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Economic Growth and Decline with Endogenous Property Rights

AARON TORNELL

Harvard University

This article introduces endogenous institutional change into a neoclassical growth model. For some parameter values, all Markov perfect equilibria involve a shift from common property to private property followed by a shift back to common property. Even in the presence of a linear production technology, this sequence of switches generates growth rates that are increasing at low levels of capital and decreasing at high levels of capital. This result rationalizes the hump-shaped growth path followed by some countries through history, as well as the conditional convergence observed in postwar data. For other parameter values, there are also equilibria in which common property prevails forever. This result rationalizes the low-growth traps in which many poor countries find themselves.

Keywords: Institutional change, Dynamic Games, Switching equilibrium

JEL classification: C73, D90, 04

1. Introduction

Differences in property rights are an important factor in explaining why growth rates differ across countries. Similarly, improvement and erosion of property rights may help explain why an economy rises and declines over time. However, there are no existing optimizing models that combine growth and changes in property rights. On the one hand, in the economic growth literature agents optimize, but the institutional framework is taken as given. On the other hand, the literature on institutional change does not offer parsimonious optimizing models. In this article we attempt to fill this gap by introducing endogenous institutional change into an optimizing growth model.

Such a model is potentially complicated because of the interactions involved. An institutional change generically occurs when the relevant state variable (such as debt or aggregate capital) reaches a certain threshold. The evolution of the state variable is a function of consumption and investment decisions, and these decisions in turn depend on the expected institutional change. This implies that the date of the future institutional change is itself a function of consumption and investment decisions. This interdependence makes the problem potentially complicated.

The model we present is a tractable version of this problem in which a closed-form solution is obtained. Within this model, for different parameter values, we can generate three stylized facts identified in the economic growth literature. The first is the celebrated conditional convergence phenomenon, according to which, within a set of countries, the growth rates are declining in the level of income per capita (Barro, 1991; Barro and Sala-i-Martin, 1995;

Baumol, 1986). The second is club convergence—namely, convergence fails to hold among poor countries, some of which remain in low-growth traps.[1] The third fact documented in an extensive literature dating back at least to Vico (1723), is that leading countries over the long-run have experienced humped-shaped growth paths. This literature includes analyses of the Roman empire in the fifth century, the Dutch republic in the seventeenth century, and Britain in the nineteenth century (see Kennedy, 1987; Landes 1969; Maddison, 1982; North, 1981; Olson, 1982; Spengler, 1932).

The model we present is a combination of an *Ak* growth model and a preemption game between two rent-seeking groups. The property rights regime can shift between *common property, private property*, and *leader-follower*. Under private property each group has access only to its own capital stock. Under common property individual property rights do not exist, and each group has access to aggregate capital. Under the leader-follower regime only one group has access to aggregate capital.

At every instant each group makes two decisions: how much to save and whether to incur a one-time loss in order to change the property-rights regime. Each group bases these decisions on the strategy of the other group.

We assume that initially common property prevails, so that both groups have access to aggregate capital. At any moment either group can displace the other group from its access to the capital stock by incurring a one-time loss (think of the cost of building a wall around an estate). If the second group does not match this move, a leader-follower regime sets in, under which the leader has access to the entire capital stock, while the follower has access to none. If the second group matches, private property emerges, and each group attains access only to its own capital stock. However, once private property is established, either group can become the leader at any time by once again incurring a one-time cost (think of the cost of destroying the other group's wall). If this act is matched by the competing group, there is a shift back to common property (there are no walls left).

One can interpret the first loss as the cost of instituting a system to enforce contracts. In the case of the leader-follower regime, the resulting system favors one group, while under private property it is impartial. The second loss can be interpreted as the cost of creating a rent-seeking organization. In addition to these one-time losses, the model also includes a stream of costs associated with defending property rights and a stream of costs associated with rent seeking.

The paths that the economy can follow are shown in Figure 1. The economy starts out under common property. This regime can last forever, or there can be a switch to either a private property or a leader-follower regime. If there is a shift to a leader-follower regime, the economy stays there forever by assumption. However, if there is a shift to private property, this regime can last forever, or there can be a shift to common property or to the leader-follower regime. The model allows a maximum of two switches.

The solution concept we use is Markov perfect equilibrium, which restricts strategies to be functions of payoff-relevant state variables.[2] Not all the paths shown in Figure 1 are Markov perfect equilibria. The key to determining which of them are equilibrium outcomes is σ (the inverse of the coefficient of relative risk aversion and the elasticity of intertemporal substitution). If $\sigma \leq 1$, as capital grows, the value of leading increases at a slower rate than the values of maintaining common property and private property forever. Since initially the

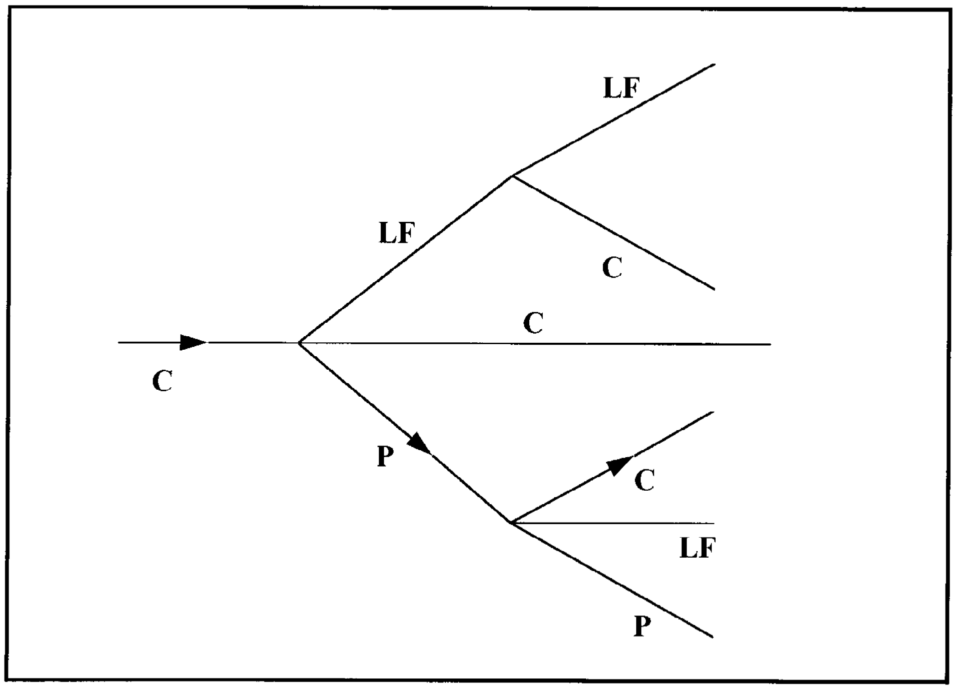

Figure 1. The paths that an economy can follow.

value of leading is lower than the value of waiting, common property (or private property) may prevail forever. However, if $\sigma > 1$, the economy must follow a cycle: when aggregate capital reaches a certain threshold, there must be a switch from common to private property. Then, when aggregate capital reaches a second threshold, there must be a switch back to common property.[3]

This cycle formalizes the observation in the economic history literature that when economies are poor, they tend to have lower growth rates and to lack institutions to protect private property rights. A shift to private property occurs when the economy becomes rich enough that groups find it worthwhile to incur the cost of creating institutions to defend private profits. Finally, as the economy becomes very rich, rent seeking becomes profitable. Thus redistributive activity increases, resistance to innovations develops, labor relations deteriorate, and fiscal deficits increase.

Notice that even though production technology in the model is linear, savings and growth rates vary over time. This variation, driven by the expectation of a regime switch, is what allows us to rationalize the three stylized facts mentioned above. First, to explain the fact that some poor countries remain in a low-growth trap, while other poor countries escape it, consider two economies that have the same initial conditions, with the difference between them being that in one $\sigma > 1$ and in the other $\sigma \leq 1$. It follows that the former economy will switch to private property and enjoy increasing growth and savings rates for a while. In contrast, the latter may get stuck under common property forever and therefore suffer from low growth forever.

Second, to explain convergence among rich countries, consider two countries, both under private property, both with $\sigma > 1$ (so that switches must take place), and with the capital stock of country A greater than that of country B. Since a switch to common property must take place when aggregate capital reaches a certain threshold, groups behave as if at a future date they will win a lottery (attain access to the other group's capital stock) and the private rate of return will fall. Therefore, as aggregate capital increases and the switch date gets closer, consumption rates increase and growth rates decline gradually. Since country B started with a lower capital stock, its growth rate will never be lower than that of country A.

Finally, to generate a hump-shaped long-run growth path, consider an economy whose equilibrium path is to switch from common to private property and then back to common property. During the first phase, even though common property prevails, the growth rate is increasing because a switch to private property is forthcoming. As this switch gets closer, groups reduce their level of appropriation of common resources because they discount less the fact that under private property each group will have access to only half of the capital stock, and the private rate of return will increase. In the second phase, although private property prevails, decreasing growth is caused by the expectation of a switch back to common property, as explained in the previous paragraph.

In the economic growth literature, growth is driven by other mechanisms, such as human capital accumulation, production externalities, and technological innovation. The channel we identify in this article should be considered as complementary to these mechanisms, not as a substitute for them.

The rest of the article is organized as follows. In Section 2 we present a brief description of the related literature. In Section 3 we present the model. In Section 4 we show how the equilibrium paths followed by the economy rationalize the three stylized facts mentioned above. In Section 5 we extend the model to include flow costs of maintaining the prevailing property rights regime. In Section 6 we present our conclusions.

2. Related Literature

This article is related to the literatures on rent seeking, switching equilibria, preemption games, revolutions, and balance-of-payments crises. We consider each in turn.

2.1. Rent Seeking

The economic growth literature has focused on the production side to explain differences in growth rates. A complementary approach is to consider property rights and rent-seeking behavior. It is often claimed that economies with poorly defined property rights tend to have low growth rates (Barro, 1996; Knack and Keefer, 1995). This is the case for poor economies that have a high physical return to capital but lack a system for impartially enforcing contracts, as well as for rich economies that possess such a system but suffer from an overwhelming level of redistributional activities on the part of interest groups. These rent-seeking activities may be reflected in high government expenditures and high inflation as in some Latin American countries or in ungovernability as observed in Britain when the Heath government fell after the miners' strike in the 1970s. Similarly, one can interpret the coefficients exhibited by African and Latin American dummies, in cross-country growth regressions, as a proxy for the absence of well-defined or well-enforced property rights (e.g., Barro, 1991).

Such arguments, however, do not explain why property rights are different across countries or why they change over time. Olson (1982) presents some explanation for these variations. He argues that the decline of leading countries like Britain was caused by the spontaneous formation of interest groups, which were able to overcome the free-rider problem due to long periods of stability. Once these groups were established, they engaged in redistributive activities, causing a de facto elimination of private property. This argument is intuitively compelling, but its microfoundations are not clear. In particular, high voracity is not a necessary consequence of the existence of competing interest groups. In fact, groups may find it profitable to limit current voracity and wait until the goose that lays the golden egg fattens. The shift from private to common property in our model and the resulting increasing appropriation rates are one formalization of Olson's argument.

2.2. Switching Equilibria and Preemption Games

I turn now to a discussion of switching equilibria, which are considered by Benhabib and Rustichini (1996) and Benhabib and Radner (1992). These papers characterize trigger strategy equilibria in economic growth models with common access to capital. They investigate under which circumstances low consumption (cooperation) will be enforced at some levels of the capital stock but not at other levels. The first paper considers a common access economy in discrete time and shows that with liner technology there exists a switching equilibrium in which the appropriation rate starts out high (no cooperation) and switches to a lower level when the capital stock reaches a certain threshold. As a consequence the growth rate is increasing in the level of capital. To generate the stylized fact that at high levels of capital the growth rate is decreasing, the paper replaces the linear technology with a Cobb-Douglas one. The second paper analyzes a similar problem in continuous time using linear preferences.

The key difference between these models and ours is that we formulate the problem as a preemption game and allow the share of capital to which each group has access to change between one regime and another, with each switch entailing a one-time loss. It is this specification that allows us to generate a hump-shaped growth rate in the presence of a linear technology and CRRA utility.[4]

Our solution method combines techniques from the preemption games literature and from the literature on differential games of joint exploitation of resources.[5] In the preemption games literature, the payoffs of leading, following, and matching are postulated as functions of time only. A novel feature of our article is that it makes these payoffs depend on a state variable (the capital stock) and makes this state variable the result of an accumulation game that takes into account the future regime switches generated by the underlying preemption game.

2.3. Revolutions

In models of conflict and revolutions such as Grossman (1991, 1994), Grossman and Kim (1996), and Skaperdas (1992), the underlying force, as in this article, is each group's desire to appropriate as many resources as possible at the expense of other groups. These models differ from ours in that they develop general equilibrium analyses of resource allocation between productive and appropriative activities, and they are static. Our model is dynamic and does not consider a multisectoral set up.

2.4. Balance-of-Payment Crises

Lastly, we turn to macroeconomic policy issues such as delayed fiscal reform and balance-of-payments crises. In both cases multiple fiscal authorities with common access to fiscal revenue follow unsustainable spending paths, aware that at some point, when debt becomes sufficiently high, a balance-of-payments crisis will take place unless there is a fiscal reform. Groups follow these paths even thought the future regime shift will make all groups worse off. If we use the stock of debt as the state variable instead of the capital stock, the framework we develop in this article can be used to rationalize this phenomenon. Velasco (1994) has modeled this phenomenon along the lines of Benhabib and Rustichini (1996).

2.5. Literature on Long-Run Growth Patterns

This subsection provides a brief examination of the historical literature on property rights and growth. According to North (1981, 1990), human communities started out as common property regimes: tribes of hunters had common access to the stock of wild animals. As these natural resources grew scarce, settled agriculture developed, and members of each tribe began to specialize in defining their group's land against outsiders. In this way, some tribes developed economies based on private property and grew into the nation-states of Babylon, Persia, Athens, and Rome. However, all these economies eventually declined and collapsed. For example, in the latter period of the Roman empire the increasing burden of military expenses and government provision of food to many citizens led to an increase in taxes paid by people without political clout and a de facto elimination of private property. As a consequence, growth in the Roman economy declined, leading to the military weakness and eventual demise of the empire in the fifth century.

North and Thomas (1973) report that during the Middle Ages law and order existed only within the boundaries of individual manors or towns, and trade was unprotected from theft and prohibitive taxation. Moreover, no impartial third party with the power to enforce agreements existed. By the end of the fifteenth century, nation-states had been created wherein a prince offered subjects security in exchange for tax revenue. In the cases of England and the Dutch republic, the prince's access to the possessions of his subjects was limited by a representative body, and private property developed. The opposite occurred in Spain and France, where absolutist states were formed. According to recent historical research, the princes of these states, to ensure their absolute rule, had to buy off the nobility by granting them monopoly rights in various industries. This was the purpose of the court, which allowed the high nobility to live near the prince. Asch and Birke (1991, p. 17) describe the situation: "For royal patronage the Court was the great market-place where all kinds of grants, privileges, and offices were haggled over .... Since the Court acted as the clearinghouse for all patronage matters its expansion was a prerequisite for political integration." Furthermore, they say in regard to the composition of the court, "The court's members were both royal officers and private entrepreneurs. Their most profitable business was brokerage" (p. 32).[6]

The Dutch republic overtook the Italian cities and was in turn overtaken by England. Meanwhile, Spain declined steadily from its sixteenth-century position of most powerful nation, suffering constant fiscal crises. The decline of each nation has been attributed to resistance to technological innovations, labor-market rigidities, and an increase in rent-seeking activity.[7]

In all cases, private property was not imposed without pain. In England, it took the Glorious Revolution to ensure the Parliament's supremacy over the king. In France, it was not until the Revolution of 1789 that medieval trade restrictions and monopoly rights held by guilds were eliminated.

3. The Model

We consider an *ak* growth model with the peculiarity that the representative agent is replaced by two infinitely living agents (i and j) who interact strategically (we refer to them as *interest groups*). They derive utility from the consumption of a single good that can be instantaneously transformed into capital. This good is produced with a linear technology using capital as the only input. We denote the capital stock of group h by k_h, aggregate capital by lowercase k, and the vector of individual capital stocks by uppercase K—that is,

$$k \equiv k_i + k_j, \qquad K \equiv (k_i, k_j).$$

Three types of property rights regimes describes the ways in which rent-seeking groups interact. First, we define *private property* as the regime that permits each group access only to its own capital stock. Under this regime, the accumulation equations are the same as those in the representative agent model:

$$\dot{k}_h(t) = ak_h(t) - c_h(t), \qquad h = i, j. \tag{1}$$

Second, there is the common property regime, under which there are no individual capital stocks, and everyone has access to the aggregate capital stock. The accumulation equation is

$$\dot{k}(t) = ak(t) - c_i(t) - c_j(t). \tag{2}$$

This type of accumulation equation has been used to study phenomena of joint exploitation of resources—for instance, in the cases of fisheries (Levhari and Mirman, 1980), labor conflicts (Lancaster, 1973), and macroeconomics (Tornell and Velasco, 1992; Tornell and Lane, 1997).[8]

Lastly, there is the leader-follower regime under which only one group (the leader) enjoys access to aggregate capital, while the other group (the follower) does not and must have zero consumption. In this case the accumulation equation is

$$\dot{k}(t) = ak(t) - c_l(t), \tag{3}$$

where the subscript l stands for leader. Under all regimes the following constraint must be satisfied:

$$k(t) \equiv k_i(t) + k_j(t) \geq 0. \tag{4}$$

The possible paths the economy can follow are illustrated in Figure 1. Initially, common property prevails. At any instant a group can displace the other group and secure exclusive access to the entire capital stock by incurring a one-time utility loss q. If the other group simultaneously undertakes the same action, the loss to each group is zero, and each attains private access to half of aggregate capital. In the first case there is a switch to the leader-follower regime. In the second case there is a switch to private property. The loss q can be interpreted as the cost of creating a legal enforcement system that is biased in favor of the leader, or it can be interpreted as a direct payment to the other group.[9]

If a switch to private property occurs, each group can regain its access to the entire capital stock, become the leader, and displace the other group by incurring another one-time utility loss r. If the other group undertakes the same action simultaneously, the loss to each group is zero, and there is a shift to common property. The loss r can be interpreted as the cost of forming a political organization or as the cost of altering the rules of a private property economy in order to induce redistribution.[10]

If a switch to a leader-follower regime occurs, the follower can never revolt against the leader and regain its access to any share of the capital stock.[11]

In Section 5 we introduce flow costs associated with maintaining a property rights regime over time, and we show that, incorporating these costs into the model, the results presented in this section remain qualitatively unchanged, except for those stated in Lemmas 3 and 9.

The problem we have just described is a preemption game in which each group has to choose its consumption policy as well as the switching times. There are three types of outcomes: a *nonswitching* one, in which neither group ever switches; a *matching* one, in which both groups always switch simultaneously; and a *leader-follower* one, in which the end result is that one group becomes the leader. The peculiarity of this game is that the payoffs to follow, lead, and match are functions of the economy's capital stock. Since this stock depends on the consumption policies, which in turn are functions of the switching rules, the problem is rather complicated. To keep matters simple and to obtain a closed-form solution we restrict the maximum number of regime switches to two.

$$\int_x^\infty \frac{\sigma}{\sigma-1} c_i(s)^{\frac{\sigma-1}{\sigma}} e^{-\delta(s-x)} ds - q(\tau_i, \tau_j) I(s, x+\tau_i) e^{-\delta(x-\tau_i)} \\ - r(T_i, T_j, \tau_i, \tau_j) I(s, x+\tau_i+T_i) e^{-\delta(x+\tau_i+T_i)}, \tag{5}$$

where σ is the reciprocal of the coefficient of relative risk aversion and also the elasticity of intertemporal substitution; $x+\tau_i$ and $x+\tau+T_i$ are the first and second switch dates of group i, I is an indicator function with $I(s, w) = 1$ if $s = w$ and zero otherwise, and

$$q(\tau_i, \tau_j) = \begin{cases} q \text{ if } \tau_i < \tau_j \\ 0 \text{ if } \tau_i = \tau_j \\ \infty \text{ if } \tau_i > \tau_j \end{cases} \quad r(T_i, T_j, \tau_i, \tau_j) = \begin{cases} r \text{ if } T_i < T_j \text{ and } \tau_i = \tau_j \\ 0 \text{ if } T_i = T_j \text{ and } \tau_i = \tau_j \\ \infty \text{ if } T_i > T_j \text{ or } \tau_i > \tau_j. \end{cases} \tag{6}$$

To capture the idea that the bigger the economy, the more costly it is for a rent-seeking group to capture it, we assume (Assumption 2) that the lower bounds on the losses r and q are increasing in the size of the economy.

Throughout the paper we use the following expressions:

$$z \equiv a(1-\sigma) + \delta\sigma, \qquad U(c) \equiv \frac{\sigma}{\sigma-1} c^{\frac{\sigma-1}{\sigma}}. \tag{7}$$

The next two assumptions finalize our description of the economy by listing the restrictions on parameters.

Assumption 1:

$$0 < a < 1, \qquad 0 < \sigma < \frac{a}{a-\delta} < 2. \tag{8}$$

This assumption includes the following restrictions: (i) $a \in (0, 1)$—that is, the rate of return on capital should be smaller than 100%; (ii) $z \equiv a(1-\sigma) + \delta\sigma > 0$, a necessary condition for the transversality condition under the leader-follower regime (note that if $\sigma = 1$, $z > 0$ is equivalent to the familiar condition $\delta > 0$); (iii) $\sigma < 2$, a necessary condition for the transversality condition under common property in the last phase of growth; (iv) $a > 2\delta$, a necessary condition for the aggregate capital stock to be increasing under common property (this is equivalent to the familiar condition of the representative agent model, $a > \delta$).

The following assumption sets lower bounds on the one-time losses r and q, ensuring that it will not be advantageous to become the leader just after a switch has taken place (see Lemmas 8 and 11).

Assumption 2: *The one-time losses r and q satisfy the following restrictions:*

$$q > U(k_0) z^{-\frac{1}{\sigma}} [1 - (2-\sigma)^{\frac{1}{\sigma}}], \qquad r > U(k^{**}) z^{-\frac{1}{\sigma}} [1 - (2-\sigma)^{\frac{1}{\sigma}}] \tag{9}$$

where k_0 is the initial aggregate capital stock and k^{**} is the level of aggregate capital at which a switch from common to private property occurs.

Since $z > 0$, $\sigma < 2$ and $U' > 0$, (9) implies that the lower bounds on r and q are increasing in k_0 and k^{**}, respectively. In other words, the larger the economy, the more costly it is for an interest group to establish a claim on it.

The state of this economy has four elements: the capital stocks of each group ($K = (k_i, k_j)$), the prevalent property rights regime (R), the number of regime switches that have taken place (N), and the time since the previous switch (x). Following Simon and Stinchombe (1989), we define a *node* as a realization of the state (K, R, N, t), where K can be any pair of nonnegative real numbers; R can be equal to p, c or l (private, common, and leader-follower, respectively); N can take the values 0, 1 or 2; and t can be any nonnegative real number.

We use Markov perfect equilibrium (MPE) as the solution concept. A strategy is Markovian if it depends solely on the payoff-relevant state variables. A pair of Markov strategies form a Markov perfect equilibrium if they are best responses to each other starting at any node in the game. In our model a strategy consists of a consumption policy and a switching rule describing the actions a group would take at every possible node. Markov strategies do not allow groups to precommit to specific switch dates, nor do they follow history-dependent strategies, such as trigger strategies. In other setups the simplicity of MPE has proven useful in reducing the number of equilibria. Therefore, it is appealing to use this equilibrium concept in order to think about long run growth and the evolution of institutions that support it. For a discussion of this solution concept see Maskin and Tirole (1994).

3.1. Summary of Results

In this section, we show that nonswitching equilibria, where common or private property prevails forever, exist if and only if $\sigma \leq 1$, and that switching equilibria exist for any σ in $(0, 2)$. Then we characterize the equilibrium paths along which there is a switch from common to private property followed by a switch back to common property. We show that there are two capital stock levels: k^{**} and k^* (with $k^{**} < k^*$) for which the following conditions hold:

(i) There is a switch from common to private property when k reaches k^{**}.

(ii) The switch to private property is followed by a reversion to common property when k reaches k^*.

(iii) For $k < k^{**}$, the growth rate is increasing. At k^{**} it jumps up, following a decreasing path thereafter until it converges to a constant as k reaches k^*, as shown in (38) and Figure 3.

The key to (i) and (ii) is that under both regimes as k goes up (1) the payoff of leading grows at a faster rate than the payoff waiting and (2) the loss that the leader has to incur is high relative to the initial stock of capital. Thus, initially no one will find it profitable to become the leader. However, as k increases, there will come a point at which it will become profitable for a group to become the leader. Knowing that the other group will switch at this point, a group will match the leader's action provided the payoff of matching is greater than the payoff of following. This will occur if the loss from a joint switch is sufficiently small, which is the case in our model.

Note that both groups end up worse off than if they had not switched a second time, from private to common property. However, the switch must occur, not because of a lack of coordination but because of the fact that leading becomes more profitable than waiting.

3.2. The Third Phase

In this section we construct Markov perfect equilibria of the continuation games starting at any node $(K, R, 2, t)$, for $R = p$, c and l. Since in the third phase two switches have already occurred and no more switches are allowed, only one of the three property rights regimes described above will prevail forever. We consider each regime in turn.

Under private property each group faces the standard representative agent problem of maximizing the first term in equation (5) subject to accumulation equation (1) and nonnegativity constraint (4). Under the leader-follower regime the leader owns the entire capital stock, while the follower has zero consumption. It follows that the leader faces the same problem as under private property, replacing accumulation equation (1) by (3). The solutions to these problems are given by the following lemma:

Lemma 1: *Starting at the nodes $(K, l, 2, t)$ or $(K, p, 2, t)$ the optimal consumption policies are given by*

$$c_p(k_h) = zk_h, \qquad c_l(k) = zk, \qquad c_f(k) = 0. \tag{10}$$

Under private property and the leader-follower regime the aggregate capital stock is

$$k(s) = k(t)e^{\sigma(a-\delta)(s-t)} \qquad \textit{for } s \geq t. \tag{11}$$

Proof. The proof is a special case of the proof of Lemma 2 and is also the same as that of the standard ak growth model with a representative agent (see Barro and Sala-i-Martin, 1995). Along the optimal path capital and consumption grow at rate $\sigma(a-\delta)$. The intuition behind this is that when the return to capital a is greater than the rate of time preference δ it pays to sacrifice current consumption in exchange for increased future consumption. The higher the elasticity of intertemporal substitution σ, the more profitable this substitution is in utility terms. ■

Substituting (10) and (11) into (5) we have that the payoffs of the continuation game starting at any node $(K, p, 2, t)$ or $(K, l, 2, t)$ are (we have defined $\alpha_h \equiv \frac{k_h}{k}$):

$$J_{p,h}(k) = \begin{cases} U(\alpha k)z^{-\frac{1}{\sigma}} & \text{if } \sigma \neq 1 \\ \frac{\log(\delta\alpha_b k)}{\delta} + \frac{\alpha-\delta}{\delta^2} & \text{if } \sigma \neq 1 \end{cases} \qquad J_l(k) = \begin{cases} U(k)z^{-\frac{1}{\sigma}} & \text{if } \sigma \neq 1 \\ \frac{\log(\delta k)}{\delta} + \frac{\alpha-\delta}{\delta^2} & \text{if } \sigma = 1 \end{cases} \tag{12}$$

Under common property both groups have access to the aggregate capital stock. The following lemma characterizes the equilibrium under this regime:

Lemma 2: *At any node $(K, c, 2, t)$ each group chooses a consumption policy in order to maximize the first term in (5) subject to accumulation equation (2), nonnegativity constraint (4), and subject to the strategy followed by the other group. The MPE of this differential game is given by the pair $\{c_c(i), c_c(k)\}$, where*

$$c_c(k) = \frac{z}{2-\sigma}k. \tag{13}$$

The path of the aggregate capital stock is

$$k_c(s) = k(t)\exp\left(\frac{\sigma[a-2\delta]}{2-\sigma}[s-t]\right) \qquad \text{for } s \geq t. \tag{14}$$

The proof of Lemma 2 is in the appendix. Here we present an heuristic argument. We refer to one group as h $(h = i, j)$ and to the other group as $-h$. As in the representative agent model (see Lemma 1) we have that (i) the consumption of each group is proportional to the capital to which it has access ($c_h = \beta_h k$, where β_h is an undetermined coefficient), and (ii) since preferences are isoelastic, it is necessary that $\dot{c}_h/c_h = \sigma[RoR_h - \delta]$. The rate of return perceived by group h (RoR_h) is simply the raw rate of return minus the share of aggregate capital that the other group, $-h$, appropriates. That is, $RoR_h = a - \beta_{-h}$. From (i) we have that $\dot{c}_h/c_h = \dot{k}/k = a - \beta_h - \beta_{-h}$, and from (ii) we have that $\dot{c}_h/c_h = \sigma[a - \beta_{-h} - \delta]$. Equalizing these two expressions we have that h's best response to β_{-h} is $\beta_h = z + (\sigma - 1)\beta_{-h}$ (recall that $z = a(1-\sigma) + \delta\sigma > 0$). That is, an increase in $-h$'s propensity to consume induces an increase in h's consumption if $\sigma > 1$(if the substitution effect dominates the income effect). An analogous argument establishes that $-h$'s best response is $\beta_{-h} = z + (\sigma - 1)\beta_h$. Lastly, the unique solution to the two equations is $\beta = \frac{z}{2-\sigma}$, as shown in (13).

Comparing (10) and (13) we can see that the aggregate marginal propensity to consume is greater under common property $(\frac{2z}{2-\sigma})$ than under private property (z). As a result the growth rate of aggregate capital falls from $\sigma(a-\delta)$ under private property to $\sigma(a-2\delta)$ under common property. This is the tragedy of the commons.

Substituting (13) and (14) into (5) we have that the value of the continuation game starting at any node $(K, c, 2, t)$ is

$$J_c(k) = U(k)\left[\frac{2-\sigma}{z}\right]^{\frac{1}{\sigma}}. \tag{15}$$

Since $z > 0$, it follows from (12) and (15) that $J_l(k) > J_c(k)$ and

$$\frac{\partial J_l(k)}{\partial k} - \frac{\partial J_c(k)}{\partial k} = [kz]^{-\frac{1}{\sigma}}[1 - (2-\sigma)^{\frac{1}{\sigma}}] = \begin{cases} > 0 & \text{if } \sigma > 1 \\ \leq 0 & \text{if } \sigma \leq 1. \end{cases} \tag{16}$$

3.3. The Second Phase

During the second phase, private property prevails and one switch has already occurred. Each group starts with a capital stock equal to $k^{**}/2$. As shown in Figure 1, at any node

$(K, p, 1, t)$ there are three possibilities: (i) one group destroys private access to the other's capital stock by incurring a loss r— it becomes the leader and the other group becomes the follower; (ii) both groups destroy private access to each other's capital, neither group incurs a loss, and there is a switch to common property; (iii) both groups wait and private property prevails. In case (i), since the leader will never find it optimal to share the aggregate capital stock with the follower and since the follower cannot revolt, the leader's payoff is $J_l(k) - r$ as given by (12). In case (ii) the payoff for both groups is $J_c(k)$ as given by (15). In case (iii) groups have the option of switching at a future date. Below, we derive the payoff associated with this strategy (see (26)).

Recall that t denotes time since the previous switch and that T is the waiting time before the next switch. At every node $(K, p, 1, t)$ group i solves the following problem:

Problem P: *Choose a consumption policy $\{c_i(s)\}_{s=t}^{t+T_i}$, a waiting time before the switch T_i, and a terminal capital $k_i(t+T_i)$ in order to maximize $\int_t^{t+T_i} U(c_i(s))e^{-\delta(s-t)}ds + e^{-\delta T_i} S_i(k(t+T_i))$, subject to accumulation equation (1), constraint (4), and j's strategy. The scrap value function is given by*

$$S_i(k_i(t+T_i)) = \begin{cases} J_l(k(t+T_i)) - r & \text{if} \quad T_i < T_j \\ J_c(k(t+T_i)) & \text{if} \quad T_i = T_j \\ 0 & \text{if} \quad T_i > T_j. \end{cases} \tag{17}$$

There are two types of equilibrium outcomes: nonswitching in which private property prevails forever, and switching in which there is a shift to common property. The type of equilibrium is determined by the value of σ, which is the inverse of the coefficient of relative risk aversion, and the elasticity of intertemporal substitution. Nonswitching outcomes exist only if $\sigma \leq 1$ (Lemma 3), while switching outcomes exist for any σ in $(0, 2)$. Proposition 1 characterizes the switching equilibria.

Lemma 3: *Private property forever is an equilibrium outcome if and only if $\sigma \leq 1$.*

Proof. If private property lasts forever, the payoff to each group is $J_{p,h}(k) = U(\alpha_h k)z^{-1/\sigma}$. For this to be an MPE, it is necessary that there are no profitable unilateral deviations along the equilibrium path. Since aggregate capital under private property is increasing (by (11)), the no-preemption condition is $J_l(k) - r < J_{p,h}(k)$ for all $k \geq k^{**}$, where k^{**} is the capital stock at the time of the switch to private property. For the no-preemption condition to hold it is necessary that (i) at the time of the switch the value of leading is smaller than the value of maintaining private property forever and that (ii) the value of maintaining private property forever increases at a faster rate than the value of leading. Condition (i) requires that $r > U(k^{**})z^{-\frac{1}{\sigma}}[1 - 2^{\frac{\sigma-1}{\sigma}}]$. It follows directly from the restriction on r (9) that this condition always holds. To determine when condition (2) holds we use (12): $\frac{\partial J_l(k)}{\partial k} - \frac{\partial J_p(k)}{\partial k} = [zk]^{-\frac{1}{\sigma}}[1 - (\alpha_h)^{\frac{\sigma-1}{\sigma}}]$, which is nonpositive if $\sigma \leq 1$, and positive if $\sigma > 1$. Thus, there exists a finite time at which it becomes profitable to deviate and become the leader if and only if $\sigma > 1$.[12] ■

Switching equilibria exist for all σ in $(0, 2)$ because the value of matching is greater than the value of following for all levels of k. The following proposition characterizes the MPEs in which there is a switch to common property:

Proposition 1: *Starting at any node* $(K, p, 1, t)$ *the pair of strategies* $\{\psi_1(k^*), \psi_1(k^*)\}$ *forms a Markov perfect equilibrium in which there is a shift to common property when aggregate capital reaches the level* k^**, if and only if*

$$k^* < \bar{k}^* = \begin{cases} U^{-1}\left(\frac{rz^{1/\sigma}}{1-(2-\sigma)^{1/\sigma}}\right) & \text{if } \sigma > 1 \\ \infty & \text{if } \sigma \leq 1. \end{cases} \tag{18}$$

The strategy $\psi_i(k^*)$ *is given by*

$$\psi_1(k^*) = \begin{cases} \text{switch if and only if } k \geq k^* \\ c_1(k, k^*) = \frac{z}{2-\sigma e^{-zT(k,k^*)}}k, \end{cases} \tag{19}$$

where the waiting time $T(k, k^*)$ *is defined by*

$$k^* = \frac{2-\sigma}{2e^{zT} - \sigma} k e^{aT}. \tag{20}$$

There are multiple equilibria indexed by the switching level of capital $k^* \in (k^{**}, \bar{k}^*]$.

Before we proceed to the proof of the above proposition, a few comments are in order. First, note that $T(k, k^*)$ is the waiting time to reach k^* starting with aggregate capital k, provided both groups use consumption policy $c_1(k, k^*)$. Since the right-hand side of (20) is strictly increasing in T (because $a > z$) it follows that $T(k, k^*)$ is unique, increasing in k^* and decreasing in k. Moreover, note that $T(k^*, k^*) = 0$.

Second, note that even when private property prevails, strategy ψ_1 instructs each group to follow a consumption policy that is a function of aggregate capital, not individual capital. Since there will be a switch to common property, the transversality condition implies that any consumption policy that is only a function of individual capital cannot belong to an MPE. The intuition behind this surprising property can be explained by expressing the consumption policy as $c_i(t) = [k_i(t) + e^{-aT}k_j(t+T)]z/[1-(\sigma-1)e^{-zT}]$.[13] The first bracketed term is group i's wealth: its own capital plus what it expects j's capital to be at the time of the switch, discounted from the time when capital becomes common property to the present at the rate a. The rest is the marginal propensity to consume. Thus, although private property prevails, each group behaves as if at a future date $t + T$ it will win a lottery and the interest rate will fall. As this date gets closer, the marginal propensity to consume increases until it reaches the level it will have after the switch to common property. It is through this channel, rather than through the production side, that our model generates the time-varying growth rates mentioned in the introduction.

We prove Proposition 1 by construction. First, for an exogenously given waiting time T, we obtain a pair of consumption policies that form an MPE. Second, we show that along the path generated by this pair of consumption policies aggregate capital reaches k^* at time $t + T$ if and only if T is set equal to $T(k, k^*)$ as defined in (20). Lastly, we show that it is optimal for each group to set its waiting time equal to $T(k, k^*)$, given that the other group is following strategy $\psi_1(k^*)$. To facilitate the exposition we summarize each of these building blocks in a lemma:

Lemma 4: *Starting at any node $(K, p, 1, t)$ consider a game in which each group solves Problem P for an exogenously given waiting time: $T_i = T_j = T$. An MPE for this game is given by the pair $\{c_1(s; T), c_1(s; T)\}$ with*

$$c_1(s; T) = \frac{ze^{\sigma(a-\delta)(s-t)}}{2 - \sigma e^{-zT}} k(t), \qquad s \in [t, t+T). \tag{21}$$

The resulting aggregate capital path is

$$k_1(s, T) = k(t)e^{a(s-t)} \frac{2e^{z(T+t-s)} - \sigma}{2e^{zT} - \sigma}, \qquad s \in [t, t+T). \tag{22}$$

The proof of Lemma 4 is in the appendix. Here we present an heuristic derivation. (i) Since private property prevails, the consumption policies must satisfy the following Euler condition:

$$c_h(s) = c_h(t)e^{\sigma(a-\delta)(s-t)} \qquad s \in [t, t+T]. \tag{23}$$

(ii) Since terminal capital is chosen optimally, consumption must be the same before and after the switch away from private property. Since the switch is to common property, we have from (13) that $c_i(t+T) = c_j(t+T) = \frac{zk(t+T)}{2-\sigma}$. To obtain $k(t+T)$ we substitute $c_i(s) = c_j(s) = c(s)$ as given be (23) in accumulation equation (1)

$$k(s) = e^{a(s-t)} \left[k(t) - \frac{2c(t)}{z} [1 - e^{-z(s-t)}] \right] \qquad s \in [t, t+T). \tag{24}$$

To derive initial consumption $c(t)$, set $s = t + T$ in (24) and plug it into the transversality condition $c(t+T) = \frac{zk(t+T)}{2-\sigma}$ to obtain $c(t+T) = \frac{z}{2-\sigma} e^{aT} [k(t) - \frac{2c(t)}{z} [1 - e^{-zT}]]$. Also, by setting $s = t + T$ in Euler condition (23) we have $c(t+T) = c(t)e^{\sigma(a-\delta)T}$. Equating these two expressions, solving for $c(t)$, and making use of the fact that $a - z = \sigma(a - \delta)$, it follows that initial consumption is $c(t) = zk(t)/[2 - \sigma e^{-zT}]$. Plugging this expression for initial consumption into (23) and (24) we obtain consumption policy (21) and the path of aggregate capital (22).

The following lemma characterizes the expression for $T(k, k^*)$ in Proposition 1:

Lemma 5: *If starting at node $(K, p, 1, t)$ both groups follow consumption policy (21), aggregate capital will reach the level k^* at time $t + T$ if and only if T is set equal to $T(k, k^*)$, as defined by (20).*

Proof. It follows from (22) that[14]

$$\frac{\partial k_1(s, T)}{\partial s} > 0, \qquad \frac{\partial k_1(s, T)}{\partial T} > 0, \qquad \frac{dk_1(t+T, T)}{dT} > 0. \tag{25}$$

Since $k_1(t+T, T)$ is strictly increasing in T, the equation $k_1(t+T, T) = k^*$ uniquely determines $T(k, k^*)$. Since $k_1(s, T)$ is strictly increasing in s, $t + T(k, k^*)$ is the first time that aggregate capital hits k^*. Hence, for each level of aggregate capital $k < k^*$ the unique waiting time before k hits k^*, along the path generated by (21), is $T(k, k^*)$. ■

Substituting consumption policy (21) into the valuation function specified in Problem P, we have that starting at node $(K, p, 1, t)$ the value of waiting a period of length $T(k, k^*)$ and then switching, given that the other group is following $\psi_1(k^*)$, is

$$W_1(k, k^*) = U\left(\frac{k}{2 - \sigma e^{-zT(k,k^*)}}\right)[1 - (\sigma - 1)e^{-zT(k,k^*)}]z^{-\frac{1}{\sigma}}. \tag{26}$$

this is the payoff obtained if, starting at node $(K, p, 1, t)$, both groups follow consumption policy $c_1(k, k^*)$ until $T(k, k^*) + t$, and then switch to common property and follow consumption policy (13) forever. Note that if we set $k = k^*$, so that $T = 0$, then $W_1(k, k^*)$ becomes equal to the payoff function associated with having common property forever (given by (15)). If we set $k^* = \infty$, so that $T = \infty$, then $W_1(k, k^*)$ becomes equal to the payoff function associated with having private property forever (given by (12)). Note also that since $W_1(k, k^*)$ is increasing in $T(k, k^*)$ and $T(k, k^*)$ is increasing in k^*, a path with a greater switching capital yields a higher payoff than a path with a smaller switching capital. For future reference we list these properties:

$$W_1(k^*, k^*) = J_c(k^*), \qquad W_1(k, k^{*'}) \geq W_1(k, k^*) \quad \forall k^{*'} > k^*. \tag{27}$$

The intuition is as follows. The higher k^*, the longer the period during which private property prevails. Since there is less pillaging under private property than under common property, consumption grows at a higher rate and for a longer period. Therefore, in the hypothetical case in which both groups could commit to not switching, both would choose to remain under private property forever if the alternative was a switch to common property.

The following lemma states that along the waiting path generated by $c(s, T)$ each group prefers waiting to switching, given that the other group will switch at $t + T(k, k^*)$.

Lemma 6: *At any node $(K, p, 1, t)$ there are no-preemption opportunities if and only if switching capital k^* is not greater than the upper bound $\bar{k}^*$ defined in $\psi_1(k^*)$.*

The proof of Lemma 6 is in the appendix. In order for one group not to have an incentive to deviate and become the leader, it is necessary that $W_1(k, k^*) > J_l(k) - r$ for all k less than the switching capital k^*. We use Figure 2 to illustrate when this condition will be satisfied. Figure 2 depicts the payoffs of group i for each level of aggregate capital for the case $\sigma > 1$. At each level of capital group i can either attack or stay put. If i attacks and the other group matches, there is a shift to common property and both groups get $J_c(k)$. This payoff is represented by the M-curve. If the other group does not match, i becomes the leader and its payoff is $J_l(k) - r$. This payoff is represented by the L-curve. If i stays put, while the other group switches, i becomes the follower and its payoff is zero. This payoff is represented by the horizontal axis. Lastly, if both groups stay put, i's payoff is $W_1(k, k^*)$.

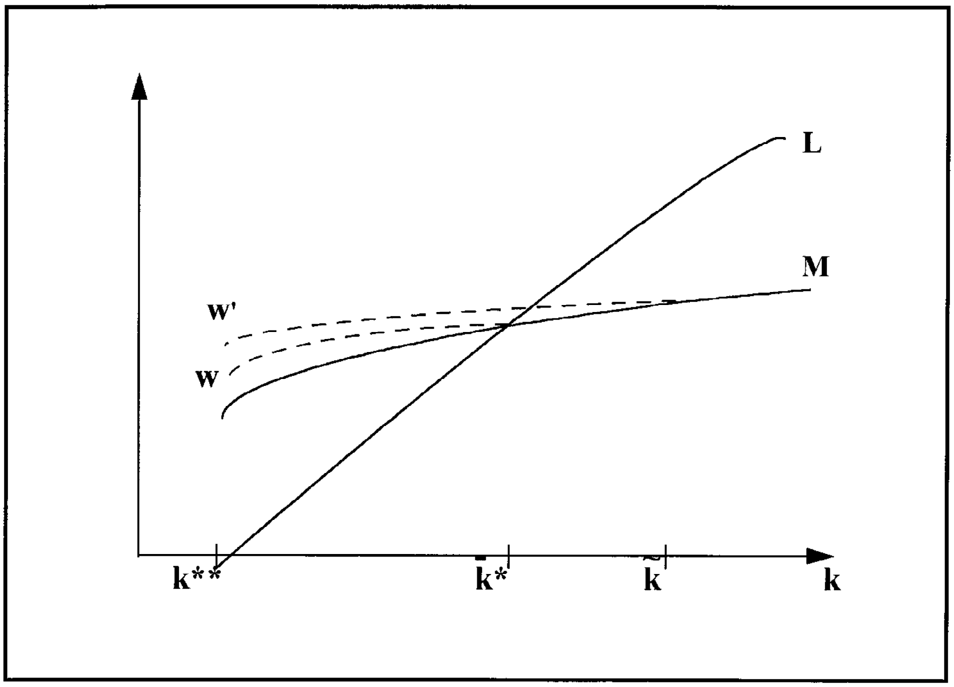

Figure 2.

The W-curve represents this payoff for the case in which switching capital k^* equals its upper bound $\bar{k}^*$. From (27) we know that the W- and M-curves coincide at $k = \bar{k}^*$ and that for lower k's the W-curve lies above the M-curve. Since the L-curve is steeper than the W-curve, the W-curve lies above the L-curve for $k^* \leq \bar{k}^*$. Thus, staying put is preferred to switching at any $k < k^*$. That is, it is not profitable to deviate and become the leader along the waiting path. When $k^* = \tilde{k} > \bar{k}^*$, the value of waiting is represented by the W'-curve. At $\tilde{k}$ the value of waiting equals that of matching, which is lower than the value of leading. Since the W'- and L-curves are continuous, it follows that there exists some interval ending in $\tilde{k}$ over which leading is preferred to waiting. Hence, the no-preemtion condition is violated for any switching capital k^* greater than $\bar{k}^*$.

Strategy $\psi_1(k^*)$ specifies that the waiting time before switching be set equal to $T(k, k^*)$. The following lemma states that this forms part of an MPE:

Lemma 7: *If one group is following strategy $\psi_i(k^*)$ and the switching level of capital k^* is no greater than the upper bound $\bar{k}^*$, then the other group will find it optimal to set its waiting time equal to $T(k, k^*)$ for any $k \leq k^*$.*

Proof. Suppose that i deviates unilaterally by setting $T_i^d \neq T(k, k^*)$ and denote by $k^d(s)$ the resulting aggregate capital stock path. Consider first the case in which $k^d(t + T_i^d) < k^*$.

Since group j is following strategy ψ_1, it would not attack at $t + T_i^d$ in this case. Thus, if i attacked at $t + T_i^d$, it would become the leader and its payoff would be $J_l(k^d(t + T_i^d)) - r$. If it did not attack, however, its payoff would be $W_1(k, k^*)$. Lemma 6 implies that i would not attack. Therefore, in this case, setting $T_i^d \neq T(k, k^*)$ would not be optimal for i. Second, consider the case $k^d(t + T_i^d) > k^*$. In this case there would exist a time $s < t + T_i^d$ such that $k^d(s) = k^*$. As a result group j would switch at time s. Thus if at time s group i stuck to its strategy, its payoff would be zero. If it attacked, however, it would get $J_c(k^*)$, a positive payoff. Again, in this case, setting $T_i^d \neq T(k, k^*)$ would not be optimal for i. Lastly, we will show that the case $k^d(t + T_i^d) = k^*$ is inconsistent with optimization by i. Since in this case j would also attack at $t + T_i^d$, transversality condition (44) implies that $c_i^d(t+T_i^d) = \frac{zk^*}{2-\sigma}$. Euler condition (23) then implies that $c_i^d(s, T_i^d) = \frac{zk^*}{2-\sigma} e^{\sigma(a-\delta)(s-T_i^d)}$ for s on $[t, t + T_i^d]$. Since the equilibrium policy satisfies $c_i(s, T(k, k^*)) = \frac{zk^*}{2-\sigma} e^{\sigma(a-\delta)(s-T(k,k^*))}$ for s on $[t, t + T(k, k^*)]$, it follows that if $T_i^d < T(k, k^*)$, then $c_i^d(s, T_i^d) > c_i(s, T(k, k^*))$ for s on $[t, t + T_i^d]$. Since j is following strategy $\psi_1(k^*)$, the fact that $c_i^d(s, T_i^d) > c_i(s, T(k, k^*))$ implies that aggregate capital grows more slowly under this deviation. Thus, $k_1(t + T_i^d) > k^d(t + T_i^d)$. Lemma 5 implies that $k_1(t + T_i^d) < k_1(t + T(k, k^*)) = k^*$. Combining the last two inequalities we have that $k^d(t + T_i^d) < k^*$, which contradicts the initial supposition that $k^d(t + T_i^d) = k^*$. If $T_i^d > T(k, k^*)$, using the same argument we would get $k^d(t + T_i^d) > k^*$, also a contradiction. Hence, we conclude that setting $T_i = T(k, k^*)$ is optimal for i. ■

The following lemma states that private property will prevail for a positive time period if the restriction on the switching cost r is satisfied:

Lemma 8: *The upper bound on switching capital $\bar{k}^*$ is greater than the aggregate capital stock at which the switch from common to private property takes place (k^{**}) if and only if the cost of switching from private to common property (r) satisfies (9).*

Proof. The upper bound on switching capital $\bar{k}^*$ is defined by $J_c(\bar{k}^*) = J_l(\bar{k}^*) - r$. Therefore, (16) implies that $\bar{k}^* > k^{**}$ if and only if $J_c(k^{**}) > J_l(k^{**}) - r$. Using payoff functions (12) and (15) we can rewrite this inequality as (9). ■

Finally, we combine Lemmas 4 to 8 to prove Proposition 1. For a given waiting time the pair of consumption policies $\{c_1(s, T), c_1(s, T)\}$, where $c_1(s, T)$ is given by (21), forms an MPE starting at any node $(K, p, 1, t)$ (Lemma 4). This pair induces an aggregate capital stock path that hits k^* at time $t + T(k, k^*)$ as defined in (20) (Lemma 5). If the switching level of aggregate capital k^* is set no greater than $\bar{k}^*$ and if one group is following $\psi_1(k^*)$, then the other group will find it optimal to set its waiting time equal to $T(k, k^*)$ for any $k < k^*$ (Lemma 7). Thus, an equilibrium consumption policy for the game in which each group solves Problem P is obtained by setting $T = T(k, k^*)$ in (21). This is precisely the consumption policy specified in ψ_1. Next, we consider the switching rule in ψ_1. Lemma 6 ensures that, for any $k < k^*$, neither group will find it profitable to preempt and become the leader. Conversely, for any $k \geq k^*$, both groups will switch because the value of matching is always greater than the value of following. Lastly, Lemma 8 states that private property will prevail for a positive amount of time if the switching cost r satisfies restriction (9).

3.4. The First Phase

During the first phase common property prevails, and no switches have yet occurred. As shown in Figure 1, at any node $(K, c, 0, t)$ there are three possibilities: (i) each group imposes private access on half of total capital and there is a switch to private property; (ii) there is a switch to the leader-follower regime—one group becomes the leader, attaining access to the entire capital stock, while the other group becomes the follower and has zero consumption; or (iii) both groups wait and common property prevails. In cases (i) and (iii) no switching costs are incurred; in case (ii) the leader incurs a loss q.

Recall that t indexes time since the beginning of the first phase and τ denotes the waiting period before the next switch. At every node $(K, c, 0, t)$, group i solves the following problem:

Problem C: *Choose a consumption policy* $\{c_i(s)\}_{s=t}^{t+r_i}$*, a waiting time* τ_i*, and a terminal aggregate capital stock* $k(t+\tau_i)$ *such that* $\int_t^{t+\tau_i} U(c_i(s))e^{-\delta(s-t)}ds + e^{-\delta\tau_i} S_i(k(t+\tau_i))$ *is maximized subject to accumulation equation (2), constraint (4), and group j's equilibrium strategy. The scrap value function is given by*

$$S_i(k(t+\tau_i)) = \begin{cases} J_l(k(t+\tau_i)) - q & \text{if } \tau_i < \tau_j \\ W_1(k(t+\tau_i), k^*) & \text{if } \tau_i = \tau_j \\ 0 & \text{if } \tau_i > \tau_j. \end{cases} \tag{28}$$

Since the leader will never find it optimal to share the aggregate capital stock with the follower and since the follower cannot revolt, the function $J_l(k)$ is given by (12). The value of matching $W_1(k(\tau_i), k^*)$ is given by the value of waiting during the second phase (26).

As in the second phase, there are two types of equilibrium outcomes: a nonswitching outcome, in which common property prevails forever, and a switching outcome, in which there is a shift to private property. The type of equilibrium is determined by σ (which is the inverse of the coefficient of relative risk aversion and the elasticity of intertemporal substitution). Nonswitching outcomes occur only if σ is not greater than one (Lemma 9). Switching equilibria exist for any value of σ on $(0, 2)$. They are characterized in Proposition 2.

Lemma 9: *Private property forever is an equilibrium outcome if and only if* $\sigma \leq 1$.

Proof. The proof is analogous to that of Lemma 3. If common property lasts forever, the payoff of each group is $J_c(k) = U(k)[\frac{2-\sigma}{z}]^{\frac{1}{\sigma}}$. Since aggregate capital under common property is increasing, it is necessary that $J_l(k) - q < J_c(k)$ for all $k \geq k_0$. For this no-preemption condition to hold it is necessary that (1) $q > J_l(k) - J_c(k) = U(k_0)z^{-\frac{1}{\sigma}}[1 - (2-\sigma)^{\frac{1}{\sigma}}]$, and (2) $0 \leq \frac{\partial J_l(k)}{\partial k} - \frac{\partial J_c(k)}{\partial k} = [zk]^{-\frac{1}{\sigma}}[1 - (2-\sigma)^{\frac{1}{\sigma}}]$. Restriction (9) implies that (i) is satisfied for any value of σ. Condition (ii) holds if and only if $\sigma \leq 1$. If $\sigma > 1$, there exists a finite time at which it becomes profitable to deviate and become the leader.

■

Proposition 2 characterizes the equilibria in which there is a switch to private property followed by a switch to common property:

Proposition 2: *Starting at any node $(K, c, 0, t)$ the pair of strategies $\{\psi_0(k^{**}), \psi_0(k^{**})\}$ forms an MPE in which there is a shift to private property when aggregate capital reaches $k^{**} > k_0$ if k^{**} satisfies (35). The strategy $\psi_0(k^{**})$ is given by*

$$\psi_0(k^{**}) = \begin{cases} \text{switch if and only if } k \geq k^{**} \\ c_0(k, k^{**}) = \frac{zk}{2-\sigma+zDe^{-z\tau(k,k^{**})}} \qquad D \equiv \frac{k^{**}}{x} + \frac{2-\sigma}{z}, \end{cases} \tag{29}$$

where the constant x is defined by (48), and waiting time $\tau(k, k^{**})$ is defined by

$$k^{**} = ke^{\frac{\sigma(a-2\delta)}{2-\sigma}\tau} \left[\frac{2-\sigma+zD}{2-\sigma+zDe^{-z\tau}}\right]^{\frac{2}{2-\sigma}}. \tag{30}$$

There exist multiple equilibria indexed by the switching level of aggregate capital k^{**}.

Note that $\tau(k, k^{**})$ is the waiting time before k^{**} is reached starting with aggregate capital k, provided both groups use consumption policy $c_0(k, k^{**})$. Since the right-hand side of (30) is strictly increasing in τ (because $a > 2\delta$, $\sigma < 2$ and $z > 0$), we have that $\tau(k, k^{**})$ is increasing in k^{**} and decreasing in k, and that $\tau(k^{**}, k^{**}) = 0$.

We prove Proposition 2 in two steps. First, we show that $c_0(k, k^{**})$ is a best response to itself given that the switching capital stock $k(t + \tau)$ is set equal to k^{**}. Second, we show that attacking when aggregate capital reaches k^{**} is optimal for one group given that the other group is following strategy ψ_0. Lemmas 10 to 12 summarize the building blocks of this proof.

Lemma 10: *Consider the differential game in which each group solves Problem C subject to the terminal condition $k(t + \tau) = k^{**}$. In this game, the pair $\{c_0(k, k^{**}), c_0(k, k^{**})\}$ (where $c_0(k, k^{**})$ is given by (29)) forms an MPE starting at any node $(K, c, 0, t)$. The resulting aggregate capital stock is*

$$k_0(s, k^{**}) = k(t)e^{\frac{\sigma(a-2\delta)}{2-\sigma}[s-t]} \left[\frac{2-\sigma+Dze^{-z(t+\tau(k,k^{**})-s)}}{2-\sigma+Dze^{-z(t+\tau(k,k^{**}))}}\right]^{\frac{2}{2-\sigma}} \qquad s \in [t, t+\tau(t, t^{**})], \tag{31}$$

where $\tau(k, k^{**})$ is defined by (30).

The proof of Lemma 10 is in the appendix. Here we present an heuristic argument. Since there is going to be a switch we consider nonstationary consumption policies. Let $c_h(s) = \gamma_h(s)k(s)$, where $\gamma_h(s)$ is an undetermined function. Using the same argument as the one we used to derive (13), we have that since the rate of return perceived by group h is $a - \gamma_{-h}(s)$, its Euler condition is $\dot{c}_h/c_h = \sigma[a - \gamma_{-h} - \delta]$. Since $c_h = \gamma_h k$, consumption must also satisfy the condition $\dot{c}_h/c_h = \dot{\gamma}_h/\gamma_h + \dot{k}/k = \dot{\gamma}_h/\gamma_h + a - \gamma_h - \gamma_{-h}$. Combining the two equations we have that the consumption rates must satisfy the following differential equations:

$$\frac{\dot{\gamma}_i}{\gamma_i} = \gamma_i + (1-\sigma)\gamma_j - z, \qquad \frac{\dot{\gamma}_j}{\gamma_j} = \gamma_j + (1-\sigma)\gamma_i - z. \tag{32}$$

Since both groups switch simultaneously when aggregate capital hits k^{**}, and each receives $k^{**}/2$ at that time, it follows that their terminal consumptions are equal: $c_i(t+\tau) = c_j(t+\tau)$. This implies that the two equations in (32) are simultaneously satisfied only if $\gamma_i(s) = \gamma_j(s)$ for all s on $[t, t+\tau]$. Thus, (32) becomes $\dot{\gamma}(s) = [2-\sigma]\gamma^2(s) + \gamma(s)$. The general solution to this differential equation is

$$\gamma(s) = z[2 - \sigma + Fze^{zs}]^{-1}, \tag{33}$$

where F is an arbitrary constant. To determine its value we set $s = t + \tau$ in (33) and use the terminal condition $c(t+\tau) = \gamma(t+\tau)k^{**}$. It follows that $F = [\frac{k^{**}}{x} + \frac{2-\sigma}{z}]e^{-z(t+\tau)}$, where x denotes the value of $c(t+\tau)$, which is defined by the transversality condition (48) in the appendix. The consumption policy in (29) is obtained by substituting F into (33) and setting $c(k) = \gamma(s)k$. To obtain the equation for the capital stock (31) we substitute $c_i(s) = c_j(s) = c_0(k, k^{**})$ in accumulation equation (2). The solution to this differential equation is in the appendix. Lastly, waiting time $\tau(k, k^{**})$ is obtained by setting $s = t+\tau$ in (31) and inverting the equation $k^{**} = k_0(t+\tau, k^{**})$ (this is (30)). Since the terminal capital stock is strictly increasing in τ, this equation uniquely determines $\tau(k, k^{**})$. Moreover, since $k_0(s, k^{**})$ is increasing in s (by (31)), $t+\tau$ is the first time that aggregate capital hits the level k^{**}.[15]

Substituting (29) and (31) in the valuation function specified in Problem C, it follows that at node $(K, c, 0, t)$ the value of remaining in the waiting path given that a switch to Private Property will take place at time $t + \tau(k, k^{**})$ is

$$\begin{aligned} W_0(k, k^{**}) \;=\; & e^{-\delta(t+\tau)}W_1(k^{**}, k^*) \\ & + \frac{U(zk)}{[2-\sigma+(D-2+\sigma)e^{-z(t+\tau)}]^{\frac{2(\sigma-1)}{\sigma(2-\sigma)}}} \\ & \times \int_t^{t+\tau} \frac{[2-\sigma+(D-2+\sigma)e^{-z(t+\tau-s)}]^{\frac{\sigma-1}{2-\sigma}}}{e^{\frac{z}{2-\sigma}s}} ds. \end{aligned} \tag{34}$$

Since there is no analytical solution for $W_0(k, k^{**})$, we cannot establish a relation between the value of waiting and the values of leading and matching as we did in (27) for the second phase. This will restrict the range of switching capital levels over which we can show analytically that the switching rule in ψ_0 is a best response to itself. The set of switching capitals (k^{**}) that we consider is the set of aggregate capital levels greater than k_0 that satisfy the following inequality:

$$J_l(k^{**}) - q \;<\; W_1(k^{**}, k^*)e^{-\delta\tau(k_0, k^{**})}, \tag{35}$$

where $J_l(k) - q$ is the payoff of leading (J_l is given by (12)) and $W_1(k^{**}, k^*)$ is the payoff of matching (W_1 is given by the payoff of waiting during the second phase (26)). The right-hand side of (35) is a lower bound on $W_0(k, k^{**})$ obtained by deleting the utility flows during the time period $(t, t+\tau(k, k^{**}))$ and by discounting as heavily as possible the value of the continuation game after the switch to private property.

The next lemma states that if the restriction on the one-time loss q is satisfied, common property will prevail for a positive time period:

Lemma 11: *If (9) is satisfied, then there exist some switching capital* (k^{**}) *greater than initial capital* k_0.

Proof. First, condition (35) is satisfied if $k^{**} = k_0$ because $W_1(k_0, k^*)e^{-\delta\tau(k_0,k_0)} = W_1(k_0, k^*) > W_1(k_0, k_0) = J_c(k_0) > J_l(k_0) - q$. The first equality holds because $\tau(k_0, k_0) = 0$ (by (30)). The first inequality and the second equality follow from (27). The last inequality is condition (9). Second, since both sides of (35) are continuous functions of k^{**}, there exist some switching capitals $k^{**} > k_0$ that satisfy (35). ■

Lemma 12 states that the switching rule specified in $\psi_0(k^{**})$ is a best response to itself:

Lemma 12: *Given that one group is following strategy* $\psi_0(k^{**})$ *and that* $k^{**} \leq \bar{k}^{**}$, *the other group will switch if and only if* $k \geq k^{**}$.

Proof. Suppose that group j is following $\psi_0(k^{**})$. In that case, if $k \geq k^{**}$ group j will attack. It follows that i will get W_1 by switching and zero by not attacking. Therefore, group i will attack if $k \geq k^{**}$. If $k < k^{**}$ group j will not attack and will follow consumption policy (29). It follows that i will not attack if and only if the following no-preemption condition is satisfied:

$$J_l(k) - q < W_0(k, k^{**}) \qquad \text{for all } k < k^{**}. \tag{36}$$

This condition is satisfied if k^{**} satisfies (35). To see why, note that if group i attacks at any $k < k^{**}$, it gets $J_l(k) - q$. However, by waiting until k reaches k^{**} group i can ensure for itself at least $W_1(k^{**}, k^*)e^{-\delta\tau(k,k^{**})} > W_1(k^{**}, k^*)e^{-\delta\tau(k_0,k^{**})} > J_l(k^{**}) - q > J_l(k) - q$ for any $k < k^{**}$. The first inequality holds because $\tau(k, k^{**})$ is decreasing in k (by (30)). The second inequality is condition (35). The last inequality follows from (12). ■

Combining Lemmas 10 to 12 proves Proposition 2. Given that both groups attack if and only if $k \geq k^{**}$, the pair of consumption policies $\{c_0(k(s), k^{**}), c_0(k(s), k^{**})\}$ (where $c_0(k(s), k^{**})$ is given by (29)) form an MPE starting at any node $(K, c, 0, t)$ (Lemma 10). If starting at node $(K, c, 0, t)$ both groups use consumption policy (29), aggregate capital hits k^{**} at time $t + \tau(k, k^{**})$, as defined in $\psi_0(k^{**})$ (Lemma 10). Given that one group is following strategy $\psi_0(k^{**})$, there are no preemption opportunities for the other group along the waiting path generated by (29) for any switching capital k^{**} that satisfies (35), and it is optimal to switch when k reaches k^{**} (Lemma 12). There exist switching capitals greater than k_0 if restriction (9) is satisfied (Lemma 11). Hence, starting at any node $(K, c, 0, t)$, if group j is using strategy $\psi_0(k^{**})$, the best response of group i is to follow $\psi_0(k^{**})$ as well.

4. Rise and Decline

In the previous section we characterized a path along which the economy switches from common to private property when it becomes rich enough ($k \geq k^{**}$) that it is worthwhile for each group to incur the one-time loss necessary to establish private property rights over the entire capital stock. However, private property does not last forever. Once this economy becomes very rich ($k \geq k^{*}$), it becomes profitable for each group to incur the one-time loss necessary to erode the private property rights of the other group, and a switch back to common property takes place. By combining Lemmas 1 and 2 and Propositions 1 and 2, it follows that the pair of strategies $\{\Psi(K, R, N), \Psi(K, R, N)\}$ forms an MPE that supports such a path:

$$\Psi(K, R, N) = \begin{cases} \psi_0(k^{**}) & \text{if } R = c \text{ and } N = 0 \\ \psi_1(k^{*}) & \text{if } R = p \text{ and } N = 1 \\ \psi_c & \text{if } R = c \text{ and } N = 2 \\ \psi_l & \text{if } R = l \text{ and } N = 1, 2, \end{cases} \tag{37}$$

where $\psi_0(k^{**})$ and $\psi_1(k^{*})$ are defined in Propositions 1 and 2, respectively, ψ_c instructs groups not to switch and to follow consumption policy (13), and ψ_l instructs groups not to switch and to follow consumption policy (10). Recall that K is the vector of individual capital stocks, R refers to the property rights regime, and N is the number of switches that have taken place. Also, note that since the economy starts at node $(K, c, 0, 0)$, it can never reach node $(K, p, 0, t)$, $(K, l, 0, t)$, $(K, c, 1, t)$, or $(K, p, 2, t)$.

Along this equilibrium path, the sequence of institutional changes generates a time-varying growth rate that is depicted in Figure 3 and is given by

$$\frac{\dot{k}(s)}{k(s)} = \begin{cases} a - \frac{2z}{2-\sigma+D(\tau)ze^{zs}} & \text{if } k < k^{**} \\ a - \frac{2z}{2-\sigma e^{-z(T+\tau-s)}} & \text{if } k^{**} \leq k < k^{*} \\ a - \frac{2z}{2-\sigma} & \text{if } k \geq k^{*}, \end{cases} \tag{38}$$

where τ is the time at which the switch from common to private property takes place, $T + \tau$ is the time at which the switch from private to common property takes place, $D(\tau)$ is a constant defined in Proposition 2, a is the marginal product of capital, σ is the elasticity of intertemporal substitution, and z is a constant defined in (7). As shown in Figure 3, the growth rate is increasing during the first phase when common property prevails. It jumps up when the shift to private property occurs (at k^{**}). Thereafter, the growth rate follows a decreasing path until the second switch back to common property occurs (at k^{*}). At this point the growth rate becomes constant.

The intuition behind this is the following. At the time of the switch from common to private property, each group experiences a sudden drop in its wealth because it loses access to half of aggregate capital. Moreover, the private rate of return increases because each group acquires exclusive access to its capital stock. Both of these effects reduce the marginal propensity to consume, causing the growth rate to jump up. Before this switch, even though common property prevails, the growth rate is increasing and higher than it would be in an economy in which common property prevails forever because in this case groups expect a

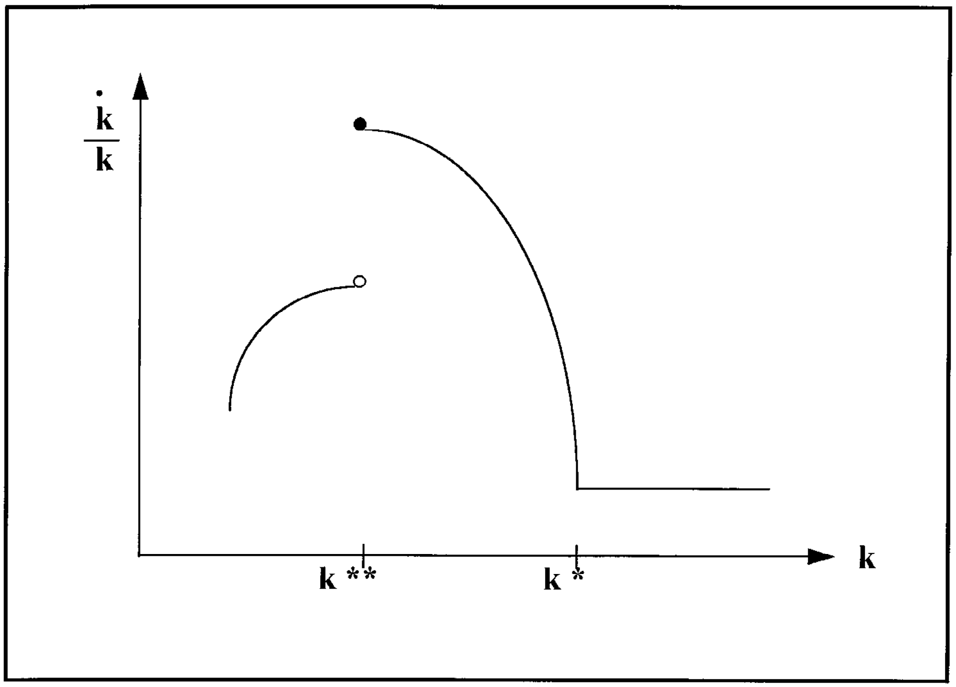

Figure 3.

switch to private property. As the anticipated switch gets closer, groups reduce their level of appropriation of common resources because they discount less the fact that each group's wealth will drop by half and the rate of return will increase. Therefore, the closer the switch date, the closer is groups' behavior that prevails under private property.

The declining growth rate after the switch to private property reflects anticipation of the next switch, back to common property. Each group behaves as if in the future it will "win a lottery" and the rate of return will fall. "Winning a lottery" means gaining access to the capital stock of the other group. The fall in the rate of return each group expects is due to the fact that the other group will appropriate a share of the now-common capital stock. The nearer the switch, the less this event is discounted. As a result, the growth rate follows a declining path.

Since production technology is linear in our model, the transition dynamics in (38) reflect only switches in property rights generated by interest-group competition. Interest-group competition affects growth in two ways: directly, by determining the relevant accumulation equation, and indirectly, through the savings rate. This channel should complement the other determinants of the growth rate identified in the literature, such as human capital accumulation, production externalities, and technological innovation.

4.1. Low-Growth Traps

The model can also rationalize the fact observed in postwar data that some poor countries have remained in low growth traps, while others with similar initial income per-capita have experienced spectacular growth. Consider two types of economies that differ only in their elasticities of intertemporal substitution σ: type A economies have $\sigma > 1$ and type B economies have $\sigma \leq 1$. Lemma 9 implies that in any type A economy, the value of leading increases at a faster rate than the value of maintaining common property forever. Therefore, in every type A economy a switch to private property must occur. This is not the case in a type B economy. Therefore, there exists a nonswitching equilibrium in which common property prevails forever. It follows that any type A economy will have an increasing growth rate and will shift away from common property, while some type B economies will never shift away from common property, and will always have the low growth rate $a - \frac{2z}{2-\sigma}$. Hence, type B economies have lower growth rates than any type A economy.

5. Costs to Maintain Institutions

So far we have only considered one-time losses associated with the creation and destruction of property rights. In principle, one could argue that the creation of institutions is not sufficient to maintain a given set of property rights but that there are additional costs of maintaining these property rights over time. In this section we discuss the implications of adding these costs to the model of Section 3. To keep the model tractable, we do not allow the choice of these flow costs to be strategic. We assume that under private property each group must spend a proportion d^P of its capital in order to enforce its property rights. Under common property, each group must spend a share d^c of aggregate capital to be able to engage in rent-seeking activities. Lastly, the leader must spend a share d^l of aggregate capital the keep the follower in check. It follows that the accumulation equations are now given by

$$\dot{k} = (a - 2d^c)k - c_i - c_j \text{ if } R = c; \qquad \dot{k} = (a - d^l)k - c_l \text{ if } R = l;$$
$$\dot{k}_h = (a - d^p)k_h - c_h \text{ if } R = p. \tag{39}$$

We restrict all costs incurred in the economy to levels lower than the productivity of capital:

$$d^l < a, \qquad 2d^p < a, \qquad 2d^c < a. \tag{40}$$

Note that all the equations for consumption, capital and value functions are obtained by simply replacing z by z^h in Section 3:

$$z^l = z - d^l(1-\sigma) > 0, \quad z^c = z - 2d^c(1-\sigma) > 0, \quad z^p = z - d^p(1-\sigma) > 0. \tag{41}$$

It follows that all the results in Section 3 remain qualitatively unchanged, except for those stated in Lemmas 3 and 9. These lemmas state that nonswitching equilibria exist only if the elasticity of intertemporal substitution σ is not greater than one. We find that if $d^l - d^c$

or $d^l - d^p$ are sufficiently large, there exist nonswitching equilibria for all values of σ. To see how the results stated in Lemma 3 change, note that

$$J_l(k) - J_p(k) = U(k)\chi, \qquad \text{where } \chi = (z^l)^{-\frac{1}{\sigma}} - \alpha_h^{-\frac{\sigma-1}{\sigma}}(z^p)^{-\frac{1}{\sigma}}.$$

Note that $U(k)$ is positive if $\sigma > 1$ and negative if $\sigma < 1$. Note also that the sign of χ is equal to the sign of $z(1-\alpha_h^{\sigma-1}) + (\sigma-1)(d^l - \alpha_h^{\sigma-1}d^p)$. Thus, when $d^l - d^p$ is sufficiently large, $\chi < 0$ if $\sigma > 1$ and $\chi > 1$, and if $\sigma < 1$. Consequently, when $d^l - d^p$ is sufficiently large, $J_l(k) - r < J_p(k)$ for all $k > 0$. Hence, private property forever is an equilibrium for any value of σ. The point is that when the costs the leader must incur to subjugate the follower are very large relative to the costs of defending private property, capital grows very slowly under the leader-follower regime. As a result, becoming the leader and owning the aggregate capital stock is not attractive.

To see how the results stated in Lemma 9 change, note that

$$J_l(k) - J_c(k) = U(k)\zeta, \qquad \text{where } \zeta = (z^l)^{-\frac{1}{\sigma}} - (2-\sigma)^{\frac{1}{\sigma}}(z^c)^{-\frac{1}{\sigma}}.$$

In this case the sign of ζ is equal to the sign of $(\sigma-1)(z + 2d^c + (2-\sigma)d^l)$. Thus, when $d^l - d^c$ is sufficiently large, $\zeta < 0$ if $\sigma > 1$, $\zeta > 0$ if $\sigma < 1$. Consequently, when $d^l - d^c$ is sufficiently large, $J_l(k) - q < J_c(k)$ for all $k > 0$. Hence, common property forever is an equilibrium for any value of σ.

The other point we want to make is that these costs are not sufficient to generate the cycles we analyzed in the previous section. To generate these cycles, one-time losses are required. To see this suppose that the one-time loss q is zero and that the economy is in the first phase, where common property prevails. As we saw above, either $J_l(k) > J_c(k)$ for all $k > 0$, in which case there is an immediate switch and common property never prevails, or $J_l(k) < J_c(k)$ for all $k > 0$ and common property can prevail forever.

6. Conclusion

In this article we have introduced endogenous institutional change into a neoclassical growth model. The model we presented allows property rights to shift back and forth between regimes of private and common property. These shifts are generated by the attempts of rent-seeking groups to secure access to a larger share of the aggregate capital stock. Regime switches do not occur frequently because they are costly for interest groups to bring about.

The model can rationalize simultaneously the low-growth traps in which several poor countries have remained, the swings of rise and decline experienced by leading economies throughout history, and the conditional convergence observed in postwar data.

We considered an economy in which common poverty prevails initially. We found that, depending on parameter values, the economy can get stuck in common property and suffer from low growth forever, or it can follow a cycle. In this cycle, a shift to private property occurs when the economy becomes rich enough that it is worthwhile for groups to incur the cost of creating institutions to defend private profits. Then, as the economy becomes very rich, rent-seeking becomes profitable, leading interest groups to erode these institutions and bringing the economy full circle back to common property. The growth rate is increasing in the first phase of this cycle, declining in the second phase, and constant at its minimum level in the third phase.

Appendix

Proof of Lemma 2

In what follows we refer to one group as h ($h = i, j$) and to the other group as $-h$. Any MPE of the game defined above is a solution to a pair of Hamiltonian problems (see Basar and Olsder, 1995). The Hamiltonian of group h is[16] $H_h = U(c_h)e^{-\delta(s-t)} + \lambda_h[ak - c_h - \hat{c}_{-h}(k)]$. To obtain the first-order conditions for group h we treat c_h as the control and $\hat{c}_{-h}$ as a function of the state (in fact, $\hat{c}_{-h}(k)$ is the equilibrium policy of $-h$). To derive an MPE we consider consumption policies of the form $c_h(k) = \beta_h k$, where β_h is an undetermined coefficient. It follows that a pair of equilibrium consumption policies must satisfy the following six first-order conditions ($h = i, j$):

$$\lambda_h(s) = c_h(s)^{-\frac{1}{\sigma}} e^{-\delta(s-t)}, \qquad \dot{\lambda}_h(s) = \lambda_h(s)[a - \beta_{-h}], \qquad \lim_{s\to\infty} \lambda_h(s)k(s) = 0. \tag{42}$$

The first two conditions imply that (i) $\dot{c}_h/c_h = \sigma[a - \beta_{-h} - \delta]$, $h = i, j$. Since $c_h = \beta_h k$, it follows that (ii) $\dot{c}_h/c_h = a - \beta_h - \beta_{-h}$, $h = i, j$. Equalizing (i) and (ii) we have that i's best response to $c_j = \beta_j k$ is $c_i = [z + (\sigma - 1)\beta_j]k$. Similarly, j's best response to $c_i = \beta_i k$ is $c_j = [z + (\sigma - 1)\beta_j]k$. The unique solution to these equations is (13). It follows that the costate variable is given by $\lambda_h(s) = [\frac{z}{2-\sigma}k(s)]^{-\frac{1}{\sigma}} e^{-\delta(s-t)} > 0$. Substituting (13) in accumulation equation (2) and solving the differential equation we obtain (14). Lastly, we verify the third condition in (42). Using (14) and the expression for $\lambda_h(s)$ it follows that $\lim_{s\to\infty} \lambda_h(s)k(s) = \lim_{s\to\infty}[\frac{2-\sigma}{z}]^{\frac{1}{\sigma}} k(t)^{\frac{\sigma-1}{\sigma}} e^{\frac{-z}{2-\sigma}(s-t)} = 0$. The last equality holds because $z > 0$ and $\sigma < 2$.

We have constructed a pair of consumption policies and an associated pair of costate variables $\{\lambda_i(s), \lambda_j(s)\}$ that satisfy the six necessary conditions in (42) and that generate a unique and continuously differentiable trajectory for aggregate capital. Since the instantaneous utility function is concave in (c, k) and the accumulation equation is linear in (c, k), the Hamiltonian is concave in (c, k). Therefore, the conditions in (42) are sufficient for a group to be maximizing its payoff given that the other group is following equilibrium consumption policy (13) (see Seierstad and Sydster, 1987). Hence, the pair $\{c_c(k), c_c(k)\}$ forms an MPE starting at any node $(K, c, 2, t)$. ■

Proof of Lemma 4

The Hamiltonian of group h is $H_h = U(c_h)e^{-\delta(s-t)} + \lambda_h[ak_h - c_h]$. The six first-order conditions for the two groups ($h = i, j$) are

$$\lambda_h(s) = c_h(s)^{-\frac{1}{\sigma}} e^{-\delta(s-t)}, \qquad \dot{\lambda}_h(s) = -\lambda_h(s)a, \tag{43}$$

$$\lambda_h(t+T_h) = e^{-\delta(t+T_h)} \frac{\partial S_h(k(t+T_h))}{\partial k_h}.$$

To find an equilibrium we must find a pair of consumption policies that satisfies the six first-order conditions. The first and second conditions imply that an optimal consumption policy must satisfy (23). The first and third conditions imply that terminal consumption must satisfy

$$c_h(t+T_h) = \begin{cases} zk(t+T_h) & \text{if } T_h < T_{-h} \\ 0 & \text{if } T_h > T_{-h} \\ \frac{zk(t+T_h)}{2-\sigma} & \text{if } T_h = T_{-h}. \end{cases} \tag{44}$$

Comparing (44) with (10) and (13), it follows that along an equilibrium path consumption must be the same before and after the switch away from private property, as mentioned in the text. Since $T_i = T_j = T$ by assumption, it follows that $c_i(t+T) = c_j(t+T) = \frac{zk(t+T)}{2-\sigma}$. The optimal consumption policies and the path of aggregate capital are derived in the text. To derive the costate variables we substitute (21) in the first condition in (43): $\lambda_h(s) = [\frac{ze^{\sigma(a-\delta)(s-t)}}{2-\sigma e^{-zT}} k(s)]^{\frac{1}{\sigma}} e^{-\delta(s-t)}$.

We have constructed a pair of consumption policies $\{c(s,T), c(s,T)\}$, given by (21), and a pair of costate variables $\{\lambda_i(s), \lambda_j(s)\}$ that satisfy the two sets of first-order conditions given by (43) and generate a unique and continuously differentiable trajectory for aggregate capital. Also, for a given waiting time, the second-order conditions are satisfied: the scrap value function $J_c(k(t+T))e^{-\delta(t+T)}$ is concave in k, and the Hamiltonian is concave in (k,c).[17] Therefore, the conditions in (43) are sufficient for a group to be maximizing its payoff, given that the other group is following equilibrium consumption policy (21) (see Seierstad and Sydster, 1987). Hence, for a given waiting time this consumption pair forms an MPE starting at any node $(K, p, 1, t)$.

Proof of Lemma 6

The no-preemption condition is

$$W_1(k,k^*) > J_l(k) - r \qquad \text{for all } k < k^*. \tag{45}$$

First, consider the case $\sigma > 1$. To show that (45) holds if $k^* \leq \bar{k}^*$, note that since $k < k^*$, (27) implies that (i) $W_1(k,k^*) > J_c(k)$. Note also that by construction $J_c(\bar{k}^*(r)) = J_l(\bar{k}^*(r)) - r$ (see (18)). Thus (16) implies that (ii) $J_c(k) > J_l(k) - r$ for all $k < \bar{k}^*(r)$. Combining inequalities (i) and (ii) it follows that the no-preemption condition holds for any $k < k^*$. To show that (45) holds only if $k^* \leq \bar{k}^*(r)$, suppose to the contrary that the switching capital is $\tilde{k} > \bar{k}^*(r)$. In this case, it follows that $J_l(\tilde{k}) - r > J_c(\tilde{k}) = W_1(\tilde{k},\tilde{k})$. Since $J_l(k)$ and $W_1(k,k^*)$ are continuous functions of k, there exists an $\epsilon > 0$ such that $J_l(\tilde{k}-\epsilon) - r > W_1(\tilde{k}-\epsilon, \tilde{k})$. Therefore, at $\tilde{k}-\epsilon$ each group will find it profitable to become the leader, violating (45).

Second we consider the case $\sigma \leq 1$. In this case $J_c(k) > J_1(k) - r$ for all $k > k^{**}$ because $J_c(k^{**}) > J_l(k^{**}) - r$ by (9), and because $J_c'(k) > J_l'(k)$ by (16). Therefore, (45) is satisfied for all $k^* > k^{**}$.[18]

Proof of Lemma 10

The Hamiltonian of group h is $H_h = U(c_h)e^{-\delta s} + \lambda_h[ak - c_h - \gamma_{-h}k]$. The six first-order conditions for the two groups $(h = i, j)$ are

$$\lambda_h(s) = c_h(s)^{-\frac{1}{\sigma}}e^{-\delta s}, \qquad \dot{\lambda}_h(s) = \lambda_h(s)[a - \gamma_{-h}(s)], \tag{46}$$

$$\begin{aligned} 0 &= U(c_h(t+\tau)) + e^{\delta\tau}\lambda_h(t+\tau)[ak^{**} - c_h(t+\tau) \\ &\quad - c_{-h}(t+\tau)] - \delta W_1(k^{**}, k^*). \end{aligned} \tag{47}$$

(47) is the transversality condition $H_h(t+\tau) + \frac{\partial(e^{-\delta\tau}S_h(k(t+\tau)))}{\partial\tau} = 0$. To obtain (47) note that since we are characterizing an equilibrium in which both groups attack simultaneously, the scrap value function is given by the value of waiting during the second phase (26) with $k = k^{**}$. Using (46) we can rewrite (47) as

$$\frac{2-\sigma}{\sigma-1}x^{\frac{\sigma-1}{\sigma}} + ak^{**}x^{-\frac{1}{\sigma}} = \delta W_1(k^{**}, k^*), \tag{48}$$

where x is $c(t+\tau)$. The four conditions in (46) imply that $\dot{c}_h/c_h = \sigma[a - \gamma_{-h} - \delta]$, $h = i, j$. The two conditions in (47) imply that $c_i(t+\tau) = c_j(t+\tau)$. The remaining steps for the derivation of the consumption policies are in the text.

In order to solve the differential equation for aggregate capital, we define $y(s) \equiv \log(k(s))$. It then follows from (2) that

$$\dot{y}(s) = a - \frac{2z}{2 - \sigma + zDe^{z(s-t-\tau)}}.$$

Thus $y(s) = as - \frac{2z}{2-\sigma}[s - \frac{\log(2-\sigma+zDe^{z(s-t-\tau)})}{2-\sigma}] + w$, where w is a constant. To obtain this constant, we use the initial condition $k(t) = k_t$. Lastly, to obtain (31) we set $k(s) = \exp(y(s))$.

We have derived a pair of consumption policies $\{c_0(k(s), k^{**}), c_0(k(s), k^{**})\}$ given by (29), that satisfies the six first-order conditions in (46) and (47). We have shown that starting at any node $(K, c, 0, t)$ this pair generates a unique, continuously differentiable trajectory for aggregate capital (31) that converges to k^{**} at time $t + \tau$. Lastly, since the scrap value function $W_1(k(t+\tau), k^*)e^{-\delta\tau}$ is concave in k and the Hamiltonian is concave in (k, c), each group is maximizing its payoff given that the other group is following strategy $y_0(k^{**})$ (see Seierstad and Sydster, 1987). Hence, this consumption pair forms an MPE starting at any node $(K, c, 0, t)$.

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Notes

1. If anything, a comparison between African and Southeast Asian countries suggests that at low levels of income per capita the growth rate is increasing in the level of income. See for instance Azariadis (1996), Azariadis and Drazen (1990), and Galor (1996).
2. This concept rules out trigger strategies and other history-dependent strategies.
3. The empirical estimates of the elasticity of intertemporal substitution are not applicable to our model because σ refers to the few powerful groups in a society, not to a representative citizen. In our model, there is never a switch to the leader-follower regime because the cost of matching is nil. Therefore the payoff of matching is greater than the payoff of following whenever it becomes profitable to become the leader.
4. In dynamic models like the one considered here, there exist switching equilibria if preferences or production are not homothetic. In this article we maintain homotheticity but introduce one-time losses associated with switching regimes.
5. For preemption games see Fudenberg and Tirole (1985), Hendricks and Wilson (1987), and Reinganum (1981). For differential games in common access economies (and fisheries) see Benhabib and Radner (1992), Haurie and Pohjola (1987), Lancaster (1973), and Tornell and Velasco (1992).
6. Common sense would indicate that French and Spanish princes should have implemented policies that would promote growth, since they were the owners of the entire capital stock. Evidence of monopoly granting and other favoritism clearly shows this was not the case. Moreover, princes were always in danger of losing their position to a rival. Paul of Russia, for example, lost the crown to his German wife, Catherine the Great.
7. In addition to these stories of "decline from within," there are "barbarians at the gate" theories about why economies collapse. In the case of Rome, economic decline has been attributed to the military power of the Barbarians. In the Dutch case, the British destroyed Dutch commercial supremacy. In the case of the Italian cities, the "barbarians at the gate" were the conquest of Constantinople by the Turks and the discovery of America.
8. In the context of labor conflicts, ak is the firm's revenue, c_i is interpreted as the wage bill, and c_j is interpreted as the amount of profits not reinvested. In a macroeconomic context, ak represents the government's net assets, and the c's are interpreted as transfers to government agencies or rent-seeking groups.
9. The following paragraph quoted in North (1990, p. 113), is illustrative: "The admission of the right of parliament to legislate, to enquire into abuses, and to share in the guidance of national policy, was practically purchased by the money granted to Edward I and Edward III" (taken from Stubbs, 1896, *The Constitutional History of England*, Vol. II).
10. For our results it is not necessary to set the costs of matching equal to zero. In the presence of positive costs of matching there would exist a path along which there would be a shift from common to private property and back to common property, if during each phase the cost of matching were sufficiently smaller than the cost of leading, so that in each phase the value of matching was greater than the value of following when the value of leading became greater than the value of waiting. The condition that the cost of matching during each phase is zero simplifies the analysis because it ensures that the payoff of matching is always greater than the payoff of following.
11. This specification rules out paths along which switches away from the leader-follower regime occur. This simplifies the computation of the payoffs associated with leading and following. Since our results depend on the value of leading becoming greater that the value of waiting, eliminating this assumption would not alter the results.
12. The intuition for why the sign of $\frac{\partial J_l(k)}{\partial k} - \frac{\partial J_p(k)}{\partial k}$ depends on the size of σ rests on a simple static argument. Aggregate capital is the same in both regimes. Thus, from (10) we know that consumption under private property is a constant fraction of the leader's consumption ($c_{p,h}(t) = \alpha_h c_l(t)$). It follows that the relation between instantaneous utilities is $U(c_{p,h}(t)) = (\alpha_h)^{\frac{\sigma-1}{\sigma}} U(c_l(t))$. Hence, the difference in marginal instantaneous utilities is $U'(c_l(t))[1 - (\alpha_h)^{\frac{\sigma-1}{\sigma}}]$. Since $J(k) = \int_t^\infty U(c(k(s)))e^{\delta(t-s)}ds$, we have that $\frac{\partial J_l(k)}{\partial k} - \frac{\partial J_p(k)}{\partial k} = \frac{\partial J_l(k)}{\partial k}[1 - (\alpha_h)^{\frac{\sigma-1}{\sigma}}]$.
13. To derive this expression note first that by substituting (23) in (1) it follows that i's capital stock at time $t + T$ is $k_i(t + T) = e^{aT}[k_i(t) - c_i(t)[1 - e^{-zT}]/z]$. Second, note that since there will be a switch to common property, (23) and (44) imply that $c_i(t) = e^{-\sigma(a-\delta)T}[k_i(t + T) + k_j(t + T)]z/(2 - \sigma)$. The expression in the text is obtained by substituting the RHS of $k_i(t + T)$ into $c_i(t)$ and rearranging terms.

14. From (7), (8) and (22) it follows that

$$\frac{\partial \log(k_1(s,T))}{\partial s} = a - \frac{2z}{2-\sigma e^{-z(T-s)}} \geq a - \frac{2z}{2-\sigma} = \frac{\sigma[a-2\delta]}{2-\sigma} > 0$$
$$\frac{\partial \log(k_1(s,T))}{\partial T} = \frac{2z}{2-\sigma e^{-z(T-s)}} - \frac{2z}{2-\sigma e^{-zT}} \geq 0.$$

The third derivative in (25) follows directly from this.

15. The signs of the derivatives follow from the restrictions on parameters: $a > 2\delta$, $0 < \sigma < 2$ and $z > 0$.
16. We disregard constraint (4). It turns out that it is not binding in equilibrium.
17. Since the instantaneous utility function is concave in c and the accumulation equation is linear in k, the Hamiltonian of each group is concave in (c, k).
18. Note that for $\sigma > 1$ we could have proved this Lemma using the fact that $\frac{\partial J_I(k)}{\partial k} > \frac{\partial W_1(k,k^*)}{\partial k}$. However, the opposite inequality cannot be established analytically for the case $\sigma < 1$. To see this, note that $\frac{dW_1}{dk} = \frac{\partial W_1}{\partial T} + \frac{\partial W_1}{\partial T}\frac{dT}{dk}$. After some algebra, we obtain

$$\frac{\partial W_1}{\partial T} = U(zk)[\sigma-1][1-e^{-zT}][2-\sigma e^{-zT}]^{\frac{1-2\sigma}{\sigma}} e^{-zT} > 0$$
$$\frac{\partial W_1}{\partial k} = [zk]^{-\frac{1}{\sigma}}[1-(\sigma-1)e^{-zT}][2-\sigma e^{-zT}]^{\frac{1-\sigma}{\sigma}} < [zk]^{-\frac{1}{\sigma}}.$$

First, to establish the inequality in the second equation, note that $E(T) \equiv [1-(\sigma-1)e^{-zT}][2-\sigma e^{-zT}]^{\frac{1-\sigma}{\sigma}} < 1$. This is because $E(0) = [2-\sigma]^{1/\sigma} < 1$, $E(\infty) = 2^{(1-\sigma)/\sigma} < 1$ and $\partial E/\partial T = z[\sigma-1]e^{-zT}[1-e^{-2T}][2-\sigma e^{-2T}]^{-\frac{1}{\sigma}} > 0$. Second, (20) implies that $dT/dk < 0$. Therefore, the two equations above imply that $\frac{dW_1}{dk} < [zk]^{-\frac{1}{\sigma}}$. Lastly, the result follows from the fact that $\frac{\partial J_I}{\partial k} = [zk]^{-\frac{1}{\sigma}}$. If $\sigma < 1$, we cannot establish analytically the sign of $\frac{dW_1}{dk} - \frac{\partial J_I}{\partial k}$, because $\frac{dW_1}{dk} > \frac{\partial J_I}{\partial k}$, but $\frac{dW_1}{dT} > 0$ and $\frac{dT}{dk} < 0$.

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