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The Pricing Formula for Commodity-Linked Bonds with Stochastic Convenience Yields and Default Risk

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Abstract. At the maturity, the owner of a commodity-linked bond has the right to receive the face value of the bond and the excess amount of spot market value of the reference commodity bundle over the prespecified exercise price. This payoff structure is an important characteristic of the commodity-linked bonds.

In this paper, we derive closed pricing formulae for the commodity-linked bonds. We assume that the reference commodity price and the value of the firm (bonds' issuer) follow geometric Brownian motions and that the net marginal convenience yield and interest rate follow Ornstein–Uhlenbech processes. In the appendix, we derive pricing formulae for bonds which are the same as the above commodity-linked bonds, except that the reference commodity price in the definition of the payoff at the maturity is replaced by the value of a special asset which depends on the convenience yield.

Key words: bond pricing, commodity-linked bond, convenience yield, default probability, PDE.

1. Introduction

1.1. REVIEWS

Schwartz (1982) introduced a general framework for pricing commodity-linked bonds where (1) the reference commodity price follows a geometric Brownian motion and the interest rate is constant. He also covered in his framework the three other cases where (2) the commodity price and the bond price (i.e., the interest rate is stochastic) follow geometric Brownian motions, (3) the commodity price and the value of the firm (bond's issuer) follow geometric Brownian motions and the interest rate is constant, and (4) the interest rate behaves stochastically as an extension to the case. There he obtained the closed pricing formulae of commodity-linked bonds for the first three cases (1), (2) and (3). Defaults at the time of the maturity of the contingent claim (or, the bonds) of the issuing firms were considered in (3), where a pricing formula was derived. But he did not derive any closed pricing formula for the case (4). In his paper there was no discussion about the convenience yields.

Carr (1987) derived a closed pricing formula for the commodity-linked bonds for an extended case of the above (3), where the bond prices follow a third geometric Brownian motion without referring to the interest rate process, which is very similar to, but a little different from the above case (4). Carr's pricing formula takes care of the default of the issuing firms. The convenience yields were not considered in his paper either.

Gibson and Schwartz (1990) is the first to consider the stochastic convenience yields for the bond pricing model. They derived the partial differential equation for the price functions of the assets defined as functions of spot commodity price and the net marginal convenience yield. They estimated parameters for the behavior of the net marginal convenience yield from market data, and calculated numerically the futures prices of the commodity.[1] Bjerksund (1991) derived a closed pricing formula for the commodity contingent claims where the commodity price follows a geometric Brownian motion, the net marginal convenience yield follows an Ornstein–Uhlenbech process, and the interest rate is constant. He did not consider the default of the issuing firms at the maturity of the commodity contingent claims. Gibson and Schwartz (1993) utilized Bejerksund's (also two other parties') pricing formula and Black's (1976) formula to fit the market prices of the crude oil futures options. Since our concerns in the present paper are the mathematical pricing formulae for the commodity contingent claims, we do not further refer their fitting results. They were able to calculate numerically the present prices of the commodity-linked bonds, but did not derive a closed analytical pricing formula.

1.2. CONVENIENCE YIELD

The owner of the commodity has the rights (this is an option) to decide how he/she will treat the commodity; sell, lend, or store it, or even consume it. As for the consumption-use commodities such as crude oil or copper, the owner may consume it for his/her own manufacturing activities, or he/she may also store it for his/her future consumption or future sell-out. The owner of the futures contracts or the other contingent claims, however, does not have this rights because of lack of storage until the maturity.

The commodities prices are also seen to change with regards to the storage level of the participants in the market. Since all the participants make their own decision taking account of their own current and future perspective of inventory levels and time intervals, the market prices will change as aggregated results of each activity conducted by them. As Duffie (1989) discusses, the convenience yield is seen as the value of the option to sell out of storage. We will thus assume that the yield will change in relation to the scarcity of the commodity in the market. A low inventory level in the market, that is, scarcity of storage, leads to be backwardation of a market where the futures prices of distant contract months are lower than those of the nearby (see, for example, Edwards and Ma, 1992). This means that backwardation occurs when there is a shortage of the available physical commodity. This shortage implies the following attitude of the holder; 'the holder of the physical commodity are unwilling to part with it, even for short period of time (Edwards and Ma, 1992)' and thus generates the convenience yields. In this respect, we may assume that there is an inverse relationship between the changes of the convenience yield and the changes of current inventory level in the market. Kaldor (1939) and Working (1948) examined and affirmed this hypothesis.[2]

A statistical analysis for the net marginal convenience yield can be done using spot and futures prices. Brennan (1991) squeezed out the net marginal convenience yield from futures prices of gold, silver, platinum, copper, No. 2 heating oil, lumber, and plywood. By analyzing those data, he showed their mean-reverting movements. Gibson and Schwartz (1990, 1993) used the relations between futures prices with different contract months to estimate parameter values in the models for the net marginal convenience yields' movements and utilized the estimated values for their numerical pricing of the contingent claims.

1.3. OUR RESULTS

In this paper, we take the approach of Gibson and Schwartz (1990, 1993), Bjerksund (1991) to express the price change of the reference commodity in relation to its convenience yield. In Appendix C, we drive a pricing formula for a special derivative. The underlying asset of this derivative itself is a derivative security, which is seen in Bjerksund (1991), that consists of a commodity and the continuously reinvested net marginal convenience yield. These two appraoches reflect two ways of treatments, that we could take, for the pricing of the commodity contingent claims. The latter one uses the value of the ownership of the commodity as its underlying variables which receive the total expected return derived from its price changes and net marginal convenience yield. Then, we see that the resulting pricing formula does not explicitly depend on the parameters related to the movements of the convenience yield. On the other hand, the former one uses the market price of the commodity where the owner of the commodity contingent claim cannot receive the convenience yield deriving from the ownership of the commodity, but receives the total expected return from its price changes.

The pricing formula in the former case includes the parameters related to the convenience yield. By using this formula, we draw several graphs of the bond prices and the default probabilities. The default occurs when the total payoff to the bond holder exceeds the value of the issuer at the maturity. The figures for the default probabilities provide us useful information to the bond issuer/holder in regard to the risk management.

1.4. ORGANIZATION OF THIS PAPER

This paper is organized as follows. In Section 2, we define the random variables, namely the commodity price, the value of the issuer, and the net marginal convenience yield, and the models for the behavior of these random variables. We also give some market conditions. Then we briefly derive the partial differential equation (in short, PDE) for the price function of the commodity-linked bonds and obtain a closed pricing formula of the bonds by solving the PDE analytically. In Section 3, we show several figures for the bond prices and default probabilities as functions of parameter values. In Section 4, we show our results, rather in detail, which is the extended case of Section 2 where the instantaneous interest rate behaves stochastically, following another Ornstein–Uhlenbech processes. In subsection 4.1 we set some additional assumptions. In subsection 4.2, we derive the PDE by using the standard no-arbitrage argument. In subsection 4.3, we derive the closed pricing formula for the commodity-linked bond $B(S_t, V_t, \delta_t, r_t, \tau)$ by applying Feynman–Kac Theorem. Some related analytical details are presented in Appendix A and B. Appendix C shows a brief mathematical derivation for the pricing function of the bond linked to the special derivative. In Appendix D, we present Mathematica's program list to calculate prices of the commodity-linked bond $B(S_t, V_t, \delta_t, \tau)$.

2. Closed Pricing Formula for the Commodity-Linked Bonds $B(S_t, V_t, \delta_t, \tau)$

In this section, we derive the pricing functions of the commodity-linked bonds, $B(S_t, V_t, \delta_t, \tau)$. To start with, we define our stochastic variables and derive the PDE. Then we obtain the closed pricing formula for the commodity-linked bonds that satisfies the derived PDE with its payoff at the maturity as the boundary condition.

Let S, V, and δ be stochastic processes. S_t is the spot price of the commodity, V_t denotes the value of the issuer (or the value of the firm), and δ_t represents the instantaneous net marginal convenience yield rate. We assume that S, V, and δ satisfy following stochastic differential equations (in short, SDE):

$$\frac{\mathrm{d}S}{S} = \alpha_S \cdot \mathrm{d}t + \sigma_S \cdot \mathrm{d}W_S \tag{1}$$

$$\frac{\mathrm{d}V}{V} = \alpha_V \cdot \mathrm{d}t + \sigma_V \cdot \mathrm{d}W_V \tag{2}$$

$$\mathrm{d}\delta = k(\mu_\delta - \delta)\mathrm{d}t + \sigma_\delta \cdot \mathrm{d}W_\delta\ , \tag{3}$$

where W_S, W_V, and W_δ are the standard Wiener processes and their correlation are such that $\mathrm{d}W_S \cdot \mathrm{d}W_V = \rho_{SV}\mathrm{d}t$, $\mathrm{d}W_S \cdot \mathrm{d}W_\delta = \rho_{S\delta}\mathrm{d}t$, and $\mathrm{d}W_V \cdot \mathrm{d}W_\delta = \rho_{V\delta}\mathrm{d}t$. We postulate that the parameters α_S, α_V, κ, μ_δ, σ_S, σ_V, σ_δ, ρ_{SV}, $\rho_{S\delta}$, and $\rho_{V\delta}$ are constants. We assume in the above that $\mathrm{d}\delta$ follows Ornstein–Uhlenbeck process that can take negative values. This is not a problem for the convenience yield, because our definition (3) is for the net marginal convenience yield. The net marginal convenience yield is defined by the differences that the gross convenience yield subtracted by the cost of carry, thus it sometimes takes negative values.

We also postulate that there is a risk free interest rate r and that this is a constant during the time interval from t to T. The length of this time interval is denoted by τ. In this paper, we assume that assets are infinitely divisible and that a short position is allowed. We also assume that there is no-arbitrage opportunity in the market.

Commodity-linked bonds have the payoff at the majority such that the owner of the bonds has right to receive, in the case of no default, in addition to the face value, the excess amount of the spot market price of the reference commodity over the prespecified exercise price. In the case where the default is considered, the payoff at the maturity is the minimum of either the payoff in the case of no default or the value of the issuer at the maturity. The total amount of the payment to the bond owner at the maturity is

$$\min[V_T, F + \max\{S_T - K, 0\}] \,. \tag{4}$$

F and K are constants, F is the face value of the bond and K is the prespecified exercise price of the reference commodity.

Also we assume that there are $N(N \geq 3)$ different assets in the market with price functions $B_i(S_t, V_t, \delta_t, \tau)$ for i-th asset, where $i = 1, 2, \cdots, N$, that have the same reference commodity. This is not an unrealistic assumption. Moreover, we postulate that for any choice of the three assets, the three vectors each of which consists of $\pi_S^i, \pi_V^i, \pi_\delta^i$ in the following equation (6) for $B_i(S_t, V_t, \delta_t, \tau)$ are linearly independent to each other: that is, the following matrices are non-singular for any choice of the three derivative assets.

$$\begin{bmatrix} \pi_S^i & \pi_V^i & \pi_\delta^i \\ \pi_S^j & \pi_V^j & \pi_\delta^j \\ \pi_S^k & \pi_V^k & \pi_\delta^k \end{bmatrix} , \tag{5}$$

where $i, j, k = 1, \cdots, N$, and $i \neq j$, $i \neq k$, and $j \neq k$.

Next, we derive the PDE for the pricing function of the commodity-linked bond, $B(S_t, V_t, \delta_t, \tau)$. By using Ito's lemma, we obtain the following equation.

$$\frac{\mathrm{d}B_i}{B_i} = \varphi_{B,i} \cdot \mathrm{d}t + \pi_S^i \cdot \mathrm{d}W_S + \pi_V^i \cdot \mathrm{d}W_V + \pi_\delta^i \cdot \mathrm{d}W_\delta \;, \tag{6}$$

where

$$\varphi_{B,i} = \frac{\left\{ \begin{array}{l} \frac{\partial B_i}{\partial S} S\alpha_S + \frac{\partial B_i}{\partial V} V\alpha_V + \frac{\partial B_i}{\partial \delta}\kappa(\mu_\delta - \delta) - \frac{\partial B_i}{\partial \tau} \\ + \frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial S^2} S^2 \sigma_S^2 + \frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial V^2} V^2 \sigma_V^2 + \frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial \delta^2} \sigma_\delta^2 \\ + \frac{\partial^2 B_i}{\partial S \partial V} SV\sigma_S\sigma_V\rho_{SV} + \frac{\partial^2 B_i}{\partial S \partial \delta} S\sigma_S\sigma_\delta\rho_{S\delta} + \frac{\partial^2 B_i}{\partial V \partial \delta} V\sigma_V\sigma_\delta\rho_{V\delta} \end{array} \right\}}{B_i} \,,$$

$$\pi_S^i = \frac{\partial B_i}{\partial S} \cdot \frac{S\sigma_S}{B_i} \,, \pi_V^i = \frac{\partial B_i}{\partial V} \cdot \frac{V\sigma_V}{B_i} \,, \pi_\delta^i = \frac{\partial B_i}{\partial \delta} \cdot \frac{\sigma_\delta}{B_i} \,.$$

We construct a portfolio W such that the portfolio consists of three different derivative assets and the commodity. We denote the weights of each assets in this portfolio as x_i $(i = 1, \cdots, 4)$ and the sum of these portfolio weights is equal to 1, i.e.,

$$\sum_{i=1}^{4} x_i = 1 \, .$$

Then the rate of return of the portfolio W is given by

$$\frac{\mathrm{d}W}{W} = x_1 \cdot \frac{\mathrm{d}B_1}{B_1} + x_2 \cdot \frac{\mathrm{d}B_2}{B_2} + x_3 \cdot \frac{\mathrm{d}B_3}{B_3} + x_4 \cdot \left(\frac{\mathrm{d}S}{S} + \delta_t \cdot \mathrm{d}t \right) \, . \tag{7}$$

This equation utilizes the property that the total rate of return of the owner of the reference commodity is the sum of the price changes of the commodity and its convenience yield.

By using the standard no-arbitrage argument, we obtain the following equations:

$$\begin{bmatrix} \varphi_{B,1} - r \\ \varphi_{B,2} - r \\ \varphi_{B,3} - r \\ \alpha_S + \delta_t - r \end{bmatrix} = \lambda_1 \cdot \begin{bmatrix} \pi_S^1 \\ \pi_S^2 \\ \pi_S^3 \\ \sigma_S \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} \pi_V^1 \\ \pi_V^2 \\ \pi_V^3 \\ 0 \end{bmatrix} + \lambda_3 \cdot \begin{bmatrix} \pi_\delta^1 \\ \pi_\delta^2 \\ \pi_\delta^3 \\ 0 \end{bmatrix} . \tag{8}$$

Note that λ_1, λ_2, and λ_3 are the market prices of risk for the commodity price, the value of the issuer, and the net marginal convenience yield, respectively. They are, logically at now, time dependent and vary with regard to the choice of the assets in the portfolio W. We show in the Appendix A that these do not depend on the choice of the assets. We assume that these λ's are constants in solving the following PDE (9). Note also that first, second, and third row of (8) are the equations for any derivative assets and the fourth row of (8) is the equation for the commodity itself. From (8) we have the following PDE.

$$\begin{aligned} &\frac{\partial B}{\partial S} S(r - \delta) + \frac{\partial B}{\partial V} V(\alpha_V - \lambda_2 \sigma_V) + \frac{\partial B}{\partial \delta} \{\kappa(\mu_\delta - \delta) - \lambda_3 \sigma_\delta\} - \\ &\quad - \frac{\partial B}{\partial \tau} - r \cdot B + \frac{1}{2} \cdot \frac{\partial^2 B}{\partial S^2} S^2 \sigma_S^2 + \frac{1}{2} \cdot \frac{\partial^2 B}{\partial V^2} V^2 \sigma_V^2 + \frac{1}{2} \cdot \frac{\partial^2 B}{\partial \delta^2} \sigma_\delta^2 + \\ &\quad + \frac{\partial^2 B}{\partial S \partial V} S V \sigma_S \sigma_V \rho_{SV} + \frac{\partial^2 B}{\partial S \partial \delta} S \sigma_S \sigma_\delta \rho_{S\delta} + \frac{\partial^2 B}{\partial V \partial \delta} V \sigma_V \sigma_\delta \rho_{V\delta} = 0 \, . \end{aligned} \tag{9}$$

Equation (9) is the PDE which every $B(S_t, V_t, \delta_t, \tau)$ must satisfy.

Next, we derive the closed form of the pricing function for the commodity-linked bonds $B(S_t, V_t, \delta_t, \tau)$ under the payoff function (4) at the bond maturity. This is done by applying Feynman–Kac Theorem (see Friedman, 1975, Chapter 6, Theorem 5.3). To calculate the expected value of the payoff function, where we write $[\tilde{S}_t, \tilde{V}_t, \tilde{\delta}_t]$ for the corresponding stochastic processes, we need to obtain the joint distribution function of $[\tilde{S}_t, \tilde{V}_t, \tilde{\delta}_t]$ based on PDE (9). The detailed derivation is shown in subsection 4.3.

From (9) we have the following SDE.

$$d\begin{bmatrix} \tilde{S}_t \\ \tilde{V}_t \\ \tilde{\delta}_t \end{bmatrix} = \begin{bmatrix} \tilde{S}_t(r-\tilde{\delta}_t) \\ \tilde{V}_t(\alpha_V - \lambda_2\sigma_V) \\ \kappa(\mu_\delta - \tilde{\delta}_t) - \lambda_3\sigma_\delta \end{bmatrix} \mathrm{d}t + \mathbf{G}\cdot d\begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \end{bmatrix}, \tag{10}$$

where the $\tilde{Z}_{1,t}, \tilde{Z}_{2,t}, \tilde{Z}_{3,t}$ are another set of independent standard Wiener processes. A choice of $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{bmatrix} \tilde{S}_t\sigma_S & 0 & 0 \\ \tilde{V}_t\sigma_V\rho_{SV} & \tilde{V}\sigma_V\cdot c & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta\cdot\bar{e} & \sigma_\delta\cdot f \end{bmatrix},$$

where $c = \sqrt{1-\rho_{SV}^2}$, $\bar{e} = \dfrac{\rho_{V\delta} - \rho_{SV}\rho_{S\delta}}{\sqrt{1-\rho_{SV}^2}}$, $f = \sqrt{1 - \dfrac{(\rho_{V\delta} - \rho_{SV}\rho_{S\delta})^2}{1-\rho_{SV}^2} - \rho_{S\delta}^2}$.
The derivation of $\mathbf{G}$ is shown in Appendix B.

Because one of the drift term of SDE (10) is the product of $\tilde{S}_t$ and $\tilde{\delta}_t$, SDE (10) is *non-linear*. However, we make the following change of variables in order to transform this *non-linear* SDE to a *linear* SDE. Then let $\tilde{P}_t = \log\tilde{S}_t$ and $\tilde{J}_t = \log\tilde{V}_t$. By applying Ito's lemma, we have

$$\begin{aligned} d\begin{bmatrix} \tilde{P}_t \\ \tilde{J}_t \\ \tilde{\delta}_t \end{bmatrix} &= \begin{bmatrix} -\tilde{\delta}_t + (r - \frac{1}{2}\sigma_s^2) \\ \alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2 \\ \kappa(\mu_\delta - \tilde{\delta}_t) - \lambda_3\sigma_\delta \end{bmatrix} \mathrm{d}t + \\ &\quad + \begin{bmatrix} \sigma_S & 0 & 0 \\ \sigma_V\rho_{SV} & \sigma_V\cdot c & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta\cdot\bar{e} & \sigma_\delta\cdot f \end{bmatrix} \cdot d\begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \end{bmatrix}. \end{aligned} \tag{11}$$

Now by using Theorem 8.2.2 in Arnold (1973, p. 129), we can solve the SDE (11) and derive the joint distribution of $[\tilde{P}_t, \tilde{J}_t, \tilde{\delta}_t]$ for the time interval $[t_0, T]$. The solution of (11) is given by

$$\begin{bmatrix} \tilde{P}_t \\ \tilde{J}_t \\ \tilde{\delta}_t \end{bmatrix} = \begin{bmatrix} \alpha(\pi) \\ \beta(\pi) \\ \gamma(\pi) \end{bmatrix} + \begin{bmatrix} \tilde{X}_t \\ \tilde{Y}_t \\ \tilde{Z}_t \end{bmatrix},$$

where $\pi = t - t_0$, $[\alpha(\pi), \beta(\pi), \gamma(\pi)]$ are deterministic functions such that

$$\begin{bmatrix} \alpha(\pi) \\ \beta(\pi) \\ \gamma(\pi) \end{bmatrix} = \begin{bmatrix} P_{t_0} + \delta_{t_0}\frac{e^{-\kappa\pi}-1}{\kappa} + \left(r - \frac{\sigma_s^2}{2}\right)\pi + (\kappa\mu_\delta - \lambda_3\sigma_\delta)\frac{1-e^{-\kappa\pi}-\kappa\pi}{\kappa^2} \\ J_{t_0} + (\alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2)\pi \\ \delta_{t_0}e^{-\kappa\pi} + (\kappa\mu_\delta - \lambda_3\sigma_\delta)\frac{1-e^{-\kappa\pi}}{\kappa} \end{bmatrix},$$

and $[\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t]$ are jointly normally distributed. Their means are zero and the variance-covariance matrix $\sum$ is given by

$$\sum = \begin{bmatrix} Var(\tilde{X}_t) & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{XY} & Var(\tilde{Y}_t) & \sigma_{YZ} \\ \sigma_{XZ} & \sigma_{YZ} & Var(\tilde{Z}_t) \end{bmatrix},$$

where

$$\begin{cases} Var(\tilde{X}_t) = \pi\left(\sigma_s^2 - 2\frac{\sigma_S\sigma_\delta\rho_{S\delta}}{\kappa} + \frac{\sigma_\delta^2}{\kappa^2}\right) + 2(1-e^{-\kappa\pi})\left(\frac{\sigma_S\sigma_\delta\rho_{S\delta}}{\kappa^2} - \frac{\sigma_\delta^2}{\kappa^3}\right) + \\ \qquad\qquad + (1-e^{-2\kappa\pi})\frac{\sigma_\delta^2}{2\kappa^3} \\ Var(\tilde{Y}_t) = \sigma_V^2\pi \\ Var(\tilde{Z}_t) = \frac{\sigma_\delta^2(1-e^{-2\kappa\pi})}{2\kappa} \\ \sigma_{XY} = \sigma_S\sigma_V\rho_{SV}\pi - \frac{\sigma_V\sigma_\delta\rho_{V\delta}\pi}{\kappa} + \frac{\sigma_V\sigma_\delta\rho_{V\delta}(1-e^{-\kappa\pi})}{\kappa^2} \\ \sigma_{XZ} = \frac{\sigma_S\sigma_\delta\rho_{S\delta}(1-e^{-\kappa\pi})}{\kappa} - \frac{\sigma_\delta^2(1-e^{-\kappa\pi})}{\kappa^2} + \frac{\sigma_\delta^2(1-e^{-2\kappa\pi})}{2\kappa^2} \\ \sigma_{YZ} = \frac{\sigma_V\sigma_\delta\rho_{V\delta}(1-e^{-\kappa\pi})}{\kappa} \end{cases}$$

The joint density function of $[\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t]$ is given by

$$f(\tilde{x}_t, \tilde{y}_t, \tilde{z}_t) = \frac{1}{(2\pi)^{3/2}\cdot\sqrt{\det\sum}}\cdot\exp\{-\tfrac{1}{2}\tilde{\mathbf{v}}^{\mathbf{T}}\textstyle\sum^{-1}\tilde{\mathbf{v}}\},$$

where

$$\tilde{\mathbf{v}} = \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \\ \tilde{z}_t \end{bmatrix}.$$

Then we calculate the present value of the expected value of the payoff function at the maturity (see Figure 1 for the payoff chart at the maturity). The solution of the PDE (9) with the boundary condition (i.e., payoff) (4) is given by

$$\begin{aligned}
B(S_t,\ V_t,\ \delta_t\ ,t) &= E\left[\min\{\tilde{V}_T,\ F+\max(\tilde{S}_T - K, 0)\}\cdot e^{-r\tau}\ \middle|\ \begin{array}{rcl}\tilde{S}_t &=& S_t\\ \tilde{V}_t &=& V_t\\ \tilde{\delta}_t &=& \delta_t\end{array}\right]\\
&= E\left[\min\{e^{\beta(\tau)+y_T},\ F+\max(e^{\alpha(\tau)+x_T} - K, 0)\}\cdot e^{-r\tau}\ \middle|\ \begin{array}{rcl}\tilde{S}_t &=& S_t\\ \tilde{V}_t &=& V_t\\ \tilde{\delta}_t &=& \delta_t\end{array}\right]\\
&= F\cdot\exp\{-r\tau\}\cdot\int_H^\infty\int_{-\infty}^R f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t +\\
&+\ \exp\{\beta(\tau)-r\tau\}\cdot\int_{-\infty}^H\int_{-\infty}^R \exp\{y_t\}f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t +\\
&+\ \exp\{\beta(\tau)-r\tau\}\cdot\int_{-\infty}^Q\int_R^\infty \exp\{y_t\}f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t +\\
&+ (F-K)\cdot\exp\{-r\tau\}\cdot\int_Q^\infty\int_R^\infty f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t +\\
&+\ \exp\{\alpha(\tau)-r\tau\}\cdot\int_Q^\infty\int_R^\infty \exp\{x_t\}f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t
\end{aligned} \tag{12}$$

where $f_{XY}(x_t,\ y_t)$ is the marginal density function of x_t and y_t and $\{H|y_T = \log\ F - \beta(\tau)\}$, $\{R|x_T = \log\ K - \alpha(\tau)\}$, and $\{Q|y_T = \log\{F + \exp\{\alpha(\tau) + x_T\} - K\} - \beta(\tau)\}$.

3. Bond Prices and Default Probabilities

In this section we show several figures for the bond prices and the default probabilities as functions of parameters. We list a Mathematica's program for computing the above pricing function (12) in Appendix D. The default probabilities are given by the following formula, that is, we calculated the probability that the total payoff to the bond holder is equal to V_T, i.e., the light gray area in the $(S_T,\ V_T)$ domain of Figure 1.

$$\begin{aligned}
\text{Default probability} &= \int_{-\infty}^H\int_{-\infty}^R f(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t +\\
&+\int_{-\infty}^Q\int_R^\infty f(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t\ .
\end{aligned} \tag{13}$$

When we use these formulae, we have to estimate each of paramters of the processes for $S_t,\ V_t,\ \delta_t$ and a constant r. For the commodity price S_t, and the

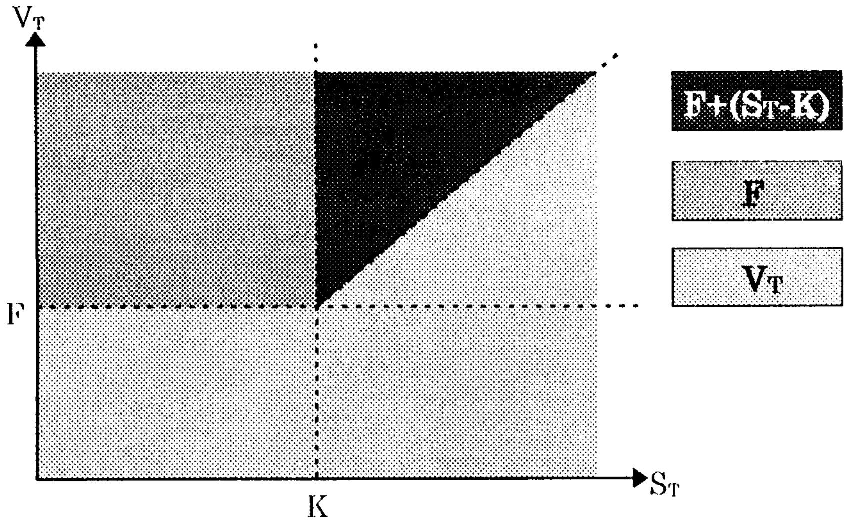

Figure 1. Payoff chart at the maturity.

interest rate r, they are quoted in the market, thus we can observe directly each of them and estimate the parameters σ_S and the constant r. But it is difficult to observe the value of the issuer V_t, and the net marginal convenience yield δ_t, because they are not quoted or reported in the market. Therefore we have to estimate each of V_t and δ_t or to use some proxies instead. For the value of the issuer, one idea to estimate each of parameters α_V and σ_V of V_t is that: we can treat stock price of this issuer firm as a call option on the value of the issuer, then we squeeze out the value of the issuer firm from its stock price. Unfortunately to accomplish this idea is not an easy task because it is difficult to know the whole cash flow of the issuer. In this section, the initial value of V_t and the parameters for the behavior of V_t are set rather subjectively. For the net marginal convenience yield, we recommend a simplified estimation procedure by Gibson and Schwartz (1990) or its revised scheme by Yamauchi (1998). Their methods use futures prices of the commodity with various delivery months. By isolating the differences between futures prices with neighboring delivery months and also by excluding interest rates effects for the futures prices, they approximately estimate one month or two months net marginal FORWARD convenience yield rate and might regard the AGGREGATED net marginal forward convenience yield rate as the net marginal covenience yield rate. Then they can be used to estimate each of parameters of δ_t .

The initial values required for the calculations of the price functions and default probabilities are set in the following way. The issuer of the commodity-linked bond has the business of producing and selling the commodity which is the underlying asset of that commodity-linked bond. The issuer wants to issue this bond with face value $F = 100$ and the maturity of 5 years. The strike price is set equal to the price of commodity at the time of issuance. The current interest rate is r

is 4% per year. The initial value of this issuer V_t is 200 which consists of this commodity-linked bond and the equity. Its expected growth ratio α_V is 2% per year and its volatility σ_V is 30% annually. These parameters of the value of the issuing firm are set so that the probability of $V_T < F$ is approximately 16% at the maturity. The current prices of commodity S_t is 20 and the price volatility σ_s is 39.2% per year. This volatility parameter is estimated from WTI crude oil prices data in NYMEX for the period from September 4, 1990 to June 20, 1994. To set the initial values of the parameters of the process of the net marginal convenience yields, we refer the detail to Yamauchi (1998). In his paper, he first calculated, from daily futures prices of different maturity, the daily values of 3 months net marginal convenience yield rates in a similar way to the one by Gibson and Schwartz (1990). Then, based on these daily values, he estimated the parameters of the process δ. The whole estimation period is from September 4, 1990 to June 22, 1993. He divided this estimation period into two periods; from September 4, 1990 to June 22, 1991 and from June 23, 1991 to June 22, 1993. From the former period, the estimated parameters were such that: $\kappa = 19.122$, $\mu_\delta = 0.324$, and $\sigma_\delta = 1.3050$. From the latter period, $\kappa = 4.547$, $\mu_\delta = 0.021$, and $\sigma_\delta = 0.2673$. The estimated parameters suggest that the convenience yields of crude oil at the former period showed a very wild movements. During the latter period, the convenience yields seem to be relatively stable. In this paper, we set two situations, namely situation A and B. We use the estimated parameters from the former period for situation A and the latter ones for situation B. We set the current level of the convenience yield rate δ_t at 0.25 for both situations. When the spot price of this commodity moves up, the convenience yield rate tends to move up in the effect according to their correlation. Their correlation $\rho_{S\delta}$ is set at 0.75 for situation A and at 0.50 for situation B. These correlation parameters are estimated from WTI crude oil prices and convenience yield rates. Since we suppose this issuing firm sells this commodity to the market, the value of this issuer is positively correlated to the changes of this commodity prices and the convenience yields. Thus the commodity prices and the value of the issuer is set to behave with correlation $\rho_{SV} = 0.50$ for both situations. Also we set the correlation parameters $\rho_{V\delta}$ between the value of the issuer and the convenience yield at 0.50 for situation A and at 0.33 for situation B.

$F = 100$	$K = 20$	$\tau = 5$	$r = 0.04$	$S_t = 20$	$\sigma_s = 0.392$
$V_t = 200$	$\alpha_V = 0.02$	$\sigma_V = 0.3$	$\lambda_2 = -0.067$	$\delta_t = 0.25$	$\rho_{SV} = 0.5$
Situation A			Situation B		
$\kappa = 19$	$\mu_\delta = 0.32$	$\sigma_\delta = 1.31$	$\kappa = 4.5$	$\mu_\delta = 0.02$	$\sigma_\delta = 0.27$
$\lambda_3 = 0.214$	$\rho_{S\delta} = 0.75$	$\rho_{V\delta} = 0.5$	$\lambda_3 = 0.074$	$\rho_{S\delta} = 0.5$	$\rho_{V\delta} = 0.33$

Figure 2 shows that the graph of the commodity-linked bond prices as a function of the speed of adjustment κ. To calculate the bond prices for Figure 2 and Figure 3,

Figure 2.

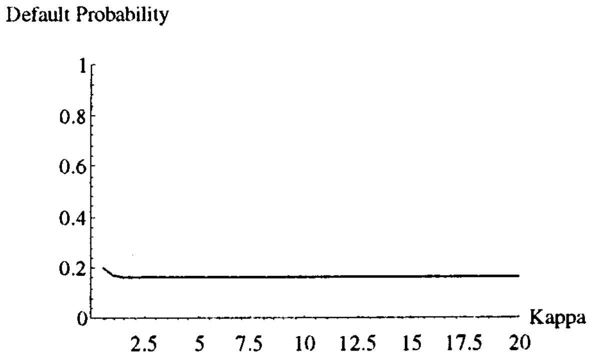

Figure 3.

we set $\mu_\delta = 0.1$, $\lambda_3 = 0.12$, $\sigma_\delta = 0.5$, $\rho_{S\delta} = 0.6$, and $\rho_{V\delta} = 0.4$ apart from situation A and B, we draw these two figures to see overall responses of bond prices and default probabilities to the values of κ. Figure 2 suggests that a smaller level of κ makes the bond prices higher than that of a larger level of κ. This means that the premium portion of bond prices decrease as κ become large, that is, movements of the convenience yield become more stable rather than that of smaller level of κ when other parameters are kept fixed. Figure 3 describes the default probabilities of the commodity-linked bond. This figure shows that the default probability become high as κ is at a smaller level. This result makes sense that the high premium, which means in part that the expected value of the payoff at the maturity is large, corresponds to the high default probability of this bond at the maturity.

Figure 4 suggests that the bond prices will increase as σ_δ increases for situation B, while the bond prices do not seem to be affected by the changes of σ_δ in situation A. This is because in a large level of κ, the convenience yield rate returns to its long term mean quickly even if σ_δ is at a large level. Consequently, the premium portion changes little as σ_δ become large. Figure 5 shows a graph of the default probabilities as a function of σ_δ.

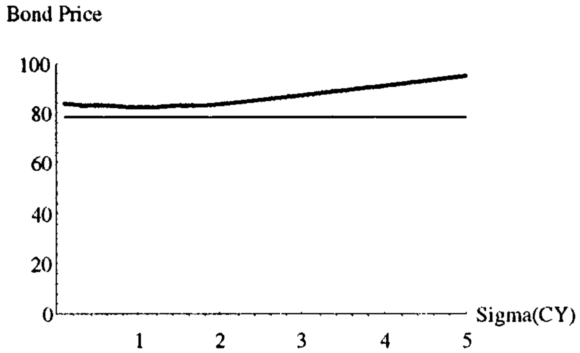

Figure 4. Thin line ______ : Situation A. Thick line ______ : Situation B.

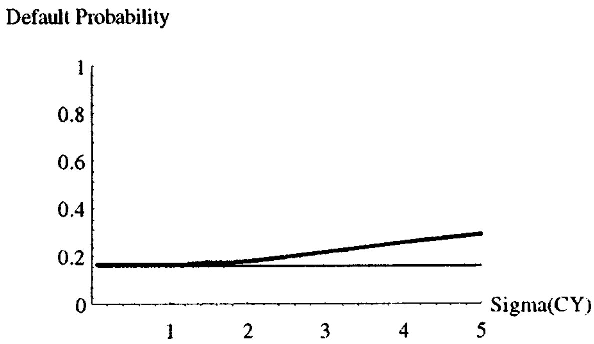

Figure 5. Thin line ______ : Situation A. Thick line ______ : Situation B.

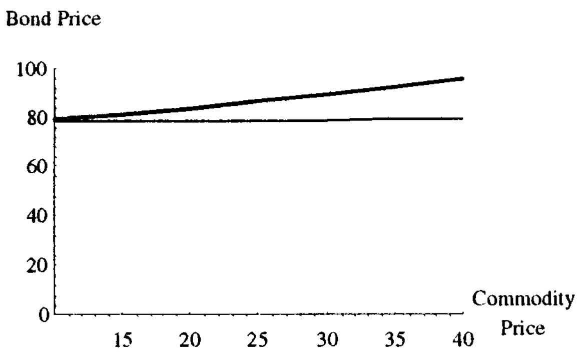

Figure 6. Thin line ______ : Situation A. Thick line ______ : Situation B.

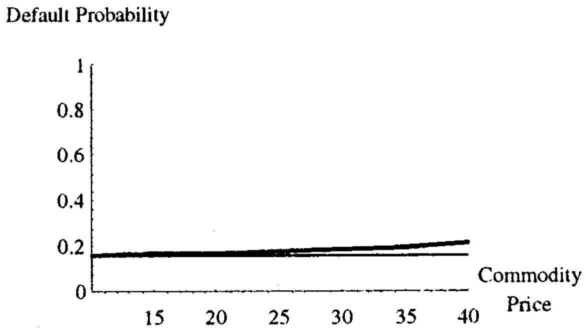

Figure 7. Thin line ———: Situation A. Thick line ▬▬▬: Situation B.

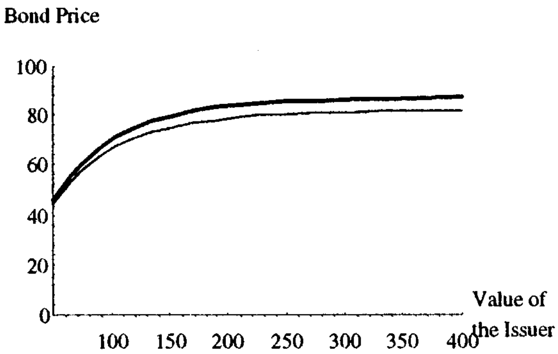

Figure 8. Thin line ———: Situation A. Thick line ▬▬▬: Situation B.

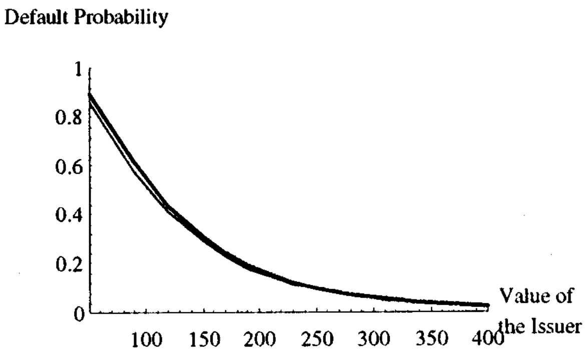

Figure 9. Thin line ———: Situation A. Thick line ▬▬▬: Situation B.

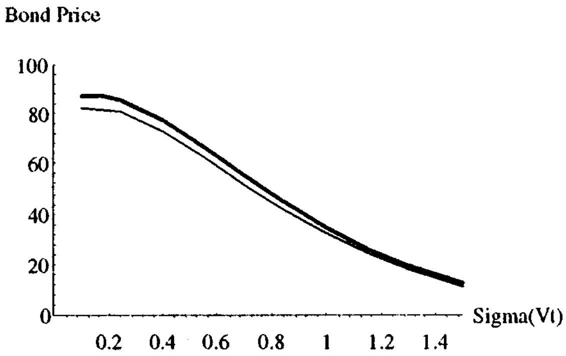

Figure 10. Thin line ——— : Situation A. Thick line ——— : Situation B.

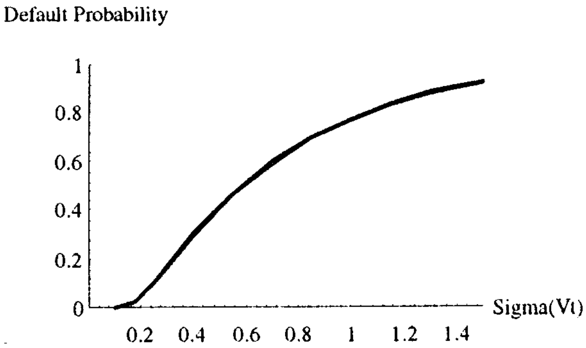

Figure 11. Thin line ——— : Situation A. Thick line ——— : Situation B.

Figure 6 shows that the higher the commodity prices S_t are, the more expensive the bond prices are. This is very natural. The strike price K is equal to 20, the premium increases as S_t moves across K from out of the money to in the money. Figure 7 is a graph of the default probabilities of the commodity-linked bond as a function of S_t. This figure also shows that the default probabilities become high as S_t becomes large which is the same as Figure 6.

Figure 8 through 11 are the graphs in relation to the value of the issuer. Figure 8 shows that the higher the value of the issuer is, the more expensive the bond price is. Figure 9 suggests that the default probability decreases as the value of the issuer V_t increases. This is very natural. If V_t is very small, the default probability of this bond at the maturity is anticipated to be high. As for Figure 10 and 11, we see that the larger the volatility of the value of the issuer σ_V is, the lower the bond price is and, at the same time, the default probability is high. These are also natural.

4. Closed Pricing Formula for the Commodity-Linked Bonds $B(S_t,\ V_t,\ \delta_t,\ r_t,\ \tau)$

In this section, we describe a straight extension of Section 2 where the instantaneous interest rate changes stochastically, following another Ornstein–Uhlenbech process.

4.1. ASSUMPTIONS FOR THE PRICING FORMULA OF COMMODITY-LINKED BONDS $B(S_t,\ V_t,\ \delta_t,\ r_t,\ \tau)$

Assume the same situation as we postulated in the Section 2 except that the interest rate behaves stochastically

$$\mathrm{d}r_t = g(\mu_r - r_t)\mathrm{d}t + \sigma_r \cdot \mathrm{d}W_r\ , \tag{14}$$

where W_r is another standard Wiener process and correlations are such that

$$\mathrm{d}W_s \cdot \mathrm{d}W_r = \rho_{Sr}\mathrm{d}t,\ \mathrm{d}W_V \cdot \mathrm{d}W_r = \rho_{Vr}\mathrm{d}t,\ \text{ and }\ \mathrm{d}W_\delta \cdot \mathrm{d}W_r = \rho_{\delta r}\mathrm{d}t\ .$$

This assumption for the behavior of the interest rate r_t might be a little problematic, because there may occur negative interest rate. However, this probability is small. In our paper, for the ease of the derivation of the pricing formula, we assume Ornstein–Uhlenbech process for r_t.

We also postulate that the following parameters are constants:

$$\alpha_S,\ \alpha_V,\ \kappa,\ \mu_\delta,\ g,\ \mu_r,\ \sigma_S,\ \sigma_V,\ \sigma_\delta,\ \sigma_r,\ \rho_{SV},\ \rho_{S\delta},\ \rho_{Sr},\ \rho_{V\delta},\ \rho_{Vr},\ \rho_{\delta r}\ .$$

We assume that there are N ($N \geq 4$) different assets with price functions $B_i(S_t,\ V_t,\ \delta_t,\ r_t,\ \tau)$, for $i = 1, 2, \cdots, N$, that have the same reference commodity in the market. For any choice of the four assets, we assume that there exists the inverse of the following matrices:

$$\begin{bmatrix} \eta_S^i & \eta_V^i & \eta_\delta^i & \eta_r^i \\ \eta_S^j & \eta_V^j & \eta_\delta^j & \eta_r^j \\ \eta_S^k & \eta_V^k & \eta_\delta^k & \eta_r^k \\ \eta_S^l & \eta_V^l & \eta_\delta^l & \eta_r^l \end{bmatrix}, \tag{15}$$

where $i, j, k, l = 1, \cdots, N$, and $i \neq j,\ i \neq k,\ i \neq l,\ j \neq k,\ j \neq l$, and $k \neq l$. These quantities η are defined later in the following equality (16).

4.2. PARTIAL DIFFERENTIAL EQUATION

In this subsection, we derive the PDE for the pricing function of the commodity-linked bond $B_i(S_t,\ V_t,\ \delta_t,\ r_t,\ \tau)$. By using Ito's lemma, we obtain the following equation for the i-th commodity-linked bond $(i = 1, 2, \cdots, N)$;

$$\frac{\mathrm{d}B_i}{B_i} = \Psi_{B,i} \cdot \mathrm{d}t + \eta_S^i \cdot \mathrm{d}W_S + \eta_V^i \cdot \mathrm{d}W_V + \eta_\delta^i \cdot \mathrm{d}W_\delta + \eta_r^i \cdot \mathrm{d}W_r\ , \tag{16}$$

where

$$\Psi_{B,i} = \frac{\left\{\begin{array}{l} \frac{\partial B_i}{\partial S} S\alpha_S + \frac{\partial B_i}{\partial V} V\alpha_V + \frac{\partial B_i}{\partial \delta}\kappa(\mu_\delta - \delta) + \frac{\partial B_i}{\partial r} g(\mu_r - r) - \frac{\partial B_i}{\partial \tau} \\ +\frac{1}{2}\cdot\frac{\partial^2 B_i}{\partial S^2} S^2\sigma_S^2 + \frac{1}{2}\cdot\frac{\partial^2 B_i}{\partial V^2} V^2\sigma_V^2 + \frac{1}{2}\cdot\frac{\partial^2 B_i}{\partial \delta^2}\sigma_\delta^2 + \frac{1}{2}\cdot\frac{\partial^2 B_i}{\partial r^2}\sigma_r^2 \\ +\frac{\partial^2 B_i}{\partial S\partial V} SV\sigma_S\sigma_V\rho_{SV} + \frac{\partial^2 B_i}{\partial S\partial \delta} S\sigma_S\sigma_\delta\rho_{S\delta} + \frac{\partial^2 B_i}{\partial S\partial r} S\sigma_S\sigma_r\rho_{Sr} \\ +\frac{\partial^2 B_i}{\partial V\partial \delta} V\sigma_V\sigma_\delta\rho_{V\delta} + \frac{\partial^2 B_i}{\partial V\partial r} V\sigma_V\sigma_r\rho_{Vr} + \frac{\partial^2 B_i}{\partial \delta\partial r}\sigma_\delta\sigma_r\rho_{\delta r} \end{array}\right\}}{B_i}\ ,$$

$$\eta_S^i = \frac{\partial B_i}{\partial S}\cdot\frac{S\sigma_S}{B_i}\ ,\quad \eta_V^i = \frac{\partial B_i}{\partial V}\cdot\frac{V\sigma_V}{B_i}\ ,$$
$$\eta_\delta^i = \frac{\partial B_i}{\partial \delta}\cdot\frac{\sigma_\delta}{B_i}\ ,\quad \eta_r^i = \frac{\partial B_i}{\partial r}\cdot\frac{\sigma_r}{B_i}\ .$$

With the same standard no-arbitrage argument used in Section 2, we obtain

$$\begin{aligned}
&\frac{\partial B}{\partial S} S_t(r_t - \delta_t) + \frac{\partial B}{\partial V} V_t(\alpha_V - \lambda_2\sigma_V) + \frac{\partial B}{\partial \delta}\{\kappa(\mu_\delta - \delta_t) - \lambda_3\sigma_\delta\} + \\
&+ \frac{\partial B}{\partial r}\{g(\mu_r - r_t) - \lambda_4\sigma_r\} + \frac{1}{2}\cdot\frac{\partial^2 B}{\partial S^2} S_t^2\sigma_S^2 + \frac{1}{2}\cdot\frac{\partial^2 B}{\partial V^2} V_t^2\sigma_V^2 + \\
&+ \frac{1}{2}\cdot\frac{\partial^2 B}{\partial \delta^2}\sigma_\delta^2 + \frac{1}{2}\cdot\frac{\partial^2 B}{\partial r^2}\sigma_r^2 + \frac{\partial^2 B}{\partial S\partial V} S_t V_t\sigma_S\sigma_V\rho_{SV} + \\
&+ \frac{\partial^2 B}{\partial S\partial \delta} S_t\sigma_S\sigma_\delta\rho_{S\delta} + \frac{\partial^2 B}{\partial S\partial r} S_t\sigma_s\sigma_r\rho_{Sr} + \frac{\partial^2 B}{\partial V\partial \delta} V_t\sigma_V\sigma_\delta\rho_{V\delta} + \\
&+ \frac{\partial^2 B}{\partial V\partial r} V_t\sigma_V\sigma_r\rho_{Vr} + \frac{\partial^2 B}{\partial \delta\partial r}\sigma_\delta\sigma_r\rho_{\delta r} - \frac{\partial B}{\partial \tau} - r_t\cdot B = 0\ ,
\end{aligned} \tag{17}$$

where λ_2, λ_3, and λ_4 are the market prices of risk for the value of the issuer, the net marginal convenience yield, and the interest rate, respectively. Equation (17) is the PDE which every $B(S_t,\ V_t,\ \delta_t,\ r_t,\ \tau)$ must satisfy.

4.3. CLOSED PRICING FORMULA

In this subsection, we derive the closed pricing formula of the commodity-linked bond $B(S_t,\ V_t,\ \delta_t,\ r_t,\ \tau)$ by applying the Feynman–Kac Theorem. In the calculation of the expected value of the payoff function (4), we need the joint distribution function of $[\tilde{S}_t,\ \tilde{V}_t,\ \tilde{\delta}_t,\ \tilde{r}_t]$ based on PDE (17). From (17), we have the following SDE:

$$d\begin{bmatrix} \tilde{S}_t \\ \tilde{V}_t \\ \tilde{\delta}_t \\ \tilde{r}_t \end{bmatrix} = \begin{bmatrix} \tilde{S}_t(\tilde{r}_t - \tilde{\delta}_t) \\ \tilde{V}_t(\alpha_V - \lambda_2\sigma_V) \\ \kappa(\mu_\delta - \tilde{\delta}_t) - \lambda_3\sigma_\delta \\ g(\mu_r - \tilde{r}_t) - \lambda_4\sigma_r \end{bmatrix} \mathrm{d}t + \\ + \begin{bmatrix} \tilde{S}_t\sigma_S & 0 & 0 & 0 \\ \tilde{V}_t\sigma_V\rho_{SV} & \tilde{V}_t\sigma_V \cdot \bar{e} & 0 & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta \cdot f & \sigma_\delta \cdot h & 0 \\ \sigma_r\rho_{Sr} & \sigma_r \cdot \bar{g} & \sigma_r \cdot i & \sigma_r \cdot j \end{bmatrix} \cdot \mathrm{d}\begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \\ \tilde{Z}_{4,t} \end{bmatrix}, \tag{18}$$

where

$$\begin{cases} \bar{e} = \sqrt{1-\rho_{SV}^2}, \quad f = \dfrac{\rho_{V\delta} - \rho_{SV}\rho_{S\delta}}{\bar{e}}, \quad \bar{g} = \dfrac{\rho_{Vr} - \rho_{SV}\rho_{Sr}}{\bar{e}} \\ h = \sqrt{1 - p_{S\delta}^2 - f^{\,2}}, \quad i = \dfrac{\rho_{\delta r} - \rho_{S\delta}\rho_{Sr} - f \cdot \bar{g}}{h}, \quad j = \sqrt{1 - \rho_{Sr}^2 - \bar{g}^2 - i^2} \end{cases}$$

and $\tilde{Z}_{1,t},\ \tilde{Z}_{2,t},\ \tilde{Z}_{3,t},\ \tilde{Z}_{4,t}$, are independent standard Wiener processes.

Next, to transform the *non-linear* SDE (18) to *linear one*, let $\tilde{P}_t = \log\ \tilde{S}_t$ and $\tilde{J}_t = \log\ \tilde{V}_t$. Then we have the following SDEs for $\tilde{P}_t$ and $\tilde{J}_t$ by applying Ito's lemma.

$$d\begin{bmatrix} \tilde{\rho}_t \\ \tilde{\tau}_t \\ \tilde{\delta}_t \\ \tilde{r}_t \end{bmatrix} = \begin{bmatrix} -\tilde{\delta}_t + r_t - \frac{1}{2}\sigma_S^2 \\ \alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2 \\ \kappa(\mu_\delta - \tilde{\delta}_t) - \lambda_3\sigma_\delta \\ g(\mu_r - \tilde{r}_t) - \lambda_4\sigma_r \end{bmatrix} \mathrm{d}t + \\ + \begin{bmatrix} \sigma_S & 0 & 0 & 0 \\ \sigma_V\rho_{SV} & \sigma_V \cdot \bar{e} & 0 & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta \cdot f & \sigma_\delta \cdot h & 0 \\ \sigma_r\rho_{Sr} & \sigma_r \cdot \bar{g} & \sigma_r \cdot i & \sigma_r \cdot j \end{bmatrix} \cdot \mathrm{d}\begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \\ \tilde{Z}_{4,t} \end{bmatrix}. \tag{19}$$

By solving the stochastic differential Equations (19), we derive the joint probability density function of $[\tilde{P}_t, \tilde{J}_t, \tilde{\delta}_t, \tilde{r}_t]$ for the time interval $[t_0, T]$ using theorem 8.2.2 in Arnold (1973, p. 129). By theorem 8.2.2, the SDE

$$\mathrm{d}\tilde{\mathbf{Q}}_t = (\mathbf{A}(t) \cdot \tilde{\mathbf{Q}}_t + \mathbf{a}(t))\mathrm{d}t + \mathbf{B}(t) \cdot \mathrm{d}\tilde{\mathbf{Z}}_t$$

has the solution

$$\tilde{\mathbf{Q}}_t = \Phi_t(\mathbf{Q}_{t_0} + \int_{t_0}^{t} \Phi_s^{-1} \cdot \mathbf{a}(s) \cdot \mathrm{d}s + \int_{t_0}^{t} \Phi_s^{-1} \cdot \mathbf{B}(s) \cdot \mathrm{d}\tilde{\mathbf{Z}}_s) \tag{20}$$

with the initial value $\mathbf{Q}_{t_0}$, where

$$\tilde{\mathbf{Q}}_t = \begin{bmatrix} \tilde{P}_t \\ \tilde{J}_t \\ \tilde{\delta}_t \\ \tilde{r}_t \end{bmatrix}, \quad \mathbf{Q}_{t_0} = \begin{bmatrix} P_{t_0} \\ J_{t_0} \\ \delta_{t_0} \\ r_{t_0} \end{bmatrix},$$

$$\mathbf{A}(t) = \mathbf{A} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & -g \end{bmatrix},$$

$$\mathbf{a}(t) = \mathbf{a} = \begin{bmatrix} -\frac{1}{2}\sigma_S^2 \\ \alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2 \\ \kappa\mu_\delta - \lambda_3\sigma_\delta \\ g\mu_r - \lambda_4\sigma_r \end{bmatrix},$$

$$\mathbf{B}(\mathbf{t}) = \mathbf{B} = \begin{bmatrix} \sigma_S & 0 & 0 & 0 \\ \sigma_V\rho_{SV} & \sigma_V \cdot \bar{e} & 0 & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta \cdot f & \sigma_\delta \cdot h & 0 \\ \sigma_r\rho_{Sr} & \sigma_r \cdot \bar{g} & \sigma_r \cdot i & \sigma_r \cdot j \end{bmatrix}, \quad \text{and } \tilde{\mathbf{Z}}_t = \begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \\ \tilde{Z}_{4,t} \end{bmatrix}.$$

In this solution (20), Φ_t stands for the fundamental matrix of $\dot{\tilde{\mathbf{Q}}}_t = \mathbf{A}(t) \cdot \tilde{\mathbf{Q}}_t \cdot \Phi_t$ and its inverse are given by

$$\Phi_t = e^{\mathbf{A}(t-t_0)} = \begin{bmatrix} 1 & 0 & \frac{1}{\kappa}(e^{-\kappa(t-t_0)} - 1) & \frac{1}{g}(1 - e^{-g(t-t_0)}) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-\kappa(t-t_0)} & 0 \\ 0 & 0 & 0 & e^{-g(t-t_0)} \end{bmatrix}, \tag{21}$$

$$\Phi_t^{-1} = \begin{bmatrix} 1 & 0 & \dfrac{1}{\kappa}(e^{\kappa(t-t_0)}-1) & \dfrac{1}{g}(1-e^{g(t-t_0)}) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\kappa(t-t_0)} & 0 \\ 0 & 0 & 0 & e^{g(t-t_0)} \end{bmatrix}, \tag{22}$$

By substituting (21) and (22) into (20), we have

$$\begin{bmatrix} \tilde{P}_t \\ \tilde{J}_t \\ \tilde{\delta}_t \\ \tilde{r}_t \end{bmatrix} = \begin{bmatrix} \alpha(\pi) \\ \beta(\pi) \\ \gamma(\pi) \\ \varepsilon(\pi) \end{bmatrix} + \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \\ \tilde{z}_t \\ \tilde{w}_t \end{bmatrix},$$

where

$$\begin{bmatrix} \alpha(\pi) \\ \beta(\pi) \\ \gamma(\pi) \\ \varepsilon(\pi) \end{bmatrix} = \begin{bmatrix} \left\{ \begin{array}{c} P_{t_0} + \delta_{t_0}\frac{e^{-\kappa\pi}-1}{\kappa} + r_{t_0}\frac{1-e^{-g\pi}}{g} - \frac{\sigma_s^2\pi}{2} \\ +(\kappa\mu_\delta - \lambda_3\sigma_\delta)\frac{1-e^{-\kappa\pi}-\kappa\pi}{\kappa^2} - (g\mu_r - \lambda_4\sigma_r)\frac{1-e^{-g\pi}-g\pi}{g^2} \end{array} \right\} \\ J_{t_0} + \left(\alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2\right)\cdot\pi \\ \delta_{t_0}\cdot e^{-\kappa\pi} + (\kappa\mu_\delta - \lambda_3\sigma_\delta)\cdot\frac{1-e^{-\kappa\pi}}{\kappa} \\ r_{t_0}\cdot e^{-g\pi} + (g\mu_r - \lambda_4\sigma_r)\cdot\frac{1-e^{-g\pi}}{\kappa} \end{bmatrix},$$

$$\begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \\ \tilde{z}_t \\ \tilde{w}_t \end{bmatrix} = \begin{bmatrix} \left\{ \begin{array}{l} \left(\sigma_s - \frac{\sigma_\delta\rho_{S\delta}}{\kappa} + \frac{\sigma_r\rho_{Sr}}{g}\right)\int_{t_0}^t \mathrm{d}\tilde{Z}_{1,s} + \frac{\sigma_\delta\rho_{S\delta}}{\kappa}\int_{t_0}^t e^{\kappa(s-t)}\mathrm{d}\tilde{Z}_{1,s} \\ -\frac{\sigma_r\rho_{Sr}}{g}\int_{t_0}^t e^{g(s-t)}\mathrm{d}\tilde{Z}_{1,s} + \left(\frac{\sigma_r\bar{g}}{g} - \frac{\sigma_\delta f}{\kappa}\right)\int_{t_0}^t \mathrm{d}\tilde{Z}_{2,s} \\ +\frac{\sigma_\delta f}{\kappa}\int_{t_0}^t e^{\kappa(s-t)}\mathrm{d}\tilde{Z}_{2,s} - \frac{\sigma_r\bar{g}}{g}\int_{t_0}^t e^{g(s-t)}\mathrm{d}\tilde{Z}_{2,s} \\ +\left(\frac{\sigma_r i}{g} - \frac{\sigma_\delta h}{\kappa}\right)\int_{t_0}^t \mathrm{d}\tilde{Z}_{3,s} + \frac{\sigma_\delta h}{\kappa}\int_{t_0}^t e^{\kappa(s-t)}\mathrm{d}\tilde{Z}_{3,s} \\ -\frac{\sigma_r i}{g}\int_{t_0}^t e^{g(s-t)}\mathrm{d}\tilde{Z}_{3,s} + \frac{\sigma_r j}{g}\int_{t_0}^t (1-e^{g(s-t)})\mathrm{d}\tilde{Z}_{4,s} \end{array} \right\} \\ \sigma_V\rho_{SV}\int_{t_0}^t \mathrm{d}\tilde{Z}_{1,s} + \sigma_V\bar{e}\int_{t_0}^t \mathrm{d}\tilde{Z}_{2,s} \\ \left\{ \begin{array}{l} \sigma_\delta\rho_{S\delta}\int_{t_0}^t e^{\kappa(s-t)}\mathrm{d}\tilde{Z}_{1,s} + \sigma_\delta f\int_{t_0}^t e^{\kappa(s-t)}\mathrm{d}\tilde{Z}_{2,s} \\ +\sigma_\delta h\int_{t_0}^t e^{\kappa(s-t)}\mathrm{d}\tilde{Z}_{3,s} \end{array} \right\} \\ \left\{ \begin{array}{l} \sigma_r\rho_{Sr}\int_{t_0}^t e^{g(s-t)}\mathrm{d}\tilde{Z}_{1,s} + \sigma_r\bar{g}\int_{t_0}^t e^{g(s-t)}\mathrm{d}\tilde{Z}_{2,s} \\ +\sigma_r i\int_{t_0}^t e^{g(s-t)}\mathrm{d}\tilde{Z}_{3,s} + \sigma_r j\int_{t_0}^t e^{g(s-t)}\mathrm{d}\tilde{Z}_{4,s} \end{array} \right\} \end{bmatrix},$$

and $\pi = t - t_0$. Note that the stochastic integrals $\int\{\cdot\}\mathrm{d}\tilde{Z}_{i,s}$ $(i = 1, 2, 3)$ are normally distributed.

Then, the pricing formula for the commodity-linked bond $B(S_t, V_t, \delta_t, r_t, \tau)$ is obtained by calculating the expected present value of the bond with payoff function (4).

$$
\begin{aligned}
&B(S_t, V_t, \delta_t, r_t, t)\\
&= E\left[\min\{\tilde{V}_T, F + \max(\tilde{S}_T - K, 0)\} \cdot \exp\left\{-\int_t^T \tilde{r}_u \mathrm{d}u\right\} \middle| \begin{array}{l} \tilde{S}_t = S_t \\ \tilde{V}_t = V_t \\ \tilde{\delta}_t = \delta_t \\ \tilde{r}_t = r_t \end{array}\right]\\
&= E\left[\min\{e^{\beta(\tau)+y_T}, F + \max(e^{\alpha(\tau)+x_T} - K, 0)\}\right.\\
&\qquad \left.\cdot \exp\left\{-\int_t^T (\varepsilon(u-t) + w_u)\mathrm{d}u\right\} \middle| \begin{array}{l} \tilde{S}_t = S_t \\ \tilde{V}_t = V_t \\ \tilde{\delta}_t = \delta_t \\ \tilde{r}_t = r_t \end{array}\right] \qquad (23)\\
&= \exp\{\eta_\beta(\tau)\} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{H}\int_{-\infty}^{R} \exp\{y_t + r_t^*\} f(x_t, y_t, r_t^*)\mathrm{d}x_t \mathrm{d}y_t \mathrm{d}r_t^* +\\
&+ F \cdot \exp\{\eta(\tau)\} \cdot \int_{-\infty}^{\infty}\int_{H}^{\infty}\int_{-\infty}^{R} \exp\{r_t^*\} f(x_t, y_t, r_t^*)\mathrm{d}x_t \mathrm{d}y_t \mathrm{d}r_t^* +\\
&+ \exp\{\eta_\beta(\tau)\} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{Q}\int_{R}^{\infty} \exp\{y_t + r_t^*\} f(x_t, y_t, r_t^*)\mathrm{d}x_t \mathrm{d}y_t \mathrm{d}r_t^* +\\
&+ (F-K) \cdot \exp\{\eta(\tau)\} \cdot \int_{-\infty}^{\infty}\int_{Q}^{\infty}\int_{R}^{\infty} \exp\{r_t^*\} f(x_t, y_t, r_t^*)\mathrm{d}x_t \mathrm{d}y_t \mathrm{d}r_t^* +\\
&+ \exp\{\eta_\alpha(\tau)\} \cdot \int_{-\infty}^{\infty}\int_{Q}^{\infty}\int_{R}^{\infty} \exp\{x_t + r_t^*\} f(x_t, y_t, r_t^*)\mathrm{d}x_t \mathrm{d}y_t \mathrm{d}r_t^*\,,
\end{aligned}
$$

where

$$
\begin{aligned}
&\{H \mid y_T = \log F - \beta(\tau)\}\,,\ \{R \mid x_T = \log K - \alpha(\tau)\}\,,\\
&\{Q \mid y_T = \log\{F + \exp\{\alpha(\tau) + x_T\} - K\} - \beta(\tau)\}\,,
\end{aligned}
$$

and

$$
\begin{bmatrix} \eta(\tau) \\ \eta_\alpha(\tau) \\ \eta_\beta(\tau) \end{bmatrix} = \begin{bmatrix} -\frac{g\mu_r - \lambda_4\sigma_r}{g}\tau + \left(\frac{g\mu_r - \lambda_4\sigma_r}{g} - r_t\right)\frac{1-e^{-g\tau}}{g} \\ P_t - \frac{\sigma_S^2}{2}\tau - \frac{\kappa\mu_\delta - \lambda_3\sigma_\delta}{\kappa}\left(\tau - \frac{1-e^{-\kappa\tau}}{\kappa}\right) - \delta_t\frac{1-e^{-\kappa\tau}}{\kappa} \\ J_t + \left(\alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2\right)\tau - \frac{g\mu_r - \lambda_4\sigma_r}{g}\left(\tau - \frac{1-e^{-g\tau}}{g}\right) - r_t\frac{1-e^{-g\tau}}{g} \end{bmatrix}.
$$

$f(x_t, y_t, r_t^*)$ is the joint density function such that

$$f(x_t, y_t, r_t^*) = \frac{1}{(2\pi)^{3/2} \cdot \sqrt{\det \sum}} \cdot \exp\left\{-\frac{1}{2}\mathbf{v}^{\mathbf{T}} \sum^{-1} \mathbf{v}\right\} ,$$

where $\mathbf{v} = \begin{bmatrix} x_t \\ y_t \\ r_t^* \end{bmatrix}$ and $\sum$ is its variance-covariance matrix which is given by

$$\sum = \begin{bmatrix} Var(x_t) & \sigma_{xy} & \sigma_{xr^*} \\ \sigma_{xy} & Var(y_t) & \sigma_{yr^*} \\ \sigma_{xr^*} & \sigma_{yr^*} & Var(r_t^*) \end{bmatrix} ,$$

where

$$\begin{cases} Var(x_t) & = \left(\sigma_S^2 + \dfrac{\sigma_\delta^2}{\kappa^2} + \dfrac{\sigma_r^2}{g^2} - 2\dfrac{\sigma_S\sigma_\delta\rho_{S\delta}}{\kappa} + 2\dfrac{\sigma_S\sigma_r\rho_{Sr}}{g} - 2\dfrac{\sigma_\delta\sigma_r\rho_{\delta r}}{\kappa g}\right)\tau + \\ & + \dfrac{\sigma_\delta^2}{2\kappa^3}(1 - e^{-2\kappa\tau}) + \dfrac{\sigma_r^2}{2g^3}(1 - e^{-2g\tau}) + \\ & + 2\dfrac{(1 - e^{-\kappa\tau})\sigma_\delta}{\kappa^2}\left(\sigma_S\rho_{S\delta} - \dfrac{\sigma_\delta}{\kappa} + \dfrac{\sigma_r\rho_{\delta r}}{g}\right) - \\ & - 2\dfrac{(1 - e^{-g\tau})\sigma_r}{g^2}\left(\sigma_S\rho_{Sr} + \dfrac{\sigma_r}{g} - \dfrac{\sigma_\delta\rho_{\delta r}}{\kappa}\right) - \\ & - 2\dfrac{\sigma_\delta\sigma_r\rho_{\delta r}}{\kappa g(\kappa + g)}(1 - e^{-(\kappa+g)\tau}) \\ Var(y_t) & = \sigma_V^2\tau \\ Var(z_t) & = \dfrac{\sigma_r^2}{g^2}\tau - 2\dfrac{\sigma_r^2}{g^3}(1 - e^{-g\tau}) + \dfrac{\sigma_r^2}{2g^3}(1 - e^{-2g\tau}) \\ \sigma_{xy} & = \left(\sigma_S\sigma_V\rho_{SV} - \dfrac{\sigma_V\sigma_\delta\rho_{V\delta}}{\kappa} + \dfrac{\sigma_V\sigma_r\rho_{Vr}}{g}\right)\tau + \\ & + \dfrac{\sigma_V\sigma_\delta\rho_{V\delta}}{\kappa^2}(1 - e^{-\kappa\tau}) - \dfrac{\sigma_V\sigma_r\rho_{Vr}}{g^2}(1 - e^{-g\tau}) \\ \sigma_{xr^*} & = \left(-\dfrac{\sigma_S\sigma_r\rho_{Sr}}{g} - \dfrac{\sigma_r^2}{g^2} + \dfrac{\sigma_\delta\sigma_r\rho_{\delta r}}{\kappa g}\right)\tau + \\ & + \dfrac{\sigma_\delta\sigma_r\rho_{\delta r}}{\kappa g(\kappa + g)}(1 - e^{-(\kappa+g)\tau}) - \dfrac{\sigma_r^2}{2g^3}(1 - e^{-2g\tau}) - \\ & - \dfrac{\sigma_\delta\sigma_r\rho_{\delta r}}{\kappa g}\left\{\dfrac{1 - e^{-\kappa\tau}}{\kappa} + \dfrac{1 - e^{-g\tau}}{g}\right\} + \\ & + \dfrac{\sigma_S\sigma_r\rho_{Sr}}{g^2}(1 - e^{-g\tau}) + 2\dfrac{\sigma_r^2}{g^3}(1 - e^{-g\tau}) \\ \sigma_{yr^*} & = -\dfrac{\sigma_V\sigma_r\rho_{Vr}}{g}\tau + \dfrac{\sigma_V\sigma_r\rho_{Vr}}{g^2}(1 - e^{-g\tau}) \end{cases}$$

The default probability is given by

$$\begin{aligned}\text{Default probability} = &\int_{-\infty}^{\infty}\int_{-\infty}^{H}\int_{-\infty}^{R} f(x_t, y_t, r_t^*)\mathrm{d}x_t\mathrm{d}y_t\mathrm{d}r_t^* \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{Q}\int_{R}^{\infty} f(x_t, y_t, r_t^*)\mathrm{d}x_t\mathrm{d}y_t\mathrm{d}r_t^* .\end{aligned} \tag{24}$$

5. Concluding Remarks

In this paper, we have derived pricing formulae for the commodity-linked bonds in three cases. In Section 2, the first pricing function $B(S_t, V_t, \delta_t, \tau)$ had three variables beside the time to maturity, namely the commodity price, the value of the issuer, and the net marginal convenience yield. We have shown several figures of the bond prices and of the default probabilities as functions of parameter values in Section 3. These figures provided us certain implication for prices of the commodity-linked bond and the default probabilities. In Section 4, the second pricing formula $B(S_t, V_t, \delta_t, r_t, \tau)$ was an extended version of the first one. This formula was the pricing function when we allow the stochastic changes for the interest rate. In Appendix C, the pricing function, $B^*(C_t, V_t, \delta_t, \tau)$, was different from the first and second ones with respect to its underlying asset. The underlying asset of $B^*(C_t, V_t, \delta_t, \tau)$ was a special asset which is a self-financing portfolio consisting of a unit of commodity and the continuously reinvested net marginal convenience yield. We note that the closed-form pricing function of this bond did not include any parameter associated with the convenience yield. In these derivation procedures, we used the standard no-arbitrage argument to derive PDEs and took the change of the variables which transformed the *non-linear* SDEs to the *linear* ones to find the joint probability distribution for calculating the pricing formulae.

As is well known, commodities and their derivative securities are one category of the risky assets for the corporate finance. We have to consider various efficient ways for their hedging schemes in regard to financial risk management. We hope that the bond pricing formulae that we derived here will provide some help to practitioners who want to hedge the risk of price fluctuations of commodities.

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Appendix A: Proof for the Independence of λ_1, λ_2, and λ_3 on the Choice of the Assets

By the standard no-arbitrage argument, we derived constants λ_1, λ_2, and λ_3 for the choice of the three assets, namely, 1st, 2nd, and 3rd bonds in Section 2. They are tentatively dependent on the choice of three assets. In this appendix, we prove its independence on the choice of the assets. First, we exchange 3rd bonds to 4th bonds. The same no-arbitrage argument is valid for this new portfolio and we obtain a new set of constants λ_1', λ_2', and λ_3' with the following equations:

$$\begin{bmatrix} \varphi_{B,1} - r \\ \varphi_{B,2} - r \\ \varphi_{B,4} - r \\ \alpha_S + \delta_t - r \end{bmatrix} = \lambda_1' \cdot \begin{bmatrix} \pi_S^1 \\ \pi_S^2 \\ \pi_S^4 \\ \sigma_S \end{bmatrix} + \lambda_2' \cdot \begin{bmatrix} \pi_V^1 \\ \pi_V^2 \\ \pi_V^4 \\ 0 \end{bmatrix} + \lambda_3' \cdot \begin{bmatrix} \pi_\delta^1 \\ \pi_\delta^2 \\ \pi_\delta^4 \\ 0 \end{bmatrix}. \tag{A.1}$$

From equations $\mathbf{D} = \lambda_1 \cdot \mathbf{A} + \lambda_2 \cdot \mathbf{B} + \lambda_3 \cdot \mathbf{C}$ and (A.1) and (5) (the assumption of non-singularity), we obtain

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \end{bmatrix} = \begin{bmatrix} \pi_S^1 & \pi_V^1 & \pi_\delta^1 \\ \pi_S^2 & \pi_V^2 & \pi_\delta^2 \\ \sigma_S & 0 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \varphi_{B,1} - r \\ \varphi_{B,2} - r \\ \alpha_S + \delta_t - r \end{bmatrix}.$$

This procedure can be iterated. The iteration is done for exchanging 2nd bond to 5th bond, and 1st bond to 6th bond. Then we have a unique time dependent constant set, λ_1, λ_2, and λ_3. Thus, these constants are independent of the choice of assets. Each of these is called the market price of risk.

Of course, we can imply the same result to the set of constants λ_1, λ_2, λ_3, and λ_4 in section 4.2 and also λ_1^*, λ_2^*, and λ_3^* in Appendix C.

Appendix B: Decomposition of $\mathbf{G} \cdot \mathbf{G}^{\mathbf{T}}$

$\mathbf{G}$ is a matrix such that

$$\mathbf{G} \cdot \mathbf{G}^{\mathbf{T}} = \begin{bmatrix} \tilde{S}_t^2 \sigma_S^2 & \tilde{S}_t \tilde{V}_t \sigma_S \sigma_V \rho_{SV} & \tilde{S}_t \sigma_S \sigma_\delta \rho_{S\delta} \\ \tilde{S}_t \tilde{V}_t \sigma_S \sigma_V \rho_{SV} & \tilde{V}_t^2 \sigma_V^2 & \tilde{V}_t \sigma_V \sigma_\delta \rho_{V\delta} \\ \tilde{S}_t \sigma_S \sigma_\delta \rho_{S\delta} & \tilde{V}_t \sigma_V \sigma_\delta \rho_{V\delta} & \sigma_\delta^2 \end{bmatrix}.$$

To get $\mathbf{G}$, the following decomposition helps:

$$\mathbf{G} \cdot \mathbf{G}^{\mathbf{T}} = \begin{bmatrix} \tilde{S}_t \sigma_S & 0 & 0 \\ 0 & \tilde{V}_t \sigma_V & 0 \\ 0 & 0 & \sigma_\delta \end{bmatrix} \cdot \begin{bmatrix} 1 & \rho_{SV} & \rho_{S\delta} \\ \rho_{SV} & 1 & \rho_{V\delta} \\ \rho_{S\delta} & \rho_{V\delta} & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{S}_t \sigma_S & 0 & 0 \\ 0 & \tilde{V}_t \sigma_V & 0 \\ 0 & 0 & \sigma_\delta \end{bmatrix}.$$

Next, we decompose the second matrix of the RHS above

$$\begin{bmatrix} 1 & \rho_{SV} & \rho_{S\delta} \\ \rho_{SV} & 1 & \rho_{V\delta} \\ \rho_{S\delta} & \rho_{V\delta} & 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & \bar{e} & f \end{bmatrix} \cdot \begin{bmatrix} a & b & d \\ 0 & c & \bar{e} \\ 0 & 0 & f \end{bmatrix}^{\mathbf{T}},$$

where

$$\begin{cases} a^2 = 1, b^2 + c^2 = 1, d^2 + \bar{e}^2 + f^2 = 1 \\ ab = \rho_{SV}, ad = \rho_{S\delta}, bd + c\bar{e} = \rho_{V\delta} \end{cases} .$$

When we assign $a = 1$, we have

$$b = \rho_{SV}, d = \rho_{S\delta} \text{ and } c^2 = 1 - \rho_{SV}^2 .$$

Now we suppose c and f be positive (of course, negative values are feasible. But we select positive values for c and f as one of the choices),

$$c = \sqrt{1 - \rho_{SV}^2}, \bar{e} = \frac{\rho_{V\delta} - \rho_{SV}\rho_{S\delta}}{\sqrt{1 - \rho_{SV}^2}}, f = \sqrt{1 - \frac{(\rho_{V\delta} - \rho_{SV}\rho_{S\delta})^2}{1 - \rho_{SV}^2} - \rho_{S\delta}^2} .$$

Then we can write **G** as follows:

$$\begin{aligned} \mathbf{G} &= \begin{bmatrix} \tilde{S}_t\sigma_S & 0 & 0 \\ 0 & \tilde{V}_t\sigma_V & 0 \\ 0 & 0 & \sigma_\delta \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & \bar{e} & f \end{bmatrix} \\ &= \begin{bmatrix} \tilde{S}_t\sigma_S & 0 & 0 \\ \tilde{V}_t\sigma_V\rho_{SV} & \tilde{V}_t\sigma_V \cdot c & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta \cdot \bar{e} & \sigma_\delta \cdot f \end{bmatrix} . \end{aligned}$$

This argument can be used to obtain a choice of **G** for the SDE (18) in relation to the PDE (17) in subsection 4.3 and the same thing applies to the SDE (C.5) in Appendix C.

Appendix C: Derivation for the Closed Pricing Formula of Commodity-Linked Bonds $B^*(C_t, V_t, \delta_t, \tau)$: One of the Underlying Asset is a Self-Financing Portfolio

In this appendix, we derive the closed pricing formula for commodity-linked bonds $B^*(C_t, V_t, \delta_t, \tau)$ with payoff at the maturity such that

$$\min[V_T, F + \max\{C_T - K, 0\}] . \tag{C.1}$$

We assume that $C(S_t, \delta_t, t)$ (in short, C_t) is the value of a self-financing portfolio, treated in Bjerksund (1991), has a unit of commodity S, and reinvests the net marginal convenience yield continuously. This has its initial value $C(S_t, \delta_t, t) = S(t)$. This portfolio was treated in Bjerksund (1991), where the evolution of its value is described by the following SDE:

$$\frac{dC}{C} = (\alpha_S + \delta_t)\mathrm{d}t + \sigma_S \cdot \mathrm{d}W_S . \tag{C.2}$$

This equation (C.2) means that the total expected return of this special portfolio comes from both the expected price changes of the commodity α_S and the net marginal convenience yield δ_t. SDEs for V_t and δ_t are the same as in Section 2. Their correlations between each other are also the same as in Section 2.

We assume here N ($N \geq 4$) different B^* in the market and the following matrices are non-singular for any choice of three assets B_i^*, B_j^*, and B_k^*.

$$\begin{bmatrix} \theta_S^i & \theta_V^i & \theta_\delta^i \\ \theta_S^j & \theta_V^j & \theta_\delta^j \\ \theta_S^k & \theta_V^k & \theta_\delta^k \end{bmatrix}, \tag{C.3}$$

where $i, j, k = 1, \ldots, N$ and $i \neq j$, $i \neq k$, and $j \neq k$.

Note that θ_S^i, θ_V^i, and θ_δ^i are defined as follows:

$$\theta_S^i = \frac{\partial B_i^*}{\partial C} \cdot \frac{C\sigma_S}{B_i^*}, \theta_V^i = \frac{\partial B_i^*}{\partial V} \cdot \frac{V\sigma_V}{B_i^*}, \theta_\delta^i = \frac{\partial B_i^*}{\partial \delta} \cdot \frac{\sigma_\delta}{B_i^*}.$$

Other assumptions are the same as in Section 2.

By using the standard no-arbitrage argument as we did this in subsection 4.2, we have the following PDE for the pricing function of this bond:

$$\begin{aligned} &\frac{\partial B^*}{\partial C} C \cdot r + \frac{\partial B^*}{\partial V} V(\alpha_V - \lambda_2^* \sigma_V) + \frac{\partial B^*}{\partial \delta}\{\kappa(\mu_\delta - \delta) - \lambda_3^* \sigma_\delta\} - \frac{\partial B^*}{\partial \tau} - r \cdot B^* + \\ &\quad + \frac{1}{2} \cdot \frac{\partial^2 B^*}{\partial C^2} S^2 \sigma_S^2 + \frac{1}{2} \cdot \frac{\partial^2 B^*}{\partial V^2} V^2 \sigma_V^2 + \frac{1}{2} \cdot \frac{\partial^2 B^*}{\partial \delta^2} \sigma_\delta^2 + \\ &\quad + \frac{\partial^2 B^*}{\partial C \partial V} C V \sigma_S \sigma_V \rho_{SV} + \frac{\partial^2 B^*}{\partial C \partial \delta} C \sigma_S \sigma_\delta \rho_{S\delta} + \frac{\partial^2 B^*}{\partial V \partial \delta} V \sigma_V \sigma_\delta \rho_{V\delta} = 0, \end{aligned} \tag{C.4}$$

where λ_2^* and λ_3^* are the market prices of risk for the value of the issuer and the net marginal convenience yield, respectively. They are dependent on time and independent on the choice of portfolio assets (see Appendix A).

This (C.4) is the PDE which every $B^*(C_t, V_t, \delta_t, \tau)$ must satisfy. This (C.4) is different from (9) in regard to the coefficients of $\partial B/\partial S$ (and $\partial B^*/\partial C$), namely, the coefficient of the first partial derivative in this PDE is '$C \cdot r$' and this does not include 'δ'.

Next, we derive the present value of the expected value of the payoff function (C.1) at the maturity by applying Feynman–Kac theorem. From (C.4), we have the following SDE:

$$\begin{aligned} \mathrm{d}\begin{bmatrix} \tilde{C}_t \\ \tilde{V}_t \\ \tilde{\delta}_t \end{bmatrix} &= \begin{bmatrix} \tilde{C}_t \cdot r \\ \tilde{V}_t(\alpha_V - \lambda_2^* \sigma_V) \\ \kappa(\mu_\delta - \tilde{\delta}_t) - \lambda_3^* \sigma_\delta \end{bmatrix} \mathrm{d}t + \\ &\quad + \begin{bmatrix} \tilde{C}_t \sigma_S & 0 & 0 \\ \tilde{V}_t \sigma_V \rho_{SV} & \tilde{V}_t \sigma_V \cdot c & 0 \\ \sigma_\delta \rho_{S\delta} & \sigma_\delta \cdot \bar{e} & \sigma_\delta \cdot f \end{bmatrix} \cdot \mathrm{d}\begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \end{bmatrix}, \end{aligned} \tag{C.5}$$

where

$$c = \sqrt{1-\rho_{SV}^2},\ \bar{e} = \frac{\rho_{V\delta} - \rho_{SV}\rho_{S\delta}}{\sqrt{1-\rho_{SV}^2}},\ f = \sqrt{1 - \frac{(\rho_{V\delta} - \rho_{SV}\rho_{S\delta})^2}{1-\rho_{SV}^2} - \rho_{S\delta}^2}\,.$$

Note that the drift terms in this SDE (C.5) are linear in the variables. This makes it easier to solve the PDE (C.4) than to solve (9). By solving the equation (C.5), the joint distribution of $[\tilde{C}_t, \tilde{V}_t, \tilde{\delta}_t]$ for the time interval $[t_0, T]$ can be derived by the theorem 8.5.2 in Arnold (1973, p. 141). The solutions are given by

$$\begin{bmatrix} \tilde{C}_t \\ \tilde{V}_t \\ \tilde{\delta}_t \end{bmatrix} = \begin{cases} C_{t_0} \cdot \exp\{\alpha(\pi) + \tilde{X}_t\} \\ V_{t_0} \cdot \exp\{\beta(\pi) + \tilde{Y}_t\} \\ \gamma(\pi) + \tilde{Z}_t \end{cases},$$

where

$$\begin{cases} \alpha(\pi) \equiv \left(r - \dfrac{1}{2}\sigma_s^2\right)\pi \\ \beta(\pi) \equiv \left(\alpha_V - \lambda_2^*\sigma_V - \frac{1}{2}\sigma_V^2\right)\pi \\ \gamma(\pi) \equiv \delta_{t_0} \cdot e^{-\kappa\pi} + (1 - e^{-\kappa\pi})\dfrac{\kappa\mu_\delta - \lambda_3^*\sigma_\delta}{\kappa} \end{cases},$$

$\pi = t - t_0$, and $[\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t]$ are jointly normal distributed with zero means and the variance-covariance matrix $\sum$ is given by

$$\sum = \begin{bmatrix} \sigma_S^2\pi & \sigma_S\sigma_V\rho_{SV}\pi & \sigma_S\sigma_\delta\rho_{S\delta}\dfrac{(1-e^{-\kappa\pi})}{\kappa} \\ \sigma_S\sigma_V\rho_{SV}\pi & \sigma_V^2\pi & \sigma_V\sigma_\delta\rho_{V\delta}\dfrac{(1-e^{-\kappa\pi})}{\kappa} \\ \sigma_S\sigma_\delta\rho_{S\delta}\dfrac{(1-e^{-\kappa\pi})}{\kappa} & \sigma_V\sigma_\delta\rho_{V\delta}\dfrac{(1-e^{-\kappa\pi})}{\kappa} & \dfrac{\sigma_\delta^2 \cdot (1-e^{-2\kappa\pi})}{\kappa} \end{bmatrix}.$$

The density function $m(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t)$ of $[\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t]$ is below.

$$m(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t) = \frac{1}{(2\pi)^{3/2} \cdot \sqrt{\det \sum}} \cdot \exp\left\{-\frac{1}{2}\tilde{\mathbf{v}}^{\mathbf{T}} \sum\nolimits^{-1} \tilde{\mathbf{v}}\right\},\ \text{where } \tilde{\mathbf{v}} = \begin{bmatrix} \tilde{X}_t \\ \tilde{Y}_t \\ \tilde{Z}_t \end{bmatrix}.$$

Finally, we derive the present value of the expected value of the payoff function at the maturity. The solution of the PDE (C.4) with initial condition (C.1) is as follows:

$$
\begin{aligned}
&B^*(C_t, V_t, \delta_t, t) = \\
&\quad = E\left[\min\{\tilde{V}_T, F + \max(\tilde{C}_T - K, 0)\} \cdot e^{-r\tau} \,\middle|\, \begin{array}{c} \tilde{C}_t = C_t \\ \tilde{V}_t = V_t \\ \tilde{\delta}_t = \delta_t \end{array} \right] \\
&\quad = E\left[\min\{V_t e^{\beta(\tau)+y_T}, F + \max(C_t e^{\alpha(\tau)+x_T} - K, 0)\} \cdot e^{-r\tau} \,\middle|\, \begin{array}{c} \tilde{C}_t = C_t \\ \tilde{V}_t = V_t \\ \tilde{\delta}_t = \delta_t \end{array} \right] \\
&\quad = V_t \cdot e^{-r\tau-\beta(\tau)} \cdot \int_{-\infty}^{H} \int_{-\infty}^{R} \exp\{y_t\} \cdot m_{XY}(x_t, y_t)\mathrm{d}x_t\mathrm{d}y_t + \\
&\quad + F \cdot e^{-r\tau} \cdot \int_{H}^{\infty} \int_{-\infty}^{R} m_{XY}(x_t, y_t)\mathrm{d}x_t\mathrm{d}y_t + \\
&\quad + V_t \cdot e^{-r\tau-\beta(\tau)} \cdot \int_{-\infty}^{Q} \int_{R}^{\infty} \exp\{y_t\} \cdot m_{XY}(x_t, y_t)\mathrm{d}x_t\mathrm{d}y_t + \\
&\quad + (F-K) \cdot e^{-r\tau} \cdot \int_{Q}^{\infty} \int_{R}^{\infty} m_{XY}(x_t, y_t)\mathrm{d}x_t\mathrm{d}y_t + \\
&\quad + C_t \cdot e^{-r\tau-\alpha(\tau)} \cdot \int_{Q}^{\infty} \int_{R}^{\infty} \exp\{x_t\} \cdot m_{XY}(x_t, y_t)\mathrm{d}x_t\mathrm{d}y_t\,,
\end{aligned}
\tag{C.6}
$$

where $m_{XY}(x_t, y_t)$ is the marginal density function of x_t and y_t and $\{H \mid y_T = \log F - \beta(\tau)\}$, $\{R \mid x_T = \log K - \alpha(\tau)\}$, and $\{Q \mid y_T = \log\{F + \exp\{\alpha(\tau) + x_T\} - K\} - \beta(\tau)\}$.

We note that the derived pricing function (C.6) does not include δ_t and the parameters related to movements of δ_t.

Appendix D: Pricing Program on Mathematica for $B(S_t, V_t, \delta_t, \tau)$

```
CLB[St_,sp_,k_,a_,ld_,sd_,DE_,Vt_,muv_,sv_,lv_,Rhosd_,Rhosv_,Rhovd_,tau_,K_,F_,r_]:=With[
{
sx    = Round[Sqrt[((sp^2)-(2*sp*sd*Rhosd)/k+(sd^2)/(k^2))*tau+2*(1-Exp[-k*tau])*((sp*sd*Rhosd)/(k^2)-
           (sd^2)/(k^3))+(1-Exp[-2*k*tau])*((sd^2)/(2*(k^3)))]*10000000]/10000000,
sy    = Round[Sqrt[(sv^2)*tau]*10000000]/10000000,
cov   = Round[(sp*sv*Rhosv*tau-(sv*sd*Rhovd*tau)/k+(sv*sd*Rhovd*(1-Exp[-k*tau]))/(k^2))*10000000]
        /10000000,
rho   = Round[(((sp*sv*Rhosv*tau-(sv*sd*Rhovd*tau)/k+(sv*sd*Rhovd*(1-Exp[-k*tau]))/(k^2)))/Sqrt[(((sp^
        2)-(2*sp*sd*Rhosd)/k+(sd^2)/(k^2))*tau+2*(1-Exp[-k*tau])*((sp*sd*Rhosd)/(k^2)-(sd^2)/(k^3))+
        (1-Exp[-2*k*tau])*((sd^2)/(2*(k^3))))*((sv^2)*tau)])*10000000]/10000000,
alpha = Round[(N[Log[St]]+(DE*(Exp[-k*tau]-1))/k+(r-(sp^2)/2)*tau+(k*a-ld*sd)*((1-Exp[-k*tau]
         -k*tau)/(k^2)))*10000000]/10000000,
beta  = Round[(N[Log[Vt]]+(muv-lv*sv-(sv^2)/2)*tau)*10000000]/10000000,
R     = Round[(N[log[K]]-N[Log[St]]+DE*(Exp[-k*tau]-1)/k+(r-(sp^2)/2)*tau+(k*a-ld*sd)*((1-Exp[-k*tau]
        -k*tau)/(k^2))))*100000]/100000,
```

```
P     = Round[(N[Log[F]]-(N[Log[Vt]]+(muv-lv*sv-(sv^2)/2)*tau))*100000]/100000
},
Q     = N[Log[GF-K+Exp[alpha+x]]]-beta;
N[1/2*Pi*sx*sy*Sqrt[1-rho^2])*0.00001*
  (Exp[beta-r*tau]*NSum[NIntegrate[Exp[y]*Exp[(((x^2)/(sx^2))+((y^2)/(sy^2))-((2*rho*x*y)/(sx*sy)))/
   ((-2)*(1-rho^2))],{y,Round[-6*sy*100]/100,P}],{x,Round[-6*sx*100]/100,R,0.00001},Method->Integrate]
  +F*Exp[-r*tau]*NSum[NIntegrate[Exp[(((x^2)/(sx^2))+((y^2)/(sy^2))-((2*rho*x*y)/(sx*sy)))/((-2)*
   (1-rho^2))],{y,P,Round[6*sy*100]/100}],{x,Round[-6*sx*100]/100,R,0.00001},Method->Integrate]
  +Exp[beta-r*tau]*NSum[NIntegrate[Exp[y]*Exp[(((x^2)/(sx^2))+((y^2)/(sy^2))-((2*rho*x*y)/(sx*sy)))/
   ((-2)*(1-rho^2))],{y,Round[-6*sy*100]/100,(Round[Q*100000]/100000)}],{x,R,Round[6*sx*100]/100,0.00001}
   ,Method->Integrate]
  +(F-K)*Exp[-r*tau]*NSum[NIntegrate[Exp[(((x^2)/(sx^2))+((y^2)/(sy^2))-((2*rho*x*y)/(sx*sy)))/((-2)*
   (1-rho^2))],{y,(Round[Q*100000]/100000),Round[6*sy*100]/100}],{x,R,Round[6*sx*100]/100,0.00001}
   ,Method->Integrate]
  +Exp[alpha-r*tau]*NSum[NIntegrate[Exp[x]*Exp[(((x^2)/(sx^2))+((y^2)/(sy^2))-((2*rho*x*y)/(sx*sy)))/
   ((-2)*(1-rho^2))],{y,(Round[Q*100000]/100000),Round[6*sy*100]/100}],{x,R,Round[6*sx*100]/100,0,00001}
   ,Method->Integrate]
  )
 ]
]
```

Note in Proof

After the final revision of this paper toward this publication, we came to know the work of K.R. Miltersen and E.S. Schwartz (1997) ‘Pricing of Options on Commodity Futures with Stochastic Term Structure of Convenience Yields and Interest Rates’. Publications from Department of Management, School of Business and Economics, Odense University. Their paper develops a model for pricing options on commodity futures in the presence of stochastic rates as well as stochastic convenience yields.

Notes

1. The spot (or sheer) price of some commodities such as crude oil are not observed in the market. We have a forward contract of the crude oil, since it takes a few weeks to deliver the oil. A device to calculate the convenience yield made by Gibson and Schwartz (1991) works without using spot prices.
2. For example, the production and transportation of crude oil cannot match with the demand of the market simultaneously, thus the shortage of crude oil costs much to the buyers of crude oil at the time being. So, if the shortage occurs in the crude oil market, then the owner of crude oil can enjoy his privilege, and the convenience yield will be recognized in the crude oil market. Moreover, in equilibrium, the convenience yield on the marginal unit of storage will be equalized among all of the potential and current owners of the commodities throughout their competition.

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