1985).
14. Hamel, G., *Theoretische Mechanik*, Springer-Verlag, Berlin, 1967.
15. Hamel, G., 'Die Lagrange–Eulersche Gleichungen der Mechanik', *Z. Phys.* **50**, 1904, 1–57.
16. Hogan, N., 'An organizing principle for a class of voluntary movements', *J. Neurosci.* **4**(11), 1984, 2745–2754.
17. Huston, R.L., 'On the equivalence of Kane's equations and Gibbs' equations for multibody dynamics formulations', *Mech. Res. Commun.* **14**(2), 1987, 123–131.
18. Jarzebowska, E., 'Variational principles and motion equations of analytical mechanics of constrained variable mass systems', Ph.D. Dissertation, Aerospace Department, Warsaw University of Technology, Warsaw, 1988 [in Polish].
19. Jarzebowska, E., 'The problem of small oscillations of mechanical systems with arbitrary order nonholonomic program constraints', *Nonlinear Vibration Problems* **24**, 1991, 141–160.
20. Jarzebowska, E. and McClamroch, N.H., 'On nonlinear control of the Ishlinsky problem as an example of a nonholonomic non-Chaplygin system', in *Proceedings of American Control Conference 2000*, Chicago, IL, IEEE, New York, 2000, 3249–3253.
21. Jarzebowska, E. and Kulinicz, K., 'Stability analysis application to selected biomechanical systems', *Biology of Sport*, Supp., No. 7, 1997, 227–232.
22. Kamman, J.W. and Huston, R.L., 'Dynamics of constrained multibody systems', *J. Appl. Mech.* **51**, 1984, 899–903.
23. Kane, T.R., *Dynamics*, Holt, Rinehart & Wilson, New York, 1968.
24. Kane, T.R. and Levinson, D.L., *Dynamics – Theory and Applications*, McGraw-Hill, New York, 1985.
25. Koh, K.C. and Cho, H.S., 'A smooth path tracking algorithm for wheeled mobile robotics with dynamic constraints', *J. Intell. Robotic Syst.* **24**, 1999, 367–385.
26. Kolmanovsky, I. and McClamroch, H.H., 'Developments in nonholonomic control problems', *IEEE Control Systems*, December 1995, 20–36.
27. Lancos, C., *The Variational Principle of Mechanics*, 4th edn., Dover Publications, New York, 1986.
28. Larin, V.B., 'Stabilizing the motion of a system With nonholonomic constraints', *Internat. J. Appl. Mech.* **34**(7), 1998, 683–693.
29. Mangeron, D. and Deleanu, S., 'Sur une classe d'équations de la mécanique analytique ou sens de Tzenoff', *Comptes Rendus de l'Academie Bulgare des Sciences* **15**, 1962, 1–9.
30. Molian, S., *The Practical Kinematics and Dynamics of Machinery*, Adlen Press, Boston, MA, 1977.
31. Moon, F.C., *Applied Dynamics*, John Wiley & Sons, New York, 1998.

32. Xu, M., Liu, C. and Huston, R.L., 'Analysis of non-linearly constrained nonholonomic multibody systems', *Internat. J. Non-Linear Mech.* **25**(5), 1990, 511–519.
33. Murray, R.M., Li, Z.X. and Sastry, S.S., *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Boca Raton, FL, 1994.
34. Nejmark, J.I. and Fufaev, N.A., *Dynamics of Nonholonomic Systems*, American Mathematical Society, Providence, RI, 1972.
35. Nielsen, J., *Vorlesungen über elementäre Mechanik*, Verlag von J. Springer, Berlin, 1935.
36. Nikravesh, P., *Computer Aided Analysis of Mechanical Systems*, Prentice Hall, Englewood Cliffs, NJ, 1988.
37. Oriolo, G. and Nakamura, Y., 'Control of mechanical systems with second-order nonholonomic constraints: Underactuated manipulators', in *Proceedings of 30th Conference on Decision and Control*, IEEE, New York, 1991, 2398–2403.
38. Passerello, C.E. and Huston, R.T., 'Another look at nonholonomic systems', *J. Appl. Mech.* **40**(1), 1973, 101–104.
39. Papastavridis, J.G., 'On the nonlinear Appell's equations and the determination of generalized reaction forces', *Internat. J. Engrg. Sci.* **26**(6), 1988, 609–625.
40. Papastavridis, J.G., 'On the Boltzmann–Hamel equations of motion: A vectorial treatment', *J. Appl. Mech.* **61**, 1994, 453–459.
41. Papastavridis, J.G., 'A panoramic overview of the principles and equations of motion of advanced engineering dynamics', *Appl. Mech. Rev.* **51**(4), 1998, 239–265.
42. Pars, L.A., *Treatise of Analytical Dynamics*, W. Heinemann, London, 1965.
43. Rong, Q., 'The equations of motion for the system of the variable mass in the non-linear non-holonomic space', *Appl. Math. Mech. (English Ed.)* **17**(4), 1996, 379–384.
44. Rong, Q., 'The fundamental equations of dynamics using representation of quasi-coordinates in the space of nonLinear non-holonomic constraints', *Appl. Math. Mech. (English Ed.)* **18**(11), 1997, 1105–1113.
45. Rong, Q., 'The method of derivation of Mac-Millan's equations for the non-linear non-holonomic systems', *Appl. Math. Mech. (English Ed.)* **17**(6), 1996, 585–587.
46. Scheuer, A. and Laugier, Ch., 'Planning sub-optimal and continuous-curvature paths for car-like robots', in *Proceedings of the 1998 IEEE/RSJ International Conference on Intelligent Robots and Systems*, IEEE, New York, 1998, 25–31.
47. Shkel, A.M. and Lumelsky, V.J., 'Motion planning for nonholonomic robots in a limited workspace', in *Proceedings of the 1998 IEEE/RSJ International Conference on Intelligent Robots and Systems*, IEEE, New York, 1998, 1473–1478.
48. Shuhong, Z. and Tianling, L., 'Another class of d'Alembert principle and a new Maggi equation for arbitrary order nonholonomic mechanical systems in derivative space', *Appl. Math. Mech. (English Ed.)* **17**(12), 1996, 1165–1169.
49. Tianling, L., 'Universal d'Alembert principle and equations of motion for any order nonholonomic systems in derivative space', *Chinese Sci. Bull.* **24**, 1992, 56–63.
50. Tzenoff, J., 'Sur les équations du mouvement des systèmes mécanique nonholonomes', *Math. Ann.* **91**, 1924, 161–168.
51. Udwadia, F. and Kalaba, R., *Analytical Dynamics – A New Approach*, Cambridge University Press, Cambridge, 1996.
52. Vershik, A.M., 'Classical and nonclassical dynamics with constraints', in *New in Global Analysis*, Gosudarstviennij Universitet, Moscow, 1984, 36–64.
53. Vershik, A.M. and Gershkovich, V.Ya., 'Nonholonomic problems and the theory of distributions', *Acta Appl. Math.* **12**, 1988, 181–209.
54. Voronets, P.V., 'On the equations of motion for nonholonomic systems', *Mat. Sb.* **22**(4), 1901 [in Russian].
55. Vranceanu, G., 'Studio geometrico dei sistemi annolonomi', *Bull. Mat.* **30**, 1929.

56. Ye-Hwa, ‘Second order constraints for equations of motion of constrained systems’, *IEEE/ASME Trans. Mechatronics* **3**(3), 1988, 240–248.
57. Zotov, Yu.K. and Tomofeyev, A.V., ‘Controllability and stabilization of programmed motions of reversible mechanical and electromechanical systems’, *J. Appl. Math. Mech.* **56**(6), 1992, 873–880.