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Lower Subdifferentiability and Integration

M. BACHIR[1], A. DANIILIDIS[2] and J.-P. PENOT[3]
[1] *UFR Sciences et Techniques, 16, route de Gray, 25030 Besançon Cedex, France. e-mail: bachir@math.univ-fcomte.fr*
[2] *Laboratoire de Mathématiques Appliquées, CNRS ERS 2055, Université de Pau et des Pays de l'Adour, Avenue de l'Université, 64000 Pau, France. e-mail: aris.daniilidis@univ-pau.fr*
[3] *Laboratoire de Mathématiques Appliquées, CNRS ERS 2055, Université de Pau et des Pays de l'Adour, Avenue de l'Université, 64000 PAU, France. e-mail: jean-paul.penot@univ-pau.fr*

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Abstract. We consider the question of integration of a multivalued operator T, that is the question of finding a function f such that $T \subseteq \partial f$. If ∂ is the Fenchel–Moreau subdifferential, the above problem has been completely solved by Rockafellar, who introduced cyclic monotonicity as a necessary and sufficient condition. In this article we consider the case where f is quasiconvex and ∂ is the lower subdifferential $\partial^<$. This leads to the introduction of a property that is reminiscent to cyclic monotonicity. We also consider the question of the density of the domains of subdifferential operators.

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1. Introduction

The integration of an operator $T\colon X \to X^*$, i.e., the question of finding a differentiable function f such that $T = \nabla f$, has attracted much interest. When the operator T is multivalued, this question is transformed into showing that for some function f one has $T \subseteq \partial f$ (for some notion of subdifferential). The above problem has been solved by Rockafellar, in case one imposes that f should be convex and takes ∂ to be the Fenchel–Moreau subdifferential of convex analysis:

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geqslant x^*(y - x),\ \forall y \in X\}. \tag{1}$$

This gave rise to the class of cyclically monotone operators. Every such operator T is included in the subdifferential ∂f_T of a l.s.c. convex function f_T (and coincides with ∂f_T if and only if T is maximal). In particular the function f_T turns out to be unique up to a constant [16].

The general question of integrating a non cyclically monotone multivalued operator $T\colon X \to 2^{X^*}$ has already been considered by several authors [3, 7, 15, 18], etc. In this article we relax the convexity requirement on f to quasiconvexity, that is convexity of its sublevel sets. The class of quasiconvex functions is much larger than the class of convex functions and appears naturally in concrete problems. A first difficulty in the question of integration arises with the choice of a subdifferential. One line of research consists in using a subdifferential of local nature generalizing the derivative (see [4, 17], e.g.). In that case, characterizations of quasiconvexity have been established by means of the concept of quasimonotonicity for multivalued operators [1, 6, 11], e.g., and references therein). In this line of research, cyclic quasimonotonicity (defined in [5]) turned out to be an intrinsic property of the subdifferentials of quasiconvex functions. Thus an analogy with the convex case appears. However, it is far from obvious to find additional assumptions ensuring that a cyclically quasimonotone operator is included in the subdifferential of a quasiconvex function.

Here we depart from this track and we work with the lower subdifferential of Plastria [14] which is an adaptation to the quasiconvex case of the Fenchel–Moreau subdifferential (1). For any $x \in X$ with $f(x) < +\infty$, the lower subdifferential $\partial^< f(x)$ is given by:

$$\partial^< f(x) = \{x^* \in X^* : f(y) - f(x) \geqslant x^*(y - x),\ \forall y \in S^<_{f(x)}\}, \tag{2}$$

where $S^<_{f(x)} := \{x' \in X : f(x') < f(x)\}$ is the strict sublevel set. Relation (2) can also take the following form:

$$\partial^< f(x) = \left\{x^* \in X^* : f(y) \geqslant \min\left\{\begin{array}{c} f(x) \\ f(x) + x^*(y - x) \end{array}\right\}, \forall y \in X\right\}. \tag{3}$$

One easily observes that, as with the Fenchel–Moreau subdifferential, $\partial^<$ is not a local notion: two functions that coincide in a neighborhood of x, may not have the same lower subdifferential at this point. We also remark that for every $x^* \in \partial^< f(x)$, we have $\{\lambda x^* : \lambda \geqslant 1\} \subseteq \partial^< f(x)$, which shows that $\partial^< f(x)$ is not bounded. (In particular ∂f and $\partial^< f$ are in general different even for convex functions.) However, under this notion, quasiconvex Lipschitz functions are characterized by the existence of a bounded selection for their lower subdifferential (see [14] for $X = \mathbb{R}^n$ and [8] for the general case). We extend these results in Section 4, while in Section 3 we consider the question of the density of the domain of the Fenchel–Moreau subdifferential of an arbitrary function f. Note that if the function f is not convex, the Fenchel–Moreau subdifferential is often empty. As we show in Section 3, its nonemptiness in a dense subset of X implies the convexity of f.

In Section 2 we review some results concerning cyclically monotone operators and Rockafellar's integration technique for the Fenchel–Moreau subdifferential. We note in particular that this integration requires a property that – a priori – seems to be weaker than cyclic monotonicity (CM), namely what we call 'cyclic monotonicity with respect to a certain point x_0' (CM(x_0)). However, these properties turn out to be equivalent. This alternative description of cyclic monotonicity motivates the introduction, in Section 5, of a new class of operators, that is operators fulfilling a certain property ($L(x_0)$) with respect to some fixed point x_0. This property represents a pointwise version of cyclic monotonicity: indeed

$(L(x_0))$ is strictly weaker than cyclic monotonicity, while an operator T is cyclically monotone if, and only if, T satisfies $(L(x))$ for all $x \in \mathrm{dom}(T)$. We also show that the lower subdifferential $\partial^< f$ of any function f restricted to the set $S^<_{f(x_0)} \cup \{x_0\}$ fulfills $(L(x_0))$. Moreover, any such operator T is included in the lower subdifferential $\partial^< f$ of some quasiconvex l.s.c. function f.

In the last section we introduce the class of operators fulfilling another property – that we denote by $(R(x_0))$ – relative to a (fixed) point x_0. This property is strictly weaker than $(L(x_0))$. It is shown that if T fulfills $(R(x))$ at every point of its domain, then it is monotone. The main result of Section 6 states that the operator T defined by $T(x) = \partial^< f(x)$, if $x \neq x_0$ and $T(x_0) = \partial f(x_0)$ satisfies $(R(x_0))$, for any f such that $\partial f(x_0) \neq \emptyset$. On the other hand, any operator of this class is always contained in the lower subdifferential of some quasiconvex l.s.c. function f. Thus we obtain a characterization of this class, which is similar to the one given for cyclic monotonicity by means of the Fenchel–Moreau subdifferential.

Let us point out that while lower semicontinuous convex functions are determined up to a constant by their Fenchel–Moreau subdifferentials, two continuous (even differentiable) quasiconvex functions having the same Plastria subdifferential may differ essentially. In fact, the Plastria subdifferential of a continuous quasiconvex function may even be empty, as shown by the example of the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^p$, where $p > 1$ is an odd integer. (More generally, $\partial^< f$ is empty whenever $\liminf_{\|x\| \to \infty} f(x)/\|x\| = -\infty$.)

Throughout this paper, we often use the following abbreviations: FM subdifferential for the Fenchel–Moreau subdifferential, l.s.c. for lower semicontinuous and CM operator for a cyclically monotone operator. Furthermore, X denotes a Banach space with dual space X^*, f a function on X with values in $\mathbb{R} \cup \{+\infty\}$, and T a multivalued operator defined on X and taking its values in the subsets of X^*. For any $x \in X$ and any $x^* \in X^*$ we denote by $x^*(x)$ the value of the functional x^* at the point x. We also use the standard notation: $B_\varepsilon(x)$ for the closed ball centered at x with radius $\varepsilon > 0$, $\mathrm{dom}(f) := \{x \in X : f(x) \in \mathbb{R}\}$ for the domain of the function f, $S_{f(x)} := \{x' \in X : f(x') \leqslant f(x)\}$ and $S^<_{f(x)} = \{x' \in X : f(x') < f(x)\}$ for the sublevel and the strict sublevel sets of f respectively and $\mathrm{dom}(T) := \{x \in X : T(x) \neq \emptyset\}$ for the domain of the multivalued operator T.

2. Integration of the Subdifferential of a Nonconvex Function

The properties we introduce and discuss in this article are defined by fixing a certain point x_0 as a base point. It is natural to ask whether this choice plays any role. In this section we shall see that this is not the case for the property of cyclic monotonicity.

DEFINITION 2.1. Let $T: X \to 2^{X^*}$ be a multivalued operator. The operator T is called

(i) cyclically monotone with respect to a point $x_0 \in \mathrm{dom}(T)$ (or alternatively T has the (CM(x_0)) property), if for any $x_1, x_2, \ldots, x_n \in X$ and any $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$ one has

$$x_n^*(x_0 - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \leqslant 0,$$

(ii) cyclically monotone (CM), if it satisfies (CM(x)) for every point x of its domain.

It is clear that Definition 2.1(ii) coincides with the standard definition of cyclic monotonicity (see Definition 2.20 in [13]), while it obviously implies Definition 2.1(i). The following proposition shows that the converse is also true.

PROPOSITION 2.2. *Every operator satisfying* (CM(x_0)) *is cyclically monotone.*

Proof. Suppose that T satisfies (CM(x_0)) and that for some $(z_i)_{i=1}^n \subset \mathrm{dom}(T)$ and $z_i^* \in T(z_i), i = 1, 2, \ldots, n$ we have $z_n^*(z_1 - z_n) + \sum_{i=1}^{n-1} z_i^*(z_{i+1} - z_i) = \alpha > 0$. For any $k \in \mathbb{N}$ and $i = 0, 1, 2, \ldots, k \cdot n$ we define $x_{i+1} = z_{i(\mathrm{mod}\, n)+1}$, $x_{i+1}^* = z_{i(\mathrm{mod}\, n)+1}^*$ (where for $i \geqslant 0$, we have $j = i \pmod n$ iff $i - j = pn$, for some $p \in \mathbb{N}$ and $0 \leqslant j < n$). Let $x_0^* \in T(x_0)$. Since T satisfies (CM(x_0)) we have:

$$x_{kn+1}^*(x_0 - x_{kn+1}) + \sum_{i=0}^{kn} x_i^*(x_{i+1} - x_i) \leqslant 0$$

which implies:

$$x_0^*(z_1 - x_0) + z_1^*(x_0 - z_1) + k\left\{ z_n^*(z_1 - z_n) + \sum_{i=1}^{n-1} z_i^*(z_{i+1} - z_i) \right\} \leqslant 0.$$

Taking the limit as $k \to +\infty$ we obtain a contradiction. □

Remark 2.3. An operator T can be cyclically monotone in a trivial way, if for instance $\mathrm{dom}(T) = \emptyset$ or if $\mathrm{dom}(T) = \{x_0\}$.

Let us observe that cyclic monotonicity of ∂f is tied to the very definition of the Fenchel–Moreau subdifferential ∂f and does not depend on the convexity of the function f. Indeed, if f is any function and $T: X \to 2^{X^*}$ any operator satisfying $T \subseteq \partial f$, then for any $x_0, x_1, \ldots, x_n \in X$ and $x_i^* \in T(x_i)$ ($i = 0, 1, \ldots, n$) relation (1) guarantees that $f(x_{i+1}) - f(x_i) \geqslant x_i^*(x_{i+1} - x_i)$. Setting $x_{n+1} := x_0$ and adding the previous inequalities yields $\sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) \leqslant 0$. Let us state this observation as a lemma for further reference.

LEMMA 2.4. *For any function f, any operator T satisfying $T \subseteq \partial f$ is cyclically monotone.*

The converse assertion dealing with the integration of cyclically monotone operators is more interesting. The proof can be found in [16] and essentially requires condition (CM(x_0)).

THEOREM 2.5. *Let T be a multivalued operator satisfying* (CM(x_0)) *at some point x_0 of its domain. Then there exists a l.s.c. convex function f_T such that $T \subseteq \partial f_T$.*

The l.s.c. convex function f_T of the above theorem has been constructed in [16] (see also [13]) by the following formula, in which c is a fixed constant:

$$f_T(x) = c + \sup\left\{ x_n^*(x - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \right\}, \tag{4}$$

where the supremum is taken over all $n \in \mathbb{N} \setminus \{0\}$, all finite sequences $\{x_1, x_2, \ldots, x_n\}$ in $\mathrm{dom}(T)$ and all $x_i^* \in T(x_i)$, for $i = 0, 1, \ldots, n$.

Let us note here that (CM(x_0)) ensures that f_T is not identically equal to $+\infty$, since $f_T(x_0) = c$.

Remark 2.6. Combining Theorem 2.5 with Lemma 2.4 we obtain an alternative way to establish Proposition 2.2.

We also recall that the second conjugate f^{**} of a proper function f is given by:

$$f^{**}(x) = \sup_{x^* \in X^*} [x^*(x) - f^*(x^*)], \tag{5}$$

where

$$f^*(x^*) = \sup_{x \in X} [x^*(x) - f(x)]. \tag{6}$$

Since the subdifferential of any function f is cyclically monotone, the l.s.c. convex function f_T given in (4) is well defined when one takes $T = \partial f$ and $\partial f(x_0) \neq \emptyset$. If in particular f is l.s.c. convex, the uniqueness of Rockafellar's integration ([16]) shows that for $c = f(x_0)$ one has $f_T = f$, so in particular $f_T = f^{**}$. If now f is not convex, a natural question arises: is f_T related to f^{**}? We provide below a positive answer in finite dimensions under a coercivity assumption on f. Let us first observe that (for $c = f(x_0)$) $f_T \leqslant f$ from which it follows $f_T \leqslant f^{**}$, since f^{**} is the greatest l.s.c. convex function majorized by f.

PROPOSITION 2.7. *Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c.,* 1*-coercive function* (*i.e.,* $\lim_{\|x\| \to \infty} f(x)/\|x\| = +\infty$)*, and let $T = \partial f$. Then for some constant c, the functions f_T and f^{**}* (*defined in* (4) *and* (5) *respectively*) *coincide.*

Proof. From our assumptions it follows that f attains its minimum at some point x_0, hence $0 \in \partial f(x_0)$. It follows that $f^{**}(x_0) = f(x_0)$. Taking $c = f(x_0)$ in (4), we conclude from (1) that $f_T \leqslant f$. Since f_T is convex l.s.c., it follows that $f_T \leqslant f^{**}$.

Let us prove the reverse inequality. Since the function f^{**} is l.s.c. and convex, it follows from Theorem B in [16] that:

$$f^{**}(x) \ = f(x_0) + \sup\left\{\sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) + x_n^*(x - x_n)\right\}, \tag{7}$$

where the supremum is taken over all $n \in \mathbb{N}$, all finite sequences $\{x_1, x_2, \ldots, x_n\}$ in $\mathrm{dom}(\partial f^{**})$ and all choices $x_i^* \in \partial f^{**}(x_i)$, for $i = 0, 1, \ldots, n$.

Using the inequality $f^{**} \leqslant f$, for any $x \in \mathbb{R}^n$ one has:

$$f(x) = f^{**}(x) \Rightarrow \partial f^{**}(x) \subseteq \partial f(x). \tag{8}$$

In particular, since $f^{**}(x_0) = f(x_0)$, one observes that

$$\partial f^{**}(x_0) \subseteq \partial f(x_0). \tag{9}$$

Fix now $x \in X$ and consider any $M < f^{**}(x)$. For some $x_1, x_2, \ldots, x_n \in X$ and $x_i^* \in \partial f^{**}(x_i)$ one has

$$M - f(x_0) < x_0^*(x_1 - x_0) + x_1^*(x_2 - x_1) + \cdots + x_n^*(x - x_n). \tag{10}$$

Since the function f is 1-coercive and is defined in a finite-dimensional space, using Theorem 3.6 of [2] we conclude that for $i \in \{1, 2, \ldots, n\}$, there exist $(y_i^j)_{j=1}^{k_i}$ in X, and $(\lambda_i^j)_{j=1}^{k_i}$in (0, 1) with $\sum_{j=1}^{k_i} \lambda_i^j = 1$ such that

$$x_i^* \in \bigcap_{j=1,2,\ldots,k_i} \partial f(y_i^j) \tag{11}$$

and

$$x_i = \sum_{j=1}^{k_i} \lambda_i^j y_i^j. \tag{12}$$

CLAIM. *There exists some $y_1^{j_1}$ such that*

$$x_0^*(y_1^{j_1} - x_0) + x_1^*(x_2 - y_1^{j_1}) \geqslant x_0^*(x_1 - x_0) + x_1^*(x_2 - x_1). \tag{13}$$

Proof. If this were not the case, then for every j we would have

$$x_0^*(y_1^j - x_0) + x_1^*(x_2 - y_1^j) < x_0^*(x_1 - x_0) + x_1^*(x_2 - x_1). \tag{14}$$

Multiplying both sides of (14) by λ_1^j and adding the resulting inequalities for $j = 1, 2, \ldots, k_1$ we get a contradiction by using (12). □

Arguing in the same way as in the proof of the above claim, we can find some $y_2^{j_2}$ such that

$$x_1^*(y_2^{j_2} - y_1^{j_1}) + x_2^*(x_3 - y_2^{j_2}) \geqslant x_1^*(x_2 - y_1^{j_1}) + x_2^*(x_3 - x_2). \tag{15}$$

It follows that

$$\begin{aligned} &x_0^*(y_1^j - x_0) + x_1^*(y_2^{j_2} - y_1^{j_1}) + x_2^*(x_3 - y_2^{j_2}) \\ &\quad \geqslant x_0^*(x_1 - x_0) + x_1^*(x_2 - x_1) + x_2^*(x_3 - x_2). \end{aligned}$$

Proceeding like this, we inductively show that

$$M - f(x_0) < x_0^*(y_1^{j_1} - x_0) + x_1^*(y_2^{j_2} - y_1^{j_1}) + \cdots + x_n^*(x - y_n^{j_n}).$$

Note that from (9) we have $x_0^* \in \partial f(x_0)$, while from (11) we get $x_i^* \in \partial f(y_i^{j_i})$, for $i = 1, 2, \ldots, n$. Now (4) guarantees that $M < f_T(x)$. Since M can be chosen to be arbitrarily close to $f^{**}(x)$, we conclude that $f_T(x) \geqslant f^{**}(x)$, hence equality holds. □

Let us remark that the above proof shows that $f_T = f^{**}$ whenever the l.s.c. function f satisfies the following condition:

(C) For any $x \in \text{dom}(\partial f^{**})$ and $x^* \in \partial f^{**}(x)$, there exist $(y_i)_{i=1}^k \subseteq X$ and $(\lambda_i)_{i=1}^k$ in $(0, 1)$ with $\sum_{i=1}^k \lambda_i = 1$, such that $x = \sum_{i=1}^k \lambda_i y_i$ and $x^* \in \bigcap_{j=1}^k \partial f(y_i)$.

The conclusion of Proposition 2.7 can be satisfied also by noncoercive functions (in infinite-dimensional spaces), as for instance by the function $f(x) = \min\{\|x\|, 1\}$.

COROLLARY 2.8 *Let f and g be two l.s.c. functions satisfying condition* (C). *If $\partial f = \partial g$, then $f^{**} = g^{**}$* (*up to a constant*).

Proof. Let $T = \partial f = \partial g$. Note that condition (C) yields $\text{dom}(T) \neq \emptyset$. Let $x_0 \in \text{dom}(T)$. The proof of Proposition 2.7 shows that $f^{**} = f_T$ when one takes $c = f(x_0)$ in (4) and that $g^{**} = f_T + g(x_0) - c$. □

3. Functions with a Dense Domain of Subdifferentiability

In the preceding section we considered operators that are (included in) the subdifferential of a nonconvex function. These operators are cyclically monotone, but this may happen in a trivial way, see Remark 2.3. The example of the function $f(x) = \min\{\|x\|, 1\}$ (also $f(x) = \sqrt{\|x\|}$) shows that one may have $f_T = f^{**}$ even if ∂f is a singleton. However this relation is more likely to be satisfied when the domain $\text{dom}(\partial f)$ is large. In this section, we shall consider the question of the density of the domain of such operators. The following proposition shows that for l.s.c. functions that do not take the value $+\infty$, the density of ∂f is equivalent to the convexity of the function.

PROPOSITION 3.1. *Let* $f: X \to \mathbb{R}$ *(i.e.,* $\mathrm{dom}(f) = X$*) be l.s.c. and such that* $\mathrm{dom}(\partial f)$ *is dense in* X*. Then* f *is convex and locally Lipschitz.*

In particular the operator ∂f *is maximal monotone and locally bounded.*

Proof. We first show that f is convex. Since $\mathrm{dom}(\partial f)$ is nonempty, we conclude that $f^{**} > -\infty$, which together with $f \geqslant f^{**}$ shows that $X = \mathrm{dom}(f) \subseteq \mathrm{dom}(f^{**})$. It follows that the l.s.c. convex function f^{**} is continuous.

We now show that the functions f and f^{**} coincide. One observes that $f(x) = f^{**}(x)$, for every $x \in \mathrm{dom}(\partial f)$. Take now any x in X. Our assumption implies the existence of a sequence $(x_n)_n$ in $\mathrm{dom}(\partial f)$ such that $(x_n) \to x$. Since $f^{**}(x_n) = f(x_n)$, for $n \in \mathbb{N}$, f is l.s.c. and f^{**} is continuous we get:

$$f^{**}(x) = \liminf_n f^{**}(x_n) = \liminf_n f(x_n) \geqslant f(x) \geqslant f^{**}(x).$$

Thus $f = f^{**}$. For the last assertion see Theorem 2.25 and Theorem 2.28 in [13], e.g. □

We do not know if the assumption $\mathrm{dom}(f) = X$ in the above proposition can be omitted. The following corollary shows that this assumption is not necessary if $X = \mathbb{R}^n$. In this case it becomes part of the conclusions.

COROLLARY 3.2. *Let* $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be l.s.c. and such that* $\mathrm{dom}(\partial f)$ *is dense in* $\mathbb{R}^n$*. Then* $\mathrm{dom}(f) = \mathbb{R}^n$ *and the function* f *is convex and locally Lipschitz.*

Proof. We have $\mathrm{dom}(\partial f) \subseteq \mathrm{dom}(f^{**})$, so $\mathrm{dom}(f^{**})$ is also dense in $\mathbb{R}^n$. Since $\mathrm{dom}(f^{**})$ is convex, it follows that $\mathrm{dom}(f^{**}) = \mathbb{R}^n$, hence f^{**} is continuous.

Arguing as in the last part of the proof of Proposition 3.1 we conclude again that f is convex and continuous. □

However the following example shows that the lower semicontinuity assumption cannot be dropped, even in the case $X = \mathbb{R}$.

EXAMPLE. Consider the indicator function i_D of any dense subset D of $\mathbb{R}$:

$$i_D(x) = \begin{cases} 0 & \text{if } x \in D, \\ +\infty & \text{if } x \notin D. \end{cases}$$

We note that this function is l.s.c. on its domain, without being l.s.c. in the whole space (unless $D = \mathbb{R}$). Moreover, for every $x \in D$, we have $\partial i_D(x) = \{0\}$, hence $D \subseteq \mathrm{dom}(\partial i_D)$. However, the function i_D is not convex.

Let us now give an infinite-dimensional version of Corollary 3.2 by means of an additional assumption on the operator ∂f. We shall say that an operator $T: X \to 2^{X^*}$ has a (*locally*) *bounded selection on its domain*, if for every $x_0 \in X$ there exists $M > 0$ and $\rho > 0$ such that:

$$\forall z \in \mathrm{dom}(T) \cap B_\rho(x_0),\ \exists z^* \in T(z): \ \|z^*\| \leqslant M. \tag{16}$$

LEMMA 3.3. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function such that* $\mathrm{dom}(\partial f)$ *is dense in* X. *If* ∂f *has a* (*locally*) *bounded selection on* $\mathrm{dom}(\partial f)$, *then* $\mathrm{dom}(f) = X$ *and* f *is* (*locally*) *Lipschitz.*

Proof. Let us first assume that ∂f has a locally bounded selection on $\mathrm{dom}(\partial f)$ and let $\rho > 0$ and $M > 0$ be as in (16). We show that the function f is Lipschitzian on the interior $\mathrm{int}\, B_\rho(x_0)$ of $B_\rho(x_0)$ with constant at most M. Indeed take any $x, y \in \mathrm{int}\, B_\rho(x_0)$. Since $\mathrm{dom}(\partial f)$ is dense on X, there exists a sequence $(x_n)_{n\in N}$ in $\mathrm{dom}(\partial f) \cap B_\rho(x_0)$ and $x_n^* \in \partial f(x_n)$, with $\|x_n^*\| \leqslant M$, such that $(x_n) \to x$. From (1) we conclude that $f(x_n) \leqslant f(y) + x_n^*(x_n - y)$. Since f is l.s.c., taking the limit as $n \to +\infty$ we get

$$f(x) \leqslant f(y) + M\|x - y\|. \tag{17}$$

Since (17) holds for all y in $\mathrm{int}\, B_\rho(x_0)$, choosing y in $\mathrm{dom}(f)$ we conclude that f is finite at x. Since x is arbitrary in $\mathrm{int}\, B_\rho(x_0)$, we conclude that $\mathrm{int}\, B_\rho(x_0) \subseteq \mathrm{dom}(f)$. It now follows easily that f is Lipschitz on $\mathrm{int} B_\rho(x_0)$.

If now ∂f has a bounded selection on $\mathrm{dom}(\partial f)$, taking $\rho = +\infty$ we conclude that f is Lipschitz. □

We now state the following corollary.

COROLLARY 3.4. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function. The following statements are equivalent:*

(i) $\mathrm{dom}(\partial f)$ *is dense in* X *and* ∂f *has a* (*locally*) *bounded selection on* $\mathrm{dom}(\partial f)$.
(ii) $\mathrm{dom}(\partial f) = X$ *and* ∂f *is* (*locally*) *bounded.*
(iii) $\mathrm{dom}(f) = X$ *and* f *is convex and* (*locally*) *Lipschitz.*

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. The implication (i) $\Rightarrow$ (iii) follows from Lemma 3.3 and Proposition 3.1. □

4. Lower Subdifferentials with a Dense Domain

In this section we endeavor to complete results of the literature concerning quasiconvex functions and their lower subdifferentials, in order to reveal analogies with the characterization of Corollary 3.4. We recall that a function $f: X \to \mathbb{R} \cup \{+\infty\}$ is called *quasiconvex*, if its sublevel sets $S_\lambda(f) := \{x \in X : f(x) \leqslant \lambda\}$ are convex for $\lambda \in \mathbb{R}$, or equivalently, if for any $x, y \in X$ and $t \in [0, 1]$ the following inequality holds:

$$f(tx + (1 - t)y) \leqslant \max\{f(x), f(y)\}.$$

We first state the following lemma concerning the lower subdifferential $\partial^<$ (defined in (2) or (3)). We omit its proof, since it is similar to the proof of Lemma 3.3.

LEMMA 4.1. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function such that* $\mathrm{dom}(\partial^< f)$ *is dense on* X. *If the operator* $\partial^< f$ *has a (locally) bounded selection on* $\mathrm{dom}(\partial^< f)$, *then* $\mathrm{dom}(f) = X$ *and* f *is (locally) Lipschitz.*

The theorem that follows is analogous to Corollary 3.4.

THEOREM 4.2. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function. The following assertions are equivalent:*

(i) $\mathrm{dom}(\partial^< f)$ *is dense on* X *and* $\partial^< f$ *has a bounded selection on* $\mathrm{dom}(\partial^< f)$.
(ii) $\partial^< f$ *has a bounded selection on* X.
(iii) f *is quasiconvex, Lipschitz and* $\mathrm{dom}(f) = X$.

Proof. The equivalence (ii) $\Leftrightarrow$ (iii) was proved in [9] (see Corollary 3.3). Implication (ii) $\Rightarrow$ (i) is obvious. For (i) $\Rightarrow$ (iii) we first apply Lemma 4.1 to conclude that f is Lipschitz. In particular the sublevel sets S_λ of f have nonempty interior, whenever $\lambda > \inf f$. It now follows from Proposition 3.1(i) of [10] that f is quasiconvex. □

The following result extends Theorem 4.2 in a non-Lipschitzian case and is comparable to Corollary 3.4. However the implication (iii) $\Rightarrow$ (ii) does not hold in general, as shown by the example below.

PROPOSITION 4.3. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function. Among the following statements one has* (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

(i) $\mathrm{dom}(\partial^< f)$ *is dense and* $\partial^< f$ *has a locally bounded selection on* $\mathrm{dom}(\partial^< f)$.
(ii) $\partial^< f$ *has a locally bounded selection on* X.
(iii) $\mathrm{dom}(f) = X$ *and* f *is quasiconvex and locally Lipschitz.*

If the restrictions of f *to its sublevel sets are Lipschitzian, then the above statements are equivalent.*

Proof. Implication (ii) $\Rightarrow$ (i) is obvious. If (i) holds, then using Lemma 4.1 we conclude that $\mathrm{dom} f = X$ and f is locally Lipschitz. From Proposition 3.1(i) of [10] it now follows that f is quasiconvex, hence (iii) holds.

Let us now assume that f is quasiconvex, continuous, $\mathrm{dom}(f) = X$ and for any $\lambda \in \mathbb{R}$ the restriction of f to $S_\lambda := \{x \in X : f(x) \leqslant \lambda\}$ is a Lipschitz function of constant k, for some $k > 0$. We show that $\partial^< f$ has a bounded selection on S_λ.

Indeed, consider any $x_0 \in S_\lambda$. If $f(x_0) = \inf f$, then $0 \in \partial^< f(x_0)$. Hence we may suppose that $f(x_0) > \inf f$. Since f is continuous, the closed convex set $S_{f(x_0)}$ has a nonempty interior. Separating $\mathrm{int}\, S_{f(x_0)}$ from $\{x_0\}$, we obtain a functional $z^* \in X^*$, with $\|z^*\| = 1$ such that $z^*(x) < z^*(x_0)$, for all $x \in \mathrm{int}\, S_{f(x_0)}$. It is easily seen that x_0 is minimizer of f on the half space $\{y \in X : z^*(y) \geqslant z^*(x_0)\}$. Set $x_0^* = k' z^*$ with $k' > k$.

CLAIM. $x_0^* \in \partial^< f(x_0)$.

Proof. Suppose that $x_0^* \notin \partial^< f(x_0)$. It follows from (2) that for some $x \in S^<_{f(x_0)}$ we have $f(x_0) - f(x) > x_0^*(x_0 - x)$. Given any $\varepsilon > 0$, we may find $y \in X$ such that $x_0^*(y) = x_0^*(x_0)$ and $x_0^*(y - x) + \varepsilon \geqslant \|x_0^*\| \|y - x\| = k'\|y - x\|$. Since f is continuous, we can find some x' in the segment $[x, y]$ such that $f(x') = f(x_0)$. We easily get that $x_0^*(x' - x) + \varepsilon \geqslant k'\|x' - x\|$. Since $f(x_0) - f(x) > x_0^*(x_0 - x) = x_0^*(y - x) > x_0^*(x' - x)$, it follows that $f(x') - f(x) > k'\|x' - x\| - \varepsilon$. Since ε is arbitrary, we have contradicted the fact that f is Lipschitz on $S_{f(x_0)}$ with constant k. □

Since x_0 is arbitrary in S_λ (and since λ is arbitrary), we have shown that $\text{dom}(\partial^< f) = X$. Moreover, the continuity assumption of (iii) ensures that for any $x \in X$ and $\lambda > f(x)$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S_\lambda$. If k is the Lipschitz constant of f on S_λ, the previous claim asserts that $\partial^< f$ has a selection on $B_\varepsilon(x)$ which is (norm) bounded by any $k' > k$. □

Remark. The claim of the preceding proof relies heavily on techniques employed in [14] (see also Corollary 4.20 in [8] or Proposition 6.2 in [12]) in order to prove the equivalence (ii) $\Leftrightarrow$ (iii) in Theorem 4.2 if $X = \mathbb{R}^n$. In finite dimensions it has been shown in Corollary 4.20 of [8] that, if condition (iii) of Proposition 4.3 holds and f is inf-compact (that is for all $\lambda \in \mathbb{R}$, the set S_λ is compact), then f is everywhere lower subdifferentiable, that is $\text{dom}(\partial^< f) = \mathbb{R}^n$. Note that the assumptions f is inf-compact and $\text{dom}(f) = X$ imply that the space X can be written as a countable union of compact sets, hence it is finite-dimensional. On the other hand, an easy compactness argument shows that if condition (iii) holds and f is inf-compact, then the restriction of f to the sublevel sets is a Lipschitz function. Hence Proposition 4.3 can be seen as an extension of Corollary 4.20 in [8] to infinite dimensions, which also establishes the existence of a locally bounded selection.

One cannot expect a characterization similar to Theorem 4.2. The following example shows that, without additional assumptions, a locally Lipschitz quasiconvex function f may have its subdifferential $\partial^< f$ everywhere empty.

EXAMPLE. Let $X = \mathbb{R}$ and consider the quasiconvex function $f: \mathbb{R} \to \mathbb{R}$, with

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0, \\ x & \text{if } x \geqslant 0. \end{cases}$$

It is easy to see that f is locally Lipschitz, but $\partial^< f(x) = \emptyset$, for all $x \in \mathbb{R}$.

5. Integration by Means of the Lower Subdifferential

In this section we consider again the problem of integrating a multivalued operator, by relaxing this time the assumption on f (to be quasiconvex instead of being convex) and by taking ∂ to be the lower subdifferential $\partial^{<}$. We replace accordingly cyclic monotonicity with a certain point-based property that we call $(L(x_0))$. This property yields the construction of a l.s.c. quasiconvex function g_T in a way reminiscent to the construction of the l.s.c. convex function f_T in (4) by means of Definition 2.1(i). We show that a cyclically monotone operator fulfills $(L(x))$ at any point $x \in \mathrm{dom}(T)$. Conversely, if an operator satisfies $(L(x))$ at every point of its domain, then it is cyclically monotone (see Proposition 5.2). Roughly speaking, property $(L(x_0))$ is to be understood as a pointwise version of cyclic monotonicity.

DEFINITION 5.1. An operator $T\colon X \to 2^{X^*}$ is said to have property $(L(x_0))$ with respect to some $x_0 \in \mathrm{dom}(T)$, if for any $n \geqslant 1$, any $x_1, x_2, \dots, x_n \in \mathrm{dom}(T)$ and any $x_i^* \in T(x_i)$ for $i = 0, 1, \dots, n$, one has:

$$\min \left\{ \begin{array}{c} x_0^*(x_1 - x_0) \\ x_1^*(x_2 - x_1) + x_0^*(x_1 - x_0) \\ \cdots \\ x_n^*(x_0 - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \end{array} \right\} \leqslant 0.$$

It follows easily that if T is cyclically monotone (see Definition 2.1(ii)), then it satisfies $(L(x))$ at every point of its domain. The following proposition shows that the converse is also true:

PROPOSITION 5.2. *If T satisfies $(L(x))$ for every $x \in \mathrm{dom}(T)$, then T is cyclically monotone.*

Proof. Suppose that T is not cyclically monotone. Then there exist $n \geqslant 2$ and $x_0, x_1, \dots, x_{n-1}$ in X and $x_0^* \in T(x_0)$, $x_1^* \in T(x_1)$,…, $x_{n-1}^* \in T(x_{n-1})$ such that (setting $x_n = x_0$)

$$\sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) > 0. \tag{18}$$

For $i = 0, 1, \dots, n-1$ and for $j = i \pmod n$ (i.e., $j = nm + i$ for some $m \in \mathbb{N}$) we set $\beta_j = x_i^*(x_{i+1} - x_i)$, so that (18) can be rewritten:

$$\sum_{j=0}^{n-1} \beta_j > 0. \tag{19}$$

Thus, there exists some $h_1 \in \{0, 1, \dots, n-1\}$ such that $\beta_{h_1} > 0$. Since the operator T satisfies $L(x_{h_1})$, there exists some $k \in \{h_1 + 1, h_1 + 2, \dots, h_1 + n\}$ such that

$$\sum_{j=h_1}^{k} \beta_j \leqslant 0. \tag{20}$$

Note that the fact that $k \neq h_1 + n$ is ensured by (18). Taking now k to be the largest integer in $\{h_1 + 1, h_1 + 2, \dots, h_1 + n - 1\}$ such that (20) is satisfied, we conclude that $\beta_{k+1} > 0$. Setting now $h_2 = k+1$ and proceeding like this, we define inductively a strictly increasing sequence $(h_q)_{q=1}^{\infty}$ such that for any $q \geqslant 1$ we have $\beta_{h_q} > 0$ and

$$\sum_{i=h_q}^{h_{q+1}-1} \beta_i \leqslant 0. \tag{21}$$

Since the sequence $(h_q \pmod n))_{q\in N}$ has an accumulating point, we can find $p > q \geqslant 1$ such that $h_p = h_q + mn$, for some $m \in \mathbb{N}$ (i.e., $h_p = h_q \pmod n$)). We thus obtain the following equality:

$$\sum_{i=h_q}^{h_{q+1}-1} \beta_i + \sum_{i=h_{q+1}}^{h_{q+2}-1} \beta_i + \cdots + \sum_{i=h_{p-1}}^{h_p-1} \beta_i = \sum_{i=h_q}^{h_p-1} \beta_i = m \sum_{i=0}^{n-1} \beta_i$$

which is not possible in view of (19) and (21). □

Remark. Considering for instance the operator $T: \mathbb{R} \to 2^{\mathbb{R}}$ given by $T(0) = \{0\}$ and $T(x) = [-1, 1]$, if $x \neq 0$, it is easy to see that T satisfies property $(L(x_0))$ for $x_0 = 0$, without being CM.

Motivated by (4) we consider the following function $g_T: X \to \mathbb{R} \cup \{+\infty\}$:

$$g_T(x) = c + \sup \min \left\{ \begin{array}{c} x_0^*(x_1 - x_0) \\ x_1^*(x_2 - x_1) + x_0^*(x_1 - x_0) \\ \cdots \\ x_n^*(x - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \end{array} \right\}, \tag{22}$$

where c is an arbitrary constant and the supremum is taken over all $n \in \mathbb{N}$, all finite sequences $(x_i)_{i=1}^{n} \in \operatorname{dom}(T)$ and all $x_i^* \in T(x_i^*)$, for $i = 0, 1, \ldots, n$. Note that the choice $n = 0$ in the above supremum yields $g_T(x) \geqslant \sup_{x_0^* \in T(x_0)} \{x_0^*(x - x_0)\} + c$. In particular $g_T(x) > -\infty$, for all $x \in X$.

Since g_T is represented as a supremum of a family of subaffine continuous functions (i.e., of functions of the form $x \to \min\{c,\ x^*(x) + b\}$, where $b, c \in \mathbb{R}$), it follows that it is quasiconvex and lower semicontinuous. Comparing (4) and (22) one notes that $g_T(x) \leqslant f_T(x)$, for every $x \in X$.

The following theorem is analogous to Theorem 2.5:

THEOREM 5.3. *If T fulfills $(L(x_0))$ then there exists a l.s.c. quasiconvex function g such that $T(x_0) \subseteq \partial g(x_0)$ and for all $x \in X$, $T(x) \subseteq \partial^{<} g(x)$.*

Proof. Set $g = g_T$. Since T fulfills $(L(x_0))$, it follows (by taking $n = 1$ and $x_1 = x_0$) that $g_T(x_0) = c$, hence as observed before, for any $x \in X$ and any $x_0^* \in T(x_0)$ we have

$$x_0^*(x - x_0) + g_T(x_0) \leqslant g_T(x)$$

which shows that $x_0^* \in \partial g_T(x_0)$.

Let $x^* \in T(x)$. For any $M < g_T(x)$, there exist $n \geqslant 0$ and (for $n > 0$) $x_1, x_2, \ldots, x_n \in X$, $x_0^* \in T(x_0)$, $x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$ such that

$$M < c + \min \left\{ \begin{array}{c} x_0^*(x_1 - x_0) \\ x_1^*(x_2 - x_1) + x_0^*(x_1 - x_0) \\ \cdots \\ x_n^*(x - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \end{array} \right\}. \tag{23}$$

In particular, setting $x_{n+1} := x$ (and considering separately the cases $n = 0$ and $n > 0$), one gets $M < \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) + c$. For any $y \in X$, and adding to both sides of this inequality the quantity $x^*(y - x)$ we obtain:

$$M + x^*(y - x) < \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) + x^*(y - x) + c. \tag{24}$$

Combining (23) and (24) and taking the minimum we obtain:

$$\min\{M, M + x^*(y - x)\} \leqslant c + \min \left\{ \begin{array}{l} x_0^*(x_1 - x_0) \\ x_1^*(x_2 - x_1) + x_0^*(x_1 - x_0) \\ \cdots \\ x_n^*(x - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \\ x^*(y - x) + \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) \end{array} \right\}$$

(with the convention $x_{n+1} := x$). As the right-hand side of the preceding inequality is always less than or equal to $g_T(y)$ and since M can be arbitrarily close to $g_T(x)$, using (3) we conclude that $x^* \in \partial^< g_T(x)$. This finishes the proof. □

Remarks. (1) If one omits the inclusion $T(x_0) \subseteq \partial g(x_0)$ in the above statement (i.e., replaces it by $T(x_0) \subseteq \partial^< g(x_0)$), then the remaining conclusion holds trivially, since one can take for g the constant function.

(2) If the operator T of Theorem 5.3 has a (locally) bounded selection at least in a dense subset of X, then the function g (of Theorem 5.3) will be (locally) Lipschitz. This is an immediate consequence of Theorem 4.2 (resp. Proposition 4.3).

We finally state the following 'converse' to Theorem 5.3.

PROPOSITION 5.4. *For any function f and any $x_0 \in \mathrm{dom}(f)$, the operator T: $S^<_{f(x_0)} \cup \{x_0\} \to 2^{X^*}$ given by $T(x) = \partial^< f(x)$ fulfills $(L(x_0))$.*

Proof. The result follows from the fact that for any $x \in S^<_{f(x_0)}$ and any $x_0^* \in T(x_0)$ one has $x_0^*(x - x_0) \leqslant 0$. □

Note that Proposition 5.4 is similar to Lemma 2.4, the difference being the domain of the operator ($S^<_{f(x_0)} \cup \{x_0\}$ instead of the whole space X).

Property $(L(x_0))$, introduced in the present section, is a logical step from cyclic monotonicity and the FM subdifferential to the lower subdifferential. Theorem 5.3 and Proposition 5.4 almost characterizes this property. However, given a function f with $\partial f(x_0) \neq \emptyset$, Proposition 5.4 (unlike Lemma 2.4) does not describe the behavior of the operator

$$T(x) = \begin{cases} \partial^{<} f(x) & \text{if } x \neq x_0, \\ \partial f(x_0) & \text{if } x = x_0, \end{cases} \tag{25}$$

on the whole space, but only on the strict level set $S^{<}_{f(x_0)}$. This is clearly shown by the following example:

EXAMPLE. Let $f:\mathbb{R} \to \mathbb{R}$ be given by:

$$f(x) = \begin{cases} -1 & \text{if } x \leqslant -1, \\ x & \text{if } x > -1. \end{cases}$$

Then the operator T defined in (25) with $x_0 = 0$ is given as follows:

$$T(x) = \begin{cases} \{1\} & \text{if } x = 0, \\ [1, +\infty) & \text{if } x \in (-1, 0) \cup (0, +\infty), \\ \mathbb{R} & \text{if } x \leqslant -1. \end{cases}$$

It is easy to see – considering the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 3/2$ – that T fails to satisfy $L(0)$.

6. Characterization of Operators which are contained in the Lower Subdifferential of a Function

In this section we introduce the property $(R(x_0))$ aiming at describing the above operator T (see (25)) in the whole space. Although this property is weaker than $(L(x_0))$, we show that operators fulfilling $(R(x_0))$ can still be 'integrated' (in the sense of Theorem 5.3). This leads to a situation similar to Lemma 2.4 and Theorem 2.5. We also show that any operator satisfying $(R(x))$ at every point of its domain, is monotone.

DEFINITION 6.1. An operator $T: X \to 2^{X^*}$ is said to have property $(R(x_0))$ with respect to some $x_0 \in \operatorname{dom}(T)$, if for any $n \geqslant 1$, for any $x_1, x_2, \ldots, x_n \in \operatorname{dom}(T)$ and any $x_i^* \in T(x_i^*)$ for $i = 0, 1, \ldots, n$, one has:

$$x_0^*(x_1 - x_0) + \sum_{i=1}^{n-1} \{x_i^*(x_{i+1} - x_i)\}^- + \{x_n^*(x_0 - x_n)\}^- \leqslant 0, \tag{26}$$

where $\{x_i^*(x_{i+1} - x_i)\}^- := \min\{x_i^*(x_{i+1} - x_i),\ 0\}$.

Definition 6.1 is in the same spirit as Definition 5.1 and Definition 2.1(i). In particular every operator that satisfies $(L(x_0))$ also satisfies $(R(x_0))$. The following example shows that the converse is not true:

EXAMPLE. Let $T: \mathbb{R} \to 2^{\mathbb{R}}$ be such that $T(0) = \{1\}$, $T(1) = \{2\}$, $T(2) = \{1\}$ and $T(x) = \emptyset$ elsewhere. One can verify that T has property $(R(x_0))$ for $x_0 = 0$, without satisfying $(L(x_0))$.

In this example one may observe that the operator T does not satisfy $(R(x))$ at every point of its domain (it fails at the point $x_0 = 1$). The following proposition (together with the fact that for one-dimensional spaces cyclic monotonicity and monotonicity coincide ([5], e.g.)) gives an explanation for this.

PROPOSITION 6.2. *If an operator T fulfills $(R(x))$ at every point of its domain, then T is monotone.*

Proof. Take any $x, y \in X$, $x^* \in T(x)$, $y^* \in T(y)$ and assume that

$$x^*(y-x) + y^*(x-y) > 0. \tag{27}$$

Interchanging the roles of x and y, we may suppose that $y^*(x-y) > 0$. Then taking $n = 1$, $x_0 = x$ and $x_n = y$, relation (26) yields that $x^*(y-x) \leqslant 0$. Taking now $n = 1$, $x_0 = y$ and $x_n = x$, relation (26) leads to a contradiction with (27). □

COROLLARY 6.3. *If $X = \mathbb{R}$, then T fulfills $(R(x))$ for all $x \in \mathrm{dom}(T)$ if, and only if, T is cyclically monotone.*

The following theorem characterizes the class of operators that satisfy property $(R(x_0))$.

THEOREM 6.4. *The operator T satisfies $(R(x_0))$ for some $x_0 \in \mathrm{dom}(T)$ if, and only if, there exists a l.s.c. quasiconvex function h_T such that $T(x_0) \subseteq \partial h_T(x_0)$ and $T(x) \subseteq \partial^< h_T(x)$, for all $x \in X$.*

Proof. (a) Let us first assume that T satisfies $(R(x_0))$ at some point x_0 of its domain. We consider the following function $h_T: X \to \mathbb{R} \cup \{+\infty\}$ given by

$$h_T(x) = c + \sup\left\{ x_0^*(x_1 - x_0) + \sum_{i=1}^{n} \{x_i^*(x_{i+1} - x_i)\}^- \right\}, \tag{28}$$

where $x_{n+1} := x$, c is an arbitrary constant and the supremun is taken over all $n \in \mathbb{N}$, all choices $x_1, x_2, \ldots, x_n \in \mathrm{dom}(T)$ and all $x_i^* \in T(x_i^*)$ for $i = 0, 1, \ldots, n$. We make here the convention that the choice $n = 0$ in the above supremum is acceptable and corresponds to the term $\sup_{x_0^* \in T(x_0)} x_0^*(x - x_0) + c$.

It is easy to see that h_T is l.s.c. and quasiconvex. From Definition 6.1 above, we conclude that $h_T(x_0) \leqslant c$, and in fact $h_T(x_0) = c$. It follows directly from (28) that for every $x \in X$ we have

$$h_T(x) \geqslant \sup_{x_0^* \in T(x_0)} x_0^*(x - x_0) + c = \sup_{x_0^* \in T(x_0)} x_0^*(x - x_0) + h_T(x_0)$$

which in view of (1) ensures that $T(x_0) \subseteq \partial h_T(x_0)$.

Let now $x \in X$ and $x^* \in T(x)$. For $M < h_T(x)$, (28) shows that there exist $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_{n+1} := x \in X$ and $x_0^* \in T(x_0)$, $x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$ such that

$$c + x_0^*(x_1 - x_0) + \sum_{i=1}^{n-1} \{x_i^*(x_{i+1} - x_i)\}^- + \{x_n^*(x - x_n)\}^- > M. \tag{29}$$

(If $n = 0$, then we have $c + x_0^*(x - x_0) > M$.) For any $y \in X$, setting $x_{n+1} := x$, adding to both sides of (29) the quantity $\{x^*(y-x)\}^-$ (and considering successively the cases $n = 0$ and $n > 0$), we obtain

$$\begin{aligned} &c + x_0^*(x_1 - x_0) + \sum_{i=1}^{n} \{x_i^*(x_{i+1} - x_i)\}^- + \{x^*(y - x)\}^- \\ &\quad > M + \{x^*(y - x)\}^-. \end{aligned} \tag{30}$$

We note that the left side of (30) is always less than or equal to $h_T(y)$. Since M can be chosen arbitrarily close to $h_T(x)$, we conclude from (30) that:

$$h_T(y) \geqslant \min \left\{ \begin{array}{c} h_T(x) \\ x^*(y - x) + h_T(x) \end{array} \right\}. \tag{31}$$

It now follows from (3) that $x^* \in \partial^< h_T(x)$. We conclude that for every $x \in X$, $T(x) \subseteq \partial^< h_T(x)$.

(b) Given any function f with $\partial f(x_0) \neq \emptyset$ we consider the multivalued operator

$$T(x) = \begin{cases} \partial^< f(x), & x \neq x_0, \\ \partial f(x_0), & x = x_0. \end{cases} \tag{32}$$

For any $x_0^* \in T(x_0)$ and any $x_1 \in \mathrm{dom}(T)$ we have:

$$f(x_1) - f(x_0) \geqslant x_0^*(x_1 - x_0). \tag{33}$$

Furthermore, for any $x_i \in \mathrm{dom}(T)$, $x_i^* \in T(x_i)$ and any $x_{i+1} \in X$, we conclude from (32) and (3) that

$$f(x_{i+1}) - f(x_i) \geqslant \min\{x_i^*(x_{i+1} - x_i),\ 0\}. \tag{34}$$

Considering any finite cycle $\{x_0, x_1, \ldots, x_n, x_{n+1} := x_0\}$ in $\mathrm{dom}(T)$ and any choice $x_i^* \in T(x_i)$, for $i = 0, 1, \ldots, n$, we conclude from (33) and (34) that:

$$x_0^*(x_1 - x_0) + \sum_{i=1}^{n} \{x_i^*(x_{i+1} - x_i)\}^- \leqslant 0 \tag{35}$$

which shows (see Definition 6.1) that T satisfies $(R(x_0))$.

The observation that property $(R(x_0))$ is inherited by smaller operators (in the sense of the inclusion of graphs) finishes the proof. □

The above theorem gives a characterization of the class of operators that satisfy $(R(x_0))$. The situation is analogous to the one corresponding to the class of cyclically monotone operators as described by Lemma 2.4 and Theorem 2.5.

Remarks. (1) Since property $(L(x_0))$ entails $(R(x_0))$, Theorem 5.3 can be deduced as a consequence of the 'only if' part of Theorem 6.4. Let us also note that, as was the case in Theorem 5.3, the inclusion $T(x_0) \subseteq \partial h_T(x_0)$ is an essential part of Theorem 6.4.

(2) Using Theorem 4.2 or Proposition 4.3, we may conclude that the quasiconvex function h_T constructed in the above proof is (locally) Lipschitz whenever the operator T has a (locally) bounded selection in a dense subset of X.

(3) If there exists $x_0 \in \mathrm{dom}(T)$ such that $T(x_0) = \{0\}$, then the above construction leads to the constant function $h_T = c$. Let us observe that this situation cannot occur if T is given by (32) unless $\partial f(x_0) = \{0\}$.

(4) One may wonder whether the analogy between $(\mathrm{CM}(x_0))$ (cyclically monotone) and $(R(x_0))$ operators can go any further. Namely, starting from an arbitrary function f with $\partial f(x_0) \neq \emptyset$, one may define an operator T of the class $(R(x_0))$ (resp. of the class $(\mathrm{CM}(x_0))$) via relation (25) (resp. $T = \partial f$) and subsequently consider the l.s.c. quasiconvex function h_T (resp. the l.s.c. convex function f_T) given by the formula (28) (resp. (4)). In both cases we have:

$$x_0^*(x - x_0) \leqslant h_T(x) \leqslant f_T(x) \leqslant f(x). \tag{36}$$

It is easily seen that if f is affine, then the functions h_T, f_T and f coincide (modulo the constant $f(x_0)$). It is also known that if f is convex and l.s.c., then f_T and f coincide [16]. However in general the function h_T does not coincide with f and in particular – unlike the convex case – the operator T defined in (25) does not uniquely determine the function f. A comparison of (4), (22) and (28) yields $h_T \leqslant g_T \leqslant f_T$. In the following example we show that if T is defined by (25), the functions h_T and g_T are in general strictly majorized by f.

EXAMPLE. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x + 1| - 1$. Then for $x_0 = 0$, the operator T in (32) is given as follows:

$$T(x) = \begin{cases} [1, +\infty) & \text{if } x \in (-1, 0) \cup (0, +\infty), \\ \{1\} & \text{if } x = 0, \\ \mathbb{R} & \text{if } x = -1, \\ (-\infty, -1] & \text{if } x < -1, \end{cases}$$

hence the constructions (22) and (28) lead to functions g_T and h_T:

$$g_T(x) = h_T(x) = \begin{cases} x & \text{if } x > -1, \\ -1 & \text{if } x \leqslant -1. \end{cases}$$

Remark. As pointed out by the referee, the results of this paragraph and the integration procedure of Rockafellar ([16]) can both be seen as particular cases of the following scheme:

Consider a general function $b: X \times X \times X^* \to \mathbb{R}$. Then for any function $f: X \to \mathbb{R} \cup \{+\infty\}$ let us define the b-subdifferential $\partial^b f: X \to 2^{X^*}$ by

$$\partial^b f(x) = \{x^* \in X^* : f(y) \geqslant f(x) + b(x, y, x^*), \text{ for all } y \in X\}. \tag{37}$$

Further, given an operator $T: X \to 2^{X^*}$ and a point x_0 in $\mathrm{dom}(T)$, define the $b(x_0)$-property as follows: For any $x_1, x_2, \ldots, x_n \in X$ and any $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$

$$\sum_{i=0}^{n} b(x_i, x_{i+1}, x_i^*) \leqslant 0, \tag{38}$$

where the convention $x_{n+1} = x_0$ is used. Then if T has this property, adapting the procedure of Rockafellar (in [16]) we can construct a function f_T in such a way that $T \subseteq \partial^b f_T$. The function f_T, being a supremum of functions of the form $b(x, y, x^*)$, will enjoy a certain property based on $b(\cdot, \cdot, \cdot)$, that we call b-convexity. In the light of this general scheme, the conclusions of Theorem 2.5 and Theorem 6.4 may read in a unified way as follows:

$$T \text{ has } b(x_0) \Leftrightarrow T \subseteq \partial^b f_T, \qquad \text{for some } b\text{-convex function } f_T.$$

Note that Theorem 2.5 corresponds to the case $b(x, y, x^*) = x^*(y - x)$, where one recovers in (37) the definition of the Fenchel–Moreau subdifferential and in (38) the definition of cyclic monotonicity (see Definition 2.1(i)). In this case, b-convexity is equivalent to convexity plus lower semicontinuity. On the other hand, Theorem 6.4 corresponds to the choice

$$b(x, y, x^*) = \begin{cases} x^*(y - x) & \text{if } x = x_0, \\ \min\{x^*(y - x), 0\} & \text{if } x \neq x_0, \end{cases}$$

where (38) is the considered $R(x_0)$ property, and b-convexity is nothing else than lower semicontinuity and quasiconvexity.

QUESTION. The class of operators fulfilling $(R(x))$ at every point of their domain is located between monotone and cyclically monotone operators (see Propositions 5.2, 6.2 and comments after Definition 6.1). However we do not know which of these inclusions is strict.

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