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THE COMPLEX PROCESS OF CONVERTING TOOLS INTO MATHEMATICAL INSTRUMENTS: THE CASE OF CALCULATORS

ABSTRACT. Transforming any tool into a mathematical instrument for students involves a complex 'instrumentation' process and does not necessarily lead to better mathematical understanding. Analysis of the constraints and potential of the artefact are necessary in order to point out the mathematical knowledge involved in using a calculator. Results of this analysis have an influence on the design of problem situations. Observations of students using graphic and symbolic calculators were analysed and categorised into profiles, illustrating that transforming the calculator into an efficient mathematical instrument varies from student to student, a factor which has to be included in the teaching process.

KEY WORDS: instrumentation process, instrumental genesis, graphic and symbolic calculators, student behaviour, conceptualisation process, limits, classroom practice

1. A PRELIMINARY AWARENESS OF A POTENTIAL NEGATIVE INFLUENCE OF GRAPHIC CALCULATORS

1.1. *The French Educational Context*

Since 1980, all types of calculators have been freely used in secondary school examinations in France. Contrary to the accepted opinion that graphic calculators are tools for teaching, Trouche and Guin (1996) have pointed out that while calculators are used by students, the French educational system has not properly acknowledged their use. However, since 1980, the importance of graphic representations and numerical methods has been growing in the French secondary mathematical curriculum: official comments emphasise the development of experimental processes, in which calculation tools should play a significant role. The use of calculators has therefore become an explicit aim in these curricula and the Ministry of Education and Technology has supported many experiments to promote the integration of new technologies into teaching. Nevertheless, no more than 15% of the teachers include graphic calculators in their teaching, in spite of the fact that all students have a graphic calculator in scientific classrooms (in the Fifth and Sixth Forms). Teachers appear to resist the

International Journal of Computers for Mathematical Learning **3:** 195–227, 1999.

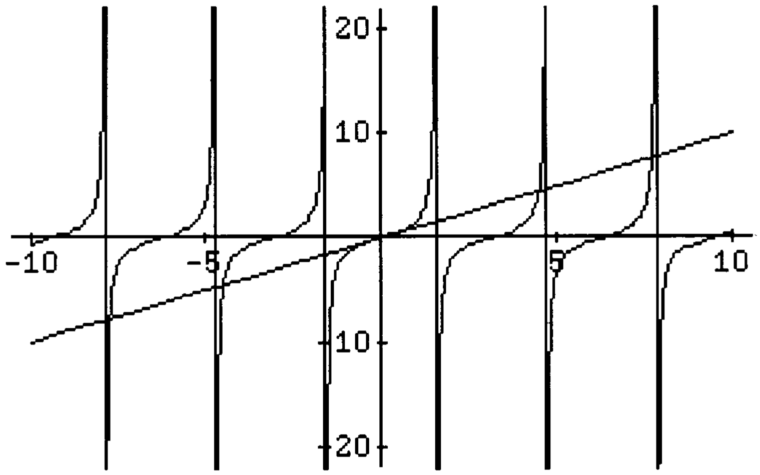

Figure 1. A representation of tan *x* on a computer screen.

integration of new technologies even at an elementary level. For example, how to obtain and read a graph (i.e., window manipulations) is missing from mathematics curriculum in France: students must acquire these types of skills by themselves outside of class time.

1.2. *Effects on Mathematical Conceptualisation*

The educational context described above may cause undesirable effects on mathematical conceptualisation. As Goldenberg (1987) has already pointed out: "students often misinterpreted what they saw in graphic representations of functions. Left alone to experiment, they could induce rules that were wrong . . . How do misconceptions distort the information that students glean from the graph?". We will present examples of student behaviour when faced with screen representations provided by graphic calculators. These examples demonstrate the insight of Goldenberg's comments.

1.2.1. *Confusion Between Mathematical Objects and Their Representation by Calculators*

In a class of 17-year-old science students (with a reference text at their disposal), when students are facing the screen image (see Figure 1), only four students evoked an infinity of solutions in their answer.

The remaining students asserted that there was a finite number of solutions (those visible on the screen). An illustrative argument was: after a certain time, the straight line $y = x$ has no intersection with the graphic representation of tan *x*. Seven students even included the intersections of

$y = x$ with the asymptotes in order to evaluate the number of solutions, reporting that: the asymptotes are part of the graph, since they appear when we ask for the graph. Of course, the presence of these asymptotes on the screen is only an outcome of the modelling of the continuum by the use of discrete tools. Finally, five students suggested an infinity of solutions at the proximity of the origin (referring to their perception of the proximity of the graphs of tan x and x), though they had seen in a theoretical lesson the tangent at 0 of the graphic representation of tan x.

1.2.2. *Conceptualisation of Mathematical Objects that Tools Cannot Show*

When students discover infinitesimal calculus at the last level of secondary school, they think that their graphic calculator can show infinity. The main idea is that graphic calculators may induce illusions as to their capacities relative to infinity. Trouche and Guin (1996) have related how the philosophy of 'seeing is reality' influences the conceptualisation of the fundamental notion of limits, particularly as there is no explicit definition of limit in the French secondary mathematics curriculum. Let us refer to a significant example, concerning the following question to 100 students 18 years old:

$$\lim_{x \to +\infty} \ell n\, x + 10 \sin x$$

All student responses were correct in the modality without the calculator (50 students). On the other hand, confronted with the rather disturbing graph produced by the calculator (see Figure 2), students could not come to terms with the inconsistency of the results displayed by the machine: in this case only 10% of the answers were correct (50 students). Most students extrapolated from what was actually visible on the screen and attempted to deduce information about the curve's behaviour towards infinity from its profile. We argue that not having mastered manipulations of the graphic calculator will lead students to revert to more primitive conceptions of limits. On the other hand, students without a calculator more often refer to strategies based on recently acquired knowledge. In this case, the reference is the text. For the majority of students, because they usually have their graphic calculator at their disposal, it is the first and often the most influential, sometimes even the only way to investigate, especially for mathematically weaker students.

1.3. *Instructional Implications*

These examples of student behaviour are significantly opposed to those described in Shoaf (1997) as "a process in which the student is silently

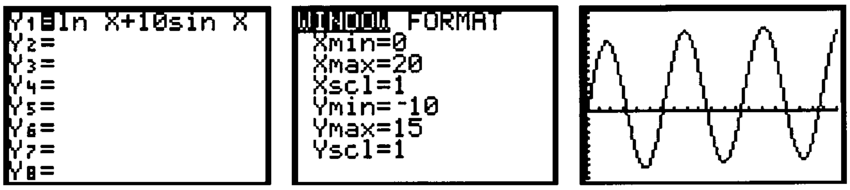

Figure 2. A representation of $\ell n\, x + 10 \sin x$ on a TI-82 screen.

conversing with himself through the calculator, asking questions of himself as he manipulates the concrete screen image. This leads him to have the knowledge to conjecture not only what is actually occurring with the image, but why it is happening. By using the graphic calculator students are more likely to construct their own mathematical understanding through conscious reflection".

Past experiments have concluded that such behaviour does not come naturally, particularly for mathematically weaker students, and that the effects of visualisation may be considerably more complex than generally believed. The challenge is therefore to find out how to achieve the aim of making the graphic calculator's visual representation of mathematical concepts both a heuristic and pedagogic tool, particularly for weaker students.

There is an unavoidable gap between 'real' mathematics and the image reflected by calculators (at the graphic level as well as at the numerical level). There is an unquestionable discretisation of the continuum which is rather disturbing for the student: screen images are a representation of reality, more precisely a display which may distort reality. For all these reasons, teachers should highlight this transformation and teach 'image'. The 'touch and see' philosophy discussed in Yerushalmy (1997) allows interesting mathematical reflection to be organised around screen images, because the chosen problem situations are well adapted to the technology. However, long-term activities on functions whose behaviour on the screen leads to a good prediction of the limit (this is the case for rational functions) may reinforce student tendencies to extrapolate a function's behaviour towards infinity. Moreover, if the conceptualisation of limits at infinity depends on the possibility of going beyond the idea of monotonic closeness widely held among students (Trouche and Guin, 1996, p. 328), then is there a risk of creating a learning obstacle to an expert conception of limits at infinity?

Considering the above-mentioned problem situations, the difficulty of elaborating situations which enhance the potential of the tool may be estimated in order to organise a new learning environment and promote more interesting mathematics tasks. Teachers should integrate the graphic calculator as a heuristic, pedagogic and cognitive tool, because this integration is not necessarily spontaneous: the calculator is not an efficient mathematical instrument *per se*, even if the transformation is quickly made for certain individuals. It is only through a complex process that students will be able to combine different available sources of information (theoretical text, a calculator, calculation by hand) to construct their own mathematical understanding.

A teacher's assistance in this process requires analysing the characteristics of the technology, taking into consideration previous research in the field of computer-based learning environments, so as to anticipate the changes it may introduce to the learning environment.

2. THE NECESSITY OF INSTRUMENTED ACTIVITY ANALYSIS

2.1. *Lessons from Computer-Based Learning Environment Studies*

Artigue, in her synthesis of the relationship between computer-based environments and learning theories (Artigue, 1997), states that the analysis of the computer's potential for mathematics learning has been strongly influenced by constructivist approaches viewing cognition as an adaptive process where knowledge is actively built by the subject. Computers serve the purpose of renewing teaching practices by managing technical computations, thus potentially promoting more conceptual understanding. In this context, the new role of the teacher is to organise and encourage interaction with the computer environment. This approach is stressed in advanced mathematical thinking: "We see the computer already proving a powerful tool in advanced mathematical thinking in mathematics education at the higher levels. The empirical evidence shows that it proves to be more successful in the educational process when it is used to enhance meaning through the use of computer environments for exploration and construction of concepts" (Dubinsky and Tall, 1991, p. 243).

Artigue pointed out an appreciable evolution in recent research with particular emphasis placed on the role played by the material tools of mathematical activities: "emphasis is put on the fact that, due to their characteristics and also due to the way they shape and constrain the possibilities of interaction with mathematical objects, they deeply condition the mathematics which can be produced and learnt" (Artigue, 1997, p. 2). In this way, by exploiting the computer as a window on the multiple ways mathematical meanings are constructed, the new model revises mathematical meanings and focuses on the individual role of mathematical meaning dependent on a given environmental context: “these tools wrap up some of the mathematical ontology of the environment and form part of the web of ideas and actions embedded in it” (Noss and Hoyles, 1996, p. 227). The authors underline the users’ difficulties, in these new ways of constructing meaning, in building connections with the official mathematics outside the microworld. These observations are confirmed by the examples cited earlier. Thus, the teacher’s role is to draw attention to the appropriate connections in the web: he or she has a crucial responsibility in shaping the relationship between the computational media and mathematical knowledge.

However, this completely revised didactic thinking requires consideration of a set of mathematical and didactic constraints which are implemented in the design of the environment: “Learning is based as much on these constraints as on the possibilities of investigations” (Dreyfus, 1993, p. 128). Dreyfus stresses the importance of the choice and the way activities are promoted by the teacher for making an effective learning tool.

One cannot avoid the new complexity caused by the introduction of technology to the classroom. Computer-based devices introduce a new source of knowledge transformation caused by the specificity and constraints of sophisticated representation systems both at the interface and inside the machine, as well as technical constraints imposed by operating systems (Balacheff, 1993, p. 147). Balacheff uses the term “computational transposition” for the process which leads to the implementation of a knowledge model. The teacher’s presence is necessary to make that knowledge compatible with the national mathematics curriculum.

In order to understand what kind of knowledge can be learnt in these computer environments, how and in what forms, it is necessary to give due consideration to student experiences. Artigue studied the adaptive processes at play in mathematical work with DERIVE (a computer algebra system). She noted that “even if perceptive processes are piloted by mathematical knowledge, mathematical knowledge engaged remains limited” (Artigue, 1997). Similarly, faced with difficulties in interpreting feed-back, instead of engaging more mathematical knowledge, students often prefer to use a trial and error strategy. Weaker students often do not question results; activities do not necessarily lead to reflective work, but may instead lead to behaviour she has called “fishing behaviour” (Artigue, 1995). The widespread idea that computer environments, because they can appear to take on technical aspects of mathematical activity, spontaneously induce mathematical activity, which is both more reflective and conceptual, must be challenged. Unfortunately, the “economy of the adaptive processes” observed during these experiments with DERIVE is also far from the desired student behaviour mentioned in Section 1.3 (Shoaf, 1997).

If, as teachers and educational researchers, we are aiming to encourage such behaviour, it is essential to attempt to understand what type of mathematical knowledge is at play in the perceptive processes of an efficient expert and how this is connected to more analytical processes. This is a prerequisite to efficient teacher intervention in the interaction process involving artefacts and students to be studied in Cognitive Ergonomy.

2.2. *Instrumented Activity in Cognitive Ergonomy*

This analysis stems from Vygotsky's hypothesis (Vygotsky, 1930), which states that artificial systems can extend man's cognitive capacities by developing his ability to act on the environment. Natural processes do not disappear, but are integrated with the instrumented act and become dependent on the instrument. As language and thought are related (see Vygotsky, 1962), Vygotsky points out the fundamental relationship between gestures and thought. Verillon and Rabardel's studies focusing on learning processes involving instruments in the area of cognitive ergonomy are based on this idea. If cognition evolves through interaction with the environment, accommodating to artefacts may have an effect on cognitive development, knowledge construction and processing, and the nature itself of the knowledge generated (Verillon and Rabardel, 1995, p. 77). They suggest models and concepts to analyse the instrumented activity of children confronted with tasks involving artefacts.

Verillon and Rabardel stress the difference between the artefact (a material object) and the instrument as a psychological construct: "The instrument does not exist in itself, it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it with his activity" (Verillon and Rabardel, 1995, p. 84). The subject has to develop the instrumental genesis and efficient procedures in order to manipulate the artefact. During this interaction process, he or she acquires knowledge which may lead to a different use of it. Similarly, the specific features of instrumented activity are specified: firstly, the constraints inherent to artefacts; secondly, the resources artefacts afford for action; and finally, the procedures linked to the use of artefacts. The subject is faced with constraints imposed by the artefact to identify, understand and manage in the course of this action: some constraints are relative to the transformations this action allows and to the way they are produced. Others imply, more or less explicitly, a prestructuration of the user's action.

The reorganisation of the activity resulting from the introduction of instruments also affords new possibilities of action which are offered to

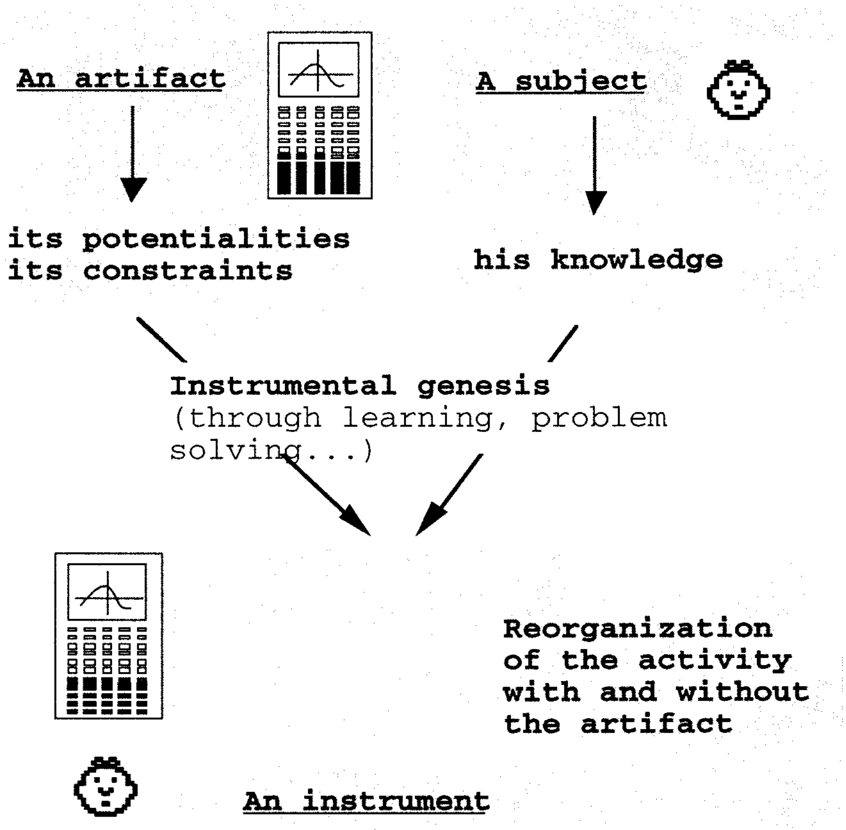

Figure 3. From an artifact to an instrument, the instrumental genesis.

the user; they may provide new conditions and new means for organising action. Thus, it can be argued that, because the instrument is not given but must be worked out by the subject (see Figure 3), the educational objectives stated above require the analysis of the instrumented activity of artefacts involved in the learning processes.

3. PRELIMINARY ANALYSIS OF THE CONSTRAINTS AND POTENTIAL OF SYMBOLIC CALCULATORS

Symbolic calculators have the potential to combine two modes of calculation: exact calculation and the approximate calculation of graphic calculators. The following will give an outline of the constraints introduced by the TI-92, a symbolic, graphic and geometric calculator, available in France since 1996 (including specific versions of DERIVE and CABRI

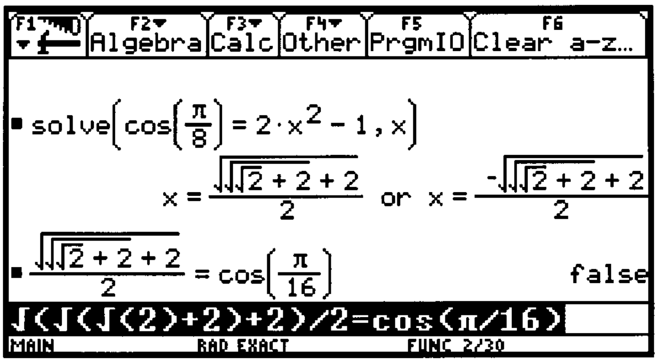

Figure 4. No recognition of the exact value of cos $\pi/16$ by the TI-92 CAS.

software). These constraints shape its potential in terms of types of actions and their management. They are specific to TI-92 but, with some modifications, they could be adapted to any calculator of the same type.

3.1. *Constraints of Symbolic Calculators*

Trouche (1996) has characterised three types of constraints which are significant elements of the computational transposition of mathematical knowledge: the internal constraints linked to the internal representation of objects and their calculation processing, the command constraints linked to the possibilities of action given to the user (choice of implemented commands) and the organisation constraints linked to the commands' access and their organisation. The last two types of constraints include constraints linked to the interface and are involved in the prestructuration of users' action. This triple net of constraints offers access to the computational transposition of mathematical knowledge.

3.1.1. *Internal Constraints*

Three types of internal constraints can be noted, all inter-related.

(1) Those caused by the inevitable limitations of the symbols available in memory (a characteristic specific to symbolic calculators). For instance, the calculator gives the exact value of cos $\pi/8$, whereas it does not recognise cos $\pi/16$. Obviously, from the formula cos $\pi/8 = 2(\cos \pi/16)^2 - 1$, the calculator may calculate the exact value of cos $\pi/16$ by solving the equation.

However, because the exact value of cos $\pi/16$ is unknown to the calculator, it will never recognise that the suggested value is the correct one (see Figure 4). The calculator's limitations may produce results whereby mathematical consistency is not easy to find.

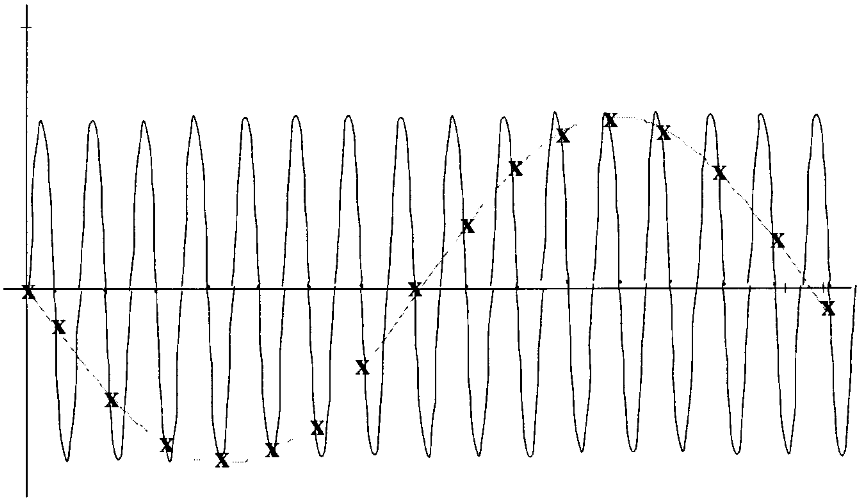

Figure 5. A diagram showing the consequence of a discrete trace: an unusual period for the sine function.

(2) the constraints linked to the discrete traces on a screen composed of a finite number of pixels. This discretization phenomenon (the same for graphic and symbolic calculators) may lead to inconsistent graphic representations depending on the chosen window. For instance, if the displayed function is periodic and if the distance between two calculated points on the *x*-axis is almost equal to the period (so that the calculated images are near), the graphic calculator joins the successive calculated points, and in this way masks the oscillations between them. Above (see Figure 5) is a diagram illustrating this process. It will be shown in Section 4 that, in spite of the presence of DERIVE, the misleading effects such images may encourage do not immediately disappear when the TI-92 takes the place of the graphic calculator.

(3) the constraints linked to the coexistence of several modes of calculation within the same tool. For example, the equation $x^2 - (1 + \sqrt{2})x + \sqrt{2} = 0$ has two roots, 1 and $\sqrt{2}$. In the exact mode, the TI-92 gives the two roots in an unsimplified form, whereas in the approximate mode the calculator gives two roots which look close to 1 and $\sqrt{2}$. The problem is that the machine refuses to identify the two roots it has found with the obvious ones (1 and $\sqrt{2}$) (see Figure 6 below).

Beyond the resolution of equations, avoiding discussion on how the computer operates internally, such answers question fundamental mathematical notions such as true or false. Many examples can be given where results may disturb students; consequently, the teacher should present situations leading to reflection on the various results from different calcu-

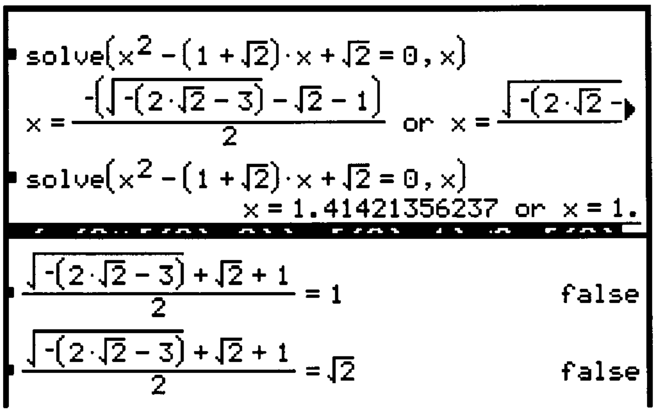

Figure 6. No recognition of the roots.

lation modes. Connections between real and complex numbers, exact and approximate calculations are not perfectly clear and require mathematical knowledge.

3.1.2. *Command Constraints*

The syntactic requirement is demanding and has to be memorised (as is usual in CAS systems), although the screen display of the input facilitates syntax control. However, even if the syntactic constraints cause difficulties for students, it may be considered as a training feature. In problem situations involving functions, the syntactic strictness of commands requires that the difference between variable, parameter, function, bound, etc., be noted.

On the other hand, the coexistence of two main modes of calculation may encourage students to bypass procedures. For instance, if they do not get a result with the limit command in the exact mode (which may occur quite frequently), they can find it by requesting the value of the function in the same mode. Another possibility is to enter the approximate mode; in that case, the limit command can also provide a result. These bypass procedures are particularly used in solving equations, because when roots are not sufficiently obvious the HOME application only solves equations of first and second degree. Thus, in the opposite case, there is a temptation to enter the approximate mode, but then some solutions may appear, and others disappear. For example, the TI-92 only gives one positive solution for the equation $e^x = x^{50}$. Moreover, if the condition $x > 100$ is added, this solution disappears, but another appears (see Figure 7 below).

We argue that, contrary to widespread opinion, the presence of symbolic commands does not necessarily exempt one from reflection,

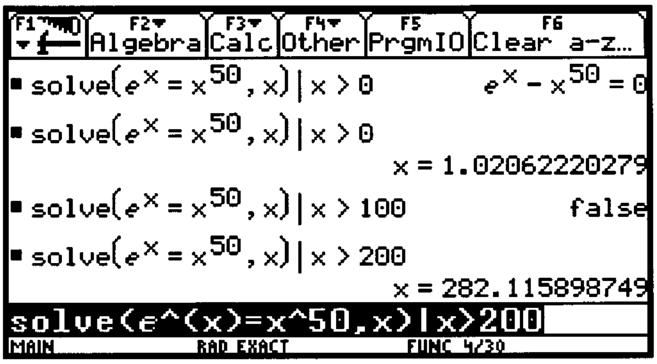

Figure 7. The resolution of $e^x = x^{50}$ on several intervals, in exact, then in approximate calculating mode.

because one cannot avoid the commands limitation. On the contrary, it may suggest questions which have not been anticipated by teachers.

3.1.3. *Organisation Constraints*

Unlike graphic calculators, calculation holds a privileged theoretical place in the TI-92, because the formal application HOME, able to handle exact and approximate calculations, provides an entry feature as simple as the graphic calculator. Actually, the situation is still more complex because it is difficult to reduce a function to a formula without seeing its graphic representation, giving some reality to the mathematical object. Moreover, the access to approximate calculation is easier because it is shortened by a special key stroke, whereas changing to the calculation mode requires a sequence of five or six key strokes. Therefore, the exact mode will not necessarily be the reference mode for students, even if it is privileged by the teacher.

3.2. *Potential of Symbolic Calculators*

The potential of learning environments has been widely described in mathematics education. Keitel and Ruthven (1993) provide a synthesis. Since their main ideas have already been discussed in Section 2.1, this section will merely recall the main point underlying research in this area: new learning environments offer the possibility of developing various new mathematical activities including interplay between the algebraic, graphic and tabular settings in order to promote experimental work. In these various settings, the mathematical objects, become new objects, called representatives by Schwartz and Dreyfus (1995). For example, in the graphic setting, a representative is obtained by choosing a viewing window.

By means of actions on these representatives, students can discover invariant properties that provide new ways to understand a concept. Handling these representatives requires new skills and a certain art, called "window shopping" by Hillel (1993). Moreover, these actions have to be verbalised and an appropriate language has to be developed to discuss the software phenomena. These new ways of constructing meaning for students by weaving their web of ideas and actions in the learning environment (Noss and Hoyles, 1996) are likely to have an impact on the conceptualisation of functions. Throughout the examples given above, it is clear that connections must be built with official mathematics, a far from easy task. The teacher interaction in this process is confirmed as crucial.

There is no doubt that reflection on the integration of new symbolic calculators and the use of computers as learning tools is closely linked. However, even if the applications of these new calculators correspond to previously available software, the main difference is that students can always have them at their disposal, facilitating, at least theoretically, their appropriation. It has already been pointed out how calculator manipulations always involve mathematical knowledge, in the handling of graphs, approximate and exact calculations and in the control of numerical approximations. The potential and constraints described above obviously involve issues concerning the implementation of mathematical work.

4. AN INSTRUMENTATION EXPERIMENT

4.1. *The Experimental Context*

4.1.1. *Guiding Ideas*

The teacher is the designer of activities where mathematics is the central focus. We know from Dorfler (1993) and Ruthven and Chaplin (1997) that experiences with calculators do not easily lead to cognitive reorganisation and that the organisation of learning activities is crucial to attaining some cognitive reorganisation. Therefore, situations have to be carefully designed in order to take advantage of the constraints and discrepancies caused by calculators, which may be considered as new learning potential. The choice of the applications involved in the activity and their articulation aims to improve investigation by enhancing varying points of view (for instance, the articulation between the algebraic and graphic settings). It is precisely because mathematical objects are not in the tangible world that the differentiation between a mathematical object and its representation is at the heart of the learning process. The two key points for learning are the functional differentiation and coordination of semiotic registers developed by Duval (1996). In addition, these coordinations constitute the threshold which, when crossed, provokes a radical change within the student's attitudes with regard to the conscious level of the cognitive process, i.e., to the initiative and command capacities. There is no doubt that the calculator deeply modifies ways of interweaving various registers and coordinating them.

One must therefore develop situations aiming to foster experimental work (investigation and anticipation) with interactions between graphic observations and theoretical calculus, and to encourage students to compare various results of different registers in order to tackle the distortion between the paper and machine environments, precisely because it is not a natural behaviour. This reflection is needed in order to seek mathematical consistency in various results and will motivate students to improve the mathematical knowledge required to overcome these contradictions (such as the distinction between approximate and exact calculation, control of numerical approximations, reflection on the unavoidable discretization of the screen and the nature of representatives and calculation algorithms).

This mathematical knowledge is necessary to understand commands as well as their limitations, to distinguish mathematical objects from calculator representations and mathematical processing from internal calculator processes, and finally to convert the symbolic calculator into a mathematical instrument. The experiment reported herein attempted to collect information on the different ways students actually make use of the calculator, in order to understand, through the evolution of their strategies, how they instrumentalise the calculator and the mathematical effects of this transformation process. It was also the aim of this experiment to determine what mathematical knowledge can underlie instrumentation.

4.1.2. *Designing a New Study Environment*

The French Ministry of Education supported an experiment in which ten classes were provided with symbolic calculators. The following describes the experimental organisation used in two of these classrooms (15/16 and 17/18-year olds). During the first three months of the school year, the older group of students used graphic calculators, then during the last six months the class went on to use symbolic calculators (TI-92). Each student had the use of a TI-92 both at school and at home. The organisation described below was used first in a graphic calculator environment then in an environment using symbolic calculators. From the beginning of the school year, for the theoretical lesson itself and for the practical

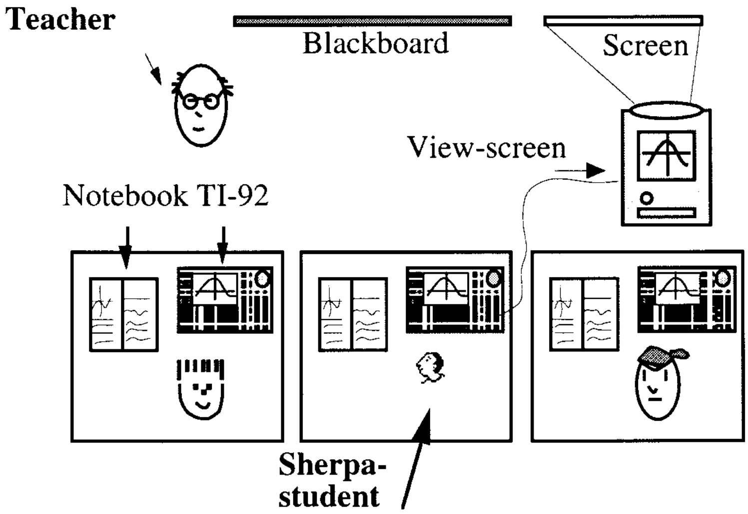

Figure 8. Organisation of a lesson in a calculator environment.

sessions, the time and spatial organisation of the learning activities were reconsidered.

Throughout the lesson both a blackboard and a screen (displaying one of the calculators) were used. This combination enabled the individual student's work, both on paper and using the calculator, to be guided by the teacher (see Figure 8 below). Each student took a turn operating the projected calculator. This student, called a 'sherpa student' played a central role in the layout of the lesson as a guide, assistant and mediator. Traditional classroom relations were altered: new classroom relations were established between the sherpa student and the other students as well as between the sherpa student and the teacher. This new context favoured classroom debates, pointed out the various student behaviours and was essential to counterbalancing the rather individualistic relationship students tend to have with a small screen. This organisation also enabled the teacher to become aware of the different steps in the student appropriation process of the instrument and reinforced the social aspect of this construction (see Section 2.2).

Such organisation entailed a rearrangement of study time: the various phases (observation, confrontation of results, testing different strategies) were longer. The teacher had to keep in mind both the potential and limits of the calculator throughout (see Section 3). This required the teacher to have a thorough knowledge of the calculator. The practical sessions were inserted as one-hour weekly sessions. The students worked in pairs (see Figure 9 below).

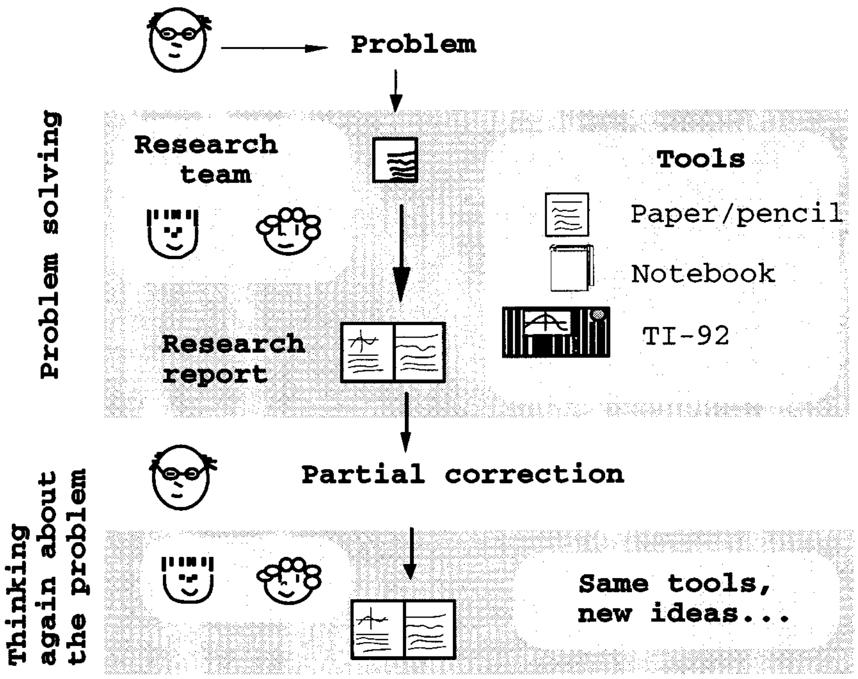

Figure 9. Practical sessions in a calculator environment.

Various problem situations were created aimed at promoting interaction between calculators, theoretical results, and handwritten calculations as an aid to conjecture, test, solve and check. After working on these problem situations in groups of two or three, each group had to explain and justify their observations or comments, noting discoveries and dead-ends in a written research report. The role of this report was twofold:

– it focused the student activity on the mathematics and not on the calculator, forcing students to give written explanations for each stage undertaken in their research (a very important step);
– it gave the teacher a better understanding of the various steps of the the students' behaviour, so as to offer appropriate assistance to help students out of deadlocks, to reinitiate reflection and to follow, week by week, the students' instrumental genesis.

Throughout this phase, the teacher was a consultant giving hints and dealing with problems as they arose in this new educational context. There is no doubt that in this case, teachers and students play a new role, as stated by Monaghan (1997): "the teacher is viewed as a technical assistant, collaborator, facilitator and as a catalyst, and students have to cooperate in group problem solving".

During the next session, a synthesis with an overhead calculator manipulated by a sherpa student, allowed the teacher to organise a group discussion and compare various approaches (different calculator and paper applications). The teacher's role was to compare different strategies, pointing out the contribution of each group, and suggesting questions designed to make students discuss the inconsistent results and seek mathematical consistency in the various results found. This collective synthesis is crucial because it allows the teacher to propose institutionalisation (recognition of student productions which are to be retained as knowledge) and decontextualisation of the desired mathematical knowledge. The teacher's role in this phase is well described in Dreyfus (1993, p. 113).

At the beginning of the experiment, the phase of the written research report was not readily accepted by the students, perhaps because they did not see the benefit of it. However, after several weeks they began to realise how this report could be a useful basis for launching experimental work, because the teacher had frequently referred to this written report throughout the synthesis phase. This new aspect of the didactic contract required a good deal of time to become an established and accepted feature of the student work. Similarly, the students' awareness of the necessity of a mathematical proof in this new educational environment was not immediate: it is only later on that students considered calculators as an aid to conjecture or checking, but not an exemption from proof. More generally, the management of experimental work in the classroom is a long-term process which requires considerable time to observe, exchange and reflect on all the collected data. Various difficulties arise when confronted with the institutional constraints, requiring that new specific rules be designed.

4.2. *An Example of Situations*

The following is one example of 17 to 18 year-old students, used to present the approach in a more concrete form. More details about these experiments may be found in Guin and Delgoulet (1996) and Trouche (1996). This problem generalizes the solving of the equation previously mentioned in Section 3.1.2.

Our purpose is to study the equations $e^x = x^{10n}$, where n is a strictly positive integer (the first equation is therefore $e^x = x^{10}$, the second one is $e^x = x^{20}$, etc.).

1. How many solutions has each of these equations got? (Prove your answer!)
2. Can you give an approximate value (10^{-5} by default) of the solutions of the first, second, third and tenth equations?
3. Can you suggest conjectures for the behaviour of the sequences of solutions when n increases?
4. Can you prove some of these conjectures?

This situation was chosen in order to suggest two main questions: first, is there always a large positive solution (the calculator does not give it for $n = 2$); and second, is it possible to organise the results into sequences of solutions? Beyond these questions, the skills aimed at were familiarisation with reference functions, dealing with various representatives, questioning the machine's results when appropriate, and finally proving the problem as soon as conjectures were formulated. Additional example situations can be found in Trouche (1996).

4.3. *Change in Student Behaviour*

Various data were collected from questionnaires focusing on mathematical tasks and the students' relationship to the calculators (called the students' barometer), the observation of some groups during the research phase, the students' written research reports and from interviews of selected students. For reasons of space, students' behaviour in terms of interviews and screen captures will not be detailed: this kind of description can be seen in Guin and Delgoulet (1996). The following gives an overview of the change in student behaviour during the experiment.

4.3.1. *Awareness of the Constraints and Potential of the Tool*

The students aged 15/16 had both discovered symbolic calculations and the graphic calculator. The manipulation difficulties in the algebra application observed at the beginning as parenthesis management or message interpretations progressively disappeared, certainly because the teacher turned his attention to them throughout the experiment. However, a correct manipulation as *Factor* or *Expand* requires that students give an explicit mathematical meaning to these commands. Consequently, manipulation difficulties reveal conceptual difficulties such as the recognition of factorized and developed forms of an algebraic expression. However, the differentiation between a mathematical object and its representation cannot be made without recognising two equivalent algebraic expressions. Although they were relieved of the technical tasks, many students could not work out strategies of comparison between expressions.

Moreover, a surprising result did not necessarily induce a question, especially for weaker students. Monaghan describes the embarrassment felt in such situations, even for experts: "We thus engaged in a successful problem solving feed-back loop with the aid of the calculator. But, the reality is not always successful: perhaps we do not get what we expected or cannot make sense of the output we get. We get an unwanted feedback loop of cognitive noise that can effectively obscure the mathematics at the heart of our use" (Monaghan, 1997, p. 2). In this context, we have to recognise that weaker students often give up the idea of understanding the command's meaning and what it does.

As a result, we observed avoidance strategies of various forms:

- automatic translations of the questions in terms of commands translating the statement word for word;
- generalising the command validity (for example, the use of the command *Solve* for solving an inequation, whereas students know this command is specific to equations);
- random trials and zapping to other commands in the same menu.

No doubt this behaviour is different from the activity in the paper/pencil environment where trying something else already requires careful thought. The main difference between the two environments is precisely that which occurs with the calculator, students lose consciousness of the task and there is little mathematical work in their activity. Once more, it is notable that the calculator does not automatically induce a more questioning and reflective mathematical attitude, as Artigue has also found (Artigue, 1997).

These students had discovered the graphic calculator simultaneously with the introduction of the function concept. Conceptual difficulties were likewise revealed and interfere with manipulations of graphs (confounding coordinates, erroneous graphic interpretations linked to definition or image domains, etc.). Beyond the difficulties linked to treatments within the graphic register, new difficulties arose when the algebraic and graphic registers interact (interpretation of the function *Sign*, distinction between function and equation required by the syntax of commands, confusion between the approximate calculations of the graphic register and the exact calculation of the algebraic register). Conceptual knowledge relative to functions had only recently begun to emerge and put obstacles in the conversion of representations from one register to another (for example conversion of an inequation to a graphic problem). Therefore, a specific exercise during several sessions was organised on the coordination between these two registers with regard to functions, equations and numbers. Even if students had not really undertaken experimental activities such as conjectures, proofs and refutations (which is not natural at all in the French educational context), they became aware of the possibilities of visualisation, anticipation and verifying offered by this interplay between several registers, even if they were not comfortable enough with the machine to take such initiative by themselves.

Focusing on the instrumental genesis, two phases become distinct. The first phase is the discovery phase of various commands, their effects and their organisation. This phase was mainly characterised by a strong dependence on the machine where students often disregarded other information sources, displayed in Figure 10 as theoretical knowledge or paper/pencil work. This phase was also revealed by students doing pioneering work associated with a wide use of available commands. Observations showed that they seldom referred to understanding tools (see Figure 10) in this phase which, nevertheless, revealed a first instrumentation level with a great diversity of strategies and techniques. As soon as commands gained mathematical meaning, students focused their action on a limited number of them. This second phase was an organisation phase which is characterised by a pruning attitude towards the first strategies and techniques. It occurred simultaneously with a progressive awareness of the effective constraints and potential uses of the calculator and a decreasing trust of the machine's results. It is also a phase in which students began to organise their actions in relation to fewer commands and to consciously coordinate them, with each other and with the other information tools, by means of what is here called the understanding tools. The student's command process is characterized by the conscious attitude to consider, with sufficient objectivity, all the information immediately available not only from the calculator, but also from other sources and to seek mathematical consistency between them. In more technical terms, it can be characterized by the propensity to choose strategies that are relevant rather than avoid those that are irrelevant (see Houdé, 1995).

The mathematical profile of students will determine how long it takes to go beyond this first level, even if learning activities are assumed to be organised by the teacher in this way. However, throughout the experiment, it was observed that the more manipulations were mastered, the more students were able to involve themselves in mathematical work.

4.3.2. *Behaviours Profiles*

In the experimental class of students 17 to 18 year-old, a great variety of behaviour was observed. In order to analyse the change in student behaviour in the instrumentation process, the following five profiles of behaviour were defined before the introduction of symbolic calculators, taking into account the mathematical profile, the relationship to the graphic calculator and the main features of student behaviour. From the observation of students using their graphic calculators, in particular during the practical sessions, and by examining their written work, especially their research report, the following profiles were defined:

- *random work method*, characterised by similar student difficulties, whether in the calculator environment or in the traditional paper/pencil environment. The tasks were carried out by means of cut and paste strategies from previously memorised solutions or hastily gener-

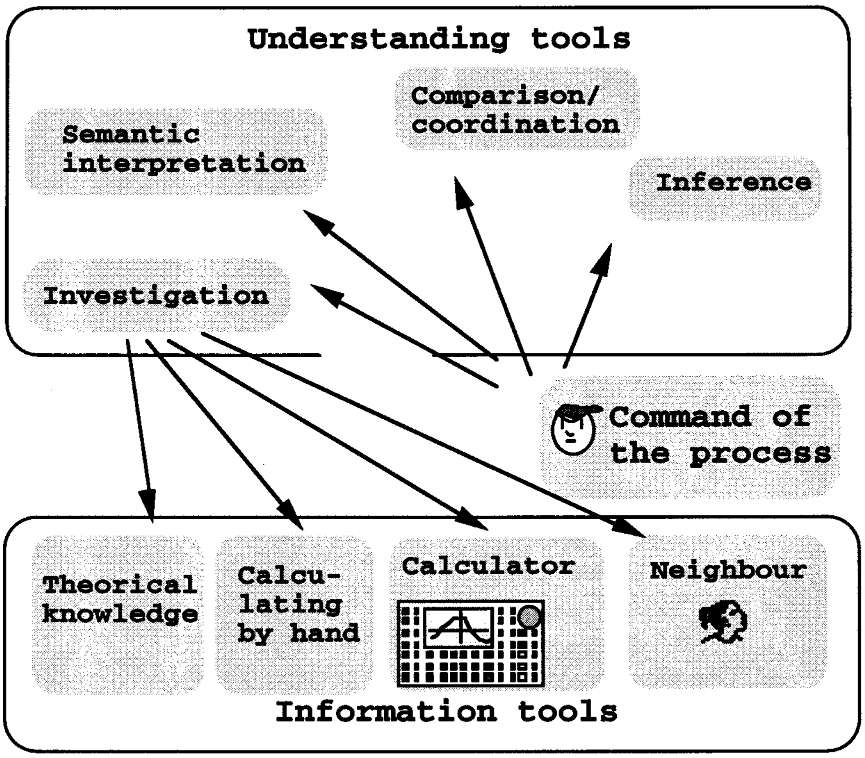

Figure 10. Calculation in a calculator environment.

alised observations. Therefore, the rather weak student's command process is revealed by trial and error procedures with very limited references to understanding tools and without verifying strategies of machine results.

- *mechanical work method*, characterised by information sources more or less restricted to the calculator investigations and simple manipulations. However, reasoning is based on the accumulation of consistent machine results. Student's command process remains rather weak, with an avoidance of mathematical references.
- *resourceful work method*, characterised by an exploration of all available information sources (calculator, but also paper/pencil work and some theoretical references). Reasoning is based on the comparison and the confrontation of this information with an average degree of student's command process. This is revealed by an investigation of a wide range of imaginative solution strategies: sometimes observations prevail, other times theoretical results predominate.

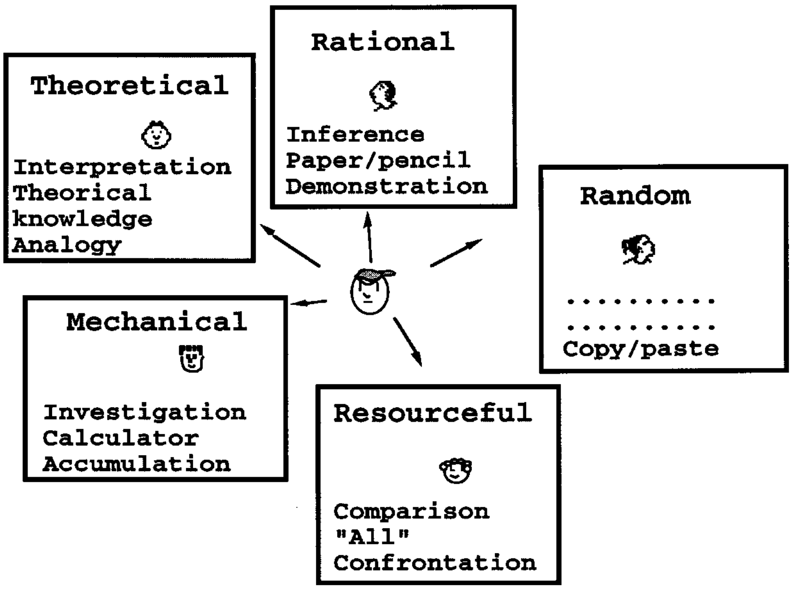

Figure 11. Typology of the different types of student behaviour.

- *rational work method*, characterised by a reduced use of the calculator, mainly working within the traditional (paper/pencil) environment. The specificity of this behaviour is a strong student's command process with an important role played by inferences in reasoning.
- *theoretical work method*, characterised by the use of mathematical reference as a systematic resource. Reasoning is essentially based on analogy and over-excessive interpretation of facts with average verifying procedures of machine results.

The summary of this typology is shown in Figure 11; for each of the profiles (for example, theoretical work methods) the following criteria were given:

– on the first line, the understanding tool most referred to (in this case, interpretation);
– on the second line, the information tool most referred to (in this case, theoretical knowledge);
– on the third line the proof method most referred to (in this case, analogy).

These categories are obviously not exclusive: a given student cannot be exactly classified in one of the given profiles. However, this above typology is very useful for two reasons:

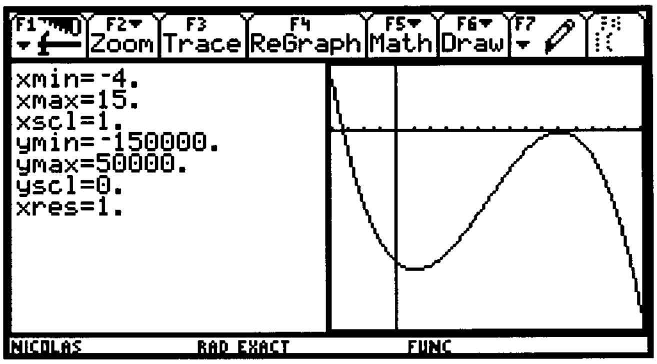

Figure 12. A strange graph.

- it establishes the different features of the overall class, which enables extreme types of behaviour to be identified;
- it allows the students' change in behaviour to be followed during the experiment. According to these different types, the instrumentation process evolved differently over a varying length of time.

4.3.3. *An Illustration of this Typology Based on Student Work*

This problem was carried out, using graphic calculators, during one of the first practical sessions of the school year.

$P(x) = 0.03\,x^4 - 300.5003x^3 + 5004.002x^2 - 10009.99x - 100100.$

a. Determine the limit in $+\infty$;

b. Determine a window which confirms your result.

The difficulty comes from the distance between the four real roots $(-10/3$, 10, 10.01 and 10000) which makes the choice of a relevant window difficult (see Figure 12 above). On the standard window, the graph which is obtained is not easy to interpret. On a fitted window, the graph does not correspond to the students' idea of a $+\infty$ limit for a function. The problem is solved differently by each type of student:

A result showing the theoretical profile. Student A applied the theorem which was learnt in class: this polynomial function has the same limit in $+\infty$ as its term of highest degree. Therefore, $\lim_{+\infty}$ P is $+\infty$. Through a drawing on paper, A indicated that he knew the overall shape of a 4 degree polynomial (see Figure 13 below).

To find an adequate window, A chose a range of abscissa (0, Xmax), adjusted Ymin and Ymax using the highest degree term: Ymax =

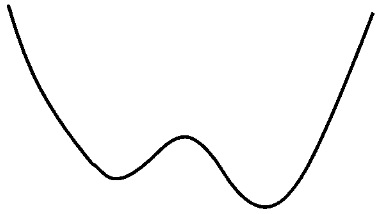

Figure 13.

$0.03(\text{Xmax})^4$. Then he increased Xmax until he obtained a window which corresponded to the expected graph. This mechanical adjustment of the ordinate and the fact that A a priori knew the general shape, allowed him to find an appropriate window quickly. It was noted that this student was able to use theoretical results to solve theoretical problems and also to solve practical problems.

A result showing the rational profile. In order to establish the limit, student B reproduced the method seen during the lesson (that is to say, demonstrate the theorem). She factorized the term of the highest degree:

$$P(x) = x^4(0.03 - 300.500\frac{1}{x} + 5004.002\frac{1}{x^2} - 10009.99\frac{1}{x^3} - \frac{100100}{x^4}).$$

B was able to give the limits of each factor of this function, then, by applying the theorems on the sums and products of limits, she found the limit of P. In order to obtain an appropriate window, B undertook a classical study of a function: the derivative of P, then the derivative of P' and the sign of $P'(x)$. Due to lack of time, B could not finish the work, which was extremely long. Her behaviour was characterized by a linear method based on the reproduction of very general calculations, not necessarily the most appropriate for this particular problem.

A result showing the random profile. Student C took a rather long time to enter the expression into the function editor of the calculator. He was unable to analyse the graph shown on the standard window or to obtain a more appropriate window, and was not able to use a more theoretical approach, which seems too difficult for such a complex object. The only relevant information was obtained from the table (values of the function, see Figure 14). The result given (after carrying out tests) was: $\lim_{+\infty} \text{P} = -\infty$.

A result showing the mechanical profile. From the beginning, student D started looking for an appropriate window for the graph and carried out various tests involving numerous commands:

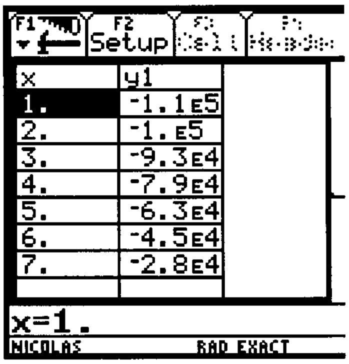

Figure 14.

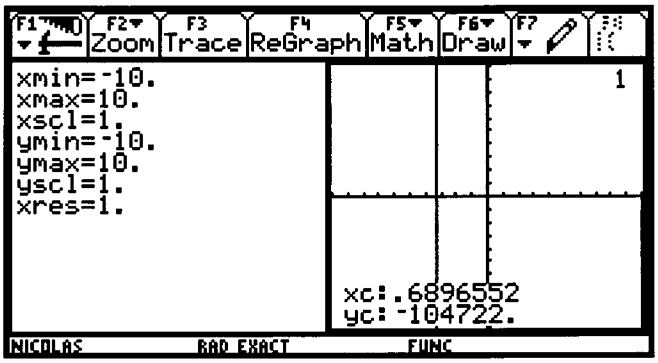

Figure 15.

- the *Trace* command led to the location of points situated outside of the screen and therefore, the student redefined the window to allow the overall graphic representation to be shown;
- secondly, the *Zoom* commands allowed quicker searches (see Figure 15).

D used all explorations possible on this calculator (using the widest possible range for the x variable). In this way, she obtained the required result using only the resources of the calculator, without any reference to theoretical results, and without putting any trace of her work on paper.

A result showing the resourceful profile. Using theorems learnt during lesson, student E was able to assert that $\lim_{+\infty} P = +\infty$. Then he looked for confirmation through a graphic representation of the function. After some concordant tests, he assumed that the graphs invalidated his first result obtained by theory: the function seemed to be strongly decreasing, even for high values of x. E, therefore, tried to solve this contradiction and tried to find a justification for the exceptional status of this function. Observing the expression of P, he noticed that the coefficient of x^3 was extremely large while the coefficient of x^4 was very small. For E, this point justified the exceptional status of this polynomial:

– for standard coefficients, it is the term with the highest degree which counts;
– in this case (a great difference between these coefficients), it is the term x^3 which counts. Therefore $\lim_{+\infty}$ P is $+\infty$. In this case, it is clear that the characterization of this behaviour was the search for coherence when confronted with the various results from different sources, validating the final results.

4.3.4. *The Change in the Different Profiles Throughout the Instrumentation Process*

The observation of students with the profiles defined above allowed the change in student behaviour to be followed in a symbolic calculator environment.

- *rational work method*: the use of symbolic calculators seemed to be mastered whereas these students were not very interested in graphic calculators. They were more attracted by symbolic calculators, probably because these tools incorporate not only approximate calculation and graphs, but also exact and formal calculations. The adaptation to the particular syntax of the machine was easy. Moreover, their behaviour was perceptibly modified with more conjectures and more partial validations, even in assessment situations.
- *resourceful work method*: these students also control their experimental processes of conjectures and validations well, but they had more difficulty in meeting the calculator's requirements, especially with regard to its new syntax. Symbolic calculators seemed to foster a rationalisation of their behaviour.
- *theoretical work method*: the effect was more contrasted for these students who found it difficult to adjust to the new syntax of symbolic calculators, referring to the mathematical syntax. Moreover, faced with the lack of an answer from the calculator, they may challenge it by trying to force the answer. This strategy may be disturbing when it occurs in a situation with time constraints (assessment situations).

However, for these three types of behaviour, the effects seem to be favourable at different levels. The situation is clearly different for the other students (random and mechanical work methods), and there is a gap between two subgroups of students: some students, who had sufficient mathematical background, had undertaken the additional work of adapting to the machine with positive effects. Conversely, some students did not have the mathematical knowledge necessary to overcome the new difficulties arising on the screen (for instance, interpretation of feed-back), and were inefficient in their analysis of the results. Often they gave up any idea of understanding, copying the formula into their notebook without any interpretation, a behaviour already described in Section 4.3.1.

Finally, with regard to the students' relationship to symbolic calculators, it is necessary to stress the change in their behaviour in terms of student's command process. For rational students, this process remained strong; for theoretical students, it increased, whereas the other profiles revealed a reduction of student's command process, which seems to be inversely proportional to the calculator power. Moreover, comparing student behaviour with and without the symbolic calculator, an unquestionable independence with regard to the calculator for rational and theoretical students was observed, i.e., these students could calculate function limits in the two environments. At the other extreme, there was a significant dependence on the part of the other students with regard to symbolic calculators, especially for random students who lost all control over results and could do nothing without their 'crutch'. The arguments which have led to all these summarised assertions can be found in Trouche (1996).

5. DISCUSSION: HOW TO SUPPORT INSTRUMENTAL GENESIS?

Most students appreciated the use of the overhead calculator, which they considered an assistance in understanding calculator manipulations, the verification of calculations and the visualisation of mathematical objects. Students also stressed the advantage of working in small groups for learning calculator manipulations, problem solving methods, and experimental research and teamwork. These opinions came about as a result of the new device set up in classrooms and were not those expressed in the first questionnaires. The awareness of the usefulness of this organisation took a rather long time, especially for the written research report, which at the beginning was seen as a pointless constraint. At the end of the experiment, most of the opinions in the two classrooms were convergent, as the following student statements illustrate:

- Finally, I have understood the mechanisms of research.
- It has modified my point of view on calculators.

- The calculator is not merely a means for calculations, it also pushes us to reflection, even if it does not exempt from proof.

These opinions also reveal an awareness of the potential and limitations of the calculator, especially with regard to the assistance they may offer within an experimental framework, even if students are not yet able to exploit them fully. However, in the two classrooms, some students think they have learnt unnecessary things, and one cannot avoid the problem of the institution's lack of recognition of the new skills acquired in the new environment. Finally, a real change for most students in their relations to mathematics and their own self-confidence was observed.

A similar experiment was carried out by Artigue in two scientific classrooms of 16/17 year-old-students (at an intermediate level between the two experimental classrooms) with special attention paid to the study of functions. For those students who were used to working with graphic calculators, a familiarity with the graphic application was quickly obtained as in the classroom described in Section 4.3.2 (17 to 18 year-old). In her analysis of the instrumentation process (Artigue et al., 1997) she identified a first phase focusing on the graphic application (table values and graphics), then a first use of HOME (the formal application) without coordination with the graphic application, and finally a reorganisation of old practices and a new coordination with the HOME application.

Artigue also stresses the role played by the relationship to the calculator in the instrumentation process beyond the mathematical level: the instrumental genesis is seldom better achieved by students who are highly attached to the machine. This assertion is confirmed by the change in the mechanical profile noted in Section 4.3.4. On the other hand, students who are more unwilling to use the machine can construct a more efficient relationship with the calculator while keeping a certain objectivity with regard to the machine, a result which is convergent with the change in behaviour identified in theoretical and rational type students.

We argue that this stage of the instrumentation process was characterised, as in the rational and resourceful method profiles, by combining all information sources (including the paper/pencil work) and cannot be attained without an explicit intervention at a conscious level (see Figure 10). The threshold of this stage is precisely the moment when the algebraic register takes priority over the graphic register, which then becomes a register for conjectures.

In cognitive psychology, Houdé points out the necessity for subjects to inhibit no relevant procedures in order to perform cognitive tasks. He argues that the deficient rationality often observed in studies about reasoning is due to a shortfall in inhibition of dangerous procedures and proves, in several experiments, that this inhibition can be learnt (Houdé, 1995). We also believe that most students in the present experiment had learnt to question calculator results, even if they tended to doubt a numeric value more readily than a graph, because a question about a graph is not spontaneous and requires significant effort (see Section 1.2).

It is hoped that this change in attitude with regard to mathematics will come about in working with new technologies which can potentially stimulate students' thinking in a conscious mode, if new work methods promote such behaviour. Nevertheless, this experiment has shown that the instrumentation process does not necessarily lead to more mathematical work. On the one hand, it may mask deficiencies in students' mathematical knowledge (a student can calculate the limit without knowing what it means); on the other hand, it may reveal such deficiencies (see Section 2.1 and Section 4.3). Sometimes, for weaker students, the calculator may induce a loss of consciousness and lead to automatic behaviours lacking reflection; sometimes, students drop the idea of understanding the command effects as described in Section 3.1.1. In this case, they do not pass through the first instrumentation level (discovery phase), because it requires specific mathematical knowledge to coordinate various semiotic representations of mathematical objects (for example, the definition of a function and its graph) and their management (e.g., management of approximate and exact calculations).

The behaviour profiles described above indicate great diversity in instrumental genesis, especially among students' abilities to interpret and coordinate calculator results. In any case, the instrumentation process is complex and slow, because it requires sufficient time to achieve a reorganisation of procedures, even for the better students who have established a relationship with the machine. The instrumentation process has a significant influence not only on students' work behaviour and its student's command process, but also on the knowledge constructing process, because of the connections between gesture and thought (see Section 2.2). It is important to stress the advantage of the learning which takes place beyond the mere instrumentation of the calculator by means of connections and reformulations they unavoidably support. No doubt they have an effective impact on the conceptualisation of numbers and functions through the multiple windows they offer for students to construct mathematical meanings to extend their web of ideas (Noss and Hoyles, 1996).

How can teachers foster the acquisition of skills throughout the instrumental genesis and facilitate an efficient relationship with the artefact? Certainly, the teaching organisation plays an essential role in supporting the development of an efficient means of task control with the instrument. How can this organisation help students pass through the discovery phase and prune unsuitable techniques in order to substitute work *with* the artefact for work *on* it? This threshold cannot be attained without a conscious coordination between efficient calculator techniques and paper/pencil work.

As in computer environments (see Section 2.1), the teacher's role has fundamentally changed, but not at all easier. The specific mathematical knowledge required for efficient instrumentation, even within simple tasks, is often underestimated by teachers: they do not acknowledge this fact because the underlying knowledge is implicit in their practice of the instrument. Moreover, the problem of the educational legitimacy of computer technologies raised in Artigue (1998) has an influence on teacher attitudes: they do not consider the instrumented techniques, which are left to students to acquire on their own, contrary to those of the paper/pencil environment. Teachers tend to resist devoting their time to helping students focus on the constraints and efficient techniques necessary to an effective management of the instrument.

Nevertheless, this knowledge, which appears to be specific to the instrumentation process, interweaves with the mathematical knowledge required by the curriculum. The on-going experiment described here also reveals that additional time devoted to emphasising efficient techniques may facilitate access to effective instrumentation and, in this way, this lost time will probably be made up during future activities.

Therefore, we argue for strong teacher involvement in the instrumentation process and full recognition of the constraints and potential of the artefact as well as various profiles of student behaviour so as to design and implement appropriate mathematical activities. Teachers have to juggle all these parameters in order to enhance students' experimental processes of combining information and understanding tools. How should teachers organise their teaching in order to turn symbolic calculators into efficient mathematical instruments? We would like to make the following suggestions:

Firstly, from an institutional point of view, include new organisation class which allow students' manipulation on the calculator to be made visible (sherpa student) and also to give them more time for research (practical sessions).

Secondly, from a technological point of view:

- introduce only a limited number of new commands in each activity, taking care not to obscure mathematical work by manipulation difficulties and limit the use of avoidance strategies (see Section 4.3.1);

- devote sufficient time learning to verify various representatives (images and numbers), emphasising their differentiation and coordination and the language relative to them;
- alternate questions in the two environments with the objective of avoiding overdependence on the machine and thus improving rather than reducing mathematical work,
- point out, with the overhead calculator, efficient strategies and instrumented techniques making the TI-92 a real mathematical instrument, i.e., provide institutionalisation of efficient instrumented techniques,
- draw attention to building connections with the national mathematics curriculum within the institutionalisation phase to suggest appropriate connections in webs of ideas.

Finally, from a psychological point of view, respect student profiles to adapt activities towards successful integration. Specific assistance must then be organised for weaker students in order to avoid work in an unconscious mode and foster instead true experimental work.

This paper has tried to underline, through the negative aspects pointed out in Section 1, how an unaccompanied acquisition of the use of calculators may be dangerous for the conceptualisation process. Teachers should consider the instrumentation process in order to articulate new techniques with older practices in the paper/pencil environment, because this reorganisation of instrumented techniques is far from spontaneous and requires spending sufficient time to reach the experimental processes. Most of these ideas have been discussed at the European Conference on symbolic and geometric calculators (Guin, 1999, to be published).

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