*Multibody System Dynamics* **1:** 3–21, 1997.



# A Geometric Unification of Constrained System Dynamics

WOJCIECH BLAJER
*Department of Mechanics, Technical University of Radom, ul. Malczewskiego 29, PL-26-600 Radom, Poland*

(Received: 2 September 1996; accepted in revised form: 8 January 1997)

**Abstract.** A unified geometric formulation of the methods used for solving constrained system problems is given. Both holonomic and nonholonomic systems are treated in like manner, and the dynamic equations are expressible in either generalized velocities or quasi-velocities. Moreover, a wide range of *unconstrained* systems are uniformly regarded as generalized particles in the multi-dimensional metric spaces relating to their configuration. The derivation is grounded on the tensor calculus formalism and appropriate geometric interpretations are reported. In its useful matrix form, the formulation turns out short, elementary and general. This unified geometric approach to constrained system dynamics may deserve to become a generally accepted method in academic and engineering applications.

**Key words:** analytical dynamics, nonholonomic systems, differential geometry.

## 1. Introduction

During the last two decades a renewed interest has been observed in the theory of constrained mechanical systems, stemmed from the rapid and simultaneous development of many technological disciplines (robotics, spacecraft, machine and vehicle dynamics, biomechanics, etc.), and stimulated by advances in computational techniques. These two factors – the needs for analysing of more and more complex machineries and the powerful investigation tool offered by numerical methods – resulted in a new "philosophy" of formulating and solving problems of analytical mechanics. Many historical approaches consequent to Lagrangian mechanics turned out to be laborious for large-scale systems and difficult to algorithmize in computer codes. This was mainly due to the arduousness of generating and differentiating the kinetic energy functions and the entanglements arising when nonco-ordinate velocity components (quasi-velocities) are involved. Sheer impenetrability of many of the classical methods, their usual derivation for systems of particles, and their strict assignment for either holonomic (H) or nonholonomic (NH) systems might have also led to some confusion. These reasons have caused intensive investigations aimed at more general, effective and suited for digital implementation approaches for handling constrained motion.

Modern methods for the dynamic analysis of constrained multibody systems fall into two main categories: the differential-algebraic equation (DAE) and ordi-

nary differential equation (ODE) formulations. The former employ a maximal set of variables to describe the constrained motion – the co-ordinates and velocities of the *unconstrained* system and the multipliers that model the constraining forces. The motion equations that follow are DAEs, often referred to as Lagrange's equations of the first kind. Many researches have preferred the DAE formulation due to its simplicity and ease of manipulation [1–4]. The approach is commonly recognized as computationally inefficient, however, and special procedures must be applied to avoid the constraint violation phenomenon. Another approach is thus, prior to numerical integration, to reduce the motion equations to a smaller (possibly minimal) set of ODEs. For open-loop (tree structure) systems a popular method of this type is the *joint co-ordinate* method; see e.g. [2, 4, 5] (for systems with closed loops, the method leads to reduced-dimension DAEs). A variety of other codes, applicable for systems of any structure, exploit the concept of an orthogonal complement matrix to the constraint Jacobian matrix, introduced in [6]. Premultiplying the constraint reaction-induced dynamic equations by the orthogonal complement results in the governing equations as ODEs, which can then be further reduced in dimension by introducing a minimal set of (independent) velocity variables. The *co-ordinate partitioning* (*LU factorization*) [3, 7], the *natural/point co-ordinates* [8, 9], the *zero eigenvalues theorem* [10] or *singular value decomposition* [11, 12], the *QR* or *Householder decomposition* [13, 14], and the *Gramm–Schmidt orthogonalization* [15, 16] are only a few examples of the computer-oriented techniques using the concepts more or less directly. Useful applications of classical Gibbs–Appell [17, 18] and Maggi's [19] equations as well as (pretended to be new) Kane's method [17] have also been demonstrated. A good review of many of the mentioned methods is provided in [20].

The prevailing direct/matrix notation used in constrained (multibody) dynamics has done some damage to our understanding of the *physics* of concerned problems, occasionally leading to inconsistencies in the mathematical modelling. Reconsidering the problems from the geometric point of view may be stimulative in amending the omissions and providing a uniform approach to constrained systems. Most of the above cited references as well as [21–23] appeal to the *geometry* of constrained motion as a direct generalization of the projection methods used in particle dynamics. However, there are only few attempts, e.g. [24–27], that set the multibody dynamics formulation precisely according to the differential geometry formalism – a powerful tool of analysis. This paper is another contribution in this field.

Important advantages of the proposed formulation are its compactness, versatility in applications, and the geometrical insight that it gives. A wide range of *uconstrained* systems can be studied, including the systems composed of unbounded particles and rigid bodies, and Lagrangian (internally constrained multi-rigid-body) systems whose equations of motion in terms of independent state variables are known. The constraints on the systems can then be either H or NH, and no attention can be paid on whether the velocity components are generalized velocities or quasi-velocities. By using the unified geometric approach, the schemes

for obtaining several types of constraint reaction-free equations of motion and the determination of associated constraint reactions are reconsidered. The other important contributions are a precise definition of *virtual* (constraint-admissible) speeds, and a novel approach to constraint violation stabilization.

The main body of the paper has been written in standard matrix notation which is the current fashion of a large segment of the multibody dynamics "community". However, most of the fundamental formulae are simultaneously written in indicial notation of tensor calculus, important for understanding the covariance/contravariance aspects and the physical (geometrical) meaning of matrix transformations. The present formulation is thus placed somewhere between these two poles apart (but virtually equivalent) formalisms, combining compactness of the former and the mathematical pertinence of the latter. For the convenience of the reader, the pivotal principles of the notation used are summarized in the Appendix.

## 2. Unconstrained System Dynamics

The methods of analytical mechanics are usually introduced for systems composed of a finite number of material points under the presence of ideal constraints. Such a procedure excludes an important segment of finite-dimensional systems with an infinite number of particles, like rigid bodies and multi-rigid-body systems with kinematic tree structure, unless some limiting procedures are carried out. From the point of view of differential geometry, however, there are no obstacles to regarding all these systems as primitive concepts like mass points and treating them as such.

In order to meet the broader sense of *unconstrained* system, consider an $n$-degree-of-freedom autonomous system characterized by $n$ generalized co-ordinates $\mathbf{p} = [p^1 \dots p^n]^T$. The governing equations of the system can be written in the following general matrix form:

$$\dot{\mathbf{p}} = \mathbf{A}(\mathbf{p})\mathbf{v} \tag{1}$$

$$\mathbf{M}(\mathbf{p})\dot{\mathbf{v}} = \mathbf{h}^*(\mathbf{p}, \mathbf{v}, t), \tag{2}$$

where $\mathbf{A}$ is an $n \times n$ invertible matrix of transformation between the $n$ velocity components $\mathbf{v} = [\nu^1 \dots \nu^n]^T$ (possibly quasi-velocities) and the generalized velocities $\dot{\mathbf{p}}$, $\mathbf{M}$ is the $n \times n$ symmetric positive definite mass matrix, $\mathbf{h}^* = [h_1 \dots h_n]^T$ represents the sum of applied forces, gyroscopic terms, centrifugal forces and Coriolis effects, $t$ is the time, and the superscript $(^*)$ is introduced after [28] to distinguish between the associated vector representations and reciprocal bases in matrix notation (see Appendix for details). The said unconstrained system can be either:

- A collection of unconstrained particles. In this case $\dot{\mathbf{p}} = \mathbf{v}, \mathbf{M}$ is a constant diagonal matrix, and $\mathbf{h}^*$ is the representation of only the applied forces.
- A collection of unconstrained rigid bodies. Equation (1) relates the time derivatives of translational and rotational co-ordinates of particular bodies and the

components of their linear and angular velocity (the latter, when expressed in a body-fixed reference frame, are quasi-velocities). Typically, (2) is an aggregation of Newton–Euler dynamic equations for individual bodies, and $\mathbf{M}$ is constant. $\mathbf{A}$ and $\mathbf{M}$ are block-diagonal matrices.

- A *Lagrangian system* – an internally constrained autonomous holonomic system whose equations of motion $\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{h}^*(\mathbf{q}, \dot{\mathbf{q}}, t)$ have been derived in independent co-ordinates $\mathbf{q}$ ($\mathbf{q} \equiv \mathbf{p}$, $\dot{\mathbf{q}} \equiv \mathbf{v}$). Equations of this type arise among others in the first step modelling of closed-loop systems, after cutting the closed loops and applying the joint co-ordinate method to the produced open-loop system. The constraints on the systems are then the closing conditions, see e.g. [2, 4, 5, 7, 20].
- A combination of systems as above.

The motion of a system described in (1) and (2) can be treated as the motion of point $\mathbf{p} = [p^1 \dots p^n]^T$ representing the system in its configuration space. The kinetic energy

$$T = \frac{1}{2}\,\mathbf{v}^T\mathbf{M}\mathbf{v} = \frac{1}{2}\,\dot{\mathbf{p}}^T\mathbf{A}^{-T}\mathbf{M}\mathbf{A}^{-1}\dot{\mathbf{p}} \quad \text{or}$$
$$T = \frac{1}{2}\,M_{ij}\nu^i\nu^j = \frac{1}{2}\,B^i_k B^j_l M_{ij}\dot{p}^k\dot{p}^l, \tag{3}$$

where $\mathbf{B} = \mathbf{A}^{-1}$ is a positive definite quadratic form on the tangent space to the configuration space at point $\mathbf{p}$. So endowed with the metric at each point $\mathbf{p}$, the configuration space becomes a Riemannian manifold $\mathsf{N}$. The space $\mathsf{E}^n$ tangent to $\mathsf{N}$ at $\mathbf{p}$ is a local $n$-dimensional Euclidean (linear vector) space [25]. The mass matrix $\mathbf{M}$ is then a metric tensor matrix of a covariant basis $\mathbf{e}_v = [\vec{\boldsymbol{e}}_{v1} \dots \vec{\boldsymbol{e}}_{vn}]^T$ of $\mathsf{E}^n$, $\mathbf{M} = \mathbf{e}_v\mathbf{e}_v^T$ ($M_{ij} = \vec{\boldsymbol{e}}_{vi} \circ \vec{\boldsymbol{e}}_{vi}$), in which the velocity (contravariant) components of $\vec{\boldsymbol{\nu}}$ are expressed, $\vec{\boldsymbol{\nu}} = \mathbf{e}_v^T\mathbf{v}$ ($\vec{\boldsymbol{\nu}} = \nu^i\vec{\boldsymbol{e}}_{vi}$). The vector $\vec{\boldsymbol{\nu}}$ has thus a unified meaning of the system velocity and momentum. Namely, by introducing a reciprocal basis to $\mathbf{e}_v$, the contravariant basis $\mathbf{e}_v^* = [\vec{\boldsymbol{e}}_v^{\,1} \dots \vec{\boldsymbol{e}}_v^{\,n}]^T = \mathbf{M}^{-1}\mathbf{e}_v$ ($\vec{\boldsymbol{e}}_v^{\,i} = G^{ij}\vec{\boldsymbol{e}}_{vj}$), where $\mathbf{G} = \mathbf{M}^{-1}$, $\vec{\boldsymbol{\nu}}$ can be variantly written as

$$\vec{\boldsymbol{\nu}} = \mathbf{e}_v^T\mathbf{v} = \mathbf{e}_v^{*T}\mathbf{v}^* \quad \text{or} \quad \vec{\boldsymbol{\nu}} = \nu^i\vec{\boldsymbol{e}}_{vi} = \nu_j\vec{\boldsymbol{e}}_v^{\,j}, \tag{4}$$

where $\mathbf{v}^* = [\nu_1 \dots \nu_n]^T = \mathbf{M}\mathbf{v}$ ($\nu_i = M_{ij}\nu^j$) is the momentum (covariant) representation of $\vec{\boldsymbol{\nu}}$ (see also Appendix). To realize the dual meaning of vectors in metric spaces is of paramount importance for understanding the further analysis.

Following the interpretation, the vector $\dot{\vec{\boldsymbol{\nu}}} = \mathbf{e}_v^T\dot{\mathbf{v}} = \mathbf{e}_v^{*T}\dot{\mathbf{v}}^*$ ($\dot{\vec{\boldsymbol{\nu}}} = \dot{\nu}^i\vec{\boldsymbol{e}}_{vi} = \dot{\nu}_j\vec{\boldsymbol{e}}_v^{\,j}$) represents both the system acceleration components $\dot{\mathbf{v}}$ in $\mathbf{e}_v$ and the system *effective* force components $\dot{\mathbf{v}}^*$ in $\mathbf{e}_v^*$, $\dot{\mathbf{v}}^* = \mathbf{M}\dot{\mathbf{v}}$ ($\dot{\nu}_i = M_{ij}\dot{\nu}^j$). The dynamic equation (2), which in indicial notation reads as $M_{ij}\dot{\nu}^j = h_i$, is then the representation of the following vector formula

$$\vec{\boldsymbol{h}} - \dot{\vec{\boldsymbol{\nu}}} = \mathbf{e}_v^{*T}(\mathbf{h}^* - \mathbf{M}\dot{\mathbf{v}}) = 0 \quad \text{or} \quad \vec{\boldsymbol{h}} - \dot{\vec{\boldsymbol{\nu}}} = (h_i - M_{ij}\dot{\nu}^j)\vec{\boldsymbol{e}}_v^{\,i} = 0 \tag{5}$$

which directly appeals to D'Alembert's form of Newton's formula $\vec{\boldsymbol{F}} - m\vec{\boldsymbol{a}} = 0$. The use of base vectors can also be seen as a substitute for the virtual displacements formalism.

The kinematic equation (1) (in indicial notation $\dot{p}^i = A^i_j \nu^j$) denotes an interdependence between the system velocity components expressed in $\mathbf{e}_v$ and $\mathbf{e}_p = [\vec{\boldsymbol{e}}_{p1} \dots \vec{\boldsymbol{e}}_{pn}]^T$ bases of $\mathsf{E}^n$. The transformation formula between these two bases is $\mathbf{e}_v = \mathbf{A}^T \mathbf{e}_p (\vec{\boldsymbol{e}}_{vj} = A^i_j \vec{\boldsymbol{e}}_{pi})$, and we can write $\dot{\vec{\boldsymbol{p}}} = \mathbf{e}_p^T \dot{\mathbf{p}} = \mathbf{e}_p^T \mathbf{A} \mathbf{v} = \mathbf{e}_v^T \mathbf{v} = \vec{\boldsymbol{\nu}}$. The equivalence $\dot{\vec{\boldsymbol{p}}} = \vec{\boldsymbol{\nu}}$ expressed in $\mathbf{e}_p$ leads to (1), while represented in $\mathbf{e}_v$ is $\mathbf{v} = \mathbf{B}\dot{\mathbf{p}}$ ($\nu^i = B^i_j \dot{p}^j$), where $\mathbf{B} = \mathbf{A}^{-1}$.

## 3. Constraints on the System

Let the system as introduced in Section 2 be subjected to $m$ independent ideal constraints, $m_{\mathrm{H}}$ H (position) and $m_{\mathrm{NH}}$ NH (nonintegrable velocity) ones, $m_{\mathrm{H}} + m_{\mathrm{NH}} = m$. Assuming the NH constraints are linear in velocities, the constraint equations are:

$$\boldsymbol{\Phi}_{\mathrm{H}}(\mathbf{p}, t) = 0 \tag{6a}$$

$$\boldsymbol{\Psi}_{\mathrm{NH}}(\mathbf{p}, \mathbf{v}, t) \equiv \mathbf{C}_{\mathrm{NH}}(\mathbf{p}, t)\mathbf{v} - \boldsymbol{\eta}_{\mathrm{NH}}(\mathbf{p}, t) = 0, \tag{6b}$$

where $\boldsymbol{\Phi}_{\mathrm{H}} = [\Phi^1_{\mathrm{H}} \dots \Phi^{m_{\mathrm{H}}}_{\mathrm{H}}]^T$, $\boldsymbol{\Psi}_{\mathrm{NH}} = [\Psi^1_{\mathrm{NH}} \dots \Psi^{m_{\mathrm{NH}}}_{\mathrm{NH}}]^T$, $\mathbf{C}_{\mathrm{NH}}$ is the $m_{\mathrm{NH}} \times n$ NH constraint matrix, and $\boldsymbol{\eta}_{\mathrm{NH}} = [\eta^1_{\mathrm{NH}} \dots \eta^{m_{\mathrm{NH}}}_{\mathrm{NH}}]^T$ vanishes for scleronomic constraints. By differentiating with respect to time the H constraints, (6) can first be unified to the velocity form $\boldsymbol{\Psi} = [\dot{\boldsymbol{\Phi}}^T_{\mathrm{H}} \quad \boldsymbol{\Psi}^T_{\mathrm{NH}}]^T$, and then, by one more differentiation of the obtained velocity constraints can be transformed to the acceleration form $\dot{\boldsymbol{\Psi}} = [\ddot{\boldsymbol{\Phi}}^T_{\mathrm{H}} \ \ \dot{\boldsymbol{\Psi}}^T_{\mathrm{NH}}]^T$, i.e.:

$$\boldsymbol{\Psi} \equiv \mathbf{C}(\mathbf{p}, t)\mathbf{v} - \boldsymbol{\eta}(\mathbf{p}, t) \equiv \begin{bmatrix} (\partial \boldsymbol{\Phi}_{\mathrm{H}}/\partial \mathbf{p})\mathbf{A} \\ \mathbf{C}_{\mathrm{NH}} \end{bmatrix} \mathbf{v} - \begin{bmatrix} -\partial \boldsymbol{\Phi}_{\mathrm{H}}/\partial t \\ \boldsymbol{\eta}_{\mathrm{NH}} \end{bmatrix} = 0 \tag{7}$$

$$\dot{\boldsymbol{\Psi}} \equiv \mathbf{C}(\mathbf{p}, t)\dot{\mathbf{v}} - \boldsymbol{\xi}(\mathbf{p}, \mathbf{v}, t) = 0, \tag{8}$$

where $\boldsymbol{\xi} \equiv -\dot{\mathbf{C}}\mathbf{v} + \dot{\boldsymbol{\eta}}$. When handling the constraint equations in the acceleration form (8), the initial values $\mathbf{p}_0$ and $\mathbf{v}_0$ must satisfy the lower-order constraint conditions, $\boldsymbol{\Phi}_{\mathrm{H}}(\mathbf{p}_0, t_0) = 0$ and $\boldsymbol{\Psi}(\mathbf{p}_0, \mathbf{v}_0, t_0) = 0$.

Assumed $m_{\mathrm{H}} > 0$, the configuration space of the constrained system, denoted $\mathsf{K}_{\mathrm{H}}$ ($k_{\mathrm{H}} = n - m_{\mathrm{H}}$), is restricted to the H constraint manifold (6a) embedded in $\mathsf{N}$ (if there are no H constraints on the system, $m_{\mathrm{H}} = 0$ and $\mathsf{K}_{\mathrm{H}} = \mathsf{N}$). The first $m_{\mathrm{H}}$ constraint vectors, represented in $\mathbf{C}$ as rows, are then the H constraint gradients, and as such are orthogonal to $\mathsf{K}_{\mathrm{H}}$. The other constraint vectors, corresponding to NH constraints, may have arbitrary directions with respect to $\mathsf{K}_{\mathrm{H}}$. Rewriting (7) as

$\Psi^j = C_i^j \nu^j = 0$ for $j = 1, \ldots, m$ and $i = 1, \ldots, n$, it can be deduced that the $j$th constraint vector is $\vec{\boldsymbol{c}}^{\,j} = C_i^j \mathbf{e}_v^i = C_i^j G^{ik} \vec{\boldsymbol{e}}_{vk}$, which in matrix notation reads

$$\mathbf{e}_c^* = [\vec{\boldsymbol{c}} \ldots \vec{\boldsymbol{c}}^{\,m}]^T = \mathbf{C}\mathbf{e}_v^* = \mathbf{C}\mathbf{M}^{-1}\mathbf{e}_v. \tag{9}$$

If $\mathrm{rank}(\mathbf{C}) = m = \max$, the constraint vectors $\vec{\boldsymbol{c}}^{\,j}$ $(j = 1, \ldots, m)$, are independent in $\mathsf{E}^n$, and as such define an $m$-dimensional subspace of $\mathsf{E}^n$, called the *constrained* subspace $\mathsf{C}^m$. The metric tensor matrix of (contravariant) basis $\mathbf{e}_c^*$ of $\mathsf{C}^m$ is $\mathbf{M}_c^{-1} = \mathbf{e}^*\mathbf{e}_c^{*T} = \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T$ ($G_c^{ij} = \vec{\boldsymbol{c}}^{\,i} \circ \vec{\boldsymbol{c}}^{\,j} = C_k^i \vec{\boldsymbol{e}}_v^{\,k} \circ \vec{\boldsymbol{e}}_v^{\,l} C_l^j = C_k^i G^{kl} C_l^j$), where $\mathbf{G}_c = \mathbf{M}_c^{-1}$ and $\mathbf{G} = \mathbf{M}^{-1}$.

For ideal constraints, the reactions of individual constraints are collinear to the corresponding constraint vectors, $\vec{\boldsymbol{r}}_J = \lambda_J \vec{\boldsymbol{c}}^{\,J}$ (do not sum for $J$). The generalized reaction force on the system (the total of the individual constraint reactions) can then be written as

$$\vec{\boldsymbol{r}} = \sum_{J=1}^{m} \vec{\boldsymbol{r}}_J = \sum_{J=1}^{m} \lambda_J \vec{\boldsymbol{c}}^{\,J} = \mathbf{e}_c^{*T} \boldsymbol{\lambda}^* = \mathbf{e}_v^{*T} (\mathbf{C}^T \boldsymbol{\lambda}^*) = \mathbf{e}_v^{*T} \mathbf{r}^*, \tag{10}$$

where $\mathbf{r}^* = [r_1 \ldots r_n]^T = \mathbf{C}^T \boldsymbol{\lambda}^*$ and $\boldsymbol{\lambda}^* = [\lambda_1 \ldots \lambda_m]^T$ (Lagrange multipliers) are the representations of $\vec{\boldsymbol{r}}$ in the bases $\mathbf{e}_v^*$ and $\mathbf{e}_c^*$, respectively. Note that $\vec{\boldsymbol{r}}$ is totally sunk in $\mathsf{C}^m$. The governing equations of the constrained system can then be written in the following form:

$$\dot{\mathbf{p}} = \mathbf{A}(\mathbf{p})\mathbf{v} \quad \text{or} \quad \dot{p}^i = a_j^i \nu^j \tag{11a}$$

$$\mathbf{M}(\mathbf{p})\dot{\mathbf{v}} = \mathbf{h}^*(\mathbf{p}, \mathbf{v}, t) + \mathbf{C}^T(\mathbf{p}, t)\boldsymbol{\lambda}^* \quad \text{or} \quad M_{ij}\dot{\nu}^j = h_i + C_i^k \lambda_k \tag{11b}$$

$$\mathbf{C}(\mathbf{p}, t)\dot{\mathbf{v}} = \boldsymbol{\xi}(\mathbf{p}, \mathbf{v}, t) \quad \text{or} \quad C_i^k \dot{\nu}^i = \xi^k \tag{11c}$$

which are often referred to as Lagrange's equations of the first kind [2, 4, 25]. With reference to (5), the geometric interpretation of (11b) is

$$\vec{\boldsymbol{h}} + \vec{\boldsymbol{r}} - \dot{\vec{\boldsymbol{\nu}}} = \mathbf{e}_v^{*T}(\mathbf{h}^* + \mathbf{C}^T\boldsymbol{\lambda}^* - \mathbf{M}\dot{\mathbf{v}}) = 0 \tag{12}$$

or in indicial notation $\vec{\boldsymbol{h}} + \vec{\boldsymbol{r}} - \dot{\vec{\boldsymbol{\nu}}} = (h_i + C_i^k \lambda_k - M_{ij}\dot{\nu}^j)\vec{\boldsymbol{e}}_v^{\,i} = 0$, which directly appeals to the geometry of simple dynamics problems (see Figure 1b). Following the geometric interpretation, the constraint velocity equation (7) can be viewed as $\boldsymbol{\eta} = \mathbf{e}_c^{*T}\vec{\boldsymbol{\nu}} = \mathbf{C}\mathbf{e}_v^*\mathbf{e}_v^T\mathbf{v} = \mathbf{C}\mathbf{v}$ ($\eta^i = \vec{\boldsymbol{c}}^{\,i} \circ \vec{\boldsymbol{\nu}} = C_J^i \vec{\boldsymbol{e}}_v^{\,j} \circ \vec{\boldsymbol{e}}_{vk}\nu^k = C_j^i \nu^j$). The term $\boldsymbol{\eta}$ is thus the projection of the system velocity $\vec{\boldsymbol{\nu}}$ into $\mathsf{C}^m$, expressed in the basis $\mathbf{e}_c$, $\vec{\boldsymbol{\eta}} = \mathbf{e}_c^T\boldsymbol{\eta}$ ($\vec{\boldsymbol{\eta}} = \eta^i \vec{\boldsymbol{c}}_i$). As such, $\boldsymbol{\eta}$ represents only that part of "constraint velocity" at $\mathbf{p}$ which is contained in $\mathsf{C}^m$, and the eventual tangent component of the constraint velocity is not "seen" in the explicit constraint equations (7) (see Figure 1a). Seemingly, the constituents of $\boldsymbol{\xi} \equiv -\dot{\mathbf{C}}\mathbf{v} + \dot{\boldsymbol{\eta}}$ in (8) and (11c) denote: $-\dot{\mathbf{C}}\mathbf{v}$ – the acceleration due to the constraint "curvature" at $\mathbf{p}$, and $\dot{\boldsymbol{\eta}}$ – the acceleration due to the constraint motion at $\mathbf{p}$ (again only those parts which are contained in $\mathsf{C}^m$).

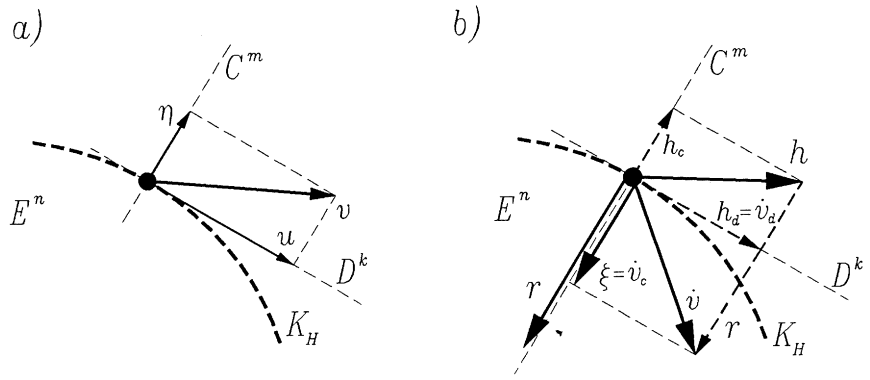



*Figure 1*. The geometry of a constrained system.

Namely, one can write $\boldsymbol{\xi} = \mathbf{v}_c^{*T}\dot{\vec{\boldsymbol{\nu}}} = \mathbf{C}\mathbf{e}_v^{*}\mathbf{e}_v^{T}\dot{\mathbf{v}} = \mathbf{C}\dot{\mathbf{v}}$ ($\xi^i = \vec{\boldsymbol{c}}^{\,i} \circ \dot{\vec{\boldsymbol{\nu}}} = C^i_j \vec{\boldsymbol{e}}^{\,j}_v \circ \vec{\boldsymbol{e}}_{vk}\dot{\nu}^k = C^i_j \delta^j_k \dot{\nu}^k = C^i_j \dot{\nu}^j$) and $\vec{\boldsymbol{\xi}} = \mathbf{e}_c^T \xi$ ($\vec{\boldsymbol{\xi}} = \xi^i \vec{\boldsymbol{c}}_i$). Note that $\boldsymbol{\eta}$ and $\xi$ are representations of $\vec{\boldsymbol{\eta}}$ and $\vec{\boldsymbol{\xi}}$ in the basis $\mathbf{e}_c$, reciprocal to $\mathbf{e}_c^*$ in which the constraint reactions are represented by $\boldsymbol{\lambda}^*$.

The DAE formulation (11) is valid for both H and NH systems, and the components of $\mathbf{v}$ can be either generalized velocities, quasi-velocities, or both. The equations can be solved directly by using a range of DAE solvers [29], or the Lagrange multiplier elimination methods can be used. A popular technique of latter type is to rewrite (11) as

$$\dot{\mathbf{p}} = \mathbf{A}\mathbf{v} \tag{13a}$$

$$\begin{bmatrix} \mathbf{M} & -\mathbf{C}^T \\ \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \boldsymbol{\lambda}^* \end{bmatrix} = \begin{bmatrix} \mathbf{h}^* \\ \boldsymbol{\xi} \end{bmatrix} \tag{13b}$$

and, since the coefficient matrix in (13b) is by assumption invertible, solve the equations as implicit ODEs in $\mathbf{v}(t)$ and $\mathbf{p}(t)$, and simultaneously determine $\lambda^*(t)$ in terms of the current state values $\mathbf{v}$ and $\mathbf{p}$.

The analysis of the constrained motion evolution by using the above governing equations is commonly evaluated as computationally inefficient, mainly due to possible large dimension of the sets of equations to handle. Moreover, when handling the constraint equations in the acceleration form, the solution may suffer from the problem of violation of the lower-order constraint equations, and special procedures (e.g. Baumgarte's method [30]) must be applied to avoid the phenomenon. The shortcomings of the dependent variable formulations have stimulated the development of methods for obtaining equations of motion in terms of a minimum number of (independent) variables. The first step of the approach is usually the projection of the dynamic equations into the *null* space of $\mathsf{C}^m$ (the space complemetary to $\mathsf{C}^m$ in $\mathsf{E}^n$) [3, 6, 9–16, 21–23, 31].

## 4. The Projection Method

As $\mathsf{C}^m$ is an $m$-subspace of the $n$-space $\mathsf{E}^n$, a $k$-dimensional ($k = n - m$) subspace $\mathsf{D}^k$ that complements $\mathsf{C}^m$ in $\mathsf{E}^n$ can be defined, $\mathsf{D}^k \cup \mathsf{C}^m = \mathsf{E}^n$ and $\mathsf{D}^n \cap \mathsf{C}^m = 0$. Note that $\mathsf{D}^k$ is the tangent space to the configuration manifold $\mathsf{K}_{\rm H}$ at $\mathbf{p} \in \mathsf{K}^H$ only for a H system ($m_{\rm NH} = 0$) – the tangent space defined as a complement of the space spanned by H constraint gradients [25]. For a NH system such that $0 < m_{\rm NH} < m$ (a system subject to both H and NH constraints), the tangent space is of dimension $k_{\rm H} = n - m_{\rm H} > k$. Finally, in the extreme case of a NH system (a system subject to only NH constraints, $m_{\rm NH} = m$), we have $\mathsf{K}_{\rm H} = \mathsf{N}$, and the tangent space is $\mathsf{E}^n$. To meet the general case of a system subjected to H and/or NH constraints, $\mathsf{D}^k$ will be called a *virtual* (velocity admissible) subspace, as it is formed by a set of unconstrained (admissible by constraints) velocity directions.

By introducing $k$ independent vectors $\vec{\boldsymbol{d}}_1, \ldots, \vec{\boldsymbol{d}}_k$ that are totally sunk in $\mathsf{D}^k$, represented in $\mathbf{e}_v$ basis as columns of an $n \times n$ matrix $\mathbf{D}(\mathbf{p}, t)$ of maximal column-rank, the (contravariant) basis of $\mathsf{D}^k$ can be defined as

$$\mathbf{e}_d = [\vec{\boldsymbol{d}}_1, \ldots \vec{\boldsymbol{d}}_k]^T = \mathbf{D}^T \mathbf{e}_{\mathbf{V}} \tag{14}$$

($\vec{\boldsymbol{d}}_i = D_i^j \vec{\boldsymbol{e}}_{vj}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$). The condition $\mathsf{D}^k \cap \mathsf{C}^m = 0$, in matrix notation, is

$$\mathbf{D}^T \mathbf{C}^T = 0 \quad \Leftrightarrow \quad \mathbf{C}\mathbf{D} = 0, \tag{15}$$

i.e. $\mathbf{D}$ is an orthogonal complement matrix to the constraint matrix $\mathbf{C}$, and (15) denotes the mutual othogonality of vectors $\vec{\boldsymbol{d}}_i$ and $\vec{\boldsymbol{c}}^{\,j}$ ($i = 1, \ldots, k$, $j = 1, \ldots m$). Using (9) and (14), this can be justified by $\vec{\boldsymbol{d}}_i \circ \vec{\boldsymbol{c}}^{\,j} = D_i^k \vec{\boldsymbol{e}}_{vk} \circ \vec{\boldsymbol{e}}_v^{\,l} C_l^j = D_i^k \delta_k^l C_l^j = D_i^k C_k^j = 0$, or in matrix form $\mathbf{e}_d \mathbf{e}_c^{*T} = \mathbf{D}^T \mathbf{e}_v \mathbf{e}_v^{*T} \mathbf{C}^T = \mathbf{D}^T \mathbf{C}^T = 0$. As it is well known, the formulation of $\mathbf{D}$ for a given $\mathbf{C}$ is not unique – different sets of $k$ independent vectors $\vec{\boldsymbol{d}}_i$ can span the same subspace $\mathsf{D}^k$.

Being independent, the $n$ vectors $\vec{\boldsymbol{d}}_1, \ldots, \vec{\boldsymbol{d}}_k, \vec{\boldsymbol{c}}^{\,1}, \ldots, \vec{\boldsymbol{c}}^{\,m}$ form a basis in $\mathsf{E}^n$. Using (9) and (14), the transformation formula between the hybrid (covariant-contravariant) basis $[\mathbf{e}_d^T \;\; \mathbf{e}_c^{*T}]^T$ and the basis $\mathbf{e}_v$ can be formulated as

$$\begin{bmatrix} \mathbf{e}_d \\ \mathbf{e}_c^* \end{bmatrix} = \begin{bmatrix} \mathbf{D}^T \\ \mathbf{C}\mathbf{M}^{-1} \end{bmatrix} \mathbf{e}_v = \mathbf{H}^T(\mathbf{p}, t) \mathbf{e}_v, \tag{16}$$

where $\mathbf{H}$ is the $n \times n$ matrix of transformation. The metric tensor matrix of $[\mathbf{e}_d^T \;\; \mathbf{e}_c^{*T}]^T$ basis is

$$\mathbf{H}^T \mathbf{M} \mathbf{H} = \begin{bmatrix} \mathbf{D}^T \mathbf{M} \mathbf{D} & 0 \\ 0 & \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T \end{bmatrix} = \begin{bmatrix} \mathbf{M}_d & 0 \\ 0 & \mathbf{M}_c^{-1} \end{bmatrix}, \tag{17}$$

where $\mathbf{M}_d(\mathbf{p}, t) = \mathbf{e}_d \mathbf{e}_d^T = \mathbf{D}^T \mathbf{M} \mathbf{D}$ ($M_{dij} = \vec{\boldsymbol{d}}_i \circ \vec{\boldsymbol{d}}_j = D_i^k \vec{\boldsymbol{e}}_{vk} \circ \vec{\boldsymbol{e}}_{vl} D_j^l = D_i^k M_{kl} D_j^l$) is the metric tensor matrix of the basis $\mathbf{e}_d$ of $\mathsf{D}^k$, and $\mathbf{G}_c(\mathbf{p}, t) = \mathbf{M}_c^{-1} = \mathbf{e}_c^* \mathbf{e}_c^{*T} =$

$\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T$ ($G_c^{ij} = \vec{\boldsymbol{c}}^{\,i} \circ \vec{\boldsymbol{c}}^{\,j} = C_k^i \vec{\boldsymbol{e}}_v^{\,k} \circ \vec{\boldsymbol{e}}_v^{\,l} C_l^j = C_k^i G^{kl} C_l^j$) is the metric tensor matrix of $\mathbf{e}_c^*$ of $\mathsf{C}^m$. From (19) it comes also (see Appendix) that

$$\mathbf{e}_v^* = \mathbf{H} \begin{bmatrix} \mathbf{e}_d^* \\ \mathbf{e}_c \end{bmatrix} = \mathbf{D}\mathbf{e}_d^* + \mathbf{M}^{-1}\mathbf{C}^T\mathbf{e}_c \tag{18}$$

($\vec{\boldsymbol{e}}_v^{\,i} = D_j^i \vec{\boldsymbol{d}}^{\,j} + G^{ik} C_k^l \vec{\boldsymbol{c}}_l$), where $\mathbf{e}_d^* = \mathbf{M}_d^{-1}\mathbf{e}_d$ ($\vec{\boldsymbol{d}}^{\,i} = G_d^{ij} \vec{\boldsymbol{d}}_j$), $\mathbf{e}_c = \mathbf{M}_c \mathbf{e}_c^*$ ($\vec{\boldsymbol{c}}_i = M_{cij} \vec{\boldsymbol{c}}^{\,l}$).

Substituting (18) into (12), and considering (15), one obtains

$$\mathbf{e}_d^{*T}(\mathbf{D}^T\mathbf{h}^* - \mathbf{D}^T\mathbf{M}\dot{\mathbf{v}}) + \mathbf{e}_c^T(\mathbf{C}\mathbf{M}^{-1}\mathbf{h}^* + \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T\boldsymbol{\lambda}^* - \mathbf{C}\dot{\mathbf{v}}) = 0 \tag{19}$$

which expresses the projections of the dynamic equation (12) into $\mathsf{D}^k$ and $\mathsf{C}^m$, respectively. By virtue of the complementarity of the subspaces in $\mathsf{E}^n$, (19) means that:

$$\mathbf{D}^T\mathbf{M}\dot{\mathbf{v}} = \mathbf{D}^T\mathbf{h}^* \tag{20a}$$

$$\mathbf{C}\dot{\mathbf{v}} = \mathbf{C}\mathbf{M}^{-1}\mathbf{h}^* + \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T\boldsymbol{\lambda}^* \tag{20b}$$

which can also be obtained by premultiplying (11b) by $\mathbf{H}^T$. The geometric interpretation of (20) is illustrated in Figure 1b, and its vector representation is:

$$\vec{\boldsymbol{h}}_d - \dot{\vec{\boldsymbol{\nu}}}_d = 0 \tag{21a}$$

$$\vec{\boldsymbol{h}}_c + \vec{\boldsymbol{r}} - \dot{\vec{\boldsymbol{\nu}}}_c = 0, \tag{21b}$$

where the subscripts $d$ and $c$ denote the projections of appropriate vectors into $\mathsf{D}^k$ and $\mathsf{C}^m$.

Since the constraint equation (11c) can be interpreted as

$$\vec{\boldsymbol{\xi}} - \dot{\vec{\boldsymbol{\nu}}}_c = \mathbf{e}_c^T(\boldsymbol{\xi} - \mathbf{C}\dot{\mathbf{v}}) = 0 \tag{22}$$

the constrained subspace projections (20b) and (21b) can be manipulated to:

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{h}^* + \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T\boldsymbol{\lambda}^* - \boldsymbol{\xi} = 0 \tag{23}$$

$$\vec{\boldsymbol{h}}_c + \vec{\boldsymbol{r}} - \vec{\boldsymbol{\xi}} = 0. \tag{24}$$

Then, the virtual subspace projections (20a) and (21a), supplied with (11c), leads to the constraint reaction-free governing equations [5, 6, 9–15, 20, 21, 31, 32]:

$$\dot{\mathbf{p}} = \mathbf{A}\mathbf{v} \tag{25a}$$

$$\begin{bmatrix} \mathbf{D}^T\mathbf{M} \\ \mathbf{C} \end{bmatrix} \dot{\mathbf{v}} = \begin{bmatrix} \mathbf{D}^T\mathbf{h}^* \\ \boldsymbol{\xi} \end{bmatrix} \tag{25b}$$

which are $2n$ ODEs in $\mathbf{v}$ and $\mathbf{p}$. Using the solution $\mathbf{v}(t)$ and $\mathbf{p}(t)$ to the ODEs, the Lagrange multipliers $\boldsymbol{\lambda}^*(t)$ can then be synthesized from the following algebraic formula

$$\boldsymbol{\lambda}^*(\mathbf{p},\mathbf{v},t) = (\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T)(\boldsymbol{\xi} - \mathbf{C}\mathbf{M}^{-1}\mathbf{h}^*). \tag{26}$$

The coefficient matrix on the left-hand side of (25b) is $\mathbf{H}^T\mathbf{M}$. Then it comes from (16) and (17) that $(\mathbf{H}^T\mathbf{M})^{-1} = [\mathbf{D}\mathbf{M}_d^{-1} \;\; \mathbf{M}\mathbf{C}^T\mathbf{M}_d]$ and $\mathbf{I} = \mathbf{M}\mathbf{D}\mathbf{M}_d^{-1}\mathbf{D}^T + \mathbf{C}^T\mathbf{M}_c\mathbf{C}\mathbf{M}^{-1}$, where $\mathbf{I}$ is the identity matrix. Using this, (25b) can be manipulated to the following equivalent form $\mathbf{M}\dot{\mathbf{v}} = \mathbf{h}^* + \mathbf{C}^T\mathbf{M}_c(\boldsymbol{\xi} - \mathbf{M}\mathbf{M}^{-1}\mathbf{h}^*)$, where $\mathbf{M}_c = (\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T)^{-1}$. The result can also be obtained directly by substituting (26) into (11b), while (26) is obtained after substituting (11b) into (11c), see e.g. [2].

Compared to DAEs (11) or (13), the dimension of ODEs (25) is reduced from $2n+m$ to $2n$. The benefit can, however, be fictitious since the $n \times n$ coefficient matrix in (25b) is a general matrix while the $(n+m)\times(n+m)$ coefficient matrix in (13b) is a sparse matrix. As special procedures can be applied to sparse matrices, the inversion of the two matrices in the process of numerical integration may thus be equally "expensive". Moreover, ODEs (25) still suffer from the constraint violation problem. Therefore, further reduction in dimension by introduction a minimal set of (independent) velocity components is usually undertaken.

## 5. The Equations of Motion in Virtual Speeds

Further developments in the description of constrained system dynamics can be achieved by introducing $k$ independent *virtual speeds* $\mathbf{u} = [u^1 \dots u^k]^T$ being the components in the basis $\mathbf{e}_d$ of the projection of $\vec{\boldsymbol{\nu}}$ into $\mathsf{D}^k$, $\vec{\boldsymbol{u}} = \mathbf{e}_d^T\mathbf{u}$ ($\vec{\boldsymbol{u}} = u^i\vec{\boldsymbol{d}}_i$, $i = 1,\dots,k$). According to the illustration in Figure 1a, we can write

$$\vec{\boldsymbol{u}} + \vec{\boldsymbol{\eta}} = \vec{\boldsymbol{\nu}}. \tag{27}$$

To find a matrix representation of this vector formula, we first write $\vec{\boldsymbol{\nu}} = \mathbf{e}_v^{*T}\mathbf{M}\mathbf{v}$, and then using (18) we have $\vec{\boldsymbol{\nu}} = (\mathbf{e}_d^{*T}\mathbf{D}^T + \mathbf{e}_c\mathbf{C}\mathbf{M}^{-1})\mathbf{M}\mathbf{v} = \mathbf{e}_d^{*T}\mathbf{D}^T\mathbf{M}\mathbf{v} + \mathbf{e}_c\mathbf{C}\mathbf{v} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{\eta}}$. On the other hand, $\vec{\boldsymbol{u}} = \mathbf{e}_d^{*T}\mathbf{u}^* = \mathbf{e}_d^{*T}\mathbf{M}_d\mathbf{u}$ and $\vec{\boldsymbol{\eta}} = \mathbf{e}_c^T\boldsymbol{\eta}$. Comparing the two formulations, we arrive at

$$\begin{bmatrix} \mathbf{D}^T\mathbf{M}\mathbf{D}\mathbf{u} \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{D}^T\mathbf{M}\mathbf{v} \\ \mathbf{C}\mathbf{v} \end{bmatrix} \equiv \mathbf{H}^T\mathbf{M}\mathbf{v}. \tag{28}$$

For scleronimic systems ($\boldsymbol{\eta} = 0$), (28) denotes that $\vec{\boldsymbol{\nu}}$ is entirely contained in $\mathsf{D}^k$ or, more specifically, the system momentum $\mathbf{v}^* = \mathbf{M}\mathbf{v}$ (the representation of $\vec{\boldsymbol{\nu}}$ in $\mathbf{e}_v^*$ basis of $\mathsf{E}^n$) projects only into $\mathsf{D}^k$ and is represented in $\mathbf{e}_d^*$ basis of the subspace as $\mathbf{u}^* = \mathbf{M}_d\mathbf{u}$. For rheonomic systems ($\boldsymbol{\eta} = 0$), $\vec{\boldsymbol{\nu}}$ is deflected from $\mathsf{D}^k$ due to the constraint "motion" perceived in $\mathsf{C}^m$ ($\vec{\boldsymbol{\eta}}$ is the component of $\vec{\boldsymbol{\nu}}$ in $\mathsf{C}^m$), see Figure 1a.

From (28) it comes directly that

$$\mathbf{u} = (\mathbf{D}^T\mathbf{M}\mathbf{D})^{-1}\mathbf{D}^T\mathbf{M}\mathbf{v} \equiv \mathbf{M}_d^{-1}(\mathbf{p},t)\mathbf{D}^T(\mathbf{p},t)\mathbf{M}(\mathbf{p})\mathbf{v}. \tag{29}$$

As the components of $\mathbf{u}$ may have no physical meaning, (29) can serve to obtain the consistent initial values $\mathbf{u}_0$ for given $\mathbf{p}_0$ and $\mathbf{v}_0$, which is of paramount importance for the succcessful initialization of the integration process of motion equations that will follow. Then, by inverting (28) we obtain

$$\mathbf{v} = \mathbf{D}\mathbf{u} + \mathbf{M}^{-1}\mathbf{C}^T(\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T)^{-1}\boldsymbol{\eta} \equiv \mathbf{D}(\mathbf{p},t)\mathbf{u} + \boldsymbol{\gamma}(\mathbf{p},t), \tag{30}$$

where $\boldsymbol{\gamma} = \mathbf{M}^{-1}\mathbf{C}\mathbf{M}_c\boldsymbol{\eta}$ denotes the representation of $\vec{\boldsymbol{\eta}}$ in $\mathbf{e}_v$ and vanishes from scleronomic constraints (note that $\boldsymbol{\eta}$ is the representation of $\vec{\boldsymbol{\eta}}$ in $\mathbf{e}_c$, i.e. $\vec{\boldsymbol{\eta}} = \mathbf{e}_v^T\boldsymbol{\gamma} = \mathbf{e}_c^T\boldsymbol{\eta}$). The formulae (29) and (30) are the direct/matrix representations of the vector equations $\vec{\boldsymbol{\nu}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{\eta}}$ and $\vec{\boldsymbol{u}} = \vec{\boldsymbol{\nu}} - \vec{\boldsymbol{\eta}}$, illustrated in Figure 1a.

The *virtual speeds* $\mathbf{u}$ are notionally equivalent to the *independent kinematic parameters* introduced in Maggi's [19, 32, 33] and Gibbs–Appell [17, 18, 32] methods, and to the *generalized speeds* used in Kane's method [7, 15, 17, 22]. The present formula states the definition more precisely from the mathematical (geometrical) point of view. Also note that by using (28) the virtual speeds $\mathbf{u}$ need not be introduced a priori – they may result from the constraint equations (7), the introduced orthogonal complement matrix $\mathbf{D}$ to the constraint matrix $\mathbf{C}$, and the system inertial properties represented in the metric of configuration space (excluding the latter from the analysis is erroneous; see the forthcoming discussion). Moreover, none of the previous methods supplies the user with an explicit general formula, such as (28) and (29), to determine explicitly the independent velocity components.

Using (30) and its differentiated form $\dot{\mathbf{v}} = \mathbf{D}\dot{\mathbf{u}} + \dot{\mathbf{D}}\mathbf{u} + \dot{\boldsymbol{\gamma}}$, it is easy to check that the velocity and acceleration constraint equations (7) and (8) are satisfied by identity, and the governing equations (25) transform to the following $n + k$ ODEs in $\mathbf{p}$ and $\mathbf{u}$:

$$\dot{\mathbf{p}} = \mathbf{A}(\mathbf{D}\mathbf{u} + \gamma) \quad \equiv \quad \dot{\mathbf{p}} = \mathbf{A}_d(\mathbf{p},t)\mathbf{u} + \mathbf{a}_d(\mathbf{p},t) \tag{31a}$$

$$\mathbf{D}^T\mathbf{M}\mathbf{D}\dot{\mathbf{u}} = \mathbf{D}^T\mathbf{h}^* - \mathbf{D}^T\mathbf{M}(\dot{\mathbf{D}}\mathbf{u} + \dot{\gamma}) \quad \equiv \quad \mathbf{M}_d(\mathbf{p},t)\dot{\mathbf{u}} = \mathbf{h}_d^*(\mathbf{p},\mathbf{u},t) \tag{31b}$$

Given $\mathbf{p}_0$ and $\mathbf{v}_0$, from (29) the initial values $\mathbf{u}_0 = \mathbf{M}_d^{-1}(\mathbf{p}_0,t_0)\mathbf{D}^T(\mathbf{p}_0,t_0)\mathbf{M}(\mathbf{p}_0)\mathbf{v}_0$. Since it comes from (25) that $\dot{\mathbf{C}}\mathbf{D} = -\mathbf{C}\dot{\mathbf{D}}$, and the $\boldsymbol{\xi} = -\dot{\mathbf{C}}\mathbf{v} + \boldsymbol{\eta} = \mathbf{C}(\dot{\mathbf{D}}\mathbf{u} + \boldsymbol{\gamma})$, the relation (26) can eventually be manipulated to

$$\boldsymbol{\lambda}^*(\mathbf{p},\mathbf{u},t) = (\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T)^{-1}\mathbf{C}(\dot{\mathbf{D}}\mathbf{u} + \dot{\boldsymbol{\gamma}} - \mathbf{M}^{-1}\mathbf{h}^*) \tag{32}$$

from which, using the solution $\mathbf{p}(t)$ and $\mathbf{u}(t)$ to ODEs (40), $\boldsymbol{\lambda}^*$ can be synthesized.

ODEs (31) provide a general procedure for obtaining the minimal-dimension equations of motion of a constrained system. The formulation requires first to determine an orthogonal complement matrix $\mathbf{D}$ to the constraint matrix $\mathbf{C}$. For

small systems this can often be done by inspection or simply guessed. For large-scale systems the determination is usually performed numerically, and numerous computer-oriented codes of this type have been proposed (see e.g. [3, 4, 6, 7, 10–16, 20]). A more complex task is to determine $\dot{\mathbf{D}}\mathbf{u}+\dot{\boldsymbol{\gamma}}$ used in the dynamic equation (31). In the co-ordinate partitioning method [3] the term is found directly from the partitioned acceleration form of constraint equations (8). Namely, from the partitioned velocity form of contraint equations, $\boldsymbol{\Psi}=\mathbf{C}\mathbf{v}+\boldsymbol{\eta}\equiv\mathbf{U}\mathbf{u}+\mathbf{W}\mathbf{w}+\boldsymbol{\eta}=0$, where $\mathbf{C}=[\mathbf{U}\ \mathbf{W}]$, $\mathbf{v}=[\mathbf{u}^T\ \mathbf{w}^T]^T$, and the $m$ *dependent* velocities $\mathbf{w}$ are chosen so that $\det(\mathbf{W})\neq 0$, (30) is obtained as $\mathbf{v}=[\mathbf{I}\ -(\mathbf{W}^{-1}\mathbf{U})^T]^T\mathbf{u}+[\mathbf{0}^T\ -(\mathbf{W}^{-1}\boldsymbol{\eta})^T]^T\equiv\mathbf{D}\mathbf{u}+\boldsymbol{\gamma}$, and $\mathbf{I}$ and $\mathbf{0}$ are the $k\times k$ and $k\times 1$ identity and zero matrices, respectively. Then, from the partitioned constraint acceleration equations, $\dot{\boldsymbol{\Psi}}=\mathbf{C}\dot{\mathbf{v}}+\xi\equiv\mathbf{U}\dot{\mathbf{u}}+\mathbf{W}\dot{\mathbf{w}}+\boldsymbol{\xi}=\mathbf{0}$, it follows that $\dot{\mathbf{v}}=[\mathbf{I}\ -(\mathbf{W}^{-1}\mathbf{U})^T]^T\dot{\mathbf{u}}+[\mathbf{0}^T\ -(\mathbf{W}^{-1}\boldsymbol{\xi})^T]^T\equiv\mathbf{D}\dot{\mathbf{u}}+\dot{\boldsymbol{\gamma}}$; see also [7, 34]. In this way, the dynamic equations (31b) in the chosen $k$ *independent* velocities $\mathbf{u}$ can be obtained. A recursive scheme for obtaining $\dot{\mathbf{D}}$ numerically, based on $\mathbf{C}$ and $\dot{\mathbf{C}}$, have also been proposed in [15] and then followed in [35]. Finally, in [26] an ingenious differentiation-free scheme (based on pure differential geometry formalism) for obtaining the dynamic equations in terms of independent variables is proposed. General and computationally cheap new codes for computing $\dot{\mathbf{D}}\mathbf{u}+\dot{\boldsymbol{\gamma}}$ are still desirable, however,

The relations (27–30) and the consequent motion equations (31) have been derived based on the *explicit* formulations (7) and (8) of constraint equations. For H systems, however, the H constraints are commonly introduced *implicitly* by a priori choice of independent co-ordinates $\mathbf{q}=[q^1\dots q^k]^T$ ($m_{\mathrm{H}}=m$, $k=n-m$) that define the system position on the H constraint manifold (the configuration space of the H constrained system, $\mathsf{K}_{\mathrm{H}}=\mathsf{K}$. The dependence

$$\mathbf{p}=\mathbf{g}(\mathbf{q},t) \tag{33}$$

stands for the $m$ H constraints on the $n$ dependent co-ordinates $\mathbf{p}$, i.e. $\boldsymbol{\Phi}_{\mathrm{H}}[\mathbf{g}(\mathbf{p},t),t]\equiv 0$. The differentiated form of (33), after considering (1), is

$$\mathbf{v}=\mathbf{D}'(\mathbf{q},t)\dot{\mathbf{q}}+\boldsymbol{\gamma}'(\mathbf{q},t), \tag{34}$$

where $\mathbf{D}'=\mathbf{A}^{-1}(\partial\mathbf{g}/\partial\mathbf{q})$ and $\boldsymbol{\gamma}'=\mathbf{A}^{-1}(\partial\mathbf{g}/\partial t)$, stands for the $m$ constraint conditions on the dependent velocities $\mathbf{v}$. Finally, using (33), (34) and $\dot{\mathbf{v}}=\mathbf{D}'\ddot{\mathbf{q}}+\dot{\mathbf{D}}'\dot{\mathbf{q}}+\dot{\boldsymbol{\gamma}}'$, the governing equations (31) transform to the following $2k$ ODEs in $\mathbf{q}$ and $\dot{\mathbf{q}}$ [2, 4, 5, 8, 9, 20, 26]

$$\mathbf{D}'^T\mathbf{M}\mathbf{D}'\ddot{\mathbf{q}}=\mathbf{D}'^T\mathbf{h}^*-\mathbf{D}'^T\mathbf{M}(\dot{\mathbf{D}}'\dot{\mathbf{q}}+\dot{\boldsymbol{\gamma}}')\ \equiv\ \mathbf{M}'_d(\mathbf{q},t)\ddot{\mathbf{q}}=\mathbf{h}^*_d(\mathbf{q},\dot{\mathbf{q}},t). \tag{35}$$

For nonautonomous systems, the virtual speeds $\mathbf{u}$ (quasi-velocities) defined in (37) and the generalized velocities $\dot{\mathbf{q}}$ as above are the representations of different (but collinear) vectors in $\mathsf{E}^n$ (see Figure 2), and the relations (30) and (34), though similar, are not equivalent. Namely, (30) expresses $\vec{\boldsymbol{\nu}}=\vec{\boldsymbol{u}}+\vec{\boldsymbol{\eta}}$, while (34) stands

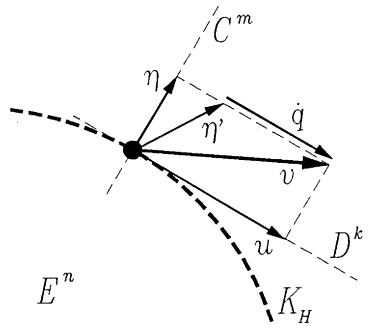



*Figure 2*. The virtual speeds **u** (quasi-velocities) and the generalized velocities $\dot{\mathbf{q}}$.

for $\vec{\boldsymbol{\nu}} = \dot{\vec{\boldsymbol{q}}} + \vec{\boldsymbol{\eta}}'$, where $\vec{\boldsymbol{\eta}}' = \mathbf{e}_v^T \boldsymbol{\gamma}'$ is the total effect of the constraint "motion" at $\mathbf{q} \in \mathsf{K}_H$ while $\vec{\boldsymbol{\eta}}$ is the projection of $\vec{\boldsymbol{\eta}}'$ in $\mathsf{C}^m$. From inverting (34) we obtain

$$\dot{\mathbf{q}} = (\mathbf{D}'\mathbf{M}\mathbf{D}')^{-1}\mathbf{D}'^T\mathbf{M}(\mathbf{v} - \boldsymbol{\gamma}'). \tag{36}$$

Evidently, for autonomous systems, **u** and $\dot{\mathbf{q}}$ are the representations of the same vector, contained in the subspace $\mathsf{D}^k$ of $\mathsf{E}^n$, $\vec{\boldsymbol{\eta}} = \vec{\boldsymbol{\eta}}' = \mathbf{0}$ and $\dot{\vec{\boldsymbol{q}}} = \vec{\boldsymbol{u}} = \vec{\boldsymbol{\nu}}$.

## 6. Elimination of Constraint Violations

The problem of constraint violation occurs when the motion equations used for simulation involve the differential forms of constraint equations. By assumption, the exact realization of only those differentiated constraint equations is assured while the lower-order (original) constraint equations may be violated by the solution (burdened with a numerical error of integration) even though the initial values of the state variables satisfy the constraint equations. The case relates mainly to DAEs (11–13) and ODEs (25), where the constraint equations are considered in the acceleration form (16). The numerical solutions $\hat{\mathbf{p}}(t)$ and $\hat{\mathbf{v}}(t)$ to these equations may violate the H constraint equation (6a) and its differentiated form as well as the NH constraint equation (6b), $\boldsymbol{\Phi}_{\mathrm{H}}(\hat{\mathbf{p}}, t) \neq \mathbf{0}$ and $\boldsymbol{\Psi}(\hat{\mathbf{p}}, \hat{\mathbf{v}}, t) \neq \mathbf{0}$. The situation is a little different when dealing with ODEs (31). By virtue of the definition of independent speeds **u**, the used relation (30) stands for the $m$ velocity constraint equations (7), $\boldsymbol{\Psi} = [\dot{\boldsymbol{\Phi}}_{\mathrm{H}}^T \;\; \boldsymbol{\Psi}_{\mathrm{NH}}^T] = \mathbf{0}$, which are by assumption satisfied by the solution $\hat{\mathbf{p}}(t)$ and $\hat{\mathbf{u}}(t)$. The formulation does not, however, protect the H constraint equation (6a) from being violated by the solution $\hat{\mathbf{p}}(t)$, $\boldsymbol{\Phi}_{\mathrm{H}}(\hat{\mathbf{p}}, t) \neq \mathbf{0}$, see also [15]. Finally, ODEs (35), derived for H systems, are released from the constraint violation problem – the used relations (33) and (34) stand for $\boldsymbol{\Phi}_{\mathrm{H}} = \mathbf{0}$ and $\dot{\boldsymbol{\Phi}}_{\mathrm{H}} \equiv \boldsymbol{\Psi} = \mathbf{0}$, and assure the exact realization of the constraint equations.

The conceptually simplest method for eliminating the H constraint violation or, in other words, for correcting the system position so that $\boldsymbol{\Phi}_{\mathrm{H}} = \mathbf{0}$ with a required numerical accuracy, is to treat $\hat{\mathbf{p}}(t)$ as a trial root of $\boldsymbol{\Phi}_{\mathrm{H}}(\mathbf{p}, t) = \mathbf{0}$ for a

given $t$, and then to solve the constraint equations for a numerically exact root. As $\dim(\boldsymbol{\Phi}_{\mathrm{H}}) = m_{\mathrm{H}} < \dim(\mathbf{p}) = n$, $n - m_{\mathrm{H}}$ supplementary equations must be added to $\boldsymbol{\Phi}_{\mathrm{H}}(\mathbf{p}, t) = \mathbf{0}$ in order to complete the process successfully, and it is essential to construct the supplementary equations so that to ensure the position correction only in the subspace defined by the $m_{\mathrm{H}}$ H constraint gradients (which is a part of $\mathsf{C}^m$ for a system subject to H and NH constraints). Such a formulation can be deduced from (28) applying $\mathbf{C}_{\mathrm{H}}\mathbf{A}^{-1}\dot{\mathbf{p}} + \boldsymbol{\eta}_{\mathrm{H}} + \dot{\hat{\boldsymbol{\Phi}}}_{\mathrm{H}} = \mathbf{0}$ instead of the velocity form of constraint equations. Then, the Newton–Raphson formula for transforming the H constraint violation $\hat{\boldsymbol{\Phi}}_{\mathrm{H}} = \boldsymbol{\Phi}_{\mathrm{H}}(\hat{\mathbf{p}}, t)$ to the position correction $\Delta\mathbf{p} = \mathbf{p}(t) - \hat{\mathbf{p}}(t)$ is as follows

$$\begin{bmatrix} \hat{\mathbf{D}}_{\mathrm{H}}^{T}\hat{\mathbf{M}} \\ \hat{\mathbf{C}}_{\mathrm{H}} \end{bmatrix} \hat{\mathbf{A}}^{-1}\Delta\mathbf{p} = -\begin{bmatrix} \mathbf{0} \\ \hat{\boldsymbol{\Phi}}_{\mathrm{H}} \end{bmatrix}, \tag{37}$$

where $\mathbf{D}_{\mathrm{H}}$ is an $n \times k_{\mathrm{H}}$ orthogonal complement matrix to the $m_{\mathrm{H}} \times n$ H constraint matrix $\mathbf{C}_{\mathrm{H}} = (\partial\boldsymbol{\Phi}_{\mathrm{H}}/\partial\mathbf{p})\mathbf{A}$, $k_{\mathrm{H}} = n - m_{\mathrm{H}}$, and the superscript $\hat{}$ denotes dependence on $\hat{\mathbf{p}}$. By inverting (37) we obtain a much more useful formula

$$\Delta\mathbf{p} = -\hat{\mathbf{A}}\hat{\mathbf{M}}^{-1}\hat{\mathbf{C}}_{\mathrm{H}}\mathbf{M}^{-1}\mathbf{C}_{\mathrm{H}}^{T})^{-1}\hat{\boldsymbol{\Phi}}_{\mathrm{H}} \tag{38}$$

released from the necessity of determining $\mathbf{D}_{\mathrm{H}}$. The elimination of violation of velocity constraint equations, $\hat{\boldsymbol{\Psi}} = \boldsymbol{\Psi}(\hat{\mathbf{p}}, \hat{\mathbf{v}}, t)$, can be done by using formulae very similar to (37) and (38). Namely, denoting $\Delta\mathbf{v} = \mathbf{v}(t) - \hat{\mathbf{v}}(t)$, it can be deduced from (35b) that

$$\begin{bmatrix} \hat{\mathbf{D}}^{T}\hat{\mathbf{M}} \\ \hat{\mathbf{C}} \end{bmatrix} \Delta\mathbf{v} = -\begin{bmatrix} \mathbf{0} \\ \hat{\boldsymbol{\Psi}} \end{bmatrix} \tag{39}$$

or in a resolved form

$$\Delta\mathbf{v} = -\hat{\mathbf{M}}^{-1}\hat{\mathbf{C}}^{T}(\hat{\mathbf{C}}\mathbf{M}^{-1}\mathbf{C}^{T})^{-1}\hat{\boldsymbol{\Psi}}. \tag{40}$$

The formulae (38) and (40) assure that the position and velocity correction are performed in $\hat{\mathsf{C}}^m$ (the system position and velocity in $\hat{\mathsf{D}}^k$ are not changed). The geometric interpretation of the correcting terms (for scleronomic constraints) is given in Figure 3.

The position and velocity corrections according to (38) and (40) can be applied after each step of integration or after a sequence of steps when the constraint violations surpass the accepted values. The formulae can also serve to modify Baumgarte's constraint stabilization method [30]. From that angle, ODEs (31) should be changed to:

$$\dot{\mathbf{p}} = \mathbf{A}(\mathbf{D}\mathbf{u} + \gamma) - \mathbf{M}^{-1}\mathbf{C}_{\mathrm{H}}^{T}(\mathbf{C}_{\mathrm{H}}\mathbf{M}^{-1}\mathbf{C}_{\mathrm{H}}^{T})^{-1}\left(\mathbf{K}_1\boldsymbol{\Phi}_{\mathrm{H}} + \mathbf{K}_0\int \boldsymbol{\Phi}_{\mathrm{H}}\,\mathrm{d}t\right) \tag{41a}$$

$$\mathbf{D}^{T}\mathbf{M}\mathbf{D}\dot{\mathbf{u}} = \mathbf{D}^{T}\mathbf{H}^{*} - \mathbf{D}^{T}\mathbf{M}(\dot{\mathbf{D}}\mathbf{u} + \dot{\gamma}), \tag{41b}$$

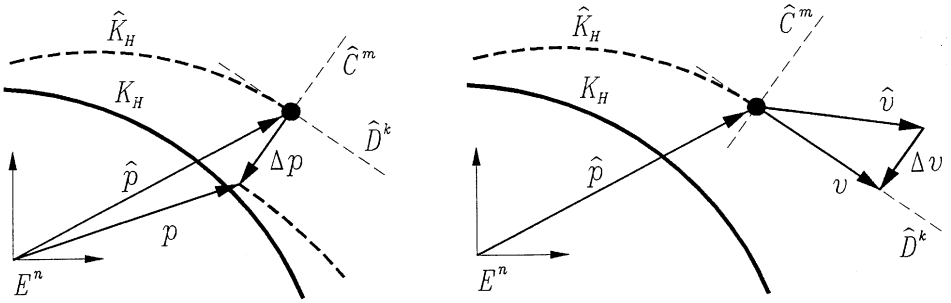



*Figure 3.* Position and velocity corrections.

while ODEs (25) will take the form:

$$\dot{\mathbf{p}} = \mathbf{A}\mathbf{v} - \mathbf{A}\mathbf{M}^{-1}\mathbf{C}_{\mathrm{H}}^{T}(\mathbf{C}_{\mathrm{H}}\mathbf{M}^{-1}\mathbf{C}_{\mathrm{H}}^{T})^{-1}\left(\mathbf{K}_1\boldsymbol{\Phi}_{\mathrm{H}} + \mathbf{K}_0\int \boldsymbol{\Phi}_{\mathrm{H}}\,\mathrm{d}t\right) \tag{42a}$$

$$\begin{bmatrix}\mathbf{D}^T\mathbf{M}\\ \mathbf{C}\end{bmatrix}\dot{\mathbf{v}} = \begin{bmatrix}\mathbf{D}^T\mathbf{h}^*\\ \boldsymbol{\xi} - (\mathbf{L}_1\boldsymbol{\Psi} + \mathbf{L}_0\int \boldsymbol{\Psi}\,\mathrm{d}t)\end{bmatrix}, \tag{42b}$$

where $\mathbf{K}_1$ and $\mathbf{K}_0$, and $\mathbf{L}_1$ and $\mathbf{L}_0$ are the $m_{\mathrm{H}} \times m_{\mathrm{H}}$ and $m \times m$ diagonal matrices of appropriate gain values [30]. Seemingly one can use the stabilized kinematic equation (42a) instead of (13a) and substitute the stabilized constraint enforcement $\boldsymbol{\xi} - (\mathbf{L}_1\boldsymbol{\Psi} + \mathbf{L}_0\int \boldsymbol{\Psi}\,\mathrm{d}t)$ instead of $\xi$ in (13b) to transform DAEs (13) into a stabilized form.

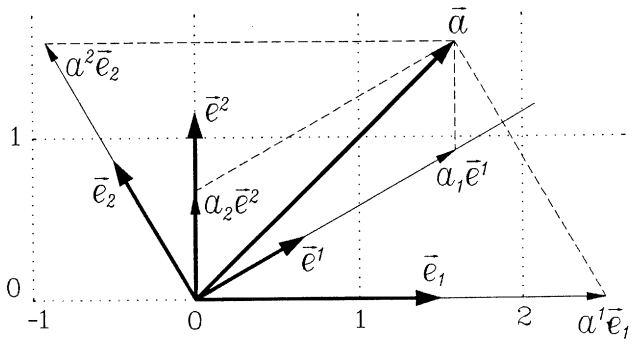
The present scheme of separate position and velocity corrections for constraint violation stabilization differs from that proposed by Baumgarte [30]. Here the position correcting terms are placed in the kinematic equations and the velocity correcting terms are situated in the dynamic equations. The position constraint violation are thus directly converted into the appropriate position corrections while the velocity corrections result from only the velocity constraint violation. In Baumgarte's method both the correcting terms are placed in the dynamic equations, and the position constraint violation is controlled indirectly by inducing an additional velocity correction – the effect in position is thus "delayed" in time.

The experience of the author is that the present method is very effective, accurate and computationally cheap. It is preferred to use the schemes (38) and (40) after each step of integration (or a sequence of steps), rather than to apply the stabilized governing equations (41) and (42). It is also important not to allow large constraint violations – the correction is performed in $\hat{C}^m$ defined for $\hat{\mathbf{p}} \in \hat{K}_{\mathrm{H}}$ (see Figure 3), and the closer $\hat{K}_{\mathrm{H}}$ to $K_{\mathrm{H}}$ the better accuracy. For larger constraint violation the formulae (38) and (40) be used recursively. In order to justify the effectiveness/accuracy of the method, it should be tested on examples and compared against other approaches. This will be done in future works. The objective of this presentation was only to show the geometric involvement of the problem and the impact it gives on the dynamic formulations.

## 7. Discussion and Conclusion

The proposed geometric formulation of constrained system dynamics appeals to intuition as a generalization of methods used in simple dynamics problems (Newtonian mechanics). The mathematical language used is that of vectors and tensors in metric spaces, a powerful and precise tool of analysis. In particular, the velocity (acceleration) and momentum (effective force) components of a system are treated as associated representations of the same vector. This implies a special meaning of the metric of system configuration space, involving the inertial properties of the system. Neglecting the inertial attributes may occasionally lead to *physical* inconsistencies in mathematical formulation. In many contributions, for instance, the velocity "norm" is introduced as $\|\vec{\boldsymbol{\nu}}\|^2 = \mathbf{v}^T\mathbf{v} = \nu_1^2 + \cdots + \nu_n^2$. This can eventually be accepted for systems of particles, due to the very specific (diagonal) metric tensor matrix of the configuration space. For a system composed of rigid bodies $\mathbf{v}$ usually consists of linear and angular velocities, however, and by using the above "norm" we sum addends characterized by different units (what we tell students never to do!). After setting the norm appropriately, $\|\vec{\boldsymbol{\nu}}\|^2 = \mathbf{v}^T\mathbf{M}\mathbf{v}$, the inconsistencies are clarified. The other example can be the matrix $\mathbf{C}^T\mathbf{C}$ introduced in the zero-eigenvalue theorem [10], the singular value decomposition [11, 12], and the QR/Householder decomposition [13, 14] methods. The entries of the matrix may also be formed by summing addends of different units. From the geometrical point of view the matrix should rather be changed to $\mathbf{C}^T(\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T)^{-1}\mathbf{C}$. The remarks do not, however, condemn the computer-oriented codes involving the physically inconsistent formulations – most of them are powerful tools of computer-aided analysis of multibody systems, and playing with numbers the units are not "seen". Nevertheless, one can never entirely shake off the feeling that something is missed in the formulations.

The proposed projective/geometric approach to solving problems of constrained system dynamics can be recommended for many reasons. It offers a comparatively simple, general and effective algorithm for obtaining constraint reaction-free equations of motion and, based on the solution to these equations, for synthesizing the associated constraint reactions. The method applies for both H and NH systems and no attention can be paid to distinguishing between generalized velocities and quasi-velocities (the entanglements arising when introducing quasi-velocities are the "Achilles heel" of Lagrangian mechanics [32]). The recent paper [36] presents several examples of application of the method. Finally, the notion of unconstrained system is extended to any autonomous Lagrangian (holonomic) system whose equations of motion are given in the form (1) and (2). All the systems can be regarded as primitive concepts like particles and treated as such (in a unified way). There are also no obstacles to analyse rheonomic and/or nonholonomic "unconstrained" systems, such as those described in (35) and (31), respectively, on which additional constraints are imposed. The configuration spaces of such systems become pseudo-




$$\mathsf{M} = \begin{bmatrix} \vec{e}_1 \circ \vec{e}_1 & \vec{e}_1 \circ \vec{e}_2 \\ \vec{e}_2 \circ \vec{e}_1 & \vec{e}_2 \circ \vec{e}_2 \end{bmatrix} = \begin{bmatrix} 2.25 & -0.75 \\ -0.75 & 1 \end{bmatrix}$$

*Figure A1*. Geometrical illustration of (A1–A3).

Riemannian, however, and the geometric interpretation as well as the mathematical description become more complex.

## Appendix

In an $n$-dimensional vector space $\mathsf{E}^n$, a vector $\vec{\boldsymbol{a}}$ can be represented either by its contravariant components $\mathbf{a} = [a^1 \dots a^n]^T$ with respect to a covariant basis $\mathbf{e} = [\vec{\boldsymbol{e}}_1 \dots \vec{\boldsymbol{e}}_n]^T$ of $\mathsf{E}^n$, or by covariant components $\mathbf{a}^* = [a_1 \dots a_n]^T$ with respect to the contravariant basis $\mathbf{e}^* = [\vec{\boldsymbol{e}}^1 \dots \vec{\boldsymbol{e}}^n]^T$ [24, 28, 37]:

$$\begin{aligned} \vec{\boldsymbol{a}} &= \mathbf{e}^T\mathbf{a} \equiv a^1\vec{\boldsymbol{e}}_1 + \cdots + a^n\vec{\boldsymbol{e}}_n \\ &= \mathbf{e}^{*T}\mathbf{a}^* \equiv a_1\vec{\boldsymbol{e}}^1 + \cdots + a_n\vec{\boldsymbol{e}}^n \end{aligned} \tag{A1}$$

or using indicial notation $\vec{\boldsymbol{a}} = a^i\vec{\boldsymbol{e}}_i = a_j\vec{\boldsymbol{e}}^j$. Using the metric tensor matrix $\mathbf{M}$ of the basis $\mathbf{e}$,

$$\mathbf{M} = \mathbf{e}\mathbf{e}^T = \begin{bmatrix} \vec{\boldsymbol{e}}_1 \circ \vec{\boldsymbol{e}}_1 & \dots & \vec{\boldsymbol{e}}_1 \circ \vec{\boldsymbol{e}}_n \\ \vdots & & \vdots \\ \vec{\boldsymbol{e}}_n \circ \vec{\boldsymbol{e}}_1 & \dots & \vec{\boldsymbol{e}}_n \circ \vec{\boldsymbol{e}}_n \end{bmatrix} \tag{A2}$$

($M_{ij} = \vec{\boldsymbol{e}}_i \circ \vec{\boldsymbol{e}}_j$), the relationships between the associated vector representations and the reciprocal bases are:

$$\begin{aligned} \mathbf{a}^* &= \mathbf{M}\mathbf{a} \quad (a_i = M_{ij}a^j), \qquad \mathbf{a} = \mathbf{M}^{-1}\mathbf{a}^* \quad (a^i = G^{ij}a_j), \\ \mathbf{e} &= \mathbf{M}\mathbf{e}^* \quad (\vec{\boldsymbol{e}}_i = M_{ij}\vec{\boldsymbol{e}}^j), \qquad \mathbf{e}^* = \mathbf{M}^{-1}\mathbf{e} \quad (\vec{\boldsymbol{e}}^i = G^{ij}\vec{\boldsymbol{e}}_j), \end{aligned} \tag{A3}$$

where $\mathbf{G} = \mathbf{M}^{-1} = \mathbf{e}^*\mathbf{e}^{*T}$ ($G^{ij} = \vec{\boldsymbol{e}}^i \circ \vec{\boldsymbol{e}}^j$, $G^{ij}M_{jk} = \delta^i_k$). A simple illustration of the formalae (A1–A3) is shown in Figure A1.

A dot product of two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ in $\mathsf{E}^n$ can be written in four different ways:

$$\vec{\boldsymbol{a}} \circ \vec{\boldsymbol{b}} = \mathbf{a}^T\mathbf{M}\mathbf{b} = \mathbf{a}^T\mathbf{b}^* = \mathbf{a}^{*T}\mathbf{b} = \mathbf{a}^{*T}\mathbf{M}^{-1}\mathbf{b}^* \tag{A4}$$

($\vec{\boldsymbol{a}} \circ \vec{\boldsymbol{b}} = a^i M_{ij} b^j = a^i b_i = a_j b^j = a_j G^{jk} b_k$), and the orthogonality condition of the two vectors is $\vec{\boldsymbol{a}} \circ \vec{\boldsymbol{b}} = 0$. Then, the "length" of a vector $\vec{\boldsymbol{a}}$ is $a = \sqrt{\vec{\boldsymbol{a}} \circ \vec{\boldsymbol{a}}}$.

The change from the basis $\mathbf{e}$ (or $\mathbf{e}^*$) to a new basis $\mathbf{e}'$ (or $\mathbf{e}'^*$) of $\mathsf{E}^n$ is associated with the following transformation formulae:

$$\begin{aligned} \mathbf{a} &= \mathbf{A}\mathbf{a}' \quad (a^i = A^i_j a'^j), \qquad \mathbf{a}'^* = \mathbf{A}^T \mathbf{a}^* \quad (a'_i = A^j_i a_j), \\ \mathbf{e}^* &= \mathbf{A}\mathbf{e}'^* \quad (\vec{\boldsymbol{e}}^{\,i} = A^i_j \vec{\boldsymbol{e}}'^{\,j}), \qquad \mathbf{e}' = \mathbf{A}^T \mathbf{e} \quad (\vec{\boldsymbol{e}}'_i = A^j_i \vec{\boldsymbol{e}}_j), \end{aligned} \tag{A5}$$

where $\mathbf{A}$ is the (inevitable) transformation matrix. The metric tensor matrix of $\mathbf{e}'$ is

$$\mathbf{M}' = \mathbf{e}'\mathbf{e}'^T = \mathbf{A}^T \mathbf{M} \mathbf{A} = (\mathbf{e}'^* \mathbf{e}'^{*T})^{-1} \quad (M'_{ij} = \vec{\boldsymbol{e}}'_j \circ \vec{\boldsymbol{e}}'_j = A^k_i M_{kl} A^l_j). \tag{A6}$$

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